On the Complexity of Polynomial Reduction

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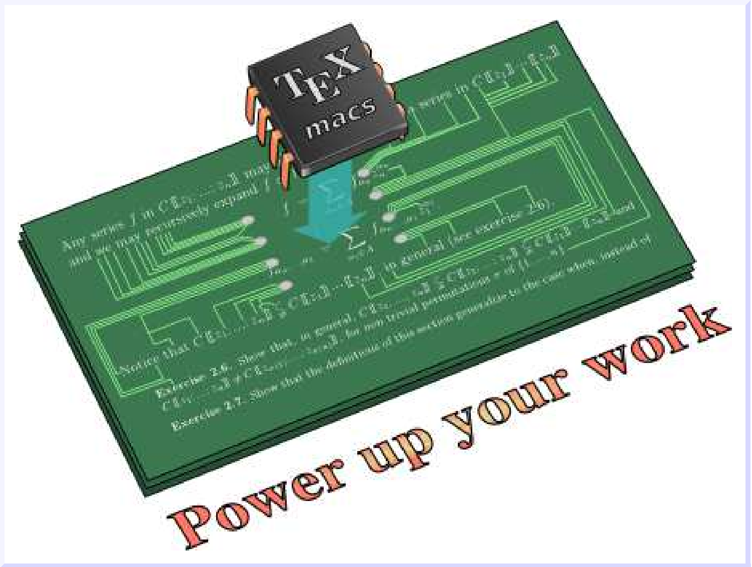

Καλαμάτα, July 23, 2015

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Complexity of univariate polynomial arithmetic

Theorem. *[Gauss, Cooley–Tukey, Schönhage–Strassen, Cantor–Kaltofen, ...]*

Two polynomials $P, Q \in \mathbb{K}[x]$ *of degree* $<n$ *over an abstract field* $\mathbb{K}$ *can be multiplied in time* $\mathsf{M}(n) = \mathcal{O}(n \log n \log\log n)$.

Theorem. *Given two polynomials* $A, B \in \mathbb{K}[x]$ *of degree* $<n$ *and* $B \neq 0$, *we may compute* $Q, R \in \mathbb{K}[x]$ *such that*

$$A = Q\,B + R, \qquad \deg R < \deg B$$

in time $\mathcal{O}(\mathsf{M}(n))$.

Complexity of multivariate polynomial arithmetic

Given $P = \sum_{i \in \mathbb{N}^n} P_i x^i \in \mathbb{K}[x] = \mathbb{K}[x_1, \ldots, x_n]$, let

$$\operatorname{supp} P = \{i \in \mathbb{N}^n : P_i \neq 0\}.$$

Theorem. *[Prony, Blahut, Ben Or–Tiwari, Canny–Kaltofen–Lakshman, ...]*

Given $P, Q \in \mathbb{K}[x]$ (with $\mathbb{K}$ of characteristic zero) such that a bound

$$\operatorname{supp}(P Q) \subseteq \operatorname{supp} P + \operatorname{supp} Q \subseteq \mathcal{S}$$

is known, we may compute $P Q$ in time $\mathcal{O}(M(s) \log s)$, where $s = |\mathcal{S}|$.

Theorem. *[vdH, vdH–Schost] If $\mathcal{S}$ is an "initial segment" of $\mathbb{N}^n$, then we may compute $P Q$ in time $\mathcal{O}(s \log s \log \log s)$.*

Question. *Given an autoreduced tuple $B = (B_1, \ldots, B_b) \in \mathbb{K}[x]^b$ and $A \in \mathbb{K}[x]$, can we use fast multiplication for efficiently computing a relation*

$$A = Q_1 B_1 + \cdots + Q_b B_b + R, \qquad R \text{ irreducible w.r.t. } B?$$

Problem. Given $f, g \in \mathbb{K}[[z]]$ and $h = fg$, compute the coefficients $h_0, \ldots, h_{n-1}$ with the extra condition that h_i must be output as soon as $f_0, g_0, \ldots, f_i, g_i$ are known for each $i < n$.

