Introduction to automatic asymptotics

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Motivation

Automatic asymptotics and transseries

– Asymptotics of non linear phenomena.

– Systematic theorie.

– Effective theory ⇒ emphasis on algebraic aspects.

– Analytic properties via *via* resummation.

Examples

– Asymptotics of the functional inverse of xe^x for $x \to \infty$. Needed for the study of Bell numbers.

– Asymptotic resolution of non linear differential equations like

$$f'f'' - f''^2 - e^{e^x} f = e^{-x^2}.$$

Short history

Theory

– Newton (± 1670): formal power series, Newton polygon.

– Puiseux, Briot, Bouquet, Fine, Smith (1850–1900): extentions and refinements of Newton polygon method.

– Hardy (1910–1911): generalized asymptotic scales, asymptotics of L-series $\longrightarrow$ Hardy fields.

– Écalle (1990–*): Transseries et resummation.

Algorithms

– Shackell (1990–*): nested forms for exp-log functions. Example:

$$e^{\log^2 x e^{e^{\log^3 x(\pi+o(1))}}}$$

– Shackell (1991): asymptotic expansions of exp-log functions and (incomplete) algorithm for Liouvillian functions.

– Gonnet, Gruntz (1992): expansions of exp-log functions.

– Richardson (1992–1996): exp-log constants.

– Salvy, Gruntz (1990–1996): implementations in MAPLE.

Outline

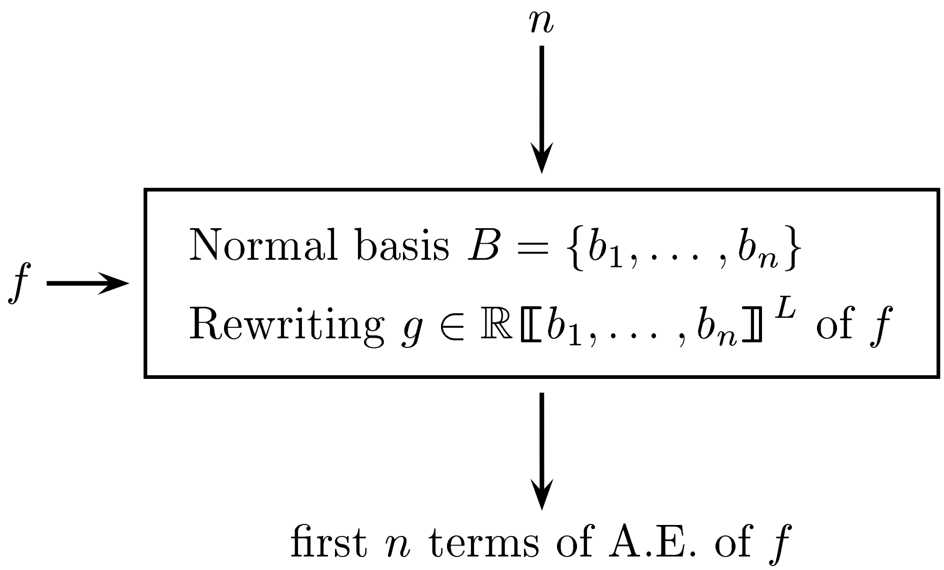
I. Asymptotics of L-functions

II. Transseries: an introduction

Asymptotics of L-functions

An L-function is a function f constructed from $\mathbb{Q}$ and x by $+, -, \times, /, \exp, \log$ and algebraic functions.

Goal: find the expansion of f for $x \to \infty$ (if f is defined at ∞).



The main problems

- Find a suitable asymptotic scale.
- Avoid indefinite cancellations:

$$f = \frac{1}{1 - x^{-1} - e^{-x}} - \frac{1}{1 - x^{-1}} \quad (x \to \infty).$$

Grid-based series

Asymptotic scales

– S: ordered group (by $\prec\!\!\prec$) of positive germs at infinity, stable under exponentiation by reals.

– S finitely generated by $B = \{b_1, \ldots, b_n\}$, if

$$S = \{b_1^{\alpha_1} \cdots b_n^{\alpha_n} | \alpha_1, \ldots, \alpha_n \in \mathbb{R}\},$$

– B basis, if $1 \prec\!\!\prec b_1 \prec\!\!\prec \cdots \prec\!\!\prec b_n$ and $\log b_1 \prec\!\!\prec \cdots \prec\!\!\prec \log b_n$.

– Example: $S = \{x^\alpha e^{x\beta} | \alpha, \beta \in \mathbb{R}\}$.

