Complex transseries and an application

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Luminy, 21/11/2002

1 Successes of analyzable functions

- First order singular (but not too singular) systems.
- Real transseries and analyzable functions.

2 The quest of new successes

- Complex transseries and analyzable function.
- Multivariate case and o-minimality.
- Partial differential equations.

3 Limits

No comments.

- Accelero-summation and well-behaved averages.
- Écalle's proof of Dulac's conjecture.
- Algorithm to solve any asymptotic algebraic differential equation

$$P(f) = 0 \quad (f \prec \mathfrak{m}).$$

- Intermediate value theorem. Example:

$$P(f) = f^7 + e^{e^x} f^3 f''' + \Gamma(\log \Gamma(x) + 1) = 0$$

- Extension to differential-difference equations

$$f(e^{\log^2 x})\, f''(x^2)\, f(q\,x) + e^{e^x} f(x)^2 + f(x+1) + \log x = 0.$$

Well-ordered power series

- Totally ordered constant field C.
- Monomial group $\mathfrak{M}$, with total ordering $\succcurlyeq$.
- [Hahn 1907] Set of well-ordered series

$$C[[\mathfrak{M}]] = \{f : \mathfrak{M} \to C \,|\, \operatorname{supp} f \text{ is well-ordered}\}$$

  forms a totally ordered field.
- $f = c_f \, \mathfrak{d}_f \, (1 + \delta_f)$
- $f \preccurlyeq g \Leftrightarrow \mathfrak{d}_f \preccurlyeq \mathfrak{d}_g$
- Canonical decomposition:

$$\begin{array}{ccccccc} f = & f^{\uparrow} & + & f^{=} & + & f^{\downarrow} \\ & \| & & \| & & \| \\ & \displaystyle\sum_{\mathfrak{m} \succ 1} f_{\mathfrak{m}} \, \mathfrak{m} & & f_1 & & \displaystyle\sum_{\mathfrak{m} \prec 1} f_{\mathfrak{m}} \, \mathfrak{m} \end{array}$$

Grid-based series

f grid-based $\Longleftrightarrow \exists\ \mathfrak{m}_1, \ldots, \mathfrak{m}_k \prec 1$ and $\mathfrak{n}$ with

$$\operatorname{supp} f \subseteq \{\mathfrak{m}_1, \ldots, \mathfrak{m}_k\}^* \mathfrak{n}.$$

$C[\![\mathfrak{M}]\!] \subseteq C[[\mathfrak{M}]]$: field of grid-based series.

1 Example

For $f = x^2 + x + 1 + x^{-1} + \cdots$, we have $\operatorname{supp} f \subseteq \{x^{-1}\}^* x^2$.

3.1 Logarithmic transseries

Start with monomial group

$$\mathfrak{L} = \mathfrak{E}_0 = \{x^{\alpha_0} (\log x)^{\alpha_1} (\log \log x)^{\alpha_2} \cdots (\log_l x)^{\alpha_l} : \alpha_0, \ldots, \alpha_l \in \mathbb{R}\}$$

and logarithm on $\mathbb{R}[[\mathfrak{L}]]^+_*$:

$$\log (c\, x^{\alpha_0} \cdots \log_l^{\alpha_l} x (1 + \delta)) =$$
$$\log c + \alpha_0 \log x + \cdots + \alpha_l \log_{l+1} x + \log (1 + \delta).$$

3.2 Inductive step

Assume $\mathfrak{E}_n$ given, with logarithm on $\mathbb{R}[\![\mathfrak{E}_n]\!]^+_*$.

$$\mathfrak{E}_{n+1} = \exp \mathbb{R}[\![\mathfrak{E}_n]\!]^\uparrow,$$

with

$$\exp f^\uparrow \succcurlyeq \exp g^\uparrow \Leftrightarrow f \geqslant g.$$

Take

$$\log (c\, e^{f^\uparrow} (1+\delta)) = \log c + f^\uparrow + \log (1+\delta).$$

Inductive limit: $\mathbb{T} = C[\![\mathfrak{E}_0 \cup \mathfrak{E}_1 \cup \cdots]\!]$.

3.3 Example

$e^{e^x (1 + \frac{1}{x} + \frac{1}{x^2} + \cdots)} \in \mathfrak{E}_2$.

