On the computation of limsups

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Motivations

Automatic asymptotics.

- Automatic expansion of solutions to very general functional equations.

- Generalized transseries expansions. Example:

$$e^{e^{e^x}+\log^{-1} x e^{e^x}+\log^{-2} x e^{e^x}+\cdots} + e^{e^{e^{\sqrt{x}}}+\log^{-1} x e^{e^{\sqrt{x}}}+\cdots} + \cdots.$$

Oscillatory behaviour

- Classically, transseries expansions are limited to strongly monotonic behaviour. Such transseries form a totally ordered field.

- We will make a first step towards the treatment of functions with explicit or hidden oscillatory behaviour.

Computation of limsups

Sign computations are important. Example:

$$\text{Expand} \quad e^{e^{\psi x}} \quad (x \to \infty).$$

And when $\psi = \psi(x)$ oscillates? Example:

$$\psi(x) = \frac{2 \sin x^2 - \sin(x^3/(x-1))}{3 + \sin e x^2 - \sin(e x^2 + 1)}.$$

$\longrightarrow$ How to compute $\liminf_{x \to \infty} \psi(x)$ and $\limsup_{x \to \infty} \psi(x)$?

Outline

I. Expansion of exp-log functions

II. A density theorem on the torus T^n

III. The algorithm

I. Expansion of exp-log functions

Definition. *An exp-log function is a function built up from $\mathbb{Q}$ and x, by the field operations, exponentiation and logarithm.*

Asymptotic scales ($x \to \infty$)

Asymptotic scale always generated by a normal basis B:

- $B = \{б_1, \cdots, б_n\}$ is a set of positive infinitesimals.
- $\log б_1 = o(\log б_2), \cdots, \log б_{n-1} = o(\log б_n)$.
- $б_1 = (\log \overset{l \text{ times}}{\cdots} \log x)^{-1}$.
- $\log б_i$ admits an expansion w.r.t. $б_1, \cdots, б_{i-1}$, for all $i > 1$.

Example: $B = \{\log^{-1} x, x^{-1}, x^{-x}, e^{-x^2/(\log x+1)}\}$.
Scale generated by B: $\log^{-\alpha x} x^{-\beta} x^{-\gamma x} e^{-\delta x^2/(\log x+1)}$.

Conjecture. (Schanuel) *If $\alpha_1, \cdots, \alpha_n$ are $\mathbb{Q}$-linearly independent complex numbers, then the transcendence degree of $\mathbb{Q}[\alpha_1, \cdots, \alpha_n, e^{\alpha_1}, \cdots, e^{\alpha_n}]$ over $\mathbb{Q}$ is at least n.*

Theorem. (Shackell, Richardson, VDH) *Assume that Schanuel's conjecture holds. Then there exists an algorithm to compute the expansion at infinity of any exp-log function w.r.t. a normal basis B.*

Example 1

$$f(x) = \frac{1}{(1-x^{-1})(1-e^{-x})} - \frac{1}{1-x^{-1}}.$$

The algorithm computes $B = \{x^{-1}, e^{-x}\}$, and returns the expansion

$$\begin{aligned} f(x) \approx \quad & \frac{1}{e^x} + \frac{1}{xe^x} + \frac{1}{x^2e^x} + \cdots + \\ & \frac{1}{e^{2x}} + \frac{1}{xe^{2x}} + \frac{1}{x^2e^{2x}} + \cdots + \\ & \vdots \end{aligned}$$

Example 2

$$f(x) = \log\log(xe^{xe^x} + 1) - \exp\exp(\log\log x + \frac{1}{x}).$$

The algorithm yields

$$B = \{\log^{-1}\log x, \log^{-1} x, x^{-1}, e^{-x}, e^{-xe^x}\},$$

with respect to which we can expand

$$\begin{aligned} f = \quad & \text{б}_3^{-1} + \text{б}_2^{-1} + \log[1 + \text{б}_3\text{б}_4[\text{б}_2^{-1} + \log(1 + \text{б}_3\text{б}_5)]] - \\ & \text{б}_3^{-1}\exp[\text{б}_2^{-1}\exp\text{б}_3 - \text{б}_2^{-1}]. \end{aligned}$$