Theorem. *[vdH] This can be done in time $\mathcal{O}(\mathsf{M}(n) \log n)$.*

Application. Assume that we want to compute $g = (1 - zf)^{-1}$, where $f \in \mathbb{K}[[z]]$. Then it suffices to evaluate

$$g \; = \; 1 + zfg,$$

where we notice that $g_0 = 1$ and

$$g_i \; = \; \sum_{j=0}^{i-1} f_{i-1-j} g_j$$

for all $i > 0$.

Remark. For the computation of the product fg in this application, the argument f is fixed. We also call this a "semi-relaxed multiplication". General relaxed multiplications can actually be reduced to semi-relaxed multiplications with constant overhead.

$$f = 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + 13z^6 + 21z^7 + \cdots$$
$$g = 1 + zfg$$

	$\uparrow$								
70	g_6	70							
29	g_5	29	29						
12	g_4	12	12	24					
5	g_3	5	5	10	15				
2	g_2	2	2	4	6	10			
1	g_1	1	1	2	3	5	8		
1	g_0	1	1	2	3	5	8	13	
		f_0	f_1	f_2	f_3	f_4	f_5	f_6	$\rightarrow$
		1	1	2	3	5	8	13	

$$g = 1 + z + 2z^2 + 5z^3 + 12z^4 + 29z^5 + 70z^6 + \cdots$$
$$fg = 1 + 2z + 5z^2 + 12z^3 + 29z^4 + 70z^5 + 169z^6 + \cdots$$

$$f = 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + 13z^6 + 21z^7 + \cdots$$
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	$\uparrow$								
70	g_6	70							
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2	g_2	2	2	4	6	10			
1	g_1	1	1	2	3	5	8		
1	g_0	1	1	2	3	5	8	13	
		f_0	f_1	f_2	f_3	f_4	f_5	f_6	$\rightarrow$
		1	1	2	3	5	8	13	

$$g = 1 + z + 2z^2 + 5z^3 + 12z^4 + 29z^5 + 70z^6 + \cdots$$
$$fg = 1 + 2z + 5z^2 + 12z^3 + 29z^4 + 70z^5 + 169z^6 + \cdots$$

$$f = 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + 13z^6 + 21z^7 + \cdots$$
$$g = 1 + zfg$$

	↑								
70	g_6	70							
29	g_5	29	29						
12	g_4	12	12	24					
5	g_3	5	5	10	15				
2	g_2	2	2	4	6	10			
1	g_1	1	1	2	3	5	8		
1	g_0	1	1	2	3	5	8	13	
		f_0	f_1	f_2	f_3	f_4	f_5	f_6	→
		1	1	2	3	5	8	13	

$$g = 1 + z + 2z^2 + 5z^3 + 12z^4 + 29z^5 + 70z^6 + \cdots$$
$$fg = 1 + 2z + 5z^2 + 12z^3 + 29z^4 + 70z^5 + 169z^6 + \cdots$$

Example, naive lazy method

$$f = 1 + z + 2\,z^2 + 3\,z^3 + 5\,z^4 + 8\,z^5 + 13\,z^6 + 21\,z^7 + \cdots$$
$$g = 1 + z\,f\,g$$

	$\uparrow$								
70	g_6	70							
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12	g_4	12	12	24					
5	g_3	5	5	10	15				
2	g_2	2	2	4	6	10			
1	g_1	1	1	2	3	5	8		
1	g_0	1	1	2	3	5	8	13	
		f_0	f_1	f_2	f_3	f_4	f_5	f_6	$\rightarrow$
		1	1	2	3	5	8	13	

$$g = 1 + z + 2\,z^2 + 5\,z^3 + 12\,z^4 + 29\,z^5 + 70\,z^6 + \cdots$$
$$f\,g = 1 + 2\,z + 5\,z^2 + 12\,z^3 + 29\,z^4 + 70\,z^5 + 169\,z^6 + \cdots$$

$$f = 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + 13z^6 + 21z^7 + \cdots$$
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	$\uparrow$								
70	g_6	70							
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5	g_3	5	5	10	15				
2	g_2	2	2	4	6	10			
1	g_1	1	1	2	3	5	8		
1	g_0	1	1	2	3	5	8	13	
		f_0	f_1	f_2	f_3	f_4	f_5	f_6	$\rightarrow$
		1	1	2	3	5	8	13	

$$g = 1 + z + 2z^2 + 5z^3 + 12z^4 + 29z^5 + 70z^6 + \cdots$$
$$fg = 1 + 2z + 5z^2 + 12z^3 + 29z^4 + 70z^5 + 169z^6 + \cdots$$