Grid-based series over a field C

– $C[\![S]\!] = C[\![b_1; \cdots; b_n]\!]$ field of series

$$f = \sigma_0 \varphi(\sigma_1, \ldots, \sigma_k),$$

where $\varphi \in C[[\sigma_1, \ldots, \sigma_k]]$, $\sigma_0, \ldots, \sigma_k \in S$ and $\sigma_i \prec\!\!\prec 1$ for $1 \leqslant i \leqslant k$.

– Example: $e^x (1 - x^{-1} - x^{-x})^{-1} \in \mathbb{R}[\![x; e^x; x^x]\!]$.

Lexicographical expansions

Lexicographical expansion of $f \in \mathbb{R}[\![b_1; \cdots; b_n]\!]$

$$\begin{aligned} f &= \sum_{\alpha_n \in \mathbb{R}} f_{\alpha_n} b_n^{\alpha_n} \\ &\vdots \\ f_{\alpha_n, \ldots, \alpha_2} &= \sum_{\alpha_1 \in \mathbb{R}} f_{\alpha_n, \ldots, \alpha_1} b_1^{\alpha_1}. \end{aligned}$$

$f_{\alpha_n, \ldots, \alpha_{i+1}}$ both in $\mathbb{R}[\![b_1; \cdots; b_i]\!]$ and $\mathbb{R}[\![b_1; \cdots; b_{i-1}]\!]\,[\![b_i]\!]$.

Observation

For each $\beta \in \mathbb{R}$, there are only a finite number of terms in the expansion of $f_{\alpha_n, \ldots, \alpha_{i+1}}$ with exponent $> \beta$ in b_i.

Example

$$\begin{aligned} \frac{1}{(1 - x^{-1})(1 - e^{-x})} &= 1 + x^{-1} + x^{-2} + x^{-3} + \cdots \\ &+ e^{-x} + x^{-1} e^{-x} + x^{-2} e^{-x} + x^{-3} e^{-x} + \cdots \\ &\vdots \end{aligned}$$

Exact representations

Avoiding indefinite cancellations

Expand lexicographically

$$f = \frac{1}{1 - x^{-1} - e^{-x}} - \frac{1}{1 - x^{-1}} \ (x \to \infty)$$

with respect to e^x next x. Keep exact representations for the coefficients of the expansion in e^{-x}.

Constant problem

Richardson: there exists a zero test for "L-constants" modulo:

Conjecture 1 (Schanuel) *If $\alpha_1, \cdots, \alpha_n \in \mathbb{C}$ are $\mathbb{Q}$-linearly independent, then*

$$\operatorname{tr} \deg_{\mathbb{Q}} \mathbb{Q}[\alpha_1, \cdots, \alpha_n, e^{\alpha_1}, \cdots, e^{\alpha_n}] \geqslant n.$$

Germs at $+\infty$

VdH: the asymptotic zero test problem for L-functions at $+\infty$ reduces to the constant problem.

Theorem 1 (VdH) *There exists an asymptotic expansion algorithm for L-functions modulo Schanuel's conjecture.*

Normal bases

L-series

- $\mathbb{R}[\![S]\!]^L$: series constructed from L-constants, monomials $b_i^{\alpha_i}$, the field operations and left composition of infinitesimal L-series by $\exp z$, $\log(1+z)$ or algebraic series.
- L-series are both expressions and series in $\mathbb{R}[\![S]\!]^{conv}$.
- Straightforward expansion algorithm for L-series.
- Iterated coefficients of L-series again L-series.

B normal basis if

B1. $b_1 = \log_l x$ is an l-th iterated logarithm.

B2. $\log b_i \in \mathbb{R}[\![b_1; \cdots; b_{i-1}]\!]^L$ for all $i > 1$.

Example:

$$B = \{\log x, x, \exp[\tfrac{x}{\log x - 1}]\},$$

but not

$$\{x, e^{e^x}\} \quad \text{nor} \quad \{x, e^{x+e^{-x^2}}, e^{x^2}\}.$$

The normal basis $B = \{b_1, \ldots, b_n\}$ is constructed gradually during the execution of the expansion algorithm. Initially, $B = \{x\}$.

Algorithm expand

Input: An L-function f.

