Lazy multiplication of power series

By Joris van der Hoeven

Lab. : LIX, École polytechnique, France

Email: vdhoeven@lix.polytechnique.fr

Web : http://lix.polytechnique.fr:80/~vdhoeven/

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Definitions

$\mathfrak{C}$: effective field of constants

$f = f_0 + f_1 z + f_2 z^2 + \cdots \in \mathfrak{C}[z]$

$g = g_0 + g_1 z + g_2 z^2 + \cdots \in \mathfrak{C}[z]$

$h = fg$

Static multiplication algorithms

Given $f_0, \cdots, f_n$ and $g_0, \cdots, g_n$, we compute $h_0, \cdots, h_{n-1}$.

Time complexity: $M(n) = O(n \log n)$.

Space complexity: $O(n)$.

Lazy multiplication algorithms

h_i is output as soon as $f_0, \cdots, f_i$ and $g_0, \cdots, g_i$ are known, where i goes from 0 to n.

Time complexity: $L(n) = O(M(n) \log n)$.

Space complexity: $O(n)$.

Applications

Functional equations

Lazy multiplication algorithms allow the coefficients of f and g to depend on the result h; i.e. f_n and g_n depend on $f_0, \cdots, f_{n-1}, g_0, \cdots, g_{n-}$ and $h_0, \cdots, h_{n-1}$.

Example: exponentiation

If $\varphi = \varphi_1 z + \varphi_2 z^2 + \cdots$, then $\psi = \exp \varphi$ satisfies

$$\psi' = \varphi' \psi \quad (\varphi_0 = 1).$$

Taking $f = \varphi', g = \psi$ and $h = \varphi' \psi$, we get

$$\psi = \int h.$$

Here $g_n = \varphi_n = \frac{1}{n} h_{n-1}$ indeed only depends on $h_0, \cdots, h_{n-1}$.

Lazy multiplication

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$g_7 z^7$	$f_0 g_7 z^7$	$f_1 g_7 z^8$	$f_2 g_7 z^9$	$f_3 g_7 z^{10}$	$f_4 g_7 z^{11}$	$f_5 g_7 z^{12}$	$f_6 g_7 z^{13}$	$f_7 g_7 z^{14}$	$\cdots$
$g_6 z^6$	$f_0 g_6 z^6$	$f_1 g_6 z^7$	$f_2 g_6 z^8$	$f_3 g_6 z^9$	$f_4 g_6 z^{10}$	$f_5 g_6 z^{11}$	$f_6 g_6 z^{12}$	$f_7 g_6 z^{13}$	$\cdots$
$g_5 z^5$	$f_0 g_5 z^5$	$f_1 g_5 z^6$	$f_2 g_5 z^7$	$f_3 g_5 z^8$	$f_4 g_5 z^9$	$f_5 g_5 z^{10}$	$f_6 g_5 z^{11}$	$f_7 g_5 z^{12}$	$\cdots$
$g_4 z^4$	$f_0 g_4 z^4$	$f_1 g_4 z^5$	$f_2 g_4 z^6$	$f_3 g_4 z^7$	$f_4 g_4 z^8$	$f_5 g_4 z^9$	$f_6 g_4 z^{10}$	$f_7 g_4 z^{11}$	$\cdots$
$g_3 z^3$	$f_0 g_3 z^3$	$f_1 g_3 z^4$	$f_2 g_3 z^5$	$f_3 g_3 z^6$	$f_4 g_3 z^7$	$f_5 g_3 z^8$	$f_6 g_3 z^9$	$f_7 g_3 z^{10}$	$\cdots$
$g_2 z^2$	$f_0 g_2 z^2$	$f_1 g_2 z^3$	$f_2 g_2 z^4$	$f_3 g_2 z^5$	$f_4 g_2 z^6$	$f_5 g_2 z^7$	$f_6 g_2 z^8$	$f_7 g_2 z^9$	$\cdots$
$g_1 z$	$f_0 g_1 z$	$f_1 g_1 z^2$	$f_2 g_1 z^3$	$f_3 g_1 z^4$	$f_4 g_1 z^5$	$f_5 g_1 z^6$	$f_6 g_1 z^7$	$f_7 g_1 z^8$	$\cdots$
g_0	$f_0 g_0$	$f_1 g_0 z$	$f_2 g_0 z^2$	$f_3 g_0 z^3$	$f_4 g_0 z^4$	$f_5 g_0 z^5$	$f_6 g_0 z^6$	$f_7 g_0 z^7$	$\cdots$
$\times$	f_0	$+ f_1 z$	$+ f_2 z^2$	$+ f_3 z^3$	$+ f_4 z^4$	$+ f_5 z^5$	$+ f_6 z^6$	$+ f_7 z^7$	$+ \cdots$