For instance, we obtain the equivalent

$$f \sim -\frac{1}{2}\text{б}_2^{-2}\text{б}_3 = -\frac{\log^2 x}{2x}.$$

II. A density theorem on the torus

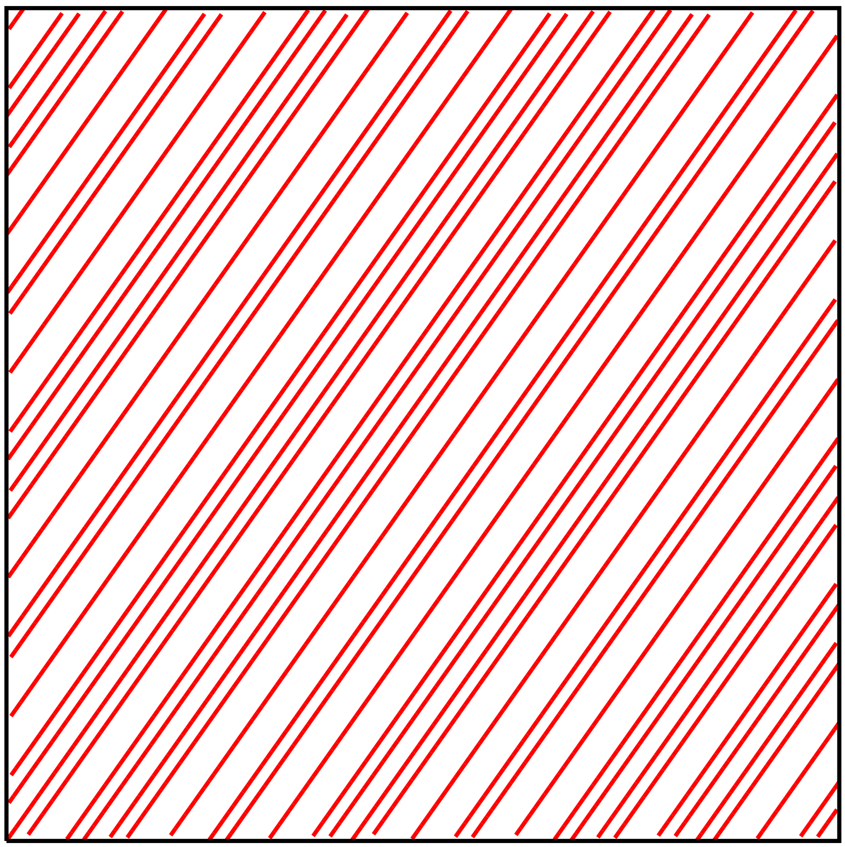


Image of $t \mapsto \overline{(t, \sqrt{2}t)} \in T^2 = \mathbb{R}^2/\mathbb{Z}^2$ for $t \in [0, 10]$.

Theorem. **(Kronecker)** *Let $\lambda_1, \cdots, \lambda_n$ be $\mathbb{Q}$-linearly independent real numbers. Then $\overline{(\lambda_1, \cdots, \lambda_n)\mathbb{R}}$ is dense on the n-dimensional torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$.*

