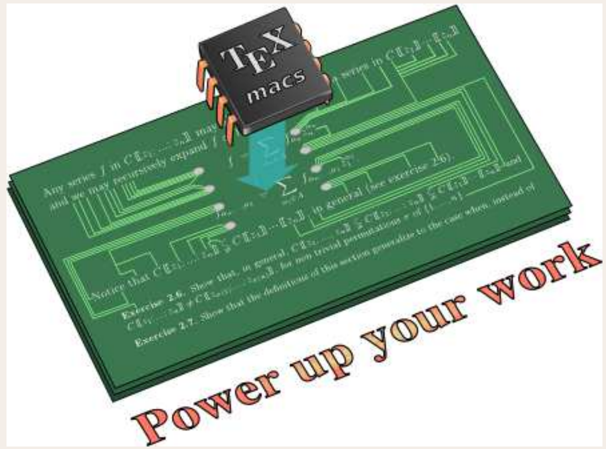
On differential Galois groups



Gecko meeting, Toulouse 2006

http://www.TeXmacs.org

Definitions

- Linear differential operator $L = \partial^r + L_{r-1}\partial^{r-1} + \cdots + L_0 \in \hat{\mathbb{Q}}(z)[\partial]$
- Fundamental system of solutions $\boldsymbol{h} = (h_1, \ldots, h_r)$ to $L h = 0$
- Picard-Vessiot extension $\mathcal{K} = \hat{\mathbb{Q}}(z)\{h_1, \ldots, h_r\}$
- $\mathcal{G}_L = \{\text{differential automorphisms } \sigma: \mathcal{K} \to \mathcal{K} \text{ over } \hat{\mathbb{Q}}(z)\}$
- $M_{\sigma,\boldsymbol{h}} = M \in \mathsf{GL}_r(\hat{\mathbb{Q}})$ such that $\sigma h_i = M_{i,1} h_1 + \cdots + M_{i,r} h_r$
- $\mathcal{G}_{L,\boldsymbol{h}} = \{M_{\sigma,\boldsymbol{h}}: \sigma \in \mathcal{G}_L\}$ Zariski closed algebraic matrix group
- Galois correspondence

Examples

- $L = \partial - 1$

  Fundamental system of solutions $\boldsymbol{h} = (\mathrm{e}^z)$

  $\mathcal{G}_{L,\boldsymbol{h}} = ((a) \colon a \in \hat{\mathbb{Q}}^{\neq})$

- $L = \partial^2 + z^{-1}\,\partial$

  Fundamental system of solutions $\boldsymbol{h} = (\log z, 1)$

  $\mathcal{G}_{L,\boldsymbol{h}} = \left(\begin{pmatrix} 1 & 2\pi\mathrm{i} \\ 0 & 1 \end{pmatrix} \colon a \in \hat{\mathbb{Q}} \right)$

  $$\begin{pmatrix} \log z + a \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \log z \\ 1 \end{pmatrix}$$

Examples (continued)

- $L = \partial^2 + (1 + z^{-1})\,\partial + z^{-1}$ (differentiate $h' + h = z^{-1}$)

  Fundamental system of solutions $\boldsymbol{h} = (\frac{1}{z} + \frac{1}{z^2} + \frac{2}{z^3} + \frac{6}{z^4} + \cdots, \mathrm{e}^{-z})$

  $\mathcal{G}_{L,\boldsymbol{h}} = \left(\left(\begin{smallmatrix} 1 & a \\ 0 & b \end{smallmatrix} \right) : a \in \hat{\mathbb{Q}}, b \in \hat{\mathbb{Q}}^{\neq} \right)$

- $L = A\,B$

  Fundamental system of solutions $\boldsymbol{h} = (B^{-1}\,\boldsymbol{h}_A, \boldsymbol{h}_B)$

  $\mathcal{G}_{L,\boldsymbol{h}} = \begin{pmatrix} \mathcal{G}_{A,\boldsymbol{h}_A} & * \\ 0 & \mathcal{G}_{B,\boldsymbol{h}_B} \end{pmatrix}$

  L factors $\Longleftrightarrow$ $\mathcal{G}_L$ admits a non-trivial invariant subspace

Outline

Important properties of L and solutions of $Lh = 0 \quad \longleftrightarrow \quad \mathcal{G}_L$

- All solutions are algebraic
- All solutions are linear combinations of exponentials
- Existence of Liouvillian solutions
- Existence of factorizations
- Integrability of Hamiltonian vector fields

1. Effective Ramis density theorem: $\mathcal{G}_L$ is generated as a closed linear algebraic group by a finite number of matrices (monodromy and Stokes matrices and generators of the local exponential groups).
2. The entries of the matrices in 1. are computable complex numbers (with fast approximation algorithms).
3. Reduce computations of/with $\mathcal{G}_L$ to linear algebra

Outline

Important properties of L and solutions of $Lh = 0 \longleftrightarrow \mathcal{G}_L$

- How to *compute* $\mathcal{G}_L$?
- How to check special properties of $\mathcal{G}_L$ (e.g. factorization) ?