Output: f rewritten as an L-series in $\mathbb{R}[\![b_1, \dots, b_n]\!]^L$.

case $f \in \mathbb{R}$ or $f = x$.
Return f.

case $f = g \,\square\, h, \quad \square \in \{+, -, \times, /\}$.
Return expand$(g) \,\square\,$ expand(h).

case $f = \log(g)$.
Set $g :=$ expand(g).
Rewrite $g = c b_1^{\alpha_1} \cdots b_n^{\alpha_n}(1+\varepsilon)$, where $c \in \mathbb{R}^*$ and $\varepsilon \prec\!\!\prec 1$.
If $\alpha_1 \neq 0$, add $\log b_1$ to B.
Return $\log c + \alpha_1 \log b_1 + \cdots + \alpha_n b_n + \log(1+\varepsilon)$.

case $f = \exp(g)$.
Set $g :=$ expand(g).
If $l = \lim g \in \mathbb{R}$, return $e^l e^{g-l}$.
Test whether $g \asymp \log b_i$ for some $2 \leqslant i \leqslant n$.
Yes $\longrightarrow$ return b_i^lexpand$(e^{g - l \log b_i})$, where $l = \lim g/(\log b_i)$.
No $\longrightarrow$

Decompose $g = g^{\uparrow} + g_0 + g^{\downarrow}$, with $g^{\uparrow} = g_{0,\dots,0}$.
Add $e^{|g^{\uparrow}|}$ to B.
Return $(e^{|g^{\uparrow}|})^{\mathrm{sign}(g)} e^{g_0} e^{g^{\downarrow}}$.

case $f = \varphi(g)$, with φ algebraic.

Set $g := \text{expand}(g)$ and $l := \lim g$.

$|l| = \infty \Rightarrow$ return $\text{expand}(\psi(g^{-1}))$, where $\psi(z) \stackrel{\text{def}}{=} \varphi(z^{-1})$.

$l \neq 0 \Rightarrow$ return $\text{expand}(\psi(g - l))$, where $\psi(z) \stackrel{\text{def}}{=} \varphi(z + l)$.

Rewrite $\varphi(z) = z^{\alpha}\psi(z^{\beta})$ with $\alpha \in \mathbb{Q}, \beta \in \mathbb{Q}_{+}^{*}, \psi \in \mathbb{R}[[z]]$.

$\psi \neq 1 \Rightarrow$ return $\text{expand}(g^{\alpha})\psi(\text{expand}(g^{\beta}))$.

Rewrite $g = c\sigma(1 + \varepsilon)$, with $c \in \mathbb{R}^{*}, \sigma \in S$ et $\varepsilon \prec\!\!\prec 1$.

Return $c^{\alpha}\sigma^{\alpha}[(1 + z)^{\alpha} \circ \varepsilon]$.

Example 1

$$f = \frac{1}{1 - x^{-1} - e^{-x}} - \frac{1}{1 - x^{-1}}.$$

Initialization: $B = \{b_1, \cdots, b_n\} := \{x\}$.

e^{-x}: since $-x \not\asymp \log b_1$, insert $b_2 := e^x \rightsquigarrow B$, whence

$$B := \{x, e^x\}.$$

f is rewrittent as

$$f = \frac{1}{1 - b_1^{-1} - b_2^{-1}} - \frac{1}{1 - b_1^{-1}}.$$

Expansion of f

We first expand with respecto to b_2:

$$f = \left(\frac{1}{1 - b_1^{-1}} - \frac{1}{1 - b_1^{-1}}\right) + \frac{b_2^{-1}}{(1 - b_1^{-1})^2} + \frac{b_2^{-2}}{(1 - b_1^{-1})^3} + \cdots.$$

The cancellation $(1 - b_1^{-1})^{-1} - (1 - b_1^{-1})^{-1} = 0$ is detected symbolically.

Transseries for f

$$\begin{array}{rcl} f &=& e^{-x} + 2x^{-1}e^{-x} + 3x^{-2}e^{-x} + \cdots \\ &+& e^{-2x} + 3x^{-1}e^{-2x} + 6x^{-2}e^{-2x} + \cdots \\ &+& e^{-3x} + 4x^{-1}e^{-3x} + 10x^{-2}e^{-3x} + \cdots \\ &+& \cdots. \end{array}$$

Example 2

$$f = \log\log(xe^{xe^x} + 1) - \exp\exp(\log\log x + \frac{1}{x})$$

Initialization: $B = \{b_1, \cdots, b_n\} := \{x\}$.