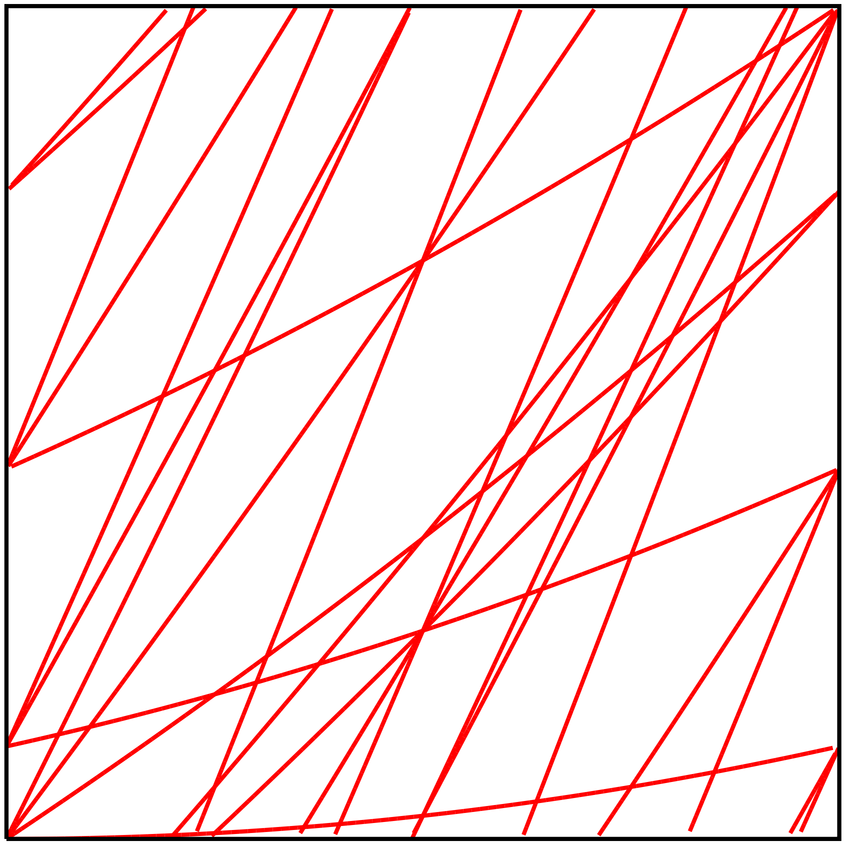


Image of $t \mapsto \overline{(3t, t^2)} \in T^2$ for $t \in [0, 4]$.

Theorem. *Assume that*

- $1 \prec\!\!\prec f_1 \prec\!\!\prec \cdots \prec\!\!\prec f_p$ *infinitely large exp-log functions.*
- $\lambda_{i,j} > 0$ *($1 \leqslant j \leqslant n_i$), such that* $\lambda_{i,1}, \cdots, \lambda_{i,n_i}$ *are* $\mathbb{Q}$*-linearly independent for each* i*.*
- $g(x) = (f_1(\lambda_{1,1}x), \cdots, f_1(\lambda_{1,n_1}x), \cdots, f_p(\lambda_{p,1}x), \cdots, f_p(\lambda_{p,n_p}x))$, *for* x *large.*

Then $\overline{\mathrm{im}\ g}$ *is dense on* T^n*, where* $n = n_1 + \cdots + n_p$*.*

Theorem. **(Bohr, Sierpiński, Weyl)** *Let* $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$ *be* $\mathbb{Q}$*-linearly independent numbers. Let*

$$X = [a_1, b_1[\times \cdots \times [a_n, b_n[\subseteq T^n$$

be an n*-dimensional block on* T^n*. Let*

$$\rho(I, X) = \frac{\mu(\{x \in I | \boldsymbol{\lambda} x \in X\})}{\mu(I)},$$

for all intervals I *of* $\mathbb{R}$*, where* μ *denotes the Lebesque mesure. Then*

$$\lim_{\mu(I) \to \infty} \rho(I, X) = \mu(X),$$

uniformly in I*.*

Theorem. *Let* $f_1, \cdots, f_p$ *and* g *be as before. Let*

$$X = [a_1, b_1[\times \cdots \times [a_n, b_n[\subseteq T^n$$

be an n*-dimensional block. Let*

$$\rho_{f,g}(I, X) = \frac{\mu(\{x \in I | g(f_1^{inv}(x)) \in X\})}{\mu(I)},$$

for all intervals I *of* $\mathbb{R}$ *(sufficiently close to infinity). Then*

$$\lim_{\mu(I) \to \infty} \rho_{f,g}(I, X) = \mu(X),$$

uniformly, for intervals sufficiently close to infinity.

