On systems of multivariate power series equations

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Prologue

The Newton polygon method

Problème 1

Consider the polynomial equation

$$P(f) = P_0 + P_1 f + \cdots + P_d f^d = 0,$$

with power series coefficients in $C[[z]]$.
We want an algorithm to compute the solutions.

Problème 2

Determine the asymptotique behaviour of $P(f)$, when $z \to 0$.

Idéa

Use the Newton polygon method.
Incorporate an idea of Smith [1875] (and [VdH 97]).

Asymptotic polynomial equations

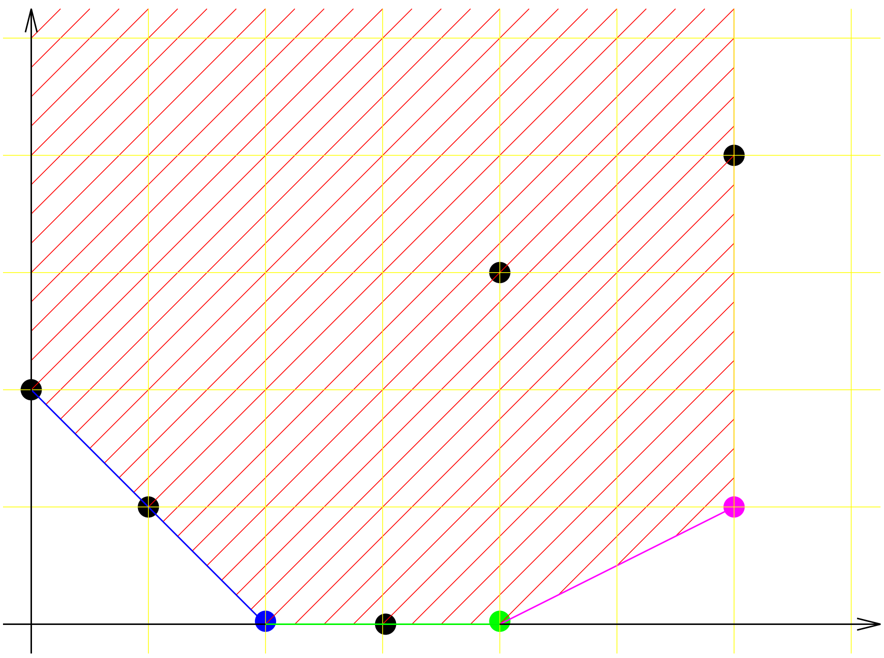
$$P(f) = P_0 + P_1 f + \cdots + P_d f^d = 0 \;\; (f \prec\!\!\prec \text{ч}).$$

Refinements

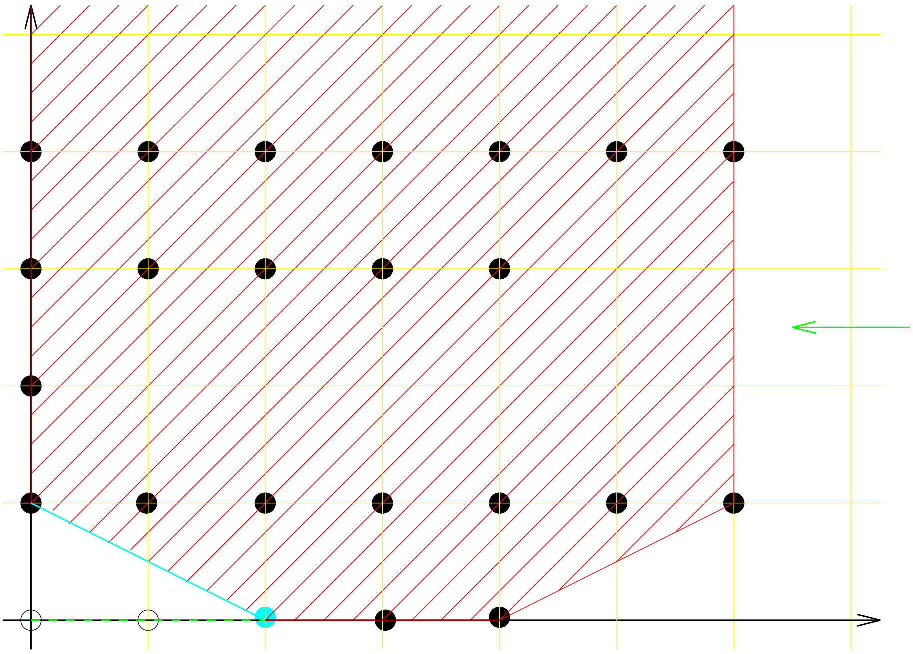
Change of variables + constraint.