More applications

Algebraic differential equations

Compute f_n, where f solution of

$$\sum_{i_0,\cdots,i_r} P_{i_0,\cdots,i_r} f^{i_0} \cdots (f^{(r)})^{i_r} = 0,$$

with suitable initial conditions.

Our result $\Rightarrow$ solution in time $O(M(n) \log n)$.

Extension to systems of algebraic differential equations.

Brent and Kung: a statical $O(M(n))$ algorithm.

Time and space complexities depend badly on r.

Harder to implement the general case.

Difference equations

$$s(z) = 1 + z\frac{s(z)^3 + 2s(z^3)}{3},$$

s_n can be computed in time $O(M(n) \log n)$.

Combinatorial interpretation: s_n is the number of stereoisomeres of alcohols of the form $C_nH_{2n+1}OH$.

Partial differential equations

$$\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y^2} + e^x f^2,$$

with $f(0, y) = \sin y$. We have

$$f = f_{0,0} + f_{1,0}x + f_{0,1}y + f_{2,0}x^2 + f_{1,1}xy + \cdots$$

The coefficients $f_{i,j}$ with $0 \leqslant i, j \leqslant n$ can be computed in time $O(M(n)^2 \log n)$ (even in time $O(M(n^2) \log n)$).

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$f_{0,9}$	$f_{1,9}$	$f_{2,9}$	$f_{3,9}$	$f_{4,9}$	$f_{5,9}$	$f_{6,9}$	$f_{7,9}$	$f_{8,9}$	$\cdots$
$f_{0,8}$	$f_{1,8}$	$f_{2,8}$	$f_{3,8}$	$f_{4,8}$	$f_{5,8}$	$f_{6,8}$	$f_{7,8}$	$f_{8,8}$	$\cdots$
$f_{0,7}$	$f_{1,7}$	$f_{2,7}$	$f_{3,7}$	$f_{4,7}$	$f_{5,7}$	$f_{6,7}$	$f_{7,7}$	$f_{8,7}$	$\cdots$
$f_{0,6}$	$f_{1,6}$	$f_{2,6}$	$f_{3,6}$	$f_{4,6}$	$f_{5,6}$	$f_{6,6}$	$f_{7,6}$	$f_{8,6}$	$\cdots$
$f_{0,5}$	$f_{1,5}$	$f_{2,5}$	$f_{3,5}$	$f_{4,5}$	$f_{5,5}$	$f_{6,5}$	$f_{7,5}$	$f_{8,5}$	$\cdots$
$f_{0,4}$	$f_{1,4}$	$f_{2,4}$	$f_{3,4}$	$f_{4,4}$	$f_{5,4}$	$f_{6,4}$	$f_{7,4}$	$f_{8,4}$	$\cdots$
$f_{0,3}$	$f_{1,3}$	$f_{2,3}$	$f_{3,3}$	$f_{4,3}$	$f_{5,3}$	$f_{6,3}$	$f_{7,3}$	$f_{8,3}$	$\cdots$
$f_{0,2}$	$f_{1,2}$	$f_{2,2}$	$f_{3,2}$	$f_{4,2}$	$f_{5,2}$	$f_{6,2}$	$f_{7,2}$	$f_{8,2}$	$\cdots$
$f_{0,1}$	$f_{1,1}$	$f_{2,1}$	$f_{3,1}$	$f_{4,1}$	$f_{5,1}$	$f_{6,1}$	$f_{7,1}$	$f_{8,1}$	$\cdots$
$f_{0,0}$	$f_{1,0}$	$f_{2,0}$	$f_{3,0}$	$f_{4,0}$	$f_{5,0}$	$f_{6,0}$	$f_{7,0}$	$f_{8,0}$	$\cdots$