III. The algorithm

Lemma. *Let* $1 \lll f_1 \lll \cdots \lll f_p$ *be exp-log functions at infinity. Let* $\lambda_{i,j} > 0$ *(*$1 \leqslant j \leqslant n_i$*) be such that* $\lambda_{i,1}, \cdots, \lambda_{i,n_i}$ *are* $\mathbb{Q}$*-linearly independent for each* i*. Denote* $U = \{x + \sqrt{-1}\, y \in \mathbb{C} | x^2 + y^2 = 1\}$ *and* $n = n_1 + \cdots + n_p$*. Let* φ *be a continuous function from* U^n *into* $\mathbb{R}$ *and let*

$$\psi(x) = \varphi(e^{\sqrt{-1}\, \lambda_{1,1} f_1(x)}, \cdots, e^{\sqrt{-1}\, \lambda_{p,n_p} f_p(x)}).$$

Then

$$\limsup_{x \to \infty} \psi(x) = \sup_{\boldsymbol{x} \in U^n} \varphi(\boldsymbol{x}).$$

Theorem. *Let $F_1, \cdots, F_q$ be exp-log functions at infinity. Let $\varphi : U^q \to \mathbb{R}$ a real algebraic function, where we consider U^q as a real algebraic variety. Assume that we have an oracle to test the $\mathbb{Q}$-linear dependence of exp-log constants. Then there exists an algorithm to compute the limsup of*

$$\psi(x) = \varphi(e^{\sqrt{-1}\, F_1(x)}, \cdots, e^{\sqrt{-1}\, F_q(x)}).$$

Idea. Reduce to the case of the lemma modulo linear combinations of the F_i's, using the rule $e^{a+b} = e^a e^b$ to rewrite φ.

Step 1. Expand $F_1, \cdots, F_q$ and order $F_1 \preccurlyeq\!\!\!\!\prec \cdots \preccurlyeq\!\!\!\!\prec F_q$.

Step 2. Reduce to the case when $F_i \asymp F_j \Rightarrow F_i = \lambda F_j$.

Step 3. Eliminate bounded F_i's (these tend to constants).

Step 4. Compute constants $\lambda_{i,j}$ and $1 \prec\!\!\!\prec f_1 \prec\!\!\!\prec \cdots \prec\!\!\!\prec f_p$, such that each F_l has the form $F_l = \lambda_{i,j} f_i$.

Step 5. Reduce to the case when $\lambda_{i,1} \cdots \lambda_{i,n_i}$ are $\mathbb{Q}$-linearly independent for each i.

Step 6. Apply lemma.

Example

$$\psi(x) = \frac{2\sin x^2 - \sin(x^3/(x-1))}{3 + \sin ex^2 - \sin(ex^2+1)}.$$

Step 1. Expansion:

$$\begin{aligned} x^2 &= x^2; \\ x^3/(x-1) &= x^2 + x + \cdots; \\ ex^2 &= ex^2; \\ ex^2+1 &= ex^2+1. \end{aligned}$$

Step 2. Make F_i's homothetic: rewrite

$$x^3/(x-1) = x^2 + x^2/(x-1)$$

and

$$e^{\sqrt{-1}\, x^3/(x-1)} = e^{\sqrt{-1}\, x^2} e^{\sqrt{-1}\, x^2/(x-1)},$$

which corresponds to the rewriting

$$\sin\frac{x^3}{x-1} = \sin x^2 \cos\frac{x^2}{x-1} + \sin\frac{x^2}{x-1}\cos x^2,$$

if we consider real and imaginary parts.

Also rewrite

$$ex^2 + 1 = (ex^2) + (1)$$

and

$$e^{\sqrt{-1}\,(ex^2+1)} = e^{\sqrt{-1}\, ex^2} e^{\sqrt{-1}},$$

which corresponds to the rewriting

$$\sin(ex^2+1) = \sin ex^2 \cos 1 + \sin 1 \cos ex^2.$$

Step 5. Eliminate $\mathbb{Q}$-linear dependencies: nothing to be done.