Series with complex coefficients

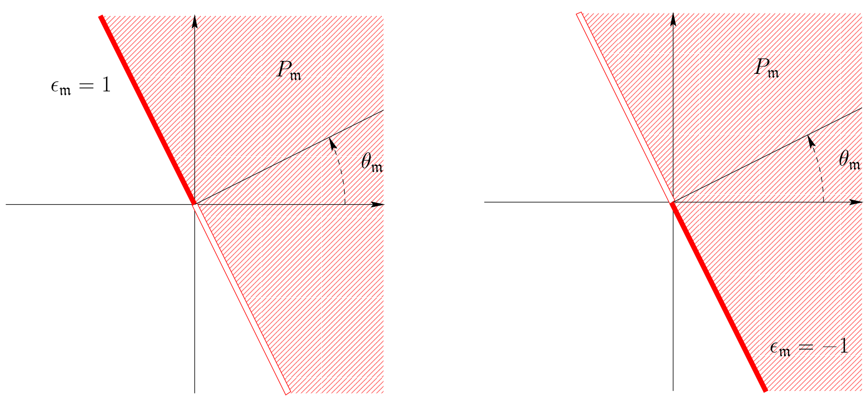
- For each $\mathfrak{m} \in \mathfrak{M}$, select a set of "positive constants"

$$P_{\mathfrak{m}} = \{c \in \mathbb{C} | (\mathrm{Re}\,(c\,e^{-i\theta_{\mathfrak{m}}}) > 0) \vee (\mathrm{Re}\,(c\,e^{-i\theta_{\mathfrak{m}}}) = 0 \wedge \mathrm{Im}(\epsilon_{\mathfrak{m}}\,c\,e^{-i\theta_{\mathfrak{m}}}) > 0)\}.$$

- For $f \in \mathbb{C}[[\mathfrak{M}]]^{\neq}$, define $f > 0 \Longleftrightarrow c_f \in P_{\mathfrak{d}(f)}$.

$\longrightarrow$ $\mathbb{C}[[\mathfrak{M}]]$ is a totally ordered (strong) vector space.

$\longrightarrow$ $\exp \mathbb{C}[[\mathfrak{M}]]^{\uparrow}$ is a monomial group.



- Many possible choices of the $\theta_{\mathfrak{m}}$ and $\epsilon_{\mathfrak{m}}$:
  - $\mathfrak{L} \longrightarrow \mathfrak{L}_{\boldsymbol{\theta},\boldsymbol{\epsilon}}$
  - $\mathfrak{E}_n \longrightarrow \mathfrak{E}_{n,\boldsymbol{\theta},\boldsymbol{\epsilon}}$
- Under the assumption that for all $i \geqslant i_0$ we have
  - $\mathfrak{d}(\log_{i+1} \mathfrak{m}) = \log \mathfrak{d}(\log_i \mathfrak{m})$.
  - $\theta_{\mathfrak{d}(\log_i \mathfrak{m})} = 0$.

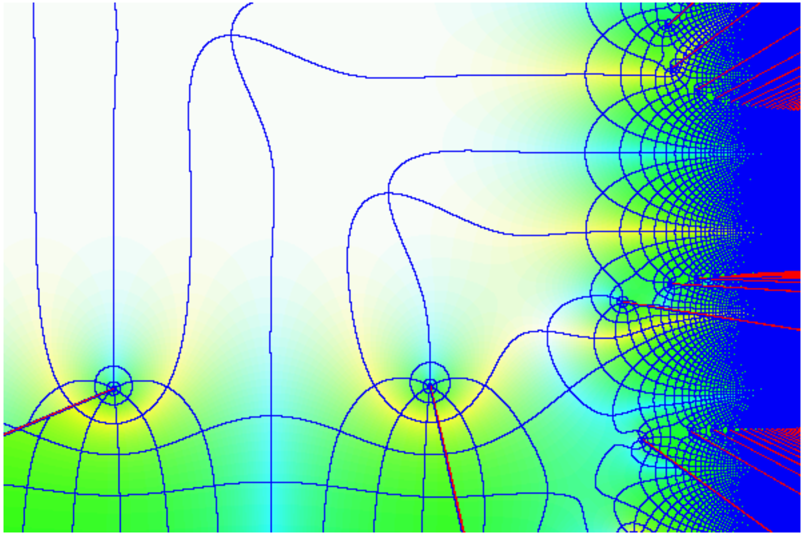
  the construction is unique modulo "turn-flips":

