# Relaxed power series solutions of differentially algebraic equations


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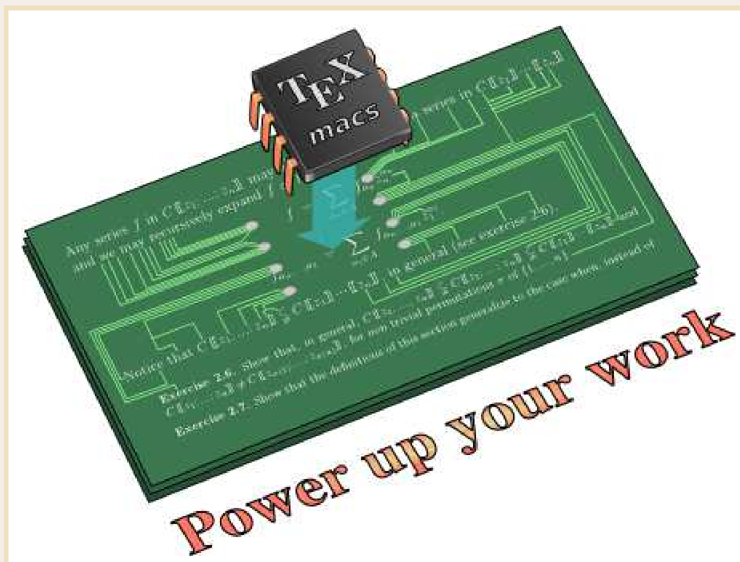


**ACA, Vlora, 2010**

http://www.TeXmacs.org

# Newton's method

**Logarithm:**

$$\log f \;=\; \int \frac{f'}{f}$$

Multiplication at order $n$: $\mathsf{M}(n) = O(n \log n \log \log n)$

Division at order $n$: $O(\mathsf{M}(n))$

Logarithm at order $n$: $O(\mathsf{M}(n))$

**Exponentiation:** [Brent & Kung 78]

$$\begin{aligned} \log \tilde{f} &= g + O(z^{\lceil n/2 \rceil}) \\ f &:= \tilde{f}\,(1 - \log \tilde{f} + g) \\ \log f &= g + O(z^n) \end{aligned}$$

Exponentiation at order $n$: $O(\mathsf{M}(n))$

## Exponentation

$$g = \exp f = \int f' \, g$$

$$g_n = \frac{1}{n} \sum_{i=0}^{n-1} (n+1-i) \, f_{n-i} \, g_i$$

## Example $f = z + z^2 + z^3 + \cdots$

$$g_0 = 1$$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $\uparrow$ | | | | | | | |
| | $g_5$ | | | | | | | |
| | $g_4$ | | | | | | | |
| | $g_3$ | | | | | | | |
| | $g_2$ | | | | | | | |
| | $g_1$ | | | | | | | |
| | $g_0$ | | | | | | | |
| | | $f'_0$ | $f'_1$ | $f'_2$ | $f'_3$ | $f'_4$ | $f'_5$ | $\rightarrow$ |
| | | 1 | 2 | 3 | 4 | 5 | 6 | |

# Lazy power series

## Exponentation

$$g = \exp f = \int f' g$$

$$g_n = \frac{1}{n} \sum_{i=0}^{n-1} (n+1-i) f_{n-i} g_i$$

## Example $f = z + z^2 + z^3 + \cdots$

$$g_0 = 1$$
$$g_1 = 1$$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $\uparrow$ | | | | | | | |
| | $g_5$ | | | | | | | |
| | $g_4$ | | | | | | | |
| | $g_3$ | | | | | | | |
| | $g_2$ | | | | | | | |
| | $g_1$ | | | | | | | |
| 1 | $g_0$ | 1 | | | | | | |
| | | $f'_0$ | $f'_1$ | $f'_2$ | $f'_3$ | $f'_4$ | $f'_5$ | $\rightarrow$ |
| | | 1 | 2 | 3 | 4 | 5 | 6 | |

# Lazy power series

## Exponentation

$$
\begin{aligned}
g &= \exp f = \int f' g \\
g_n &= \frac{1}{n} \sum_{i=0}^{n-1} (n+1-i)\, f_{n-i}\, g_i
\end{aligned}
$$

