Machine computations with transseries

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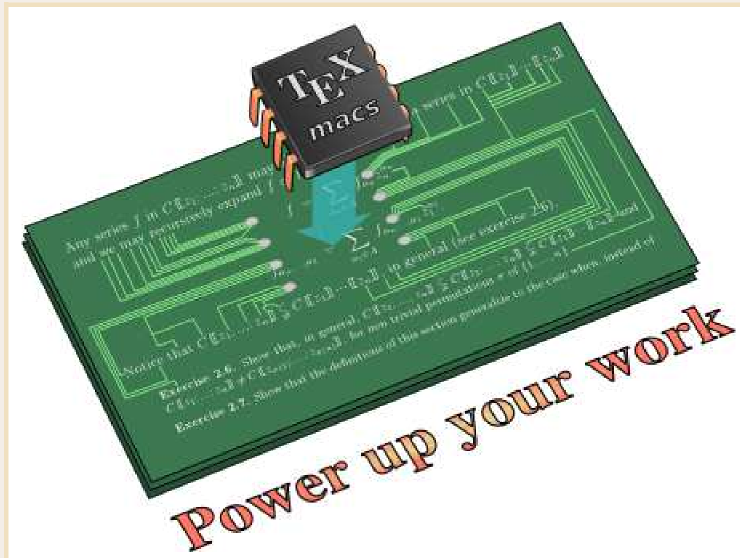

Fields institute at Toronto, 2009

http://www.TeXmacs.org

Challenges

- **Asymptotic behaviour for $x \to \infty$**

$$f = \log \log \left(x\, \mathrm{e}^{x\, \mathrm{e}^{x}} + 1\right) - \exp \exp \left(\log \log x + \frac{1}{x}\right)$$

- **Asymptotic resolution of differential equations**

$$(\log x)\, f^2 - f' + f + \mathrm{e}^{-x} = 0$$

- **Asymptotic resolution of functional equations**

$$f = \frac{1}{x} + f'(x^2) + f(\mathrm{e}^{\log^2 x})$$

Examples

$x \to \infty$

$$
\begin{aligned}
\frac{1}{1-x^{-1}-x^{-\mathrm{e}}} &= 1+x^{-1}+x^{-2}+x^{-\mathrm{e}}+x^{-3}+x^{-\mathrm{e}-1}+\cdots \\
\frac{1}{1-x^{-1}+\mathrm{e}^{-x}} &= 1+\frac{1}{x}+\frac{1}{x^2}+\cdots+\mathrm{e}^{-x}+2\,\frac{\mathrm{e}^{-x}}{x}+\cdots+\mathrm{e}^{-2x}+\cdots \\
-\mathrm{e}^{x}\int\frac{\mathrm{e}^{-x}}{x} &= \frac{1}{x}-\frac{1}{x^2}+\frac{2}{x^3}-\frac{6}{x^4}+\frac{24}{x^5}-\frac{120}{x^6}+\cdots \\
\Gamma(x) &= \frac{\sqrt{2\,\pi}\,\mathrm{e}^{x\,(\log x-1)}}{x^{1/2}}+\frac{\sqrt{2\,\pi}\,\mathrm{e}^{x\,(\log x-1)}}{12\,x^{3/2}}+\frac{\sqrt{2\,\pi}\,\mathrm{e}^{x\,(\log x-1)}}{288\,x^{5/2}}+\cdots \\
\zeta(x) &= 1+2^{-x}+3^{-x}+4^{-x}+\cdots \\
\varphi(x) &= \frac{1}{x}+\varphi(x^{\pi})=\frac{1}{x}+\frac{1}{x^{\pi}}+\frac{1}{x^{\pi^2}}+\frac{1}{x^{\pi^3}}+\cdots \\
\psi(x) &= \frac{1}{x}+\psi(\mathrm{e}^{\log^2 x})=\frac{1}{x}+\frac{1}{\mathrm{e}^{\log^2 x}}+\frac{1}{\mathrm{e}^{\log^4 x}}+\frac{1}{\mathrm{e}^{\log^8 x}}+\cdots
\end{aligned}
$$

Incomplete transbasis theorem

- **Field of grid-based transseries**
  - $\mathbb{T} = \mathbb{R}[[\mathfrak{T}]]$, where transmonomial group $\mathfrak{T}$ ordered by $\preccurlyeq$.
  - Given $\mathfrak{m} \in \mathfrak{T}$, we have either $\mathfrak{m} = \log_k x$ or $\mathfrak{m} \in \exp \mathbb{R}[[\mathfrak{T}]]_{\succ}$

$$\mathrm{e}^x + x^{-1}\mathrm{e}^x + x^{-2}\mathrm{e}^x + \cdots + x^2 + x + 1 + x^{-1} + \mathrm{e}^{-x} + \mathrm{e}^{-2x} + \cdots$$

- **Transbasis $\mathfrak{B}$ for grid-based series**

TB0. $\mathfrak{B} = (\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$ with $1 \prec \mathfrak{b}_1 \prec \cdots \prec \mathfrak{b}_n$.

TB1. $\mathfrak{b}_1 = \log_k z$ for some $k \in \mathbb{Z}$.

TB2. $\log \mathfrak{b}_i \in \mathbb{R}[[\mathfrak{b}_1; \ldots; \mathfrak{b}_{i-1}]] = \mathbb{R}[[\mathfrak{b}_1^{\mathbb{R}} \cdots \mathfrak{b}_{i-1}^{\mathbb{R}}]]$ for $i > 1$.

- **Examples**
  - $\mathfrak{B} = (\log x, x, x^x, \mathrm{e}^{x^2/(1-x^{-1})})$
  - $\mathfrak{B} \neq (\log \log x, \log x, x, \mathrm{e}^{x + \mathrm{e}^{-x^2}}, \mathrm{e}^{x^2})$

Incomplete transbasis theorem

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TB2. $\log \mathfrak{b}_i \in \mathbb{R}[[\mathfrak{b}_1; \ldots; \mathfrak{b}_{i-1}]] = \mathbb{R}[[\mathfrak{b}_1^{\mathbb{R}} \cdots \mathfrak{b}_{i-1}^{\mathbb{R}}]]$ for $i > 1$.

