A new zero-test for formal power series

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Written using GNU TeXmacs (www.texmacs.org)

The problem

Testing functional identities

- $\sin^2 x + \cos^2 x = 1$
- $\log(x^{x^x} + e^{x \log x}) - x^x \log x = \log\left(1 + x^{x(1 - x^{x-1})}\right)$

functional identities = constant identities + power series identities

Testing constant identities

- $\sqrt[3]{\sqrt[5]{32/5} - \sqrt[5]{27/5}} = (1 + \sqrt[5]{3} - \sqrt[5]{9}) / \sqrt[5]{25}$
- $\int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x = \sqrt{\pi}$

Testing power series identities

- $Q \in \mathcal{R}[F, F' \ldots, F^{(r)}] \subseteq \mathcal{R}\{F\}$, $\mathcal{R} = \mathcal{K}[z]$, $Q \notin \mathcal{R}$
- $f \in \mathcal{K}[[z]]$ such that $Q(f, f', \ldots, f^{(r)}) = 0$
- Given $P \in \mathcal{R}\{F\}$, do we have $P(f) = 0$?
- Towers: replace $\mathcal{R}$ by $\mathcal{R}[f, \ldots, f^{(r)}, S_Q(f)^{-1}]$ and continue.

Existing algorithms

Structural approches.

Ax, Risch, Richardson

Bounding the valuation.

Khovanskii, Shackell/vdH, witness conjectures

Model theoretical approach.

Dedef/Lipschitz

Groebner bases and saturation.

Shackell, Péladan-Germa/vdH

Varying the initial conditions.

Péladan-Germa

Generalized solution approach.

Shackell, Shackell/vdH, vdH

Normalization of the equation

1. Ensure that $\frac{\partial Q}{\partial F^{(i)}}(f) \neq 0$ for some $i \in \{0, \ldots, r\}$
   (modulo replacing Q by $\frac{\partial Q}{\partial F^{(i)}}$)

2. Work with derivation $\delta = \frac{z \partial}{\partial z}$ and reduce to the case when

   $$Q = L F + z M$$

   with $L \in \mathcal{K}[\delta]$ and $M \in \mathcal{R}\{F\}$
   (modulo a transformation $f \rightarrow f_0 + \cdots + f_k z^k + \tilde{f} z^{k+1}$)

3. We now have a recurrence relation for the coefficients of f:

   $$f_k = -\frac{1}{\Lambda(k)} (M(f))_{k-1},$$

   where $\Lambda \in \mathcal{K}[k]$ is obtained by substituting $\delta \rightarrow k$ in L

4. Let s be the largest root of Λ in $\mathbb{N}$
   f unique solution to $Q(f) = 0$ with fixed $f_0, \ldots, f_s$

The algorithm

Algorithm $P \equiv 0$

INPUT: a differential polynomial $P \in \mathcal{R}\{F\}$

OUTPUT: true if and only if $P \equiv 0$

Step 1 [Initialize]

$H := 1$, $R := P$, $\mathrm{reducing} := \mathrm{true}$

Step 2 [Reduction]

while reducing [invariant: $H \not\equiv 0$ and $P \equiv 0 \Leftrightarrow R \equiv 0$]

- **if** $R \in \mathcal{R}$ **then return** $R = 0$
- **else if** $I_R \equiv 0$ **then** $R := R - I_R V_R$
- **else if** $S_R \equiv 0$ **then** $H := I_R H, R := R \operatorname{rem} S_R$
- **else if** $Q \operatorname{rem} R \neq 0$ **then** $H := I_R S_R H, R := Q \operatorname{rem} R$
- **else** $H := I_R S_R H, \mathrm{reducing} := \mathrm{false}$

[Final test]

let k be minimal with $\deg_{\prec z^k} H_{+f_0 + \cdots + f_k z^k} = 0$

$k := \max\{k, s\}$

return $\deg_{\prec z^k} R_{+f_0 + \cdots + f_k z^k} \neq 0$

Asymptotic algebraic equations

The Puiseux theorem

Let $A \in \mathcal{K}[[z]][F]^*$. Then

$$A(f) = 0$$

admits $\deg A$ solutions in $\mathcal{K}^{\mathrm{alg}}[[z^{\mathbb{Q}}]]$.

The Puiseux theorem for asymptotic algebraic equations [vdH 1997]

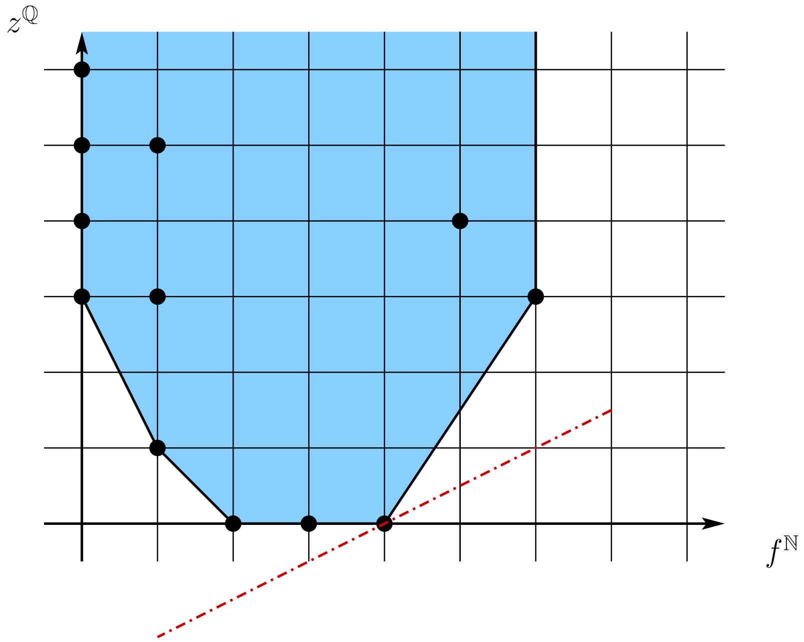
Let $A \in \mathcal{K}[[z]][F]^*$ and $\nu \in \mathbb{R} \cup \{-\infty\}$. Then

$$A(f) = 0 \qquad (f \prec z^{\nu})$$

admits $\deg_{\prec z^{\nu}} A$ solutions in $\mathcal{K}^{\mathrm{alg}}[[z^{\mathbb{Q}}]]$,