## Example $f = z + z^2 + z^3 + \cdots$

$$
\begin{aligned}
g_0 &= 1 \\
g_1 &= 1 \\
g_2 &= \tfrac{3}{2}
\end{aligned}
$$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $\uparrow$ | | | | | | | |
| | $g_5$ | | | | | | | |
| | $g_4$ | | | | | | | |
| | $g_3$ | | | | | | | |
| | $g_2$ | | | | | | | |
| 1 | $g_1$ | 1 | | | | | | |
| 1 | $g_0$ | 1 | 2 | | | | | |
| | | $f'_0$ | $f'_1$ | $f'_2$ | $f'_3$ | $f'_4$ | $f'_5$ | $\rightarrow$ |
| | | 1 | 2 | 3 | 4 | 5 | 6 | |

# Lazy power series

## Exponentation

$$g = \exp f = \int f' g$$
$$g_n = \frac{1}{n} \sum_{i=0}^{n-1} (n+1-i) f_{n-i} g_i$$

## Example $f = z + z^2 + z^3 + \cdots$

$$g_0 = 1$$
$$g_1 = 1$$
$$g_2 = \frac{3}{2}$$
$$g_3 = \frac{13}{6}$$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $\uparrow$ | | | | | | | |
| | $g_5$ | | | | | | | |
| | $g_4$ | | | | | | | |
| | $g_3$ | | | | | | | |
| $\frac{3}{2}$ | $g_2$ | $\frac{3}{2}$ | | | | | | |
| 1 | $g_1$ | 1 | 2 | | | | | |
| 1 | $g_0$ | 1 | 2 | 3 | | | | |
| | | $f'_0$ | $f'_1$ | $f'_2$ | $f'_3$ | $f'_4$ | $f'_5$ | $\rightarrow$ |
| | | 1 | 2 | 3 | 4 | 5 | 6 | |

## Exponentation

$$g = \exp f = \int f' \, g$$

$$g_n = \frac{1}{n} \sum_{i=0}^{n-1} (n+1-i) \, f_{n-i} \, g_i$$

## Example $f = z + z^2 + z^3 + \cdots$

$$g_0 = 1$$
$$g_1 = 1$$
$$g_2 = \tfrac{3}{2}$$
$$g_3 = \tfrac{13}{6}$$
$$g_4 = \tfrac{73}{24}$$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $\uparrow$ | | | | | | | |
| | $g_5$ | | | | | | | |
| | $g_4$ | | | | | | | |
| $\frac{13}{6}$ | $g_3$ | $\frac{13}{6}$ | | | | | | |
| $\frac{3}{2}$ | $g_2$ | $\frac{3}{2}$ | 3 | | | | | |
| 1 | $g_1$ | 1 | 2 | 3 | | | | |
| 1 | $g_0$ | 1 | 2 | 3 | 4 | | | |
| | | $f'_0$ | $f'_1$ | $f'_2$ | $f'_3$ | $f'_4$ | $f'_5$ | $\rightarrow$ |
| | | 1 | 2 | 3 | 4 | 5 | 6 | |

# Lazy power series

## Exponentation

$$g = \exp f = \int f' \, g$$

$$g_n = \frac{1}{n} \sum_{i=0}^{n-1} (n+1-i) \, f_{n-i} \, g_i$$

## Example $f = z + z^2 + z^3 + \cdots$

$$g_0 = 1$$
$$g_1 = 1$$
$$g_2 = \frac{3}{2}$$
$$g_3 = \frac{13}{6}$$
$$g_4 = \frac{73}{24}$$
$$g_5 = \frac{167}{40}$$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $\uparrow$ | | | | | | | |
| | $g_5$ | | | | | | | |
| $\frac{73}{24}$ | $g_4$ | $\frac{73}{24}$ | | | | | | |
| $\frac{13}{6}$ | $g_3$ | $\frac{13}{6}$ | $\frac{13}{3}$ | | | | | |
| $\frac{3}{2}$ | $g_2$ | $\frac{3}{2}$ | 3 | $\frac{9}{2}$ | | | | |
| 1 | $g_1$ | 1 | 2 | 3 | 4 | | | |
| 1 | $g_0$ | 1 | 2 | 3 | 4 | 5 | | |
| | | $f'_0$ | $f'_1$ | $f'_2$ | $f'_3$ | $f'_4$ | $f'_5$ | $\rightarrow$ |
| | | 1 | 2 | 3 | 4 | 5 | 6 | |

