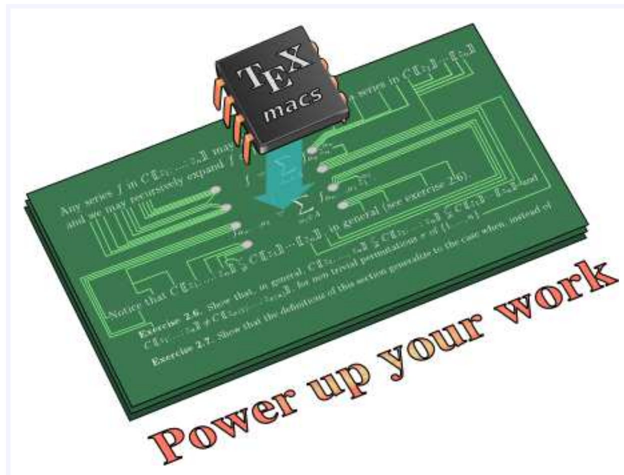
Zero-tests for transcendental functions



Seoul 2007
http://www.TeXmacs.org

Examples

$$\cos^2 x + \sin^2 x \overset{?}{=\!=} 1$$

$$\mathrm{Li}_{2,1}\frac{t-1}{t} \overset{?}{=\!=} -\mathrm{Li}_3\, t + \log t\, \mathrm{Li}_2\, t - \frac{1}{2}\log^2 t\, \mathrm{Li}_1\, t - \frac{1}{6}\log^3 t + \zeta(3)$$

$$\log\log\left(x\, e^{x\, e^x} + 1\right) - \exp\exp\left(\log\log x + \frac{1}{x}\right) \overset{?}{>} 0, \qquad x \to \infty$$

Examples

$$\frac{\pi}{4} \stackrel{?}{=} 8 \arctan \frac{1}{10} - \arctan \frac{1}{100} - \arctan \frac{1}{515} - \arctan \frac{3583}{371498882}$$

$$\zeta(6) \stackrel{?}{=} 12\,\zeta(5,1) + 6\,\zeta(4,2)$$

$$e^{e^{e^{10}} + e^{-e^{e^{10}}}} - e^{e^{e^{10}}} - 1 \stackrel{?}{>} 0$$

Different classes of functions

- Algebraic functions: $y^5 = x\,y^2 - (x+1)\,y + 3$
- Exp-log functions: $\mathrm{e}^{x+\mathrm{e}^{x\log(y+3)}}\log(x+\mathrm{e}^y)$
- Holonomic functions: $x\,f'''(x) = (x^2-3)\,f''(x) + 2\,f(x)$
- Pfaffian functions: $f'(x) = \frac{3\,x\,f(x)^2}{f(x)^3 + x^2}$
- Differentially algebraic functions, $f''(x)\,f'(x)^6 = 3\,f(x)^4 + x^3$
- Solutions to more general functional equations

$$\Gamma(x+1) = x\,\Gamma(x)$$

$$\Gamma(x) = \int_0^\infty t^x\,\mathrm{e}^{-t}\mathrm{d}t$$

Functions

$f \in \mathcal{K}[[z]]$ with $\mathcal{K} \subseteq \mathbb{C}$ given by $(\text{equation}, \text{initial conditions})$:

- $Q \in \mathcal{K}[F, \ldots, F^{(r)}]$ with $Q(f, \ldots, f^{(r)}) = 0$
- $f_0 = f(0), \ldots, f_r = \frac{1}{r!} f^{(r)}(0) \in \mathcal{K}$ with $(\partial Q / \partial F^{(r)})(f(0), \ldots, f^{(r)}(0)) \neq 0$

Computation of coefficients:

$$L F = z M, \qquad L \in \mathcal{K}[\delta], M \in \mathcal{K}[F, \ldots, F^{(r)}]$$

$$f_k = -\frac{1}{\Lambda(k)} (M(f))_{k-1}, \qquad \forall k > r, \Lambda(k) \neq 0$$

More generally: $\forall k > s, \Lambda(k) = 0$ for some $s \geqslant r$.