Theorem. *Given a transbasis $\mathfrak{B}_0$ and $f \in \mathbb{T}$, there exists a transbasis $\mathfrak{B} = (\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$ which extends $\mathfrak{B}_0$ and such that $f \in \mathbb{R}[[\mathfrak{b}_1; \ldots; \mathfrak{b}_n]]$.*

Exact computations with transseries

- **Problem with infinite cancellations**

$$\frac{1}{1-\frac{1}{x}}\frac{1}{1-\frac{1}{e^x}}-\frac{1}{1-\frac{1}{x}} = 1+\frac{1}{x}+\frac{1}{x^2}+\cdots+\frac{1}{e^x}+\frac{1}{x\,e^x}+\cdots+\frac{1}{e^{2x}}+\cdots$$
$$-1-\frac{1}{x}-\frac{1}{x^2}+\cdots$$

- **vdH/Richardson/Salvy/Shackell/96**
  - **Setting:** $\mathcal{E}$ class of exp-log expressions in x, assume zero-test.
  - **Problem:** find an asymptotic expansion of $f\in\mathcal{E}$.
  - **Algorithm:**
    - Induction on subformulas g of f
    - Progressive computation of transbasis $\mathfrak{B}$ for g
    - Ensure that $[\mathfrak{b}_n^{\alpha_n}\cdots\mathfrak{b}_k^{\alpha_k}]\,g\in\mathcal{E}$

Example

$$f = \log(e^{-x} + x + 1) - \log(x + 1)$$

$$\begin{aligned}
\mathfrak{B} &:= \{x\} \\
x &= x \\
-x &= -x \\
e^{-x} &= \cdots \\
\mathfrak{B} &:= \{x, e^x\} \\
\cdots &= e^{-x} \\
e^{-x} + x + 1 &= e^{-x} + x + 1 = [x + 1] + [e^{-x}] \\
\log(e^{-x} + x + 1) &= \log\left(x\left(\frac{e^{-x} + x + 1}{x}\right)\right) = \cdots \\
\mathfrak{B} &:= \{\log x, x, e^x\} \\
\cdots &= \log x + \log\left(\frac{e^{-x} + x + 1}{x}\right) \\
\log(x + 1) &\rightsquigarrow \log x + \log\left(\frac{x + 1}{x}\right) \\
f &= \log x + \log\left(\frac{e^{-x} + x + 1}{x}\right) - \left(\log x + \log\left(\frac{x + 1}{x}\right)\right) \\
&= \left[\log x + \log\left(\frac{x + 1}{x}\right) - \left(\log x + \log\left(\frac{x + 1}{x}\right)\right)\right] + \\
&\quad \left[\frac{1}{x + 1} e^{-x}\right] - \left[\frac{1}{2\,(x + 1)^2} e^{-2x}\right] + \cdots
\end{aligned}$$

Local communities

Cartesian representation of $f \in \mathbb{R}[\![\mathfrak{b}_1; \ldots; \mathfrak{b}_n]\!]$ (not unique)

$$\begin{aligned} f &= \check{f}(\mathfrak{m}_1, \ldots, \mathfrak{m}_k) \\ \check{f} &\in \mathbb{R}[[z_1, \ldots, z_k]]\, z_1^{\mathbb{Z}} \cdots z_k^{\mathbb{Z}} \\ \mathfrak{m}_i &\in (\mathfrak{b}_1^{\mathbb{R}} \cdots \mathfrak{b}_n^{\mathbb{R}})^{\prec} \end{aligned}$$

Cartesian algebra: family of subalgebras $L_k \subseteq \mathbb{R}[[z_1, \ldots, z_k]]$ with

1. $z_1, \ldots, z_k \in L_k$.
2. L_k stable under division by z_1 when defined.
3. L_k stable under substitution when defined.

Example: smallest Cartesian algebra L^{el} which contains e^{z_1} and $\log(1 + z_1)$.

Local community:

- Cartesian algebra with implicit function theorem.
- Power series analogue of o-minimal structure.

Zero-tests

- **A naive algorithm (Shackell/91, Péladan-Germa/vdH/96)**
  - Let $R_g = \mathbb{R}[g_1, \ldots, g_d]$ stable under $\partial_{z_1}, \ldots, \partial_{z_k}$, with $g_1, \ldots, g_d \in E_k^{\mathrm{el}}$.
  - Example: $R_g = \mathbb{R}[z_1, z_2, \mathrm{e}^{z_1}, \log(1 + z_2 \mathrm{e}^{z_1}), (1 + z_2 \mathrm{e}^{z_1})^{-1}]$.
  - Any $f \in E^{\mathrm{el}}$ is in R_g for suitable subexpressions $g_1, \ldots, g_d$.

  Algorithm for testing $f = 0$:

  1. Let $G := \{f\}$
  2. If $\varphi(0) \neq 0$ for some $\varphi \in G$, then return false
  3. Let $G' := \mathrm{RedGB}(G \cup \partial_{z_1} G \cup \cdots \cup \partial_{z_k} G)$
  4. If $G' = G$ then return true, else go to step 2

- **Zero-tests for constants**
  - **Richardson/97.** $\rightsquigarrow$ Schanuel's conjecture
  - Become a number theorist?
  - Become a model theorist?
  - Ignore the problem?

Meta-expansion of transseries

- **Approximators**

$$\begin{aligned} f &= \underset{n \to \infty}{\operatorname{cwlim}} \check{f}_{;n} \\ \check{f}_{;n} &\in \mathbb{R}[\mathfrak{T}] \end{aligned}$$

$$\begin{aligned} f &= \exp \frac{1}{x} \\ \check{f}_{;n} &= 1 + \frac{1}{x} + \frac{1}{2\,x^2} + \cdots + \frac{1}{n!\,x^n} \end{aligned}$$

$$\begin{aligned} f &= g\,h \\ f_{;n} &= g_{;n}\,h_{;n} \end{aligned}$$

- **Expanders**

$$\begin{aligned} f &= \check{f}_0 + \check{f}_1 + \check{f}_2 + \cdots = \check{f}(1) \\ \check{f}_n &\in \mathbb{R}[\mathfrak{T}] \\ \check{f} &= \check{f}_0 + \check{f}_1\, z + \check{f}_2\, z^2 + \cdots \end{aligned}$$

$$\begin{aligned} f &= \exp \frac{1}{x} \\ \check{f} &= \exp \frac{z}{x} \end{aligned}$$

$$\begin{aligned} f &= g\, h \\ \check{f} &= \check{g}\, \check{h} \end{aligned}$$