# Lazy power series

## Exponentation

$$g \;=\; \exp f \;=\; \int f'\, g$$

$$g_n \;=\; \frac{1}{n} \sum_{i=0}^{n-1} (n+1-i)\, f_{n-i}\, g_i$$

## Example $f = z + z^2 + z^3 + \cdots$

$$g_0 = 1$$
$$g_1 = 1$$
$$g_2 = \tfrac{3}{2}$$
$$g_3 = \tfrac{13}{6}$$
$$g_4 = \tfrac{73}{24}$$
$$g_5 = \tfrac{167}{40}$$
$$g_6 = \tfrac{4051}{720}$$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $\uparrow$ | | | | | | | |
| $\frac{167}{40}$ | $g_5$ | $\frac{167}{40}$ | | | | | | |
| $\frac{73}{24}$ | $g_4$ | $\frac{73}{24}$ | $\frac{73}{12}$ | | | | | |
| $\frac{13}{6}$ | $g_3$ | $\frac{13}{6}$ | $\frac{13}{3}$ | $\frac{13}{2}$ | | | | |
| $\frac{3}{2}$ | $g_2$ | $\frac{3}{2}$ | 3 | $\frac{9}{2}$ | 6 | | | |
| 1 | $g_1$ | 1 | 2 | 3 | 4 | 5 | | |
| 1 | $g_0$ | 1 | 2 | 3 | 4 | 5 | 6 | |
| | | $f'_0$ | $f'_1$ | $f'_2$ | $f'_3$ | $f'_4$ | $f'_5$ | $\rightarrow$ |
| | | 1 | 2 | 3 | 4 | 5 | 6 | |

## Anticipation $\rightsquigarrow$ acceleration

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\uparrow$ | | | | | | | | |
| | $g_6$ | | | | | | | | |
| | $g_5$ | | | | | | | | |
| | $g_4$ | | | | | | | | |
| | $g_3$ | | | | | | | | |
| | $g_2$ | | | | | | | | |
| | $g_1$ | | | | | | | | |
| 1 | $g_0$ | 1 | | | | | | | |
| | | $f'_0$ | $f'_1$ | $f'_2$ | $f'_3$ | $f'_4$ | $f'_5$ | $f'_6$ | $\rightarrow$ |
| | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | |

# Relaxed power series

## Anticipation $\rightsquigarrow$ acceleration

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\uparrow$ | | | | | | | | |
| | $g_6$ | | | | | | | | |
| | $g_5$ | | | | | | | | |
| | $g_4$ | | | | | | | | |
| | $g_3$ | | | | | | | | |
| | $g_2$ | | | | | | | | |
| 1 | $g_1$ | 1 | | | | | | | |
| 1 | $g_0$ | 1 | 2 | | | | | | |
| | | $f_0'$ | $f_1'$ | $f_2'$ | $f_3'$ | $f_4'$ | $f_5'$ | $f_6'$ | $\rightarrow$ |
| | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | |

# Relaxed power series

## Anticipation $\rightsquigarrow$ acceleration

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\uparrow$ | | | | | | | | |
| | $g_6$ | | | | | | | | |
| | $g_5$ | | | | | | | | |
| | $g_4$ | | | | | | | | |
| | $g_3$ | | | | | | | | |
| $\frac{3}{2}$ | $g_2$ | $\frac{3}{2}$ | $3$ | $\frac{9}{2}$ | | | | | |
| $1$ | $g_1$ | $1$ | $2$ | $3$ | | | | | |
| $1$ | $g_0$ | $1$ | $2$ | $3$ | | | | | |
| | | $f'_0$ | $f'_1$ | $f'_2$ | $f'_3$ | $f'_4$ | $f'_5$ | $f'_6$ | $\rightarrow$ |
| | | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | |

## Anticipation $\rightsquigarrow$ acceleration

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\uparrow$ | | | | | | | | |
| | $g_6$ | | | | | | | | |
| | $g_5$ | | | | | | | | |
| | $g_4$ | | | | | | | | |
| $\frac{13}{6}$ | $g_3$ | $\frac{13}{6}$ | | | | | | | |
| $\frac{3}{2}$ | $g_2$ | $\frac{3}{2}$ | $3$ | $\frac{9}{2}$ | | | | | |
| $1$ | $g_1$ | $1$ | $2$ | $3$ | | | | | |
| $1$ | $g_0$ | $1$ | $2$ | $3$ | $4$ | | | | |
| | | $f'_0$ | $f'_1$ | $f'_2$ | $f'_3$ | $f'_4$ | $f'_5$ | $f'_6$ | $\rightarrow$ |
| | | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | |

# Relaxed power series

## Anticipation $\rightsquigarrow$ acceleration

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\uparrow$ | | | | | | | | |
| | $g_6$ | | | | | | | | |
| | $g_5$ | | | | | | | | |
| $\frac{73}{24}$ | $g_4$ | $\frac{73}{24}$ | $\frac{73}{12}$ | $\frac{73}{8}$ | | | | | |
| $\frac{13}{6}$ | $g_3$ | $\frac{13}{6}$ | $\frac{13}{3}$ | $\frac{13}{2}$ | | | | | |
| $\frac{3}{2}$ | $g_2$ | $\frac{3}{2}$ | $3$ | $\frac{9}{2}$ | $6$ | $\frac{15}{2}$ | | | |
| $1$ | $g_1$ | $1$ | $2$ | $3$ | $4$ | $5$ | | | |
| $1$ | $g_0$ | $1$ | $2$ | $3$ | $4$ | $5$ | | | |
| | | $f'_0$ | $f'_1$ | $f'_2$ | $f'_3$ | $f'_4$ | $f'_5$ | $f'_6$ | $\rightarrow$ |
| | | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | |

# Relaxed power series

## Anticipation ⇝ acceleration

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\uparrow$ | | | | | | | | |
| | $g_6$ | | | | | | | | |
| $\frac{167}{40}$ | $g_5$ | $\frac{167}{40}$ | | | | | | | |
| $\frac{73}{24}$ | $g_4$ | $\frac{73}{24}$ | $\frac{73}{12}$ | $\frac{73}{8}$ | | | | | |
| $\frac{13}{6}$ | $g_3$ | $\frac{13}{6}$ | $\frac{13}{3}$ | $\frac{13}{2}$ | | | | | |
| $\frac{3}{2}$ | $g_2$ | $\frac{3}{2}$ | 3 | $\frac{9}{2}$ | 6 | $\frac{15}{2}$ | | | |
| 1 | $g_1$ | 1 | 2 | 3 | 4 | 5 | | | |
| 1 | $g_0$ | 1 | 2 | 3 | 4 | 5 | 6 | | |
| | | $f'_0$ | $f'_1$ | $f'_2$ | $f'_3$ | $f'_4$ | $f'_5$ | $f'_6$ | $\rightarrow$ |
| | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | |

## Anticipation $\rightsquigarrow$ acceleration

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\uparrow$ | | | | | | | | |
| $\frac{4051}{720}$ | $g_6$ | $\frac{4051}{720}$ | $\frac{4051}{360}$ | $\frac{4051}{240}$ | $\frac{4051}{190}$ | $\frac{4051}{144}$ | $\frac{4051}{120}$ | $\frac{28357}{720}$ | |
| $\frac{167}{40}$ | $g_5$ | $\frac{167}{40}$ | $\frac{167}{20}$ | $\frac{501}{40}$ | $\frac{167}{10}$ | $\frac{167}{8}$ | $\frac{501}{20}$ | $\frac{1169}{40}$ | |
| $\frac{73}{24}$ | $g_4$ | $\frac{73}{24}$ | $\frac{73}{12}$ | $\frac{73}{8}$ | $\frac{73}{6}$ | $\frac{365}{24}$ | $\frac{73}{4}$ | $\frac{511}{24}$ | |
| $\frac{13}{6}$ | $g_3$ | $\frac{13}{6}$ | $\frac{13}{3}$ | $\frac{13}{2}$ | $\frac{26}{3}$ | $\frac{65}{6}$ | $13$ | $\frac{91}{6}$ | |
| $\frac{3}{2}$ | $g_2$ | $\frac{3}{2}$ | $3$ | $\frac{9}{2}$ | $6$ | $\frac{15}{2}$ | $9$ | $\frac{21}{2}$ | |
| 1 | $g_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | |
| 1 | $g_0$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | |
| | | $f_0'$ | $f_1'$ | $f_2'$ | $f_3'$ | $f_4'$ | $f_5'$ | $f_6'$ | $\rightarrow$ |
| | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | |

# Relaxed power series

## Anticipation ⇝ acceleration

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\uparrow$ | | | | | | | | |
| $\frac{4051}{720}$ | $g_6$ | $\frac{4051}{720}$ | $\frac{4051}{360}$ | $\frac{4051}{240}$ | $\frac{4051}{190}$ | $\frac{4051}{144}$ | $\frac{4051}{120}$ | $\frac{28357}{720}$ | |
| $\frac{167}{40}$ | $g_5$ | $\frac{167}{40}$ | $\frac{167}{20}$ | $\frac{501}{40}$ | $\frac{167}{10}$ | $\frac{167}{8}$ | $\frac{501}{20}$ | $\frac{1169}{40}$ | |
| $\frac{73}{24}$ | $g_4$ | $\frac{73}{24}$ | $\frac{73}{12}$ | $\frac{73}{8}$ | $\frac{73}{6}$ | $\frac{365}{24}$ | $\frac{73}{4}$ | $\frac{511}{24}$ | |
| $\frac{13}{6}$ | $g_3$ | $\frac{13}{6}$ | $\frac{13}{3}$ | $\frac{13}{2}$ | $\frac{26}{3}$ | $\frac{65}{6}$ | $13$ | $\frac{91}{6}$ | |
| $\frac{3}{2}$ | $g_2$ | $\frac{3}{2}$ | $3$ | $\frac{9}{2}$ | $6$ | $\frac{15}{2}$ | $9$ | $\frac{21}{2}$ | |
| $1$ | $g_1$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | |
| $1$ | $g_0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | |
| | | $f'_0$ | $f'_1$ | $f'_2$ | $f'_3$ | $f'_4$ | $f'_5$ | $f'_6$ | $\rightarrow$ |
| | | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | |

$$\begin{aligned} \mathsf{R}(n) &= O(2\,\mathsf{M}(n/2) + 4\,\mathsf{M}(n/4) + 8\,\mathsf{M}(n/8) + \cdots) \\ &= O(\mathsf{M}(n) \log n). \end{aligned}$$

# Solving differential equations

## Algebraic dynamical system

$$F = F_0 + \int \Phi(F), \qquad \Phi \in \mathbb{K}[F]^r, F \in \mathbb{K}[[z]]^r, s = \text{size}(\Phi)$$

## Relaxed resolution

$$\mathsf{T}(n) = s\,\mathsf{R}(n)$$

$$\mathsf{R}(n) = O(\mathsf{M}(n)\log n) \qquad \text{[vdH 97]}$$

$$\mathsf{R}(n) = O(\mathsf{M}(n)\,\mathrm{e}^{2\sqrt{\log\log n}}), \text{ nice } \mathbb{K} \qquad \text{[vdH 03]}$$

$$\mathsf{R}(n) = \mathsf{M}(n), \text{ naive or Karatsuba model} \qquad \text{[vdH 97]}$$

Also works for functional equations and partial differential equations.

## Newton's method

$$\mathsf{T}(n) = \mathsf{M}(n)\,(s\,r + \tfrac{8}{3}r^2 + \tfrac{4}{3}r) + \cdots \qquad \text{[Sedoglavic 01]}$$

$$\mathsf{T}(n) = \mathsf{M}(n)\,(s + \tfrac{4}{3}r) + \cdots, \text{ if } r = O(\log n), \text{ nice } \mathbb{K} \qquad \text{[vdH 06]}$$

# Implicit equations

## Algebraic differential equation

$$\Phi(F) = 0, \qquad \Phi \in \mathbb{K}[F, \ldots, F^{(r)}]^d, F \in \mathbb{K}[[z]]^d$$

Assume unique solution for given $F_0, \ldots, F_k \in \mathbb{K}^d$.

## Assume ultimate linearity

$$\Phi(F + \delta F) = \frac{\partial \Phi}{\partial F}(F)\,\delta F + \cdots + \frac{\partial \Phi}{\partial F^{(r)}}(F)\,(\delta F)^{(r)}.$$

Assume $\frac{\partial \Phi}{\partial F^{(j)}} \neq 0$ for some $j$.

## Extraction of coefficients

$$\begin{aligned} \Phi(F)_n &\in \mathbb{K}[n] + \mathbb{K}[n]\,F_{n-v} + \cdots + \mathbb{K}[n]\,F_{n-v-(i-1)} \\ v &: \text{shift} \\ i &: \text{index} \end{aligned}$$

for sufficiently large $n$, say $n > k$.