Constants

- Evaluations of f as above at $z = 1$, where $\mathcal{K} = \mathbb{Q}[\mathrm{i}]$.
- More generally: limits of f at $z = 1$.

Solutions to equations with initial conditions

Our framework. Functions are multi-valued using analytic continuation.

Example: $\sqrt{(-1)^2} = 1$ and $\sqrt{(-1)} = -1$ depending on initial conditions.

Solutions to equations without initial conditions

I.e. differential field extensions such as $\mathcal{K}\{f\}/(f' - f)$ for $f = \mathrm{e}^z$.

Example: $\sqrt{(-1)^2} \neq 1$, since $\sqrt{(-1)^2} = X \in \mathcal{K}[X]/(X^2 - 1)$.

Functions with branchcuts

Functions are single-valued on a chosen domain of $\mathbb{C}$.

Example: $\sqrt{(-1)^2} = 1$, since $\sqrt{\cdot}$ has a branchcut on $\mathbb{R}^{\leqslant}$.

Reduction to normal form

- Groebner bases.
- Risch structure theorem (see also Dan Richardson's talk).
- Lie algebra of MZVs:

$$\begin{aligned} \zeta_{0,0,1} ш \zeta_{0,0,1} &= 2\zeta_{0,0,1,0,0,1} + 6\,\zeta_{0,0,0,1,0,1} + 12\,\zeta_{0,0,0,0,1,1} \\ \zeta_3^2 &= 2\,\zeta_{3,3} + 6\,\zeta_{4,2} + 12\,\zeta_{5,1} \\ \zeta_3^2 &= 2\,\zeta_{3,3} + \zeta_6 \end{aligned}$$

Model theoretic approach

- Denef/Lipschitz

Examine consequences of $\mathrm{Equations} = f = 0$

- Saturation (Shackell, Péladan-Germa, VdH).
- Variation of initial conditions (Péladan-Germa).
- Generalized series solutions (Shackell, VdH).

Expansion / multiple-precision evaluation

1. May require asymptotic expansions:

$$\mathrm{e}^{\mathrm{e}^{\mathrm{e}^{10}}+\mathrm{e}^{-\mathrm{e}^{\mathrm{e}^{10}}}} - \mathrm{e}^{\mathrm{e}^{\mathrm{e}^{10}}} - 1 = \frac{1}{2}\,\mathrm{e}^{-\mathrm{e}^{\mathrm{e}^{10}}} + \frac{1}{6}\,\mathrm{e}^{-2\mathrm{e}^{\mathrm{e}^{10}}} + \cdots.$$

2. Required order can be worse than polynomial in expression size:

$$\Phi(z) = z - 2\log(1 - \log(1 - z/2)) = \frac{-1}{24}\,z^3 + O(z^4).$$

$$(\Phi \circ \overset{s}{\cdots} \circ \Phi)(z) = O(z^{3^s})$$

3. $\sin(10^{10^{10^{10}}}) \overset{?}{>} 0$.

Evaluate at random points or primes

- $(\Phi \circ \overset{s}{\cdots} \circ \Phi)(1000\,s)$.
- Evaluate a polynomial d.a.g. modulo many primes $p_1, \ldots, p_n$.

Prove and disprove in parallel

Quickly disprove and postpone proofs

Algorithm 1: saturation

- $\mathcal{R} = \mathcal{K}[f_1, \ldots, f_n]$
- $\partial : \mathcal{R} \to \mathcal{R}$ derivation (one can also take several $\partial_1, \ldots, \partial_n$)
- $\varepsilon : \mathcal{R} \to \mathcal{K}$ evaluation in 0.

Example: $\mathcal{R} = \mathcal{K}[\cos z, \sin z]$, $\varepsilon(P(\cos z, \sin z)) = P(1, 0)$.