$$\begin{aligned}\varphi\colon \sum_{\mathfrak{m}} f_{\mathfrak{m}} \mathfrak{m} \quad &\longmapsto \quad \sum_{\mathfrak{m}} e^{i\xi_{\mathfrak{m}}} \iota_{\epsilon_{\mathfrak{m}} \varsigma_{\mathfrak{m}}}(f_{\mathfrak{m}} e^{-i\theta_{\mathfrak{m}}}) \mathfrak{m};\\ \hat{\varphi}\colon \sum_{\mathfrak{m}} f_{\mathfrak{m}} \mathfrak{m} \quad &\longmapsto \quad \sum_{\mathfrak{m}} e^{i\xi_{\mathfrak{m}}} \iota_{\epsilon_{\mathfrak{m}} \varsigma_{\mathfrak{m}}}(f_{\mathfrak{m}} e^{-i\theta_{\mathfrak{m}}}) e^{\varphi(\log \mathfrak{m})},\end{aligned}$$

  where $\iota_1(z) = z$ and $\iota_{-1}(z) = \bar{z}$.

Expansion of the exp-log function

$$f = \log\left(e^{e^z + iz} + e^{ie^z}\right)$$



Example of a complex transseries

If $e^z \succ 1$ and $e^{e^z + iz} \succ e^{ie^z}$, then

$$\begin{aligned} f &= e^z + i\,z + \log\left(1 + e^{(i-1)e^z - iz}\right) \\ &= e^z + i\,z + e^{(i-1)e^z - iz} + \frac{1}{2}\,e^{2(i-1)e^z - 2iz} + \cdots \in \mathbb{C}[\![z; e^z; e^{e^z + iz}]\!]. \end{aligned}$$

If $e^z \succ 1$ and $e^{e^z + iz} \prec e^{ie^z}$, then

$$\begin{aligned} f &= i\,e^z + \log\left(1 + e^{(1-i)e^z + iz}\right) \\ &= i\,e^z + e^{(1-i)e^z + iz} + \frac{1}{2}\,e^{2(1-i)e^z + 2iz} + \cdots \in \mathbb{C}[\![z; e^z; e^{e^z + iz}]\!]. \end{aligned}$$

If $e^z \prec 1$ and $e^{iz} \succ 1$, then

$$\begin{aligned} f &= i\,z + \log\left(1 + (e^{e^z} - 1) + e^{-iz}\,e^{ie^z}\right) \\ &= i\,z + e^z + e^{-iz} + (i-1)\,e^{(1-i)z} - \frac{1}{2}\,e^{-2iz} + \cdots \in \mathbb{C}[\![z; e^z]\!]. \end{aligned}$$

If $e^z \prec 1$ and $e^{iz} \prec 1$, then

$$\begin{aligned} f &= \log\left(1 + (e^{ie^z} - 1) + e^{iz}\,e^{e^z}\right) \\ &= i\,e^z - e^{iz} + (1+i)\,e^{(1+i)z} - \frac{1}{2}\,e^{2iz} + \cdots \in \mathbb{C}[\![z; e^z]\!]. \end{aligned}$$

3.3.1 The Puiseux theorem

Let $A \in \mathcal{K}[[z]][F]^*$. Then

$$A(f) = 0$$

admits $\deg A$ solutions in $\mathcal{K}^{\mathrm{alg}}[[z^{\mathbb{Q}}]]$.

3.3.2 The Puiseux theorem for asymptotic algebraic equations [vdH 1997]

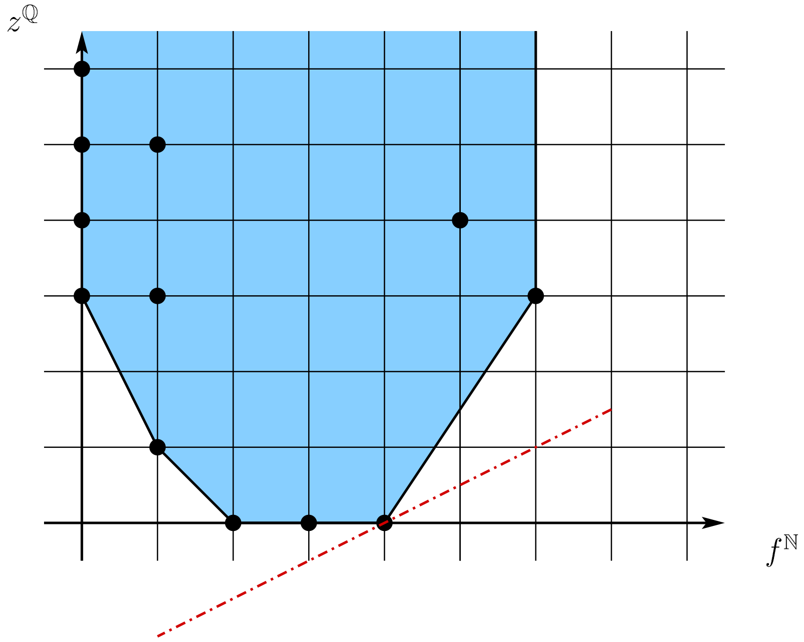
Let $A \in \mathcal{K}[[z]][F]^*$ and $\nu \in \mathbb{R} \cup \{-\infty\}$. Then