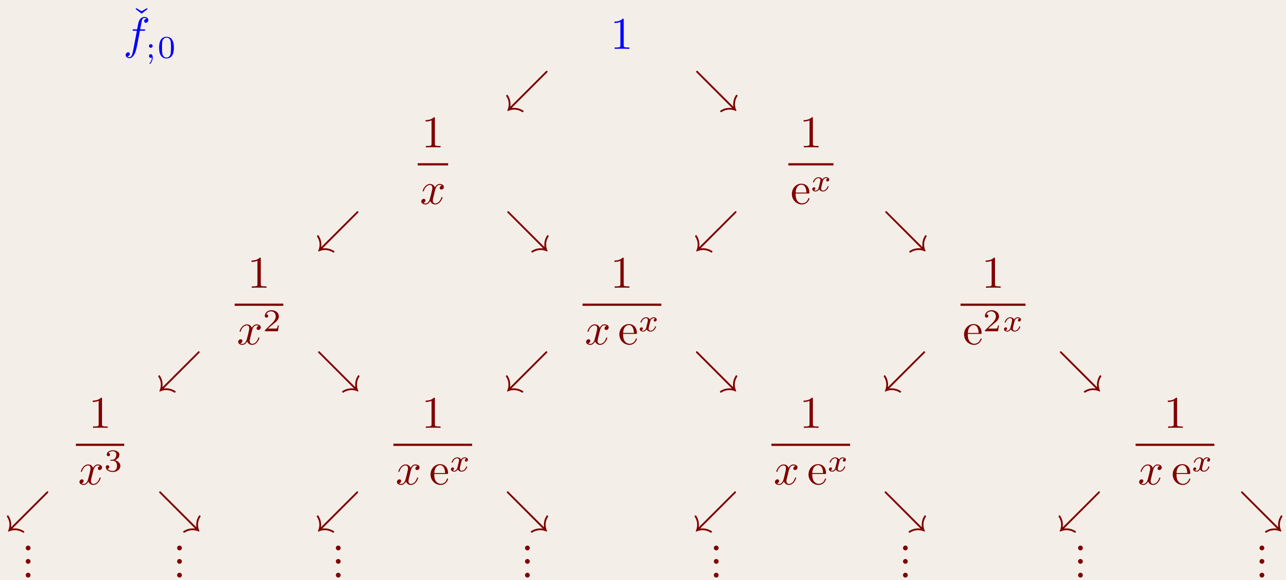
Meta-expansion of transseries

- **Comparison**

$$\begin{aligned}\check{f}_{;n} &= \check{f}_0 + \cdots + \check{f}_n\\ \check{f}_n &= \check{f}_{;n} - \check{f}_{;n-1}\end{aligned}$$