Algorithm $P \equiv 0$

INPUT: a polynomial $P \in \mathcal{R}$

OUTPUT: **true** if and only if $P \equiv 0$, i.e. $\forall i, \varepsilon(P^{(i)}) = 0$

Step 1 [Initialize]

$\mathcal{G} := \{P\}$

Step 2 [Loop]

If $\exists G \in \mathcal{G}, \varepsilon(G) \neq 0$ then return **false**

If $\forall G \in \mathcal{G}, \partial G \xrightarrow{\mathcal{G}} 0$ then return **true**

$\mathcal{G} := \mathrm{GB}(\mathcal{G} \cup \partial \mathcal{G})$ and repeat step 2

Example: If $\mathcal{G} := \{\cos^2 z + \sin^2 z - 1\}$, then $\varepsilon(\mathcal{G}) = \{0\}$ and $\partial \mathcal{G} = \{0\}$.

Reminders on Ritt reduction

- $\mathcal{K}\{F\} = K[F, F', \ldots]$
- Given $\mathcal{A} = \{A_1, \ldots, A_n\} \subseteq \mathcal{K}\{F\}$, $P \xrightarrow{\mathcal{A}} R$ implies

$$H_Q^{\alpha} P = \sum_{i=1}^{n} \sum_{j=0}^{\infty} C_{i,j} A_i^{(j)} + R.$$

- Groebner basis $\longrightarrow$ Coherent autoreduced set.

Algorithm 2: variation of initial conditions

- $\mathcal{R} = \mathcal{K}[f_1, \ldots, f_r] = \mathcal{D} / [\mathcal{A}]$, $\mathcal{D} = \mathcal{K}\{\varphi_1, \ldots, \varphi_k\}$
- $\varepsilon_c \colon \mathcal{R} \to \mathcal{K}$ characterized by $c = (c_1, \ldots, c_r) = (\varepsilon(f_1), \ldots, \varepsilon(f_r))$.
- Fixed initial conditions c_0 with $\varepsilon_{c_0}(\mathcal{H}_{\mathcal{A}}) \neq 0$

Algorithm $\{P_1, \ldots, P_p\} \equiv 0$

INPUT: differential polynomials $P_1, \ldots, P_p \in \mathcal{D}$

OUTPUT: **true** if and only if $P_1 \equiv \cdots \equiv P_p \equiv 0$

Step 1 [Ritt reduction]

$\mathcal{B} := \mathrm{AutoRedCoh}(\mathcal{A} \cup \{P_1, \ldots, P_p\})$

Modulo recursive zero-tests, assume that $\mathcal{H}_{\mathcal{B}} \not\equiv 0$.

Step 2 [Test]

If $\varepsilon_{c_0}(\mathcal{H}_{\mathcal{B}}) \neq 0$ then return **true**

$\mathcal{V} := \{c \colon \varepsilon_c(\mathcal{B}) = 0\}$

$\mathcal{W} := \{c \colon \varepsilon_c(\mathcal{H}_{\mathcal{B}}) = 0\}$

Return $c_0 \in \overline{\mathcal{V} - \mathcal{W}}$

The Puiseux theorem

Let $A \in \mathcal{K}[[z]][F]^*$. Then

$$A(f) = 0$$

admits $\deg A$ solutions in $\mathcal{K}^{\mathrm{alg}}[[z^{\mathbb{Q}}]]$.