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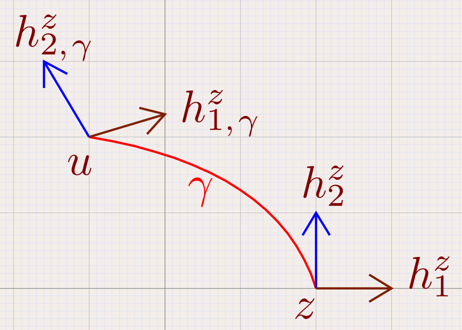
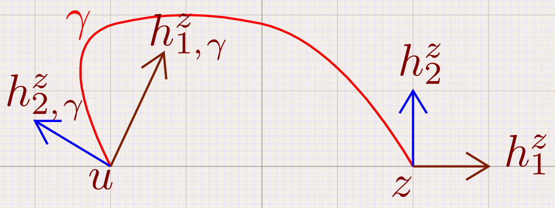
Monodromy



$$\begin{pmatrix} h^z_{1,\gamma} \\ h^z_{2,\gamma} \end{pmatrix} = \Delta_\gamma \begin{pmatrix} h^u_1 \\ h^u_2 \end{pmatrix}$$

Monodromy



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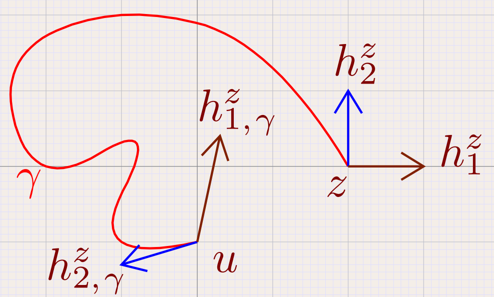

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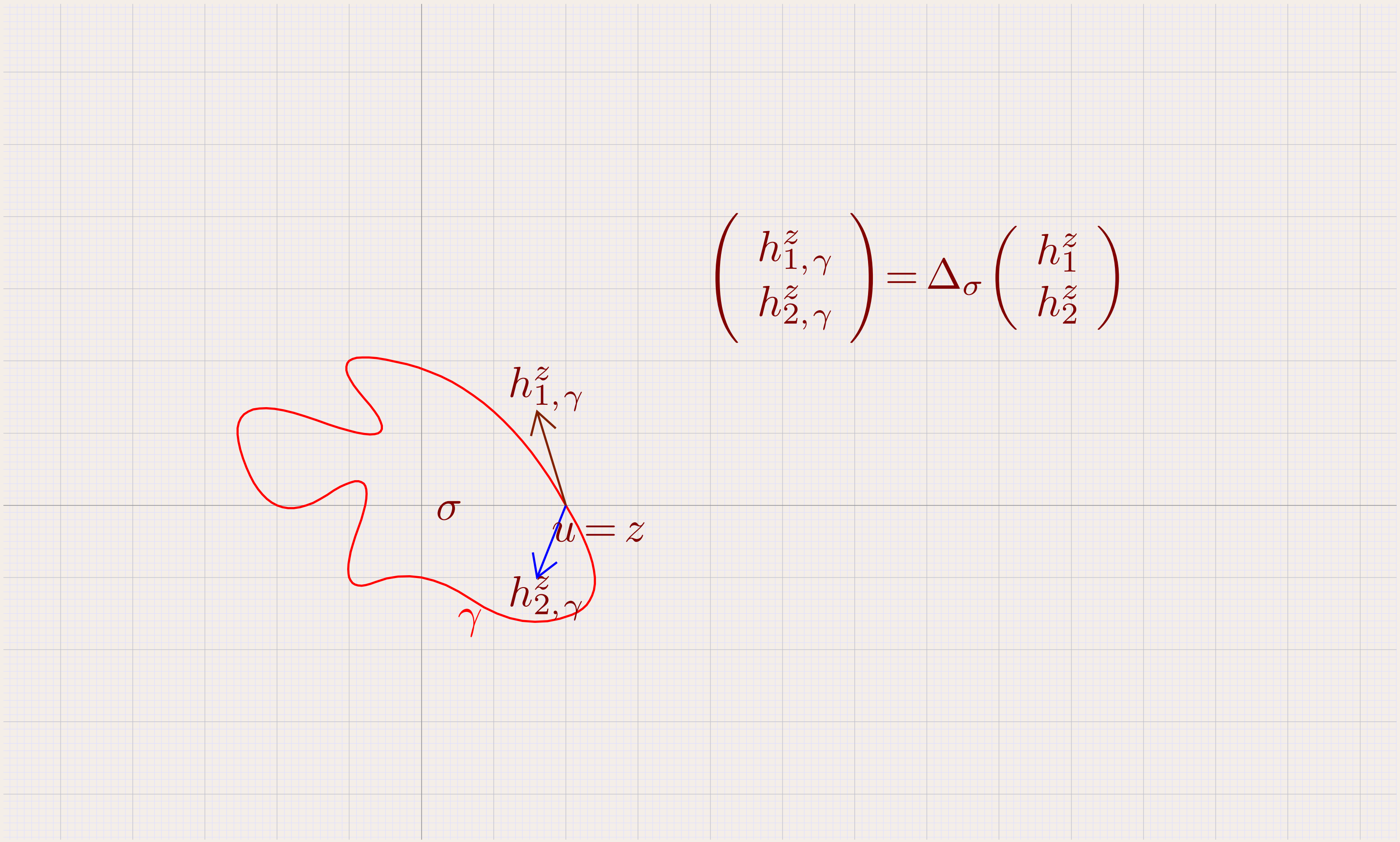
Monodromy