$$f = \varphi + \tilde{f} \;\; (\tilde{f} \prec\!\!\prec \tilde{\text{ч}}), \text{ où } \tilde{\text{ч}} \prec\!\!\prec \text{ч}.$$

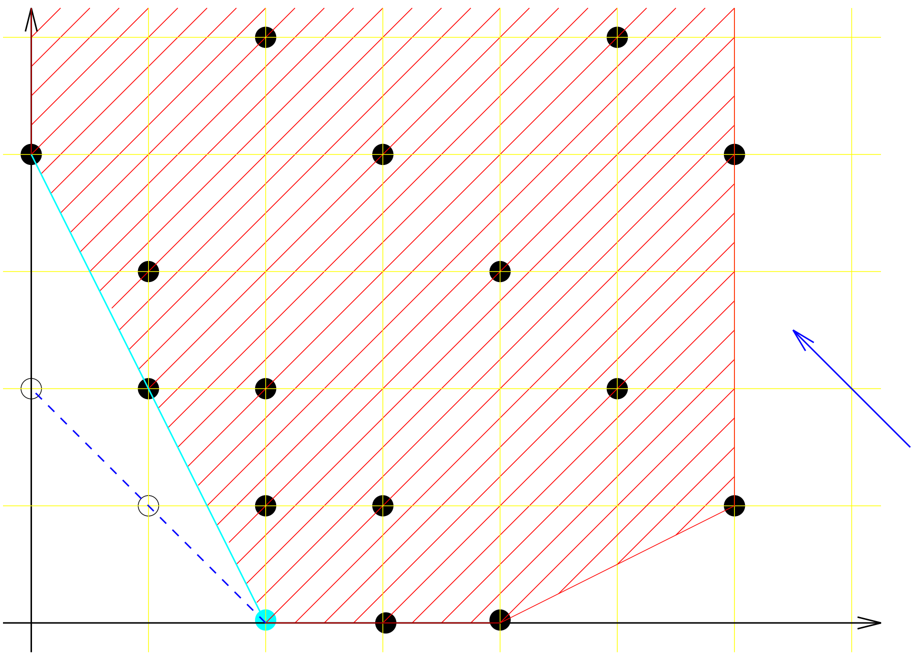
Admissible refinement $\longleftrightarrow$ one step of the Newton polygon method.



Exemple : $\quad z^2 - 2zf + f^2 - 2f^3 + (1+z^3)f^4 - (z^2+z^5)f^6 = 0,$
où $z \prec\!\!\prec 1$.



Raffinement : $f = 1 + \tilde{f}$ $(\tilde{f} \prec\!\!\prec 1)$.

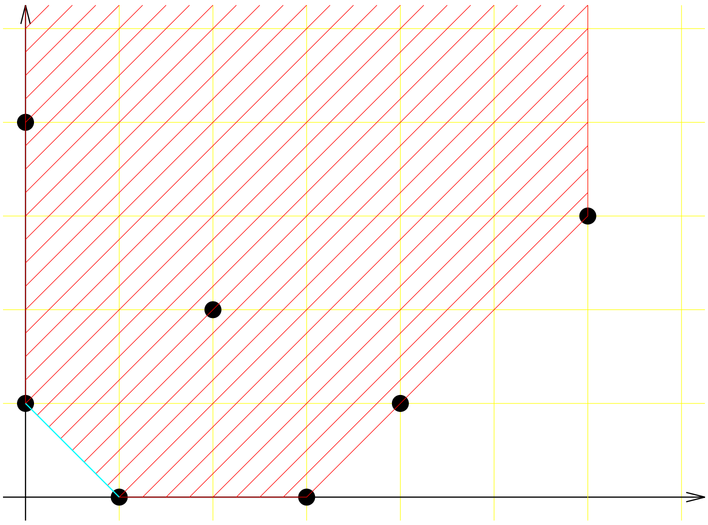


Raffinement : $f = z + \tilde{f}$ $(\tilde{f} \prec\!\!\prec z)$.