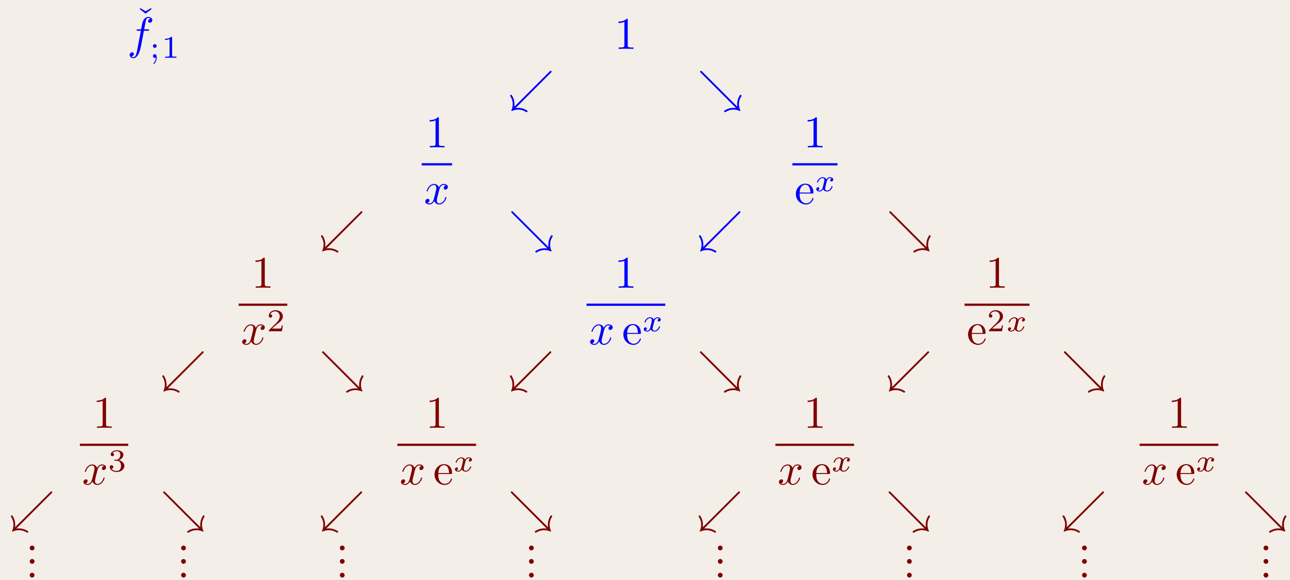
Meta-expansion of transseries

- **Comparison**



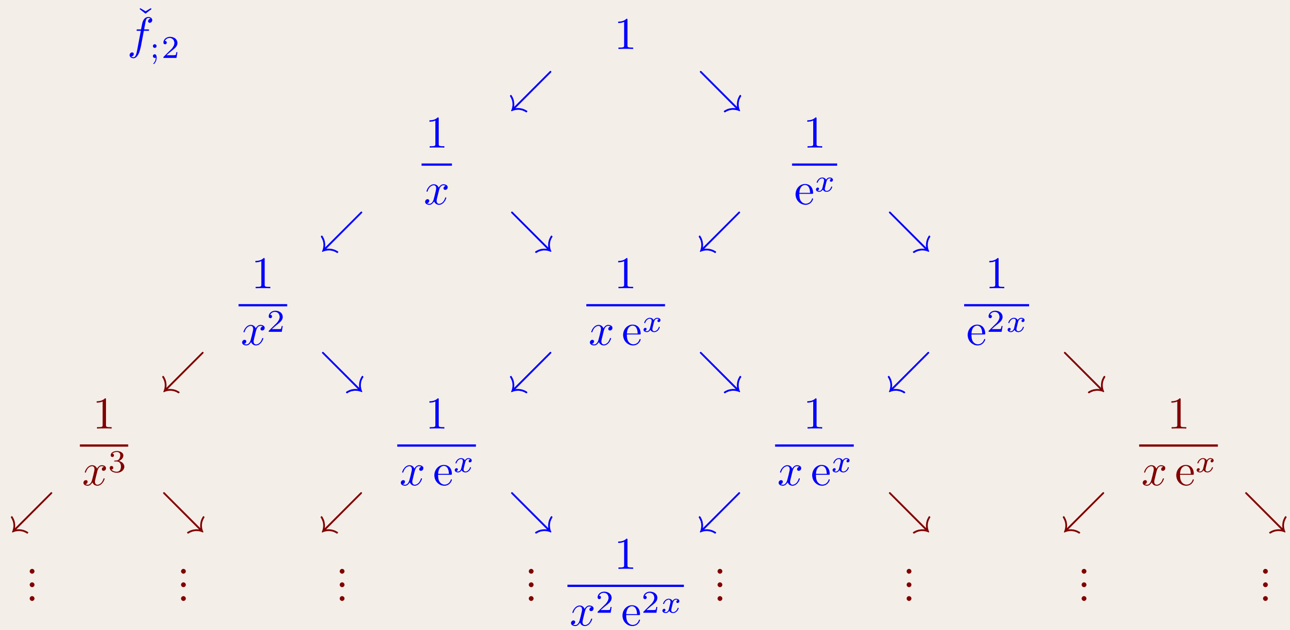
Meta-expansion of transseries

- **Comparison**

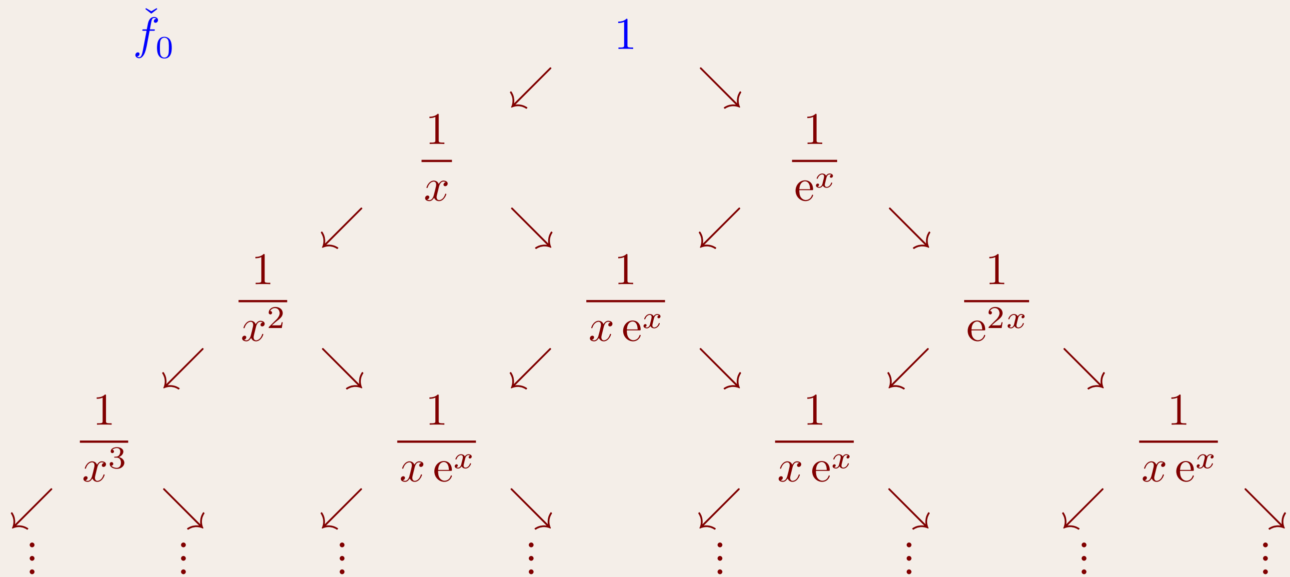


Meta-expansion of transseries

- **Comparison**



- **Comparison**


$\check{f}_0$
1
$\frac{1}{x}$
$\frac{1}{e^x}$
$\frac{1}{x^2}$
$\frac{1}{x\,e^x}$
$\frac{1}{e^{2x}}$
$\frac{1}{x^3}$
$\frac{1}{x\,e^x}$
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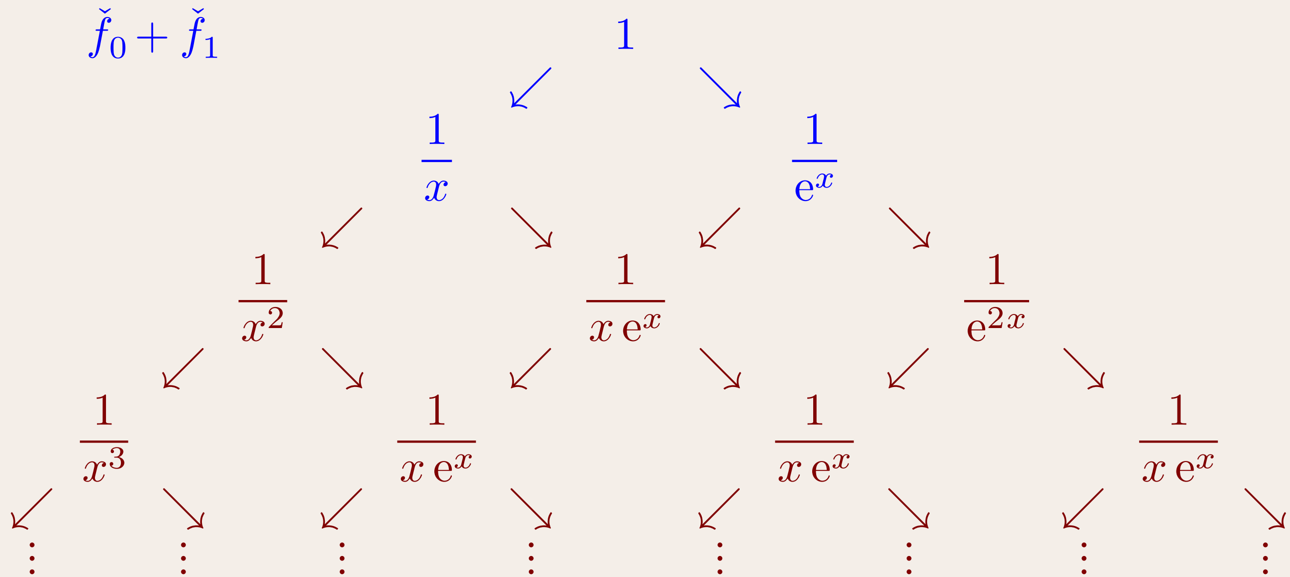
Meta-expansion of transseries