$$\begin{pmatrix} h^z_{1,\gamma} \\ h^z_{2,\gamma} \end{pmatrix} = \Delta_\sigma \begin{pmatrix} h^z_1 \\ h^z_2 \end{pmatrix}$$

Schlesinger density theorem

Fundamental system of formal solutions $\boldsymbol{h}^0$ at 0 of the form

$$h = \left(\sum_{0 \leqslant i < r} h_r(z^{1/k}) \log^i z \right) z^{\sigma} \, \mathrm{e}^{P(z^{-1/k})}$$

Three types of points $z \in \mathbb{C}$:

1. Non singular points (basis $\boldsymbol{h}^z$ of convergent power series solutions)
2. Regular singular points (basis $\boldsymbol{h}^z$ of convergent solutions with logs)
3. Irregular singular points (basis $\boldsymbol{h}^z$ with divergent or exponential els)

Schlesinger: in absence of irregular singular points (also consider $z = \infty$), the monodromy matrices generate $\mathcal{G}_L$

Effective Ramis density theorem

Extra matrices

Local exponential group $\longleftrightarrow$ $\mathbb{Q}$-linear relations between exponential parts

Stokes matrices $\longleftrightarrow$ divergent counterpart of monodromy matrices

Resummation

$$\tilde{f} = \frac{1}{z} + \frac{1}{z^2} + \frac{2}{z^3} + \frac{6}{z^4} + \cdots$$

$$\hat{f}(\zeta) = (\tilde{\mathcal{B}} f)(\zeta) = 1 + \zeta + \zeta^2 + \zeta^3 + \cdots = \frac{1}{1-\zeta}$$

$$f(z) = (\mathcal{L}_\theta \hat{f})(z) = \int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} \frac{\mathrm{e}^{-z\zeta}}{1-\zeta} \mathrm{d}\zeta$$

Stokes matrix at $\theta = 0$: “change” between $\mathcal{L}_{0^+} \hat{f}$ and $\mathcal{L}_{0^-} \hat{f}$

Effective Ramis density theorem

Extra matrices

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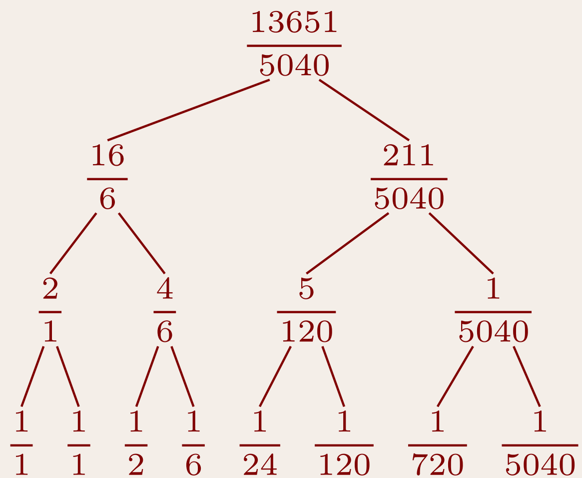
Stokes matrices $\longleftrightarrow$ divergent counterpart of monodromy matrices