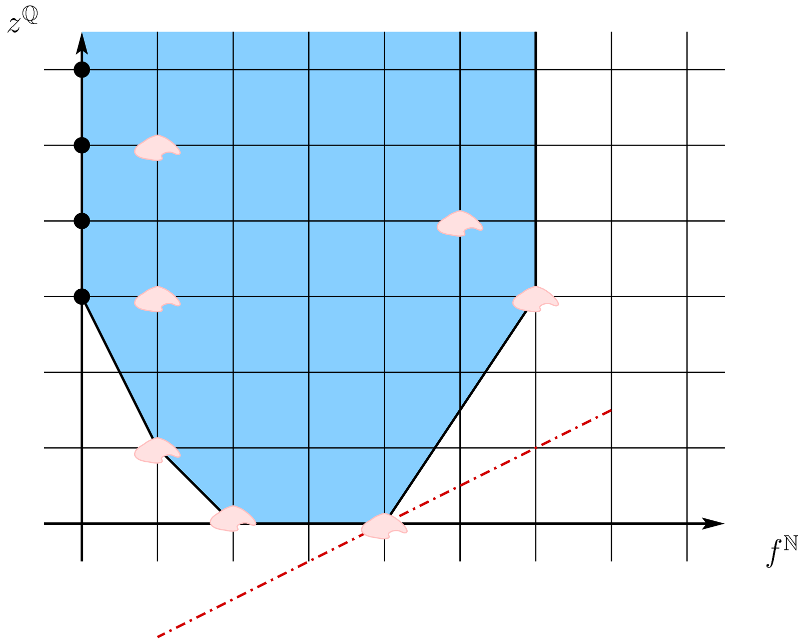
where $\deg_{\prec z^{\nu}} A$ is the Newton degree.

Definition of the Newton degree



$$\frac{z^3}{1-z}+\frac{z}{1-z^2}\,f+f^2-2\,f^3+f^4+z^4\,f^5+z^3\,f^6=0$$

Differential Newton degree



$$\frac{z^3}{1-z}+\frac{z}{1-z^2}(f+f')+(f+f'')^2+(f+f')^4+$$
$$z^4(f'+f''')^5+z^3(f^6+f\,f'\,f''\,f'''\,f''''\,f''''')=0$$

Logarithmic transseries solutions of aade's

1 Logarithmic transseries

Generalized series in $z, \log z, \log \log z, \ldots, \log_l z$ for some l

Example: $z + (\log z)\, z + 2!\, (\log z)^2\, z^2 + 3!\, (\log z)^3\, z^3 + \cdots$

Notation: $\mathbb{L}$ field of grid-based logarithmic transseries

2 Theorem for the resolution of aade's in $\mathbb{L}$ [vdH 2001]

Let $A \in \mathbb{L}\{F\}$ and $\nu \in \mathbb{R} \cup \{-\infty\}$. Then

$$A(f) = 0 \qquad (f \prec z^\nu)$$

admits at least $\deg_{\prec z^\nu} A$ solutions in $\mathbb{L}$.

3 Also

- There exist no solutions if $\deg_{\prec z^\nu} A = 0$.
- f remains the unique solution to $Q(f) = 0$ in $\mathbb{L}$ modulo $\prec z^s$.

4 Negative case

If $\deg_{\prec z^k} R_{+f_0+\cdots+f_k z^k} = 0$, then $R \not\equiv 0$ and $P \not\equiv 0$.

5 Positive case

- Assume that $\deg_{\prec z^k} R_{+f_0+\cdots+f_k z^k} \neq 0$.
- There exists an $\tilde{f} \in \mathbb{L}$ with $R(\tilde{f}) = 0$ and $\tilde{f} - f \prec z^k$.
- Since $Q \operatorname{rem} R = 0$ and $I_R S_R | H$, we have a relation of the form

$$H^\beta Q = X_0 R + \cdots + X_t R^{(t)}.$$

- Since $R(\tilde{f}) = 0$ and $H(\tilde{f}) \neq 0$, we have $Q(\tilde{f}) = 0$.
- But f was the unique solution to $Q(f) = 0$ modulo $\prec z^s$.
- Hence $f = \tilde{f}$ and $R \equiv P \equiv 0$.

Advantages of the new algorithm

- Differential equations of arbitrary order.
- Accomodates divergent power series solutions and
  "diveregence does not get worse during the algorithm".
- Better understanding of previous work by Shackell
  using the differential algebra setting.
- More efficient?
  Witness conjectures...
- Generalizations to partial differential equations?

Witness conjecture

(convergent case)

In order to test whether $P(f) = 0$, it suffices to test whether

$$P(f)_0 = \cdots = P(f)_{\varpi(\sigma)} = 0,$$

where

- σ is the "total input size" (in order to describe the problem).
- The witness function $\varpi(\sigma)$ is linear (probably $\varpi(\sigma) = 2\,\sigma$ is OK).