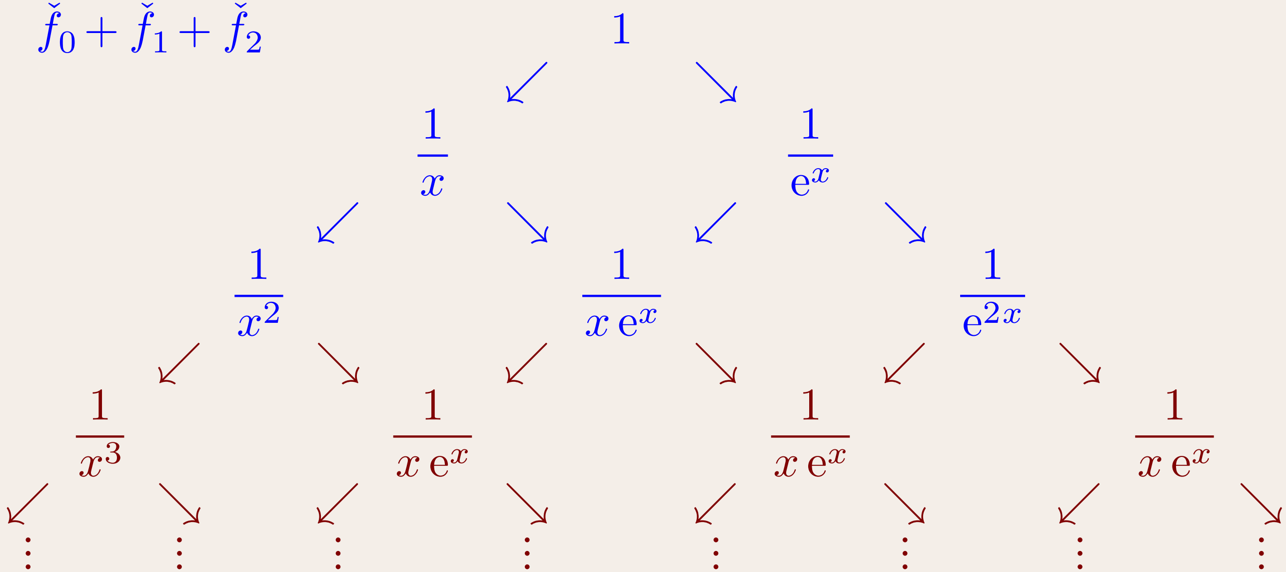
- **Comparison**



Meta-expansion of transseries

- **Comparison**

$\check{f}_0 + \check{f}_1 + \check{f}_2$



Basic operations

- **Left composition with a power series**

$$\begin{aligned} g &= \varphi \circ f & \varphi \in \mathbb{R}[[t]], f \prec 1 \\ \check{g}(z) &= \varphi(z \check{f}(z)) \end{aligned}$$

- **Restriction of support and exponentiation**

$$\begin{aligned} g &= f_{\succ} \\ \check{g}(z) &= \check{f}_{\succ}(z) & \check{g}_n = \check{f}_{n,\succ} \\ \mathrm{e}^f &= \mathrm{e}^{f_{\succ}} \mathrm{e}^{f_{\asymp}} \mathrm{e}^{f_{\prec}} & \text{assuming } \mathfrak{T} \text{ effective} \end{aligned}$$

- **Logarithm and auto-adaptation**

$$\begin{aligned} g &= \log f \\ \check{f}_{;n} &= c_n \mathfrak{d}_n (1 + \check{\varepsilon}_n) \\ \ell^{(n)} &= \log \mathfrak{d}_n + \log c_n + \log (1 + \varepsilon_n) \\ \check{g}_{;n} &= \ell^{(n)}_{;n} \end{aligned}$$

Examples in Mathemagix

```
Mmx] use "symbolix"; use "multimix";
Mmx] x == infinity ('x);
Mmx] 1 / (x + 1)
Mmx] foo == (1 / (1 - 1/x)) * (1 / (1 - exp (-x)))
Mmx] foo[5]
Mmx] 1 / (x + log x + log log x)
Mmx] lengthen (log log (x * exp (x * exp x) + 1) -
               exp exp (log log x + (1/x)), 1)
Mmx]
```

Meta-operations on expanders

$$\begin{aligned} \check{g} &= \Phi(\check{f}) \\ g &= f \end{aligned}$$

- **Shortening and lengthening**

$$\begin{aligned} \check{f} &\longmapsto z\check{f} && \text{Shorten} \\ \check{f} &\longmapsto \check{f}_0 + \frac{1}{z}(\check{f} - \check{f}_0) && \text{Lengthen} \end{aligned}$$

- **Dominant bias**

- **Stabilization**

$$(\mathrm{stab}\, f)_{;n} = \sum_{\substack{\mathfrak{m} \in \mathrm{supp}\, \check{f}_{;n} \\ \check{f}_{;n+1,\mathfrak{m}} = \check{f}_{;n,\mathfrak{m}}}} \check{f}_{;n,\mathfrak{m}} \mathfrak{m} \qquad \text{Stabilize}$$

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- **Dominant bias**

$$\begin{aligned} \check{f}_0\, z^0 &= \tau_{0,0}\, z^0 + \tau_{0,1}\, z^0 + \cdots + \tau_{0,l_0}\, z^0 \\ \check{f}_1\, z^1 &= \tau_{1,0}\, z^1 + \tau_{1,1}\, z^1 + \cdots + \tau_{1,l_1}\, z^1 \\ \check{f}_2\, z^2 &= \tau_{2,0}\, z^2 + \tau_{2,1}\, z^2 + \cdots + \tau_{2,l_2}\, z^2 \\ &\vdots \end{aligned}$$