e^x: since $x \not\asymp \log b_1$, insertion $e^x \rightsquigarrow B$, whence

$$B := \{x, e^x\}.$$

e^{xe^x}: Test whether $xe^x = b_1 b_2 \asymp \log b_2 = b_1 = x$.
No, so $e^{xe^x} \rightsquigarrow B$ and

$$B := \{x, e^x, e^{xe^x}\}.$$

$\log(xe^{xe^x} + 1)$: We have $xe^{xe^x} + 1 = b_1 b_3 + 1$.
The exponent of b_1 in $b_1 b_3$ does not vanish, so $\log x \rightsquigarrow B$.
We get

$$B := \{\log x, x, e^x, e^{xe^x}\}$$

and

$$\log(xe^{xe^x} + 1) = b_2 b_3 + b_1 + \log(1 + b_2^{-1} b_4^{-1}).$$

$\log\log(xe^{xe^x} + 1)$: treated similarly. B remains invariant and

$$\log\log(xe^{xe^x}+1) = b_2+b_1+\log[1+b_2^{-1}b_3^{-1}[b_1+\log(1+b_2^{-1}b_4^{-1})]].$$

$\log x$: $\log x = b_1$.

$\log \log x$: Insertion $\log \log x \rightsquigarrow B$;

$$B := \{\log \log x, \log x, x, e^x, e^{xe^x}\}.$$

$\exp(\log \log x + x^{-1})$: $\log \log x + x^{-1} = b_1 + b_3^{-1} \to \infty$.
We have $b_1 \asymp \log b_2$, whence:

$$\exp(\log \log x + \tfrac{1}{x}) = b_2 e^{b_3^{-1}},$$

avec $b_3^{-1} \to 0$.

$\exp \exp(\log \log x + 1/x)$: $b_2 e^{b_3^{-1}} \asymp \log b_3$ and

$$\exp \exp(\log \log x + \tfrac{1}{x}) = b_3 \exp[b_2 \exp b_3^{-1} - b_2],$$

with $b_2 \exp b_3^{-1} - b_2 \to 0$.

Expansion for f:

$$\begin{aligned} f &= b_3 + b_2 + \log[1 + b_3^{-1} b_4^{-1}[b_2 + \log(1 + b_3^{-1} b_5^{-1})]] \\ &- b_3 \exp[b_2 \exp b_3^{-1} - b_2], \end{aligned}$$

$$\boxed{f = -\frac{\log^2 x}{2x} - \frac{\log x}{2x} - \frac{\log^3 x}{6x^2} - \frac{\log^2 x}{2x^2} + O\left(\frac{\log x}{x^2}\right)}.$$

Transseries: an introduction

Theorem: VdH/Marker, Macintyre, van den Dries

The asymptotic inverse of $\log x \log\log x$ is not asymptotic to an L-function.

Goal

Construct a formal field of grid-based series $\mathbb{R}[\![x]\!] = \mathbb{R}[\![\mathfrak{T}]\!]$ with an exponential $\exp : \mathbb{R}[\![x]\!] \to \mathbb{R}[\![x]\!]$ and a logarithm $\log : \mathbb{R}[\![x]\!]^{+}_{*} \to \mathbb{R}[\![x]\!]$.

Analysis: through resummation.

Examples

$$
\begin{aligned}
f_1 = 1 + x^{-1} + x^{-2} + \cdots +\\
e^{-x} + x^{-1}e^{-x} + \cdots +\\
e^{-2x} + \cdots\cdots
\end{aligned}
$$

$$
\begin{aligned}
f_2 = -\frac{e^{-x^2}}{2x} + \frac{e^{-x^2}}{4x^3} - \frac{e^{-x^2}}{8x^5} + \cdots\\
-\frac{e^{-3x^2}}{6x} + \frac{e^{-3x^2}}{36x^3} - \cdots\\
-\frac{e^{-5x^2}}{10x} + \cdots\cdots
\end{aligned}
$$

$$
\begin{aligned}
f_3 = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^4} + \cdots +\\
\frac{1}{e^{\log^2 x}} + \frac{1}{e^{2\log^2 x}} + \frac{2}{e^{\log^2 x}} + \frac{2}{e^{8\log^2 x}} \cdots\\
\frac{1}{e^{\log^4 x}} + \cdots\cdots
\end{aligned}
$$

Idea behind construction

Closure under exponentiation

Starting with $\mathbb{R}[\![E_0]\!] = \mathbb{R}[\![x^{-\mathbb{R}}]\!]$, construct a sequence

$$\mathbb{R}[\![E_0]\!] \hookrightarrow \mathbb{R}[\![E_1]\!] \hookrightarrow \mathbb{R}[\![E_2]\!] \hookrightarrow \cdots$$

of fields, such that the exponential of each element in $\mathbb{R}[\![E_i]\!]$ is defined by $\mathbb{R}[\![E_{i+1}]\!]$. Direct limit $\longrightarrow$ the field $\mathbb{R}^{alog}[\![\![x]\!]\!] = \mathbb{R}[\![L_0]\!]$ of alogarithmic transseries with a total exponentiation.