# Series with undetermined coefficients

## The $\mathbb{K}$-vector space of "tangent numbers" $\mathbb{D}$

$$\begin{aligned} c &= (f_c, n_c, i_c, c^*, c^{(0)}, \ldots, c^{(i_c-1)}) \\ f_c &: \text{pointer to relaxed solution of } \Phi(F)=0 \\ n_c, i_c &\in \mathbb{N} \\ c^*, c^{(0)}, \ldots, c^{(i_c-1)} &\in \mathbb{K} \\ c &\rightsquigarrow c^* + c^{(0)} F_{n_c} + \cdots + c^{(i_c-1)} F_{n_c-(i_c-1)} \end{aligned}$$

## One step substitution

$$\begin{aligned} \mathbb{D}_i &= \{c \in \mathbb{D} : i_c \leqslant i\} \\ \mathbb{D}_0 &\cong \mathbb{K} \\ \tau : \mathbb{D}_{i+1} &\rightarrow \mathbb{D}_i \\ \tau(c) &\rightsquigarrow [c^* + c^{(i)} f_{n_c-i}] + c^{(0)} F_{n_c} + \cdots + c^{(i-1)} F_{n_c-(i-1)} \end{aligned}$$

# Substitution product

| | | | | | |
|---|---|---|---|---|---|
| | | | | | |
| | | | | | |
| | | | $\tilde{\tilde{a}}_3\,\tilde{\tilde{b}}_2$ | $\tilde{\tilde{a}}_4\,\tilde{\tilde{b}}_2$ | $\tilde{\tilde{a}}_5\,\tilde{\tilde{b}}_2$ |
| | | | $[a_3^{(0)}\,F_5 + \tilde{a}_3^*]\,b_1$ | $[a_4^{(0)}\,F_6 + \tilde{a}_4^*]\,b_1$ | $[a_5^{(0)}\,F_7 + \tilde{a}_5^*]\,b_1$ |
| | | | $[a_3^{(0)}\,F_5 + a_3^{(1)}\,F_4 + a_3^*]\,b_0$ | $[a_4^{(0)}\,F_6 + a_4^{(1)}\,F_5 + a_4^*]\,b_0$ | $[a_5^{(0)}\,F_7 + a_5^{(1)}\,F_6 + a_5^*]\,b_0$ |

$$
\begin{aligned}
a *_{i+1} b &= a_0\,b_0 + a_0\,(b - b_0) + (a - a_0)\,b_0 + \tilde{a} *_i \tilde{b},\\
\tilde{a} &= \tau(a - a_0)\\
\tilde{b} &= \tau(b - b_0)\\
a *_0 b &= a\,b
\end{aligned}
$$

# Relaxed resolution of implicit equations

## Successive systems of coefficients

$$\begin{aligned} \Sigma_n &:= \{\Phi(F)_j = 0 \colon j \leqslant n\} \\ \text{trigonalization } \Sigma_n &\rightsquigarrow F_0, \ldots, F_{n-v} \text{ and system } \Sigma'_n \text{ in } F_{n-v+1}, \ldots, F_v \\ \text{trigonalization } \Sigma_n &\Leftrightarrow \text{trigonalization } \Sigma_n \cup \{\Phi(F)_n = 0\} \end{aligned}$$

## Complexity analysis

Substitution product in $\mathbb{D}_i[[z]]$: $O(\mathsf{R}(n) + i\, r\, n)$

Trigonalization linear systems $\Sigma_n$: $O(i^2\, r^3\, n)$

Total cost: $O(s\, \mathsf{R}(n) + (s + i\, r^2)\, i\, r\, n)$

## Algebraic case

Coefficients $a_n^{(j)}$ do not depend on $n$

Total cost: $O(s\, \mathsf{R}(n) + (s + r)\, i\, r\, n)$

# Example in MATHEMAGIX

```
Mmx] use "algebramix";
Mmx] z: Series Rational == series (0, 1);
```

```
Mmx] Z: Series Unknown Rational == z;
```

```
Mmx] eq (f) == z + f - f * f + derive (z^3 * f);
```

```
Mmx] sol == implicit_series (eq, 1 :> Rational)
```

$1 + z + 2\,z^2 + 6\,z^4 - 12\,z^5 + 42\,z^6 - 132\,z^7 + 438\,z^8 - 1524\,z^9 + O(z^{10})$

```
Mmx] eq (sol)
```

$O(z^{10})$

```
Mmx] sol[1000]
```

1587009312385076787686959166342859223672549364354245675973269253623241913773709937482225\
7523028060113733705473909420019020860469524170097285036148971247503534381935356229147 94\
9854331343836157421192143092475065519119935749003664296008264451695796769060432328962 95\
1929578329775017287012333954428539664154050961746766388931943158521109673413513629571 31\
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