Quasi-lineair equations

Equation of Newton degree one.

Unique solution : implicit function theorem.



Équation quasi-linéaire :

$$z + z^3 + f - 7z^2 f^2 + 5f^3 + zf^4 - z^3 f^6 \ \ (f \lll 1).$$

Smith's approch + improvement

$$f^2 - \frac{2}{1-z} f + \frac{1}{(1-z)^2} = z^{10000}.$$

5000 steps necessary before root separation.

Idea : solve the "derived" equation

$$2\varphi - \frac{2}{1-z} = 0,$$

and refine $f = \varphi + \tilde{f}$ $(\tilde{f} \lll 1)$; one obtains $\tilde{f}^2 = z^{10000}$.

Asymptotic behaviour of $P(f)$

Déterminer the dominant term of $P(f)$.

Example, if $P(f) = f^2 - z^3$, the dominant term is

$$\begin{cases} f^2, \text{ if } f \succ\!\!\succ z^{3/2}; \\ z^3, \text{ if } f \prec\!\!\prec z^{3/2}; \\ (c^2-1)z^3, \text{ if } f = z^{3/2}(c+\tilde{f}), \text{ with } c^2 \neq 1 \text{ and } \tilde{f} \prec\!\!\prec 1; \\ 2c\tilde{f}z^{3/2}, \text{ if } f = z^{3/2}(c+\tilde{f}), \text{ with } c^2 = 1 \text{ and } \tilde{f} \prec\!\!\prec 1; \\ 0, \text{ if } f = z^{3/2}. \end{cases}$$

Algorithm

Newton polygon method + (improved) Smith's trick.

Algorithm is non deterministic.

Multivariate series

Notations

C : constant field.

$\check{C}$: dynamic extension of C by finite # of parameters satisfying polynomial constraints.

$X = \{x_1, \cdots, x_p\}$.

S_X : set of monomials $x_1^{\alpha_1} \cdots x_p^{\alpha_p}$.

$f \in \check{C}[[S_X]]$ formal power series (generalized exponents).

Σ : set of constraints of the form

$$\begin{cases} x_1^{\alpha_1} \cdots x_p^{\alpha_p} \prec\!\!\prec 1 & \text{ou} \\ x_1^{\alpha_1} \cdots x_p^{\alpha_p} \asymp 1. \end{cases}$$

R : region of C^p determined by Σ.

Problem

Determine the possible asymptotic behaviours of f modulo a subdivision

$$R = R_1 \amalg \cdots \amalg R_r$$

of R into a finite number of regions, determined by sets of contraints $\Sigma_1, \cdots, \Sigma_p$ as above, and a series of refinements.

Idea

Introduce an elimination ordering

$$x_1 >^{elim} \cdots >^{elim} x_p.$$

Use the Newton polygon method in a lexicographical way.
We will only consider refinements of the form

$$\begin{cases} x_q = ц(\varphi + x'_q) \ \ (x'_q \prec\!\!\!\prec 1); \\ x_q = ц\varphi, \end{cases}$$

where $ц \in S_{x_{q+1},\cdots,x_p}$ et $\varphi \asymp 1$ is a regular series in $x_{q+1}, \cdots, x_p$.
In the last case, the variable x_q is eliminated from X.