Accelero-summation (Écalle)

$$\begin{array}{ccccccccc}
\tilde{f} & & & & & & & & f \\
\tilde{\mathcal{B}}_{z_1}\big\downarrow & & & & & & & & \big\uparrow \hat{\mathcal{L}}_{z_p}^{\theta_p} \\
\hat{f}_1 & \underset{\hat{\mathcal{A}}_{z_1 \to z_2}^{\theta_1}}{\longrightarrow} & \hat{f}_2 & \longrightarrow & \cdots & \longrightarrow & \hat{f}_{p-1} & \underset{\hat{\mathcal{A}}_{z_{p-1} \to z_p}^{\theta_{p-1}}}{\longrightarrow} & \hat{f}_n
\end{array}$$

Fast evaluation of holonomic functions

- Brent (e), Chudnovsky^2, Karatsuba, VdH, Haible-Papanikolaou



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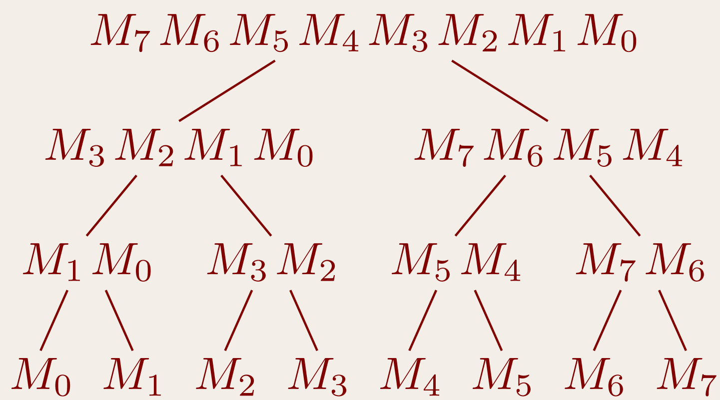


$$\begin{pmatrix} f_{n+1} \\ \vdots \\ f_{n+r} \end{pmatrix} = M_n \begin{pmatrix} f_n \\ \vdots \\ f_{n+r-1} \end{pmatrix}$$

Complexity results

- VdH-97: certified continuation along non singular paths
- VdH-98: regular singular connection matrices
- VdH-05: irregular singular connection matrices

Idea: initial conditions w.r.t. canonical basis of solutions

series of type	evaluation in $z \in \hat{\mathbb{Q}}$	evaluation in general z
$\sum_{n=0}^{\infty} \frac{f_n}{(n!)^{\kappa}} z^n$	$O(M(n) \log n)$	$O(M(n) \log n \log \log n)$
$\sum_{n=0}^{\infty} f_n z^n$	$O(M(n) \log^2 n)$	$O(M(n) \log^2 n \log \log n)$
$\sum_{n=0}^{\infty} f_n (n!)^{\kappa} z^n$	$O(M(n) \log^3 n)$	$O(M(n) \log^3 n)$

Factorization of differential operators

1. Compute generators $M_1, \ldots, M_m$ of $\mathcal{G}_L$ at a non-singular point
2. Fix a precision p for zero-testing
3. Try to compute a non-trivial invariant subspace V of for $M_1, \ldots, M_m$
4. If no such V exists then return "fail"
5. From V, reconstruct a candidate factorization $L = A\,B$
6. If $L = A\,B$ holds, then return (A, B)
7. Double the precision and go to step 3

Better complexity than van Hoeij, Cluzeau, etc. ???