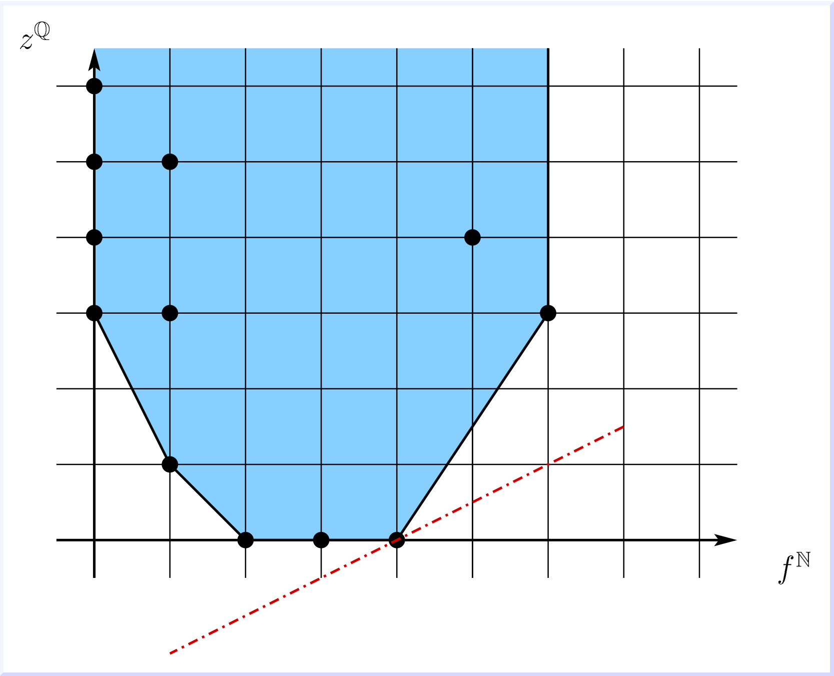
The Puiseux theorem for asymptotic algebraic equations

Let $A \in \mathcal{K}[[z]][F]^*$ and $\nu \in \mathbb{R} \cup \{-\infty\}$. Then

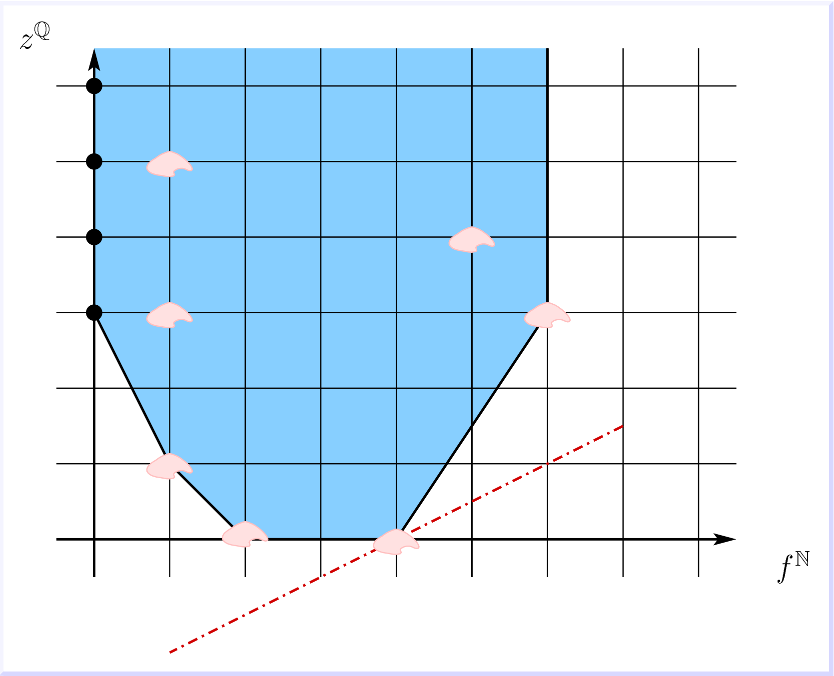
$$A(f) = 0 \qquad (f \prec z^{\nu})$$

admits $\deg_{\prec z^{\nu}} A$ solutions in $\mathcal{K}^{\mathrm{alg}}[[z^{\mathbb{Q}}]]$,

where $\deg_{\prec z^{\nu}} A$ is the Newton degree.



$$\frac{z^3}{1-z} + \frac{z}{1-z^2}\, f + f^2 - 2\, f^3 + f^4 + z^4\, f^5 + z^3\, f^6 = 0$$



$$\frac{z^3}{1-z} + \frac{z}{1-z^2}(f+f') + (f+f'')^2 + (f+f')^4 + z^4(f'+f''')^5 + z^3(f^6 + f\,f'\,f''\,f'''\,f''''\,f''''') = 0$$