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$$\begin{aligned} \check{f}_0\, z^0 &\rightsquigarrow \tau_{0,0}\, z^0 + \tau_{0,1}\, z^1 + \cdots + \tau_{0,l_0}\, z^{l_0} \\ \check{f}_1\, z^1 &\rightsquigarrow \tau_{1,0}\, z^1 + \tau_{1,1}\, z^2 + \cdots + \tau_{1,l_1}\, z^{l_1+1} \\ \check{f}_2\, z^2 &\rightsquigarrow \tau_{2,0}\, z^2 + \tau_{2,1}\, z^3 + \cdots + \tau_{2,l_2}\, z^{l_2+2} \\ &\vdots \end{aligned}$$

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- **Stabilization**

$$(\operatorname{stab} f)_{;n} = \sum_{\substack{\mathfrak{m} \in \operatorname{supp} \check{f}_{;n} \\ \check{f}_{;n+1,\mathfrak{m}} = \check{f}_{;n,\mathfrak{m}}}} \check{f}_{;n,\mathfrak{m}}\, \mathfrak{m} \qquad \text{Stabilize}$$

Example of stabilization

$$f = \log\log\left(x\,\mathrm{e}^{x\,\mathrm{e}^{x}} + 1\right) - \exp\exp\left(\log\log x + \tfrac{1}{x}\right).$$

$$\begin{aligned}
\check{f}_{;0} &= \log x \\
\check{f}_{;1} &= \log x + \frac{\log x}{x\,\mathrm{e}^{x}} + \frac{1}{x\,\mathrm{e}^{x}} \\
\check{f}_{;2} &= \frac{\log x}{x\,\mathrm{e}^{x}} + \frac{1}{x\,\mathrm{e}^{x}} - \frac{\log^2 x}{2\,x^2\,\mathrm{e}^{2x}} - \frac{\log x}{x^2\,\mathrm{e}^{2x}} - \frac{1}{2\,x^2\,\mathrm{e}^{2x}} \\
\check{f}_{;3} &= -\frac{\log x}{2\,x} + \frac{\log x}{x\,\mathrm{e}^{x}} + \frac{1}{x\,\mathrm{e}^{x}} - \frac{\log^2 x}{2\,x^2\,\mathrm{e}^{2x}} - \frac{\log x}{x^2\,\mathrm{e}^{2x}} - \frac{1}{2\,x^2\,\mathrm{e}^{2x}} + \\
&\quad \frac{\log^3 x}{3\,x^3\,\mathrm{e}^{3x}} + \frac{\log^2 x}{x^3\,\mathrm{e}^{3x}} + \frac{\log x}{x^3\,\mathrm{e}^{3x}} + \frac{1}{3\,x^3\,\mathrm{e}^{3x}}
\end{aligned}$$

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$$\begin{aligned}
\check{f}_{;0} &= \log x\\
\check{f}_{;1} &= \frac{\log x}{x\,\mathrm{e}^{x}} + \frac{1}{x\,\mathrm{e}^{x}}\\
\check{f}_{;2} &= \frac{\log x}{x\,\mathrm{e}^{x}} + \frac{1}{x\,\mathrm{e}^{x}} - \frac{\log^2 x}{2\,x^2\,\mathrm{e}^{2x}} - \frac{\log x}{x^2\,\mathrm{e}^{2x}} - \frac{1}{2\,x^2\,\mathrm{e}^{2x}}\\
\check{f}_{;3} &= -\frac{\log x}{2\,x} + \frac{\log x}{x\,\mathrm{e}^{x}} + \frac{1}{x\,\mathrm{e}^{x}} - \frac{\log^2 x}{2\,x^2\,\mathrm{e}^{2x}} - \frac{\log x}{x^2\,\mathrm{e}^{2x}} - \frac{1}{2\,x^2\,\mathrm{e}^{2x}} +\\
&\quad \frac{\log^3 x}{3\,x^3\,\mathrm{e}^{3x}} + \frac{\log^2 x}{x^3\,\mathrm{e}^{3x}} + \frac{\log x}{x^3\,\mathrm{e}^{3x}} + \frac{1}{3\,x^3\,\mathrm{e}^{3x}}
\end{aligned}$$

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$$\begin{aligned}
\check{f}_{;0} &= \\
\check{f}_{;1} &= \frac{\log x}{x\,\mathrm{e}^{x}} + \frac{1}{x\,\mathrm{e}^{x}} \\
\check{f}_{;2} &= \frac{\log x}{x\,\mathrm{e}^{x}} + \frac{1}{x\,\mathrm{e}^{x}} - \frac{\log^2 x}{2\,x^2\,\mathrm{e}^{2x}} - \frac{\log x}{x^2\,\mathrm{e}^{2x}} - \frac{1}{2\,x^2\,\mathrm{e}^{2x}} \\
\check{f}_{;3} &= -\frac{\log x}{2\,x} + \frac{\log x}{x\,\mathrm{e}^{x}} + \frac{1}{x\,\mathrm{e}^{x}} - \frac{\log^2 x}{2\,x^2\,\mathrm{e}^{2x}} - \frac{\log x}{x^2\,\mathrm{e}^{2x}} - \frac{1}{2\,x^2\,\mathrm{e}^{2x}} + \\
&\quad \frac{\log^3 x}{3\,x^3\,\mathrm{e}^{3x}} + \frac{\log^2 x}{x^3\,\mathrm{e}^{3x}} + \frac{\log x}{x^3\,\mathrm{e}^{3x}} + \frac{1}{3\,x^3\,\mathrm{e}^{3x}}
\end{aligned}$$

Example of stabilization

$$f = \log\log\left(x\,\mathrm{e}^{x\,\mathrm{e}^x} + 1\right) - \exp\exp\left(\log\log x + \tfrac{1}{x}\right).$$

$$\begin{aligned}
\check{f}_{;4} \;=\;& -\frac{\log^2 x}{2\,x} - \frac{\log x}{2\,x} - \frac{\log x}{6\,x^2} + \frac{\log x}{x\,\mathrm{e}^x} + \frac{1}{x\,\mathrm{e}^x} - \frac{\log^2 x}{2\,x^2\,\mathrm{e}^{2x}} - \frac{\log x}{x^2\,\mathrm{e}^{2x}} - \frac{1}{2\,x^2\,\mathrm{e}^{2x}} + \frac{\log^3 x}{3\,x^3\,\mathrm{e}^{3x}} + \\
& \frac{\log^2 x}{x^3\,\mathrm{e}^{3x}} + \frac{\log x}{x^3\,\mathrm{e}^{3x}} + \frac{1}{3\,x^3\,\mathrm{e}^{3x}} - \frac{\log^4 x}{4\,x^4\,\mathrm{e}^{4x}} - \frac{\log^3 x}{x^4\,\mathrm{e}^{4x}} - \frac{3\log^2 x}{2\,x^4\,\mathrm{e}^{4x}} - \frac{\log x}{x^4\,\mathrm{e}^{4x}} - \frac{1}{4\,x^4\,\mathrm{e}^{4x}}
\end{aligned}$$