$$A(f) = 0 \qquad (f \prec z^{\nu})$$

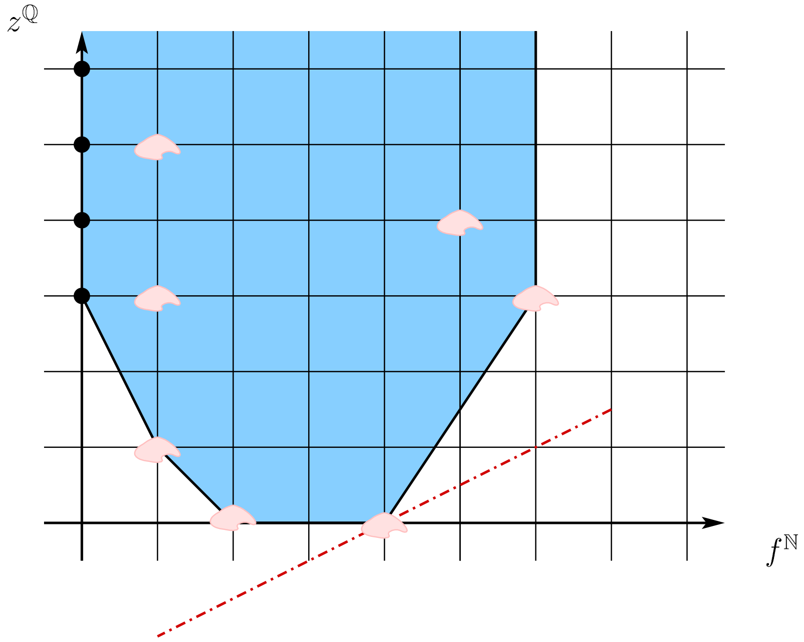
admits $\deg_{\prec z^{\nu}} A$ solutions in $\mathcal{K}^{\mathrm{alg}}[[z^{\mathbb{Q}}]]$,

where $\deg_{\prec z^{\nu}} A$ is the Newton degree.

Newton degree



Differential Newton degree



$\mathbb{T}$ good candidate for an existentially closed H-field (without ordering):

Theorem 1. *Consider an asymptotic algebraic differential equation*

$$P(f)=0 \quad (f \prec \mathfrak{m}) \tag{1}$$

of Newton degree d*, with coefficients in* $\mathbb{C}[[\mathfrak{b}_1; \ldots; \mathfrak{b}_n]] \subseteq \mathbb{T}$*. Then there exist at least* d *solutions when counting with multiplicities. Moreover, these solutions are all in* $\mathbb{C}[[\log_l \mathfrak{b}_1; \ldots, \log \mathfrak{b}_1; \mathfrak{b}_1; \ldots; \mathfrak{b}_n]]$ *for some* l*.*

Corollary 2. *The field of complex transseries is Picard-Vessiot closed (but not differentially closed).*

Theorem 3. *There exists an algorithm to find the general solution to* (1) *in the field of complex transseries (which depends on parameters satisfying real algebraic constraints). The logarithmic depth of this general solution is uniformly bounded in terms of the complexity of the equation.*

3.3.3 Testing functional identities

- $\sin^2 x + \cos^2 x = 1$
- $\log(x^{x^x} + e^{x \log x}) - x^x \log x = \log\left(1 + x^{x(1 - x^{x-1})}\right)$

functional identities = constant identities + power series identities

3.3.4 Testing constant identities

- $\sqrt[3]{\sqrt[5]{32/5} - \sqrt[5]{27/5}} = (1 + \sqrt[5]{3} - \sqrt[5]{9}) / \sqrt[5]{25}$
- $\int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x = \sqrt{\pi}$

3.3.5 Testing power series identities

- $Q \in \mathcal{R}[F, F' \ldots, F^{(r)}] \subseteq \mathcal{R}\{F\}$, $\mathcal{R} = \mathcal{K}[z]$, $Q \notin \mathcal{R}$
- $f \in \mathcal{K}[[z]]$ such that $Q(f, f', \ldots, f^{(r)}) = 0$
- Given $P \in \mathcal{R}\{F\}$, do we have $P(f) = 0$?
- Towers: replace $\mathcal{R}$ by $\mathcal{R}[f, \ldots, f^{(r)}, S_Q(f)^{-1}]$ and continue.