Step 6. At this stage, we have

$$\varphi(a, \hat{a}, b, \hat{b}, c, \hat{c}) = \frac{2a - a\hat{c} - c\hat{a}}{3 + b - b\cos 1 - \hat{b}\sin 1}$$

with

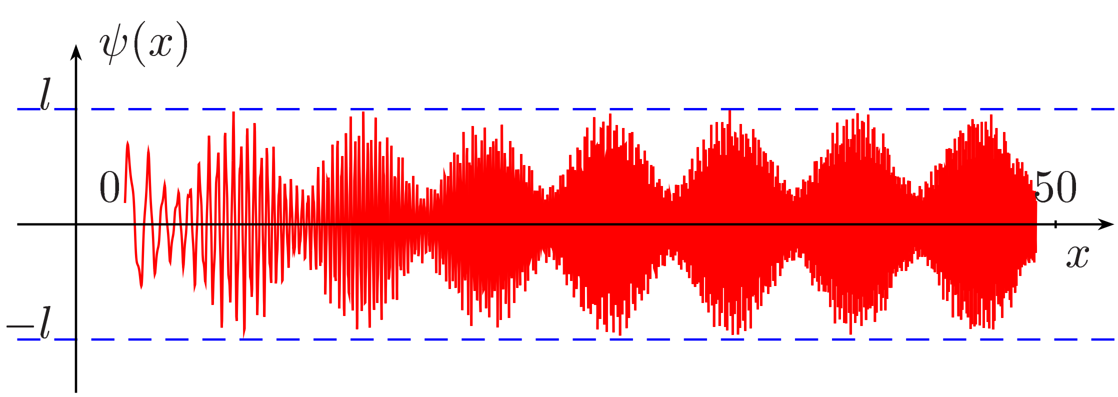
$$\begin{aligned} a &= \sin x^2 \quad , \quad \hat{a} = \cos x^2; \\ b &= \sin ex^2 \quad , \quad \hat{b} = \cos ex^2; \\ c &= \sin(x^2(x-1)), \quad \hat{c} = \cos(x^2(x-1)). \end{aligned}$$

The maximum of φ on U^3 is attained for

$$a = 1, \hat{a} = 0, b = -1/2, \hat{b} = \sqrt{3}/2, c = 0, \hat{c} = -1.$$

Hence

$$\begin{aligned} \limsup_{x\to\infty} \psi(x) &= \frac{6}{5 + \cos 1 - \sqrt{3}\ \sin 1} = l. \\ \liminf_{x\to\infty} \psi(x) &= \frac{-6}{5 + \cos 1 - \sqrt{3}\ \sin 1} = -l. \end{aligned}$$



Conclusion

Main ideas

– Oscillating components $\longrightarrow$ parameters.

– Density theorems $\longrightarrow$ constraints satisfied by parameters.

Generalizations

– Exp-log functions $\longrightarrow$ any class of strongly monotonic functions with automatic expansion algorithm.

– Algebraic functions $\longrightarrow$ any class of functions with effective maximum computation.

– Complete asymptotic expansions.

Complete expansions

Oscillating components $\longrightarrow$ parameters

$$f(x) = \exp\exp(x \sin x)$$

Leads to three cases:

$$f(x) = \begin{cases} 1 + e^{x \sin x} + \cdots & (\sin x < 0, \sin x \not\succ x^{-1}); \\ e^{e^{x \sin x}} & (\sin x \preccurlyeq x^{-1}); \\ e^{e^{x \sin x}} & (\sin x > 0, \sin x \not\succ x^{-1}). \end{cases}$$

Degenerate case

No constraint checking "possible" for parameters:

$$3 - \sin x - \sin x^2 - \sin x^3 \geqslant_\infty \frac{1}{e^x}?$$

Linked to Diophantine approximation:

$$2 - \sin x - \sin e x^2 \geqslant_\infty \frac{1}{\Gamma(x+2)}.$$

$\longrightarrow$ intuitionistic approach: constraints may very well be undecidable. If they are to hard to check, let them like they are.

Other example

$$f(x) = \exp\exp((\sin 10^{10^{10^{10}}})x).$$