$$\begin{aligned}
\check{f}_{;5} \;=\;& -\frac{\log^2 x}{2\,x} - \frac{\log x}{2\,x} - \frac{\log^2 x}{2\,x^2} - \frac{\log x}{6\,x^2} - \frac{\log x}{24\,x^3} + \frac{\log x}{x\,\mathrm{e}^x} + \frac{1}{x\,\mathrm{e}^x} - \frac{\log^2 x}{2\,x^2\,\mathrm{e}^{2x}} - \frac{\log x}{x^2\,\mathrm{e}^{2x}} - \\
& \frac{1}{2\,x^2\,\mathrm{e}^{2x}} + \frac{\log^3 x}{3\,x^3\,\mathrm{e}^{3x}} + \frac{\log^2 x}{x^3\,\mathrm{e}^{3x}} + \frac{\log x}{x^3\,\mathrm{e}^{3x}} + \frac{1}{3\,x^3\,\mathrm{e}^{3x}} - \frac{\log^4 x}{4\,x^4\,\mathrm{e}^{4x}} - \frac{\log^3 x}{x^4\,\mathrm{e}^{4x}} - \frac{3\log^2 x}{2\,x^4\,\mathrm{e}^{4x}} - \\
& \frac{\log x}{x^4\,\mathrm{e}^{4x}} - \frac{1}{4\,x^4\,\mathrm{e}^{4x}} + \frac{\log^5 x}{5\,x^5\,\mathrm{e}^{5x}} + \frac{\log^4 x}{x^5\,\mathrm{e}^{5x}} + \frac{2\log^3 x}{x^5\,\mathrm{e}^{5x}} + \frac{2\log^2 x}{x^5\,\mathrm{e}^{5x}} + \frac{\log x}{x^5\,\mathrm{e}^{5x}} + \frac{1}{5\,x^5\,\mathrm{e}^{5x}}
\end{aligned}$$

Example of stabilization

$$f = \log\log\left(x\,\mathrm{e}^{x\,\mathrm{e}^{x}} + 1\right) - \exp\exp\left(\log\log x + \tfrac{1}{x}\right).$$

$$\begin{aligned}
\check{f}_{;4} \;=\;& -\frac{\log^2 x}{2\,x} - \frac{\log x}{2\,x} - \frac{\log x}{6\,x^2} + \frac{\log x}{x\,\mathrm{e}^{x}} + \frac{1}{x\,\mathrm{e}^{x}} - \frac{\log^2 x}{2\,x^2\,\mathrm{e}^{2x}} - \frac{\log x}{x^2\,\mathrm{e}^{2x}} - \frac{1}{2\,x^2\,\mathrm{e}^{2x}} + \frac{\log^3 x}{3\,x^3\,\mathrm{e}^{3x}} + \\
& \frac{\log^2 x}{x^3\,\mathrm{e}^{3x}} + \frac{\log x}{x^3\,\mathrm{e}^{3x}} + \frac{1}{3\,x^3\,\mathrm{e}^{3x}} - \frac{\log^4 x}{4\,x^4\,\mathrm{e}^{4x}} - \frac{\log^3 x}{x^4\,\mathrm{e}^{4x}} - \frac{3\log^2 x}{2\,x^4\,\mathrm{e}^{4x}} - \frac{\log x}{x^4\,\mathrm{e}^{4x}} - \frac{1}{4\,x^4\,\mathrm{e}^{4x}}
\end{aligned}$$

$$\begin{aligned}
\check{f}_{;5} \;=\;& -\frac{\log^2 x}{2\,x} - \frac{\log x}{2\,x} + O\!\left(\frac{\log^2 x}{2\,x^2}\right) - \frac{\log x}{6\,x^2} + O\!\left(\frac{\log x}{24\,x^3}\right) + \frac{\log x}{x\,\mathrm{e}^{x}} + \frac{1}{x\,\mathrm{e}^{x}} - \frac{\log^2 x}{2\,x^2\,\mathrm{e}^{2x}} - \\
& \frac{\log x}{x^2\,\mathrm{e}^{2x}} - \frac{1}{2\,x^2\,\mathrm{e}^{2x}} + \frac{\log^3 x}{3\,x^3\,\mathrm{e}^{3x}} + \frac{\log^2 x}{x^3\,\mathrm{e}^{3x}} + \frac{\log x}{x^3\,\mathrm{e}^{3x}} + \frac{1}{3\,x^3\,\mathrm{e}^{3x}} - \frac{\log^4 x}{4\,x^4\,\mathrm{e}^{4x}} - \frac{\log^3 x}{x^4\,\mathrm{e}^{4x}} - \\
& \frac{3\log^2 x}{2\,x^4\,\mathrm{e}^{4x}} - \frac{\log x}{x^4\,\mathrm{e}^{4x}} - \frac{1}{4\,x^4\,\mathrm{e}^{4x}} + O\!\left(\frac{\log^5 x}{5\,x^5\,\mathrm{e}^{5x}}\right)
\end{aligned}$$

Strongly linear operations

- **Extension by strong linearity**

$$L\colon \mathfrak{T} \longrightarrow \mathbb{R}[[\mathfrak{T}]]$$

$$\begin{aligned} L\colon \mathbb{R}[[\mathfrak{T}]] &\longrightarrow \mathbb{R}[[\mathfrak{T}]] \\ \sum_{\mathfrak{m}\in\mathfrak{T}} f_{\mathfrak{m}}\,\mathfrak{m} &\longmapsto \sum_{\mathfrak{m}\in\mathfrak{T}} f_{\mathfrak{m}}\,L\mathfrak{m} \end{aligned}$$

$$(\widecheck{Lf})_n = \sum_{p+q=n} \sum_{\mathfrak{m}\in\operatorname{supp}\check{f}_p} \check{f}_{p,\mathfrak{m}}\,(\widecheck{L\mathfrak{m}})_q$$

- **Differentiation**

$$\begin{aligned} (\log_n x)' &= \frac{1}{x\log x\cdots\log_{n-1} x} \\ (\mathrm{e}^f)' &= f'\,\mathrm{e}^f \end{aligned}$$