Preparation of the equation

1. Ensure that $\frac{\partial Q}{\partial F^{(i)}}(f) \neq 0$ for some $i \in \{0, \ldots, r\}$

   (modulo replacing Q by $\frac{\partial Q}{\partial F^{(i)}}$)

2. Work with derivation $\delta = \frac{z\,\partial}{\partial z}$ and reduce to the case when

   $$Q = L F + z M$$

   with $L \in \mathcal{K}[\delta]$ and $M \in \mathcal{R}\{F\}$

   (modulo a transformation $f \rightarrow f_0 + \cdots + f_k z^k + \tilde{f} z^{k+1}$)

3. We now have a recurrence relation for the coefficients of f:

   $$f_k = -\frac{1}{\Lambda(k)} (M(f))_{k-1},$$

   where $\Lambda \in \mathcal{K}[k]$ is obtained by substituting $\delta \rightarrow k$ in L

4. Let s be the largest root of Λ in $\mathbb{N}$

   f unique solution to $Q(f) = 0$ with fixed $f_0, \ldots, f_s$

Algorithm $P \equiv 0$

INPUT: a differential polynomial $P \in \mathcal{R}\{F\}$

OUTPUT: true if and only if $P \equiv 0$

Step 1 [Initialize]

$H := 1$, $R := P$, $\mathrm{reducing} := \mathrm{true}$

Step 2 [Reduction]

while $\mathrm{reducing}$ [invariant: $H \not\equiv 0$ and $P \equiv 0 \Leftrightarrow R \equiv 0$]

if	$R \in \mathcal{R}$	**then return**	$R = 0$
else if	$I_R \equiv 0$	**then**	$R := R - I_R V_R$
else if	$S_R \equiv 0$	**then**	$H := I_R H, R := R \operatorname{rem} S_R$
else if	$Q \operatorname{rem} R \neq 0$	**then**	$H := I_R S_R H, R := Q \operatorname{rem} R$
else			$H := I_R S_R H, \mathrm{reducing} := \mathrm{false}$

[Final test]

let k be minimal with $\deg_{\prec z^k} H_{+f_0 + \cdots + f_k z^k} = 0$

$k := \max\{k, s\}$

return $\deg_{\prec z^k} R_{+f_0 + \cdots + f_k z^k} \neq 0$

3.3.6 Negative case

If $\deg_{\prec z^k} R_{+f_0+\cdots+f_k z^k} = 0$, then $R \not\equiv 0$ and $P \not\equiv 0$.

3.3.7 Positive case

- Assume that $\deg_{\prec z^k} R_{+f_0+\cdots+f_k z^k} \neq 0$.
- There exists an $\tilde{f} \in \mathbb{L}$ with $R(\tilde{f}) = 0$ and $\tilde{f} - f \prec z^k$.
- f is the unique solution in $\mathbb{L}$ to $Q(f) = 0$ modulo $\prec z^s$.
- k is sufficiently big, such that $H(\tilde{f}) \neq 0$ for *all* $\tilde{f}$ with $\tilde{f} - f \prec z^k$.
- Since $Q \operatorname{rem} R = 0$ and $I_R S_R | H$, we have a relation of the form

$$H^\beta Q = X_0 R + \cdots + X_t R^{(t)}.$$

- Since $R(\tilde{f}) = 0$ and $H(\tilde{f}) \neq 0$, we have $Q(\tilde{f}) = 0$.
- But f was the unique solution to $Q(f) = 0$ modulo $\prec z^s$.
- Hence $f = \tilde{f}$ and $R \equiv P \equiv 0$.

- $\mathcal{E}^{\mathrm{expr}}$ set of exp-log constant expressions.
- $\bar{\cdot}: \mathcal{E}^{\mathrm{expr}} \rightarrow \mathcal{E}$ value of exp-log expression as exp-log constant.
- $s: \mathcal{E}^{\mathrm{expr}} \rightarrow \mathbb{N}$ size of exp-log expression.
- $\mathcal{E}_1^{\mathrm{expr}}$ set of $f \in \mathcal{E}^{\mathrm{expr}}$ such that for each subexpression of e^g of f, we have $|\bar{g}| \leqslant 1$.