Imposition of a constraint $ц \asymp щ$

Either add $цщ^{-1} \asymp 1$ to Σ,
or apply the following algorithm `constraint`:

Introduce a new parameter $0 \neq \lambda \in \check{C}$.
Write $цщ^{-1} = x_q^{\alpha_q} \cdots x_p^{\alpha_p}$ and set $ч = \sqrt[\alpha_q]{x_{q+1}^{\alpha_{q-1}} \cdots x_p^{\alpha_p}}$.
Separate two cases and refine

$$\begin{cases} x_q = ц(\lambda + x'_q) \ \ (x'_q \prec\!\!\!\prec 1); \\ x_q = \lambda ц. \end{cases}$$

One step of the Newton polygon method in x_q

ч : a monomial in $x_1, \cdots, x_{q-1}$.
$[\text{ч}]f$ is Newton prepared if

- $[\text{ч}]f$ is a formal *power* series in x_q.
- There exist $\text{щ}, \text{ц}$ so that the dominant monomials of $[\text{ч}]f$ are of the form $\text{щ}(x_q/\text{ц})^\alpha$.

Newton polynomial

$$P(\lambda) = \sum_{\alpha \in \mathbb{N}} ([\text{чщ}(x_q/\text{ц})^\alpha]f)\ \lambda^\alpha.$$

Newton degree : degree of P.

Algorithme `Newton_step`

Input : $[ч]f$ Newton prepared.

Action : refinement $x_q = ц(\varphi + x'_q)$ $(x'_q \prec\!\!\prec 1)$ or elimination $x_q = ц\varphi$, where $ц\varphi$ is a first approximation of the solution to $[ч]f = 0$ in x_q.

Case when P has several roots

$\lambda \neq 0$ new parameter in $\breve{C}$ with $P(\lambda) = 0$.

Separate two cases and refine

$$\begin{cases} x_q = ц(\lambda + x'_q) \ (x'_q \prec\!\!\prec 1); \\ x_q = \lambda ц. \end{cases}$$

Case when P has a unique root

Compute unique infinitesimal root $ц\varphi$ of

$$\frac{\partial^{\deg P - 1}([ч]f)}{\partial x_q^{\deg P - 1}} = 0$$

in x_q, separate two cases and refine

$$\begin{cases} x_q = ц(\varphi + x'_q) \ (x'_q \prec\!\!\prec 1); \\ x_q = ц\varphi. \end{cases}$$

Computation of the dominant monomial

While f is not regular, impose a face $F \subseteq S_X$ (finite) as being the face of dominant monomials and execute $\texttt{dom\_sub}(f, 1, F)$.

The algorithm $\texttt{dom\_sub}(f, ч, F)$

Input : f, monomial ч in $x_1, \cdots, x_{q-1}$ and F with $F_{ч} = [ч]F \neq \emptyset$.
Output : either the dominant monomial м of $[ч]f$, or the exception "recommence".

Case 1 : $\exists$ unique α with $F_{ч x_q^\alpha} \neq \emptyset$.
Return $\texttt{dom\_sub}(f, ч x_q^\alpha, F) x_q^\alpha$.

Case 2 : f is not a formal *power* series in x_q.
Choose ц $\in F$ and execute $\texttt{constraint}(ц \asymp щ)$ for each other щ $\in F_{ч}$.

Otherwise, separate cases 3 and 4 :

Case 3 : non singular case.

Choose ц arbitrary in F.

Execute `constraint`(ц $\asymp$ щ) for each other щ $\in F$.

Let м be the unique element of $F_{\text{ч}}$ after rewriting.

Return м.

Case 4 : singular case.

(Idea : one step of Newton polygon method)

For each α with $F_{\text{ч}x_q^\alpha} \neq \emptyset$, execute `dom_sub`($f$, ч$x_q^\alpha$, F).

Let n be the # of times `dom_sub` does not return "recommence".

If $n = 0$, return "recommence".

If $n = 1$, kill the process.

Otherwise, choose ц $=$ чx_q^αщ and ц$'$ $=$ ч$x_q^{\alpha'}$щ$'$ $\in F_{\text{ч}}$ with $\alpha' \neq \alpha$.

Execute `constraint`(щ$^{\alpha''-\alpha'}$щ$'^{\alpha-\alpha''}$щ$''^{\alpha'-\alpha}$ $\asymp 1$) for each ц$''$ $=$ ч$x_q^{\alpha''}$щ$''$ $\in F_{\text{ч}}$.

Execute `Newton_step`([ч]f) and return "recommence".