- **Composition**

$$\begin{aligned} (\log_n x)\circ g &= \log_n g \\ (\mathrm{e}^f)\circ g &= \mathrm{e}^{f\circ g} \end{aligned}$$

Integration

- **Trace of the integration**

$$T\,\mathfrak{m} = \tau_{\int \mathfrak{m}} \qquad \text{(dominant term)}$$
$$= \begin{cases} \dfrac{\mathfrak{m}^2}{\tau_{\mathfrak{m}'}} & \text{if } \log \mathfrak{m} \succcurlyeq x \\ [T((x\,\mathfrak{m}) \circ \exp)] \circ \log & \text{otherwise} \end{cases}$$

Extend by strong linearity

- **Integration**

$$\Delta = 1 - \partial T$$
$$\int = T\,(1 + \Delta + \Delta^2 + \cdots)$$

$$\check{\int}(z) = \sum_{n \geqslant 0} T\,(1 - \Delta)^n\, z^n$$
$$(\widecheck{\int f})(z) = \check{\int}(z)\,\check{f}(z) = \sum_{n \geqslant 0} \sum_{k \geqslant 0} (\check{\int}_n\, \check{f}_k)\, z^{n+k}$$

- **Composition**

$$(\log_n x) \circ g = \log_n g$$
$$(\mathrm{e}^f) \circ g = \mathrm{e}^{f \circ g}$$

Demonstration inside Mathemagix

```
Mmx] use "symbolix"; use "multimix";
```

```
Mmx] x == infinity ('x);
```

```
Mmx] 1 / (1 - 1/x - exp (-x))
```

$$1+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+O\left(\frac{1}{x^4}\right)+\frac{1}{\mathrm{e}^{x}}+\frac{2}{x\,\mathrm{e}^{x}}+\frac{3}{x^2\,\mathrm{e}^{x}}+O\left(\frac{1}{x^3\,\mathrm{e}^{x}}\right)+\frac{1}{\mathrm{e}^{2x}}+\frac{3}{x\,\mathrm{e}^{2x}}+O\left(\frac{1}{x^2\,\mathrm{e}^{2x}}\right)+\frac{1}{\mathrm{e}^{3x}}+O\left(\frac{1}{x\,\mathrm{e}^{3x}}\right)$$

```
Mmx] lengthen fixed_point_expander (f :-> (exp x) * integrate (exp (-x) * ((log x) * f *
     f + exp (-x)), x))
```

$$\frac{-1}{2\,\mathrm{e}^{x}}-\frac{\log(x)}{12\,\mathrm{e}^{2x}}-\frac{1}{36\,x\,\mathrm{e}^{2x}}+\frac{1}{108\,x^2\,\mathrm{e}^{2x}}-\frac{1}{162\,x^3\,\mathrm{e}^{2x}}+\frac{1}{162\,x^4\,\mathrm{e}^{2x}}+O\left(\frac{1}{x^5\,\mathrm{e}^{2x}}\right)-\frac{\log(x)^2}{48\,\mathrm{e}^{3x}}-\frac{5\log(x)}{288\,x\,\mathrm{e}^{3x}}+\frac{23\log(x)}{3456\,x^2\,\mathrm{e}^{3x}}-\frac{5}{1152\,x^2\,\mathrm{e}^{3x}}+O\left(\frac{\log(x)}{x^3\,\mathrm{e}^{3x}}\right)+\frac{53}{13824\,x^3\,\mathrm{e}^{3x}}+O\left(\frac{1}{x^4\,\mathrm{e}^{3x}}\right)$$

```
Mmx] fixed_point_expander (f :-> 1/x + derive (f@(x^2), x) + f@exp((log x)^2))
```

$$\frac{1}{x}-\frac{2}{x^3}+\frac{12}{x^7}-\frac{168}{x^{15}}+O\left(\frac{1}{x^{31}}\right)+\frac{1}{\mathrm{e}^{\log(x)^2}}-\frac{2}{\mathrm{e}^{3\log(x)^2}}-\frac{8\log(x)}{x\,\mathrm{e}^{4\log(x)^2}}+\frac{12}{\mathrm{e}^{7\log(x)^2}}+\frac{48\log(x)}{x\,\mathrm{e}^{12\log(x)^2}}+O\left(\frac{1}{\mathrm{e}^{15\log(x)^2}}\right)+\frac{512\log(x)^2}{x^3\,\mathrm{e}^{16\log(x)^2}}+\frac{32\log(x)}{x^3\,\mathrm{e}^{16\log(x)^2}}-\frac{16}{x^3\,\mathrm{e}^{16\log(x)^2}}+O\left(\frac{\log(x)}{x\,\mathrm{e}^{28\log(x)^2}}\right)+\frac{1}{\mathrm{e}^{\log(x)^4}}-\frac{2}{\mathrm{e}^{3\log(x)^4}}-\frac{8\log(x)^2}{\mathrm{e}^{4\log(x)^4+\log(x)^2}}+O\left(\frac{1}{\mathrm{e}^{7\log(x)^4}}\right)-\frac{64\log(x)^3}{x\,\mathrm{e}^{16\log(x)^4}}+O\left(\frac{\log(x)^4}{\mathrm{e}^{16\log(x)^4+3\log(x)^2}}\right)+\frac{1}{\mathrm{e}^{\log(x)^8}}+O\left(\frac{1}{\mathrm{e}^{3\log(x)^8}}\right)$$

```
Mmx] lengthen (product (x, x), 8)
```

$$\mathrm{e}^{x\log(x)-x-\frac{\log(x)}{2}}+\frac{\mathrm{e}^{x\log(x)-x-\frac{\log(x)}{2}}}{12\,x}+\frac{\mathrm{e}^{x\log(x)-x-\frac{\log(x)}{2}}}{288\,x^2}-\frac{139\,\mathrm{e}^{x\log(x)-x-\frac{\log(x)}{2}}}{51840\,x^3}-\frac{571\,\mathrm{e}^{x\log(x)-x-\frac{\log(x)}{2}}}{2488320\,x^4}+\frac{163879\,\mathrm{e}^{x\log(x)-x-\frac{\log(x)}{2}}}{209018880\,x^5}+O\left(\frac{\mathrm{e}^{x\log(x)-x-\frac{\log(x)}{2}}}{x^6}\right)$$

```
Mmx]
```