Closure under logarithm

Next, construct a second sequence

$$\mathbb{R}[\![L_0]\!] \hookrightarrow \mathbb{R}[\![L_1]\!] \hookrightarrow \mathbb{R}[\![L_2]\!] \hookrightarrow \cdots$$

of fields, such that the logarithm of each element in $\mathbb{R}[\![L_i]\!]^{+}_{*}$ is defined in $\mathbb{R}[\![L_{i+1}]\!]$. Direct limit $\longrightarrow$ $\mathbb{R}[\![\![x]\!]\!]$.

Construction of $\mathbb{R}[[E_1]]$

Each series $f \in \mathbb{R}[[x^{-1}]]$ can be decomposed as

$$\begin{array}{rclcccc} f & = & f^{\uparrow} & + & f^{c} & + & f^{\downarrow} = \\ & & \sum_{\alpha<0} f_\alpha x^{-\alpha} & + & f_0 & + & \sum_{\alpha>0} f_\alpha x^{-\alpha}. \end{array}$$

We take

$$E_1 = x^{-\mathbb{R}} \exp \mathbb{R}[[x^{-1}]].$$

So each $f \in \mathbb{R}[[x^{-1}]]$ can be written as

$$f = \sum_{\substack{\alpha \\ g=\sum_{\beta<0} g_\beta x^{-\beta}}} f_{x^{-\alpha} e^g} x^{-\alpha} e^g.$$

We take the lexicographical ordering on E_1:

$$x^{-\alpha} e^g \prec\!\!\prec 1 \Leftrightarrow g < 0 \lor (g = 0 \land \alpha > 0).$$

For each $f \in \mathbb{R}[[x^-1]]$:

$$\exp f = \exp f^{\uparrow} + \exp f^{c} \exp f^{\downarrow}.$$

Construction of $\mathbb{R}[[E_{i+1}]]$

Each series $f \in \mathbb{R}[[E_i]]$ can be decomposed as

$$\begin{array}{rclcccc} f & = & f^{\uparrow} & + & f^{c} & + & f^{\downarrow} = \\ & & \sum_{\text{ц} \succ\!\!\succ 1} f_{\text{ц}} \text{ц} & + & f_1 & + & \sum_{\text{ц} \prec\!\!\prec 1} f_{\text{ц}} \text{ц}. \end{array}$$

We take

$$E_{i+1} = x^{-\mathbb{R}} \exp \mathbb{R}[[E_i]],$$

with the lexicographical ordering. Finally,

$$L_0 = E_\infty = E_0 \cup E_1 \cup E_2 \cup \cdots.$$

Construction of $\mathbb{R}[\![L_1]\!]$

We know how to construct $\mathbb{R}^{alog}[\![\![x]\!]\!]$.

Formally, we can also construct $\mathbb{R}^{alog}[\![\![\log x]\!]\!]$.

Question: how to embed

$$\mathbb{R}^{alog}[\![\![x]\!]\!] \hookrightarrow \mathbb{R}^{alog}[\![\![\log x]\!]\!].$$

Consider the formal isomorphism

$$\begin{array}{rcl} \mathbb{R}^{alog}[\![\![x]\!]\!] & \to & \mathbb{R}^{alog}[\![\![\log x]\!]\!]. \\ f & \mapsto & f \circ \log. \end{array}$$

The embedding ι restricted to $\mathbb{R}[\![E_0]\!]$ is given by

$$\iota(x^\alpha) = \exp(\alpha \log x) \in \mathbb{R}[\![E_1 \circ \log]\!]$$

on monomials and extended by linearity. For $x^\alpha e^f \in \mathbb{R}[\![E_i]\!]$:

$$\iota(x^\alpha e^f) = \exp(\alpha \log x + \iota(f)) \in \mathbb{R}[\![E_{i+1} \circ \log]\!],$$

and we again extend by linearity.

Construction de $\mathbb{R}[\![\![x]\!]\!]$

We take the inductive sequence of the sequence

$$\mathbb{R}[\![x]\!] \hookrightarrow \mathbb{R}[\![\log x]\!] \hookrightarrow \mathbb{R}[\![\log_2 x]\!] \hookrightarrow \cdots.$$

Remark

Alternatively, one first closes under log and next under exp.