Conjecture 4. *There exists a witness function $\varpi(s) = K\,s$, such that*

$$\bar{f} = 0 \iff |\bar{f}| < e^{-s(f)}$$

for all $f \in \mathcal{E}_1^{\mathrm{expr}}$.

3.4 Exp-log case

- $K[[z]]$ ring of power series over a constant field.
- $\mathcal{E}^{\mathrm{expr}}$ expressions build from $K, z, +, \times, 1/(1+\cdot), \log(1+\cdot)$ and $\exp$.
- $\bar{\cdot}: \mathcal{E}^{\mathrm{expr}} \to \mathcal{E} \subseteq K[[z]]$ value of exp-log expression as power series.

Conjecture 5. *There exists a witness function $\varpi(s) = K\,s$, such that*

$$\bar{f} = 0 \iff v(\bar{f}) > \varpi(s(f))$$

for all $f \in \mathcal{E}^{\mathrm{expr}}$.

3.5 Differentially algebraic case

Conjecture 6. *There exists a witness function $\varpi(s) = K\,s$, such that*

$$P(f) = 0 \iff v(P(f)) > \varpi(s(\mathrm{Problem})),$$

where $s(\mathrm{Problem})$ is the "total input size" (in order to describe the problem).

The multiple Stokes phenomenon

Consider the linear differential equation

$$f' = e^{-e^{(1-i)z}}\left(1 + e^{-e^{(1+i)z}}\right) + f^2$$

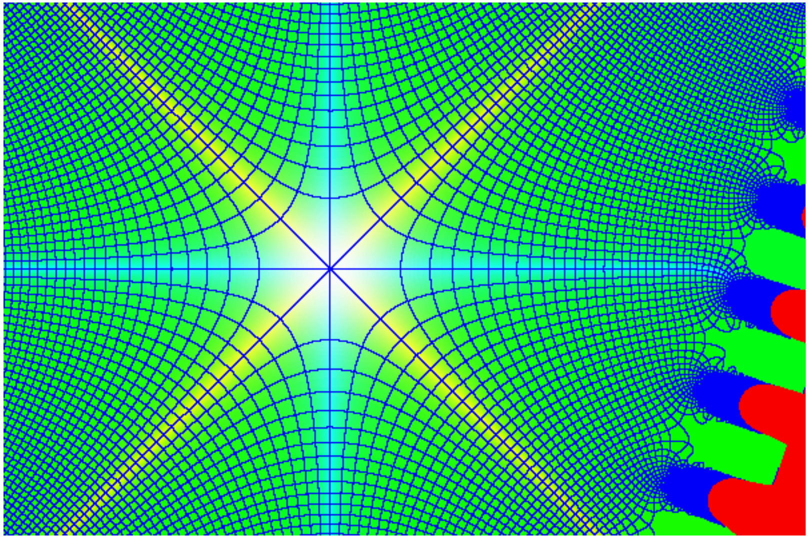
Four types of regions:

	$e^{e^{(1-i)z}} \succ 1$	$e^{e^{(1-i)z}} \prec 1$
$e^{e^{(1+i)z}} \succ 1$	R_1	R_2
$e^{e^{(1+i)z}} \prec 1$	R_3	R_4

?

Numerical example of the multiple Stokes phenomenon

$$f'' = e^z f' + e^{(2+i)z} f + 1$$



4 Accelero-summation

$$\begin{array}{ccccc} \tilde{f}(z) = \tilde{f}(z_1) & & & & f(z_n) = f(z) \\ \downarrow \tilde{\mathcal{B}} & & & & \uparrow \mathcal{L} \\ \hat{f}(\zeta_1) & \xrightarrow{\mathcal{A}} & \cdots & \xrightarrow{\mathcal{A}} & \hat{f}(\zeta_n) \end{array}$$

5 Parallel summation

$$\begin{array}{ccccc} & \nearrow & & \searrow & \\ \tilde{f}(z) = \tilde{f}(z_1, \ldots, z_n) & \xrightarrow{\tilde{\mathcal{B}}} & \hat{f}(\zeta_1, \ldots, \zeta_n) & \xrightarrow{\mathcal{L}} & f(z_1, \ldots, z_n) = f(z) \\ & \searrow & & \nearrow & \end{array}$$