Differential Galois groups

..., Kovacic, Singer, Ulmer, van Hoeij & Weil, Singer & Compoint, ...

$$\mathcal{G} = \mathcal{F}\,\mathrm{e}^{\mathcal{L}} \qquad (\forall N \in \mathcal{F}, N\,\mathrm{e}^{\mathcal{L}} = \mathrm{e}^{\mathcal{L}}\,N)$$

Ingredients:

1. Computation of $\langle M \rangle$ for a single matrix
2. Testing whether $M \in \mathcal{F}\,\mathrm{e}^{\mathcal{L}}$ for given $\mathcal{F}$ and $\mathcal{L}$

The algorithm

Step 1. [Initialize algorithm]

Compute $\langle M_i \rangle = \mathcal{F}_i \, \mathrm{e}^{\mathcal{L}_i}$ for each $i \in \{1, \ldots, m\}$

Let $\mathcal{F} := \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_m$ (notice that $1 \in \mathcal{F}$)

Let $\mathcal{L} := \mathsf{Lie}(\mathcal{L}_1 + \cdots + \mathcal{L}_m)$

Step 2. [Closure]

While there exists an $N \in \mathcal{F} \setminus \{1\}$ with $N \mathcal{L} N^{-1} \nsubseteq \mathcal{L}$

Let $\mathcal{L} := \mathsf{Lie}(\mathcal{L} + N \mathcal{L} N^{-1})$

While there exists an $N \in \mathcal{F} \setminus \{1\}$ with $N \in \mathrm{e}^{\mathcal{L}}$ set $\mathcal{F} := \mathcal{F} \setminus \{N\}$

While there exists $N \in \mathcal{F}^2$ with $N \notin \mathcal{F} \, \mathrm{e}^{\mathcal{L}}$ do

Compute $\langle N \rangle = \mathcal{F}' \, \mathrm{e}^{\mathcal{L}'}$

If $\mathcal{L}' \nsubseteq \mathcal{L}$ then set $\mathcal{L} := \mathsf{Lie}(\mathcal{L} + \mathcal{L}')$, quit loop and repeat step 2

Otherwise, set $\mathcal{F} := \mathcal{F} \cup \{N\}$

Return $\mathcal{F} \, \mathrm{e}^{\mathcal{L}}$

Faster computations with the discrete part

More compact representation of elements in $\mathcal{H} = \mathcal{G} / \mathrm{e}^{\mathcal{L}}$

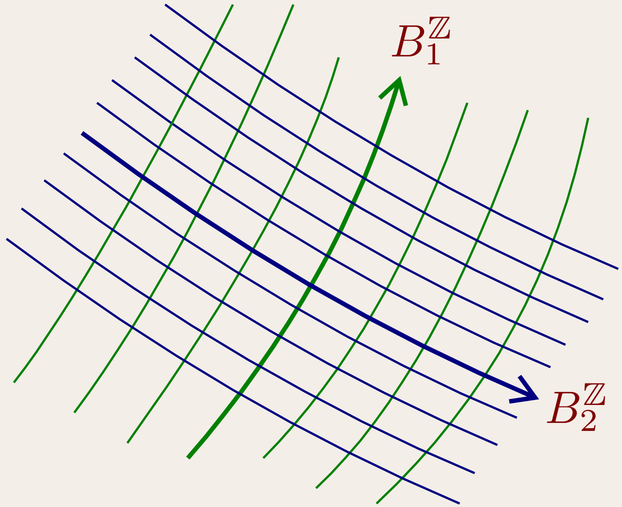
- Reduce to the case when $\mathcal{G} \subseteq \mathrm{Norm}(\mathrm{e}^{\mathcal{L}})^o$
- First basis element $M = B_1 = \mathrm{e}^X$ with
  - $M\,\mathrm{e}^{\mathcal{L}} \in \mathcal{H}$
  - $M\,\mathrm{e}^{\mathcal{L}}$ generates $(\mathrm{e}^{\mathbb{C}X} \cap \mathcal{G}) / \mathrm{e}^{\mathcal{L}}$
  - M has maximal order q with these properties
- Set $\mathcal{H}' := \{N \in \mathcal{H} : [M, N] = 0\}$, $\mathcal{L}' := \mathcal{L} \oplus \mathbb{C}\,X$, so that

$$\mathcal{H} = \{1, \ldots, M^{q-1}\}\,\mathcal{H}' / \mathrm{e}^{\mathcal{L}'}.$$

- Other basis elements $B_2, \ldots, B_b$ by induction, with

$$\|B_1\|_{\mathcal{L}} \leqslant \cdots \leqslant \|B_b\|_{\mathcal{L}}$$

Non commutative basis reduction



- If $[B_i, B_j] = 0$ reduce using LLL.
- If $[B_i, B_j] \neq 0$, then $\|[B_i, B_j]\|_{\mathcal{L}} = O(\|B_i\|_{\mathcal{L}} \|B_j\|_{\mathcal{L}})$ $\rightsquigarrow$ new elements