Logarithmic transseries

Generalized series in $z, \log z, \log \log z, \ldots, \log_l z$ for some l

Example: $z + (\log z)\, z + 2!\, (\log z)^2\, z^2 + 3!\, (\log z)^3\, z^3 + \cdots$

Notation: $\mathbb{L}$ field of grid-based logarithmic transseries

Theorem for the resolution of aade's in $\mathbb{L}$ [vdH 2001]

Let $A \in \mathbb{L}\{F\}$ and $\nu \in \mathbb{R} \cup \{-\infty\}$. Then

$$A(f) = 0 \qquad (f \prec z^{\nu})$$

admits at least $\deg_{\prec z^{\nu}} A$ solutions in $\mathbb{L}$.

If $\deg_{\prec z^{\nu}} A = 0$, then there are no solutions.

Setup

- $Q \in \mathcal{R}[F, F' \ldots, F^{(r)}] \subseteq \mathcal{R}\{F\}$, $\mathcal{R} = \mathcal{K}[z]$, $Q \notin \mathcal{R}$
- $f \in \mathcal{K}[[z]]$ such that $Q(f, f', \ldots, f^{(r)}) = 0$
- Given $P \in \mathcal{R}\{F\}$, do we have $P(f) = 0$?
- Towers: recursively replace $\mathcal{R}$ by $\mathcal{R}[f, \ldots, f^{(r)}, S_Q(f)^{-1}]$.
- $Q = L F + z M$, with $L \in \mathcal{K}[\delta]$ and $M \in \mathcal{K}[F, \ldots, F^{(r)}]$.
- $f_k = -\frac{1}{\Lambda(k)} (M(f))_{k-1}$ for $k > s$.

Algorithm $P \equiv 0$

INPUT: a differential polynomial $P \in \mathcal{R}\{F\}$

OUTPUT: true if and only if $P \equiv 0$

Step 1 [Initialize]

$H := 1$, $R := P$, $\mathrm{reducing} := \mathrm{true}$

Step 2 [Reduction]

while $\mathrm{reducing}$ [invariant: $H \not\equiv 0$ and $P \equiv 0 \Leftrightarrow R \equiv 0$]

if	$R \in \mathcal{R}$	**then return**	$R = 0$
else if	$I_R \equiv 0$	**then**	$R := R - I_R V_R$
else if	$S_R \equiv 0$	**then**	$H := I_R H, R := R \operatorname{rem} S_R$
else if	$Q \operatorname{rem} R \neq 0$	**then**	$H := I_R S_R H, R := Q \operatorname{rem} R$
else			$H := I_R S_R H, \mathrm{reducing} := \mathrm{false}$

Step 3 [Final test]

let k be minimal with $\deg_{\prec z^k} H_{+f_0+\cdots+f_k z^k} = 0$

$k := \max\{k, s\}$

return $\deg_{\prec z^k} R_{+f_0+\cdots+f_k z^k} \neq 0$

Negative case

If $\deg_{\prec z^k} R_{+f_0+\cdots+f_k z^k} = 0$, then $R \not\equiv 0$ and $P \not\equiv 0$.

Positive case

- Assume that $\deg_{\prec z^k} R_{+f_0+\cdots+f_k z^k} \neq 0$.
- There exists an $\tilde{f} \in \mathbb{L}$ with $R(\tilde{f}) = 0$ and $\tilde{f} - f \prec z^k$.
- Since $Q \operatorname{rem} R = 0$ and $I_R S_R | H$, we have a relation of the form

$$H^\beta Q = X_0 R + \cdots + X_t R^{(t)}.$$

- Since $R(\tilde{f}) = 0$ and $H(\tilde{f}) \neq 0$, we have $Q(\tilde{f}) = 0$.
- But f was the unique solution to $Q(f) = 0$ modulo $\prec z^s$ (also in $\mathbb{L}$).
- Hence $f = \tilde{f}$ and $R \equiv P \equiv 0$.

Conclusion

Thank you