Structure theorem

$B \subseteq \mathfrak{L}$ normal basis if

B1. $b_1 = \log_l x$ is an l-th iterated logarithm.

B2. $\log b_i \in \mathbb{R}[\![b_1; \cdots; b_{i-1}]\!]$ for all $i > 1$.

Theorem 2 (Structure theorem) *Let f be a transseries and let B_0 be a normal basis. Then there exists a normal basis $B \supseteq B_0$, such that $f \in \mathbb{R}[\![B]\!]$.*

Closure properties

$\mathbb{R}[\![[x]\!]]$ closed under

- Differentiation.
- Integration.
- Functional composition.
- Functional inversion.

Derivation with respect to x

Case $f \in \mathbb{R}\llbracket E_0 \rrbracket$

$$f' = \left(\sum_{\alpha \in \mathbb{R}} f_\alpha x^{-\alpha} \right)' = \frac{-1}{x} \sum_{\alpha \in \mathbb{R}} \alpha f_\alpha x^{-\alpha}.$$

Case $f \in \mathbb{R}\llbracket E_{k+1} \rrbracket \quad (k = 0, 1, \cdots)$

$$f' = \left(\sum_{\text{ц} \in x^{-\mathbb{R}} \exp(\mathbb{R}\llbracket E_k \rrbracket^{\uparrow})} f_{\text{ц}} \text{ц} \right)' = \sum_{\text{ц} \in \exp(\mathbb{R}\llbracket E_k \rrbracket^{\uparrow})} f_{\text{ц}} (\log \text{ц})' \text{ц}.$$

f' is well defined:

$$\begin{aligned} \operatorname{supp} f &\subseteq \text{ц}_1^{\mathbb{N}-p} \cdots \text{ц}_n^{\mathbb{N}-p} = \text{Ц} \quad \Rightarrow \\ \operatorname{supp} f' &\subseteq (\operatorname{supp} \log \text{ц}_1 \cup \cdots \cup \operatorname{supp} \log \text{ц}_n) \text{Ц}. \end{aligned}$$

General case

Extend the derivation to $\mathbb{R}[\![\![x]\!]\!]$ using

$$(f \circ \log x)' = \frac{f' \circ \log x}{x}.$$

Yes, we got a derivation

Linearity: trivial.

Observation: $(e^f)' = f' e^f$ for transmonomials e^f.

Hence, for transmonomials e^f, e^g:

$$(e^f e^g)' = (e^{f+g})' = (f+g)' e^{f+g} = f' e^f e^g + g' e^f e^g = (e^f)' e^g + e^f (e^g)'.$$

General case: bilinearity.

Linear differential equations

$$Lh = L_r h^{(r)} + \cdots + L_0 h = 0.$$

Classical result: $L_0, \ldots, L_r \in \mathbb{C}[[z]]$
$\exists$ basis of formal solutions of the form

$$h = (h_{r-1} \log^{r-1} z + \cdots + h_0) e^{\alpha z} e^{P(\sqrt[p]{z^{-1}})},$$

where $p \in \mathbb{N}^*, h_0, \ldots, h_r \in \mathbb{C}[[\sqrt[p]{z}]], \alpha \in \mathbb{C}$ and
$P \in \mathbb{C}[\sqrt[p]{z^{-1}}]$ without constant term.

Generalization: $L_0, \ldots, L_r \in \mathbb{C}^{alog}[[[x]]]$
Notation: $\mathbb{C}^{alog}[[[x]]] = \mathbb{R}^{alog}[[[x]]] + i\mathbb{R}^{alog}[[[x]]]$.
$\exists$ basis of formal solutions of the form

$$h = (h_{r-1} \log^{r-1} x + \cdots + h_0) e^{\varphi},$$

where $h_0, \ldots, h_r \in \mathbb{C}^{alog}[[[x]]]$ and $\varphi \in i\mathbb{R}^{alog}[[[x]]]$.

Algebraic differential equations

Intermediate value theorem

P: algebraic differential polynomial with coefficients in $\mathbb{R}[[[x]]]$.
Assume that $P(f) < 0$ and $P(g) > 0$ for $f < g$ in $\mathbb{R}[[[x]]]$.
Then $\exists h \in \mathbb{R}[[[x]]]$ with $f < h < g$ and $P(h) = 0$.