Related results

Functional composition and reversion

Brent and Kung:

- Static $O(M(n)\sqrt{n \log n})$ composition and reversion algorithms in characteristic zero.
- $O(M(n))$ algorithm for static left composition with differential algebraic function.

van der Hoeven:

- Static $O(M(n) \log n)$ right composition with algebraic power series.
- Lazy $O(L(n) \log n)$ right composition with algebraic power series.
- Lazy $O(L(n)\sqrt{n \log n})$ composition and reversion algorithms in characteristic zero.

Premature computations

If the first 2^{p+1} coefficients of f and g are known, then the multiplication

$$\begin{aligned}\Pi_{2^p,2^p} &= (f_{2^p} z^{2^p} + \cdots + f_{2^{p+1}-1} z^{2^{p+1}-1}) \\ &\quad (g_{2^p} z^{2^p} + \cdots + g_{2^{p+1}-1} z^{2^{p+1}-1})\end{aligned}$$

can be performed prematurely.

If the first $n = (k+1)2^p$ coefficients of f and g are known, with $k \in \{2, 3, \cdots\}$ and $p \geqslant 1$, then the multiplications

$$\begin{aligned}\Pi_{2^p,k2^p} &= (f_{2^p} z^{2^p} + \cdots + f_{2^{p+1}-1} z^{2^{p+1}-1}) \\ &\quad (g_{k2^p} z^{k2^p} + \cdots + g_{(k+1)2^p-1} z^{(k+1)2^p-1})\end{aligned}$$

and

$$\begin{aligned}\Pi_{k2^p,2^p} &= (f_{k2^p} z^{k2^p} + \cdots + f_{(k+1)2^p-1} z^{(k+1)2^p-1}) \\ &\quad (g_{2^p} z^{2^p} + \cdots + g_{2^{p+1}-1} z^{2^{p+1}-1})\end{aligned}$$

can be performed prematurely.

Algorithm C. Input $n \in \mathbb{N}$. Output h_n.
A: extendable array which contains $h_0, h_1, \cdots$ whose entries are initialized by 0. We assume that $h_0, \cdots, h_{n-1}$ have been computed.

C1. [Border]
If $n = 0$, then set $A[0] := f_0 g_0$.
Otherwise, set $A[n] := A[n] + f_0 g_n + f_n g_0$.

C2. [Diagonal]
If $n = 2^{p+1}$ for some $p \geqslant 0$, then compute $\Pi_{2^p,2^p}$ and set $A[i] := A[i] + \Pi_{2^p,2^p,i}$ for all $2^{p+1} \leqslant i \leqslant 2^{p+2} - 2$.

C3. [Main]
For each $k \geqslant 2$ and $p \geqslant 0$ such that $n = (k+1)2^p$, do the following:

- Compute $\Pi_{2^p,k2^p}$ and set $A[i] := A[i] + \Pi_{2^p,k2^p,i}$ for all $(k+1)2^p \leqslant i \leqslant (k+3)2^p - 2$.
- Compute $\Pi_{k2^p,2^p}$ and set $A[i] := A[i] + \Pi_{k2^p,2^p,i}$ for all $(k+1)2^p \leqslant i \leqslant (k+3)2^p - 2$.