Example, naive lazy method

$$f = 1 + z + 2\,z^2 + 3\,z^3 + 5\,z^4 + 8\,z^5 + 13\,z^6 + 21\,z^7 + \cdots$$
$$g = 1 + z f g$$

	$\uparrow$								
70	g_6	70							
29	g_5	29	29						
12	g_4	12	12	24					
5	g_3	5	5	10	15				
2	g_2	2	2	4	6	10			
1	g_1	1	1	2	3	5	8		
1	g_0	1	1	2	3	5	8	13	
		f_0	f_1	f_2	f_3	f_4	f_5	f_6	$\rightarrow$
		1	1	2	3	5	8	13	

$$g = 1 + z + 2\,z^2 + 5\,z^3 + 12\,z^4 + 29\,z^5 + 70\,z^6 + \cdots$$
$$f g = 1 + 2\,z + 5\,z^2 + 12\,z^3 + 29\,z^4 + 70\,z^5 + 169\,z^6 + \cdots$$

$$f = 1 + z + 2\,z^2 + 3\,z^3 + 5\,z^4 + 8\,z^5 + 13\,z^6 + 21\,z^7 + \cdots$$
$$g = 1 + z\,f\,g$$

	$\uparrow$								
70	g_6	70							
29	g_5	29	29						
12	g_4	12	12	24					
5	g_3	5	5	10	15				
2	g_2	2	2	4	6	10			
1	g_1	1	1	2	3	5	8		
1	g_0	1	1	2	3	5	8	13	
		f_0	f_1	f_2	f_3	f_4	f_5	f_6	$\rightarrow$
		1	1	2	3	5	8	13	

$$g = 1 + z + 2\,z^2 + 5\,z^3 + 12\,z^4 + 29\,z^5 + 70\,z^6 + \cdots$$
$$f\,g = 1 + 2\,z + 5\,z^2 + 12\,z^3 + 29\,z^4 + 70\,z^5 + 169\,z^6 + \cdots$$

Example, naive lazy method

$$f = 1 + z + 2\,z^2 + 3\,z^3 + 5\,z^4 + 8\,z^5 + 13\,z^6 + 21\,z^7 + \cdots$$
$$g = 1 + z f g$$

$\vdots$	$\uparrow$	$\vdots$							
70	g_6	70	$\iddots$						
29	g_5	29	29	$\iddots$					
12	g_4	12	12	24	$\iddots$				
5	g_3	5	5	10	15	$\iddots$			
2	g_2	2	2	4	6	10	$\iddots$		
1	g_1	1	1	2	3	5	8	$\iddots$	
1	g_0	1	1	2	3	5	8	13	$\cdots$
		f_0	f_1	f_2	f_3	f_4	f_5	f_6	$\rightarrow$
		1	1	2	3	5	8	13	$\cdots$

$$g = 1 + z + 2\,z^2 + 5\,z^3 + 12\,z^4 + 29\,z^5 + 70\,z^6 + \cdots$$
$$f g = 1 + 2\,z + 5\,z^2 + 12\,z^3 + 29\,z^4 + 70\,z^5 + 169\,z^6 + \cdots$$

$$f = 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + 13z^6 + 21z^7 + \cdots$$
$$g = 1 + zfg$$

	$\uparrow$								
	g_6								
	g_5								
	g_4								
	g_3								
	g_2								
	g_1								
1	g_0	1							
		f_0	f_1	f_2	f_3	f_4	f_5	f_6	$\rightarrow$
		1	1	2	3	5	8	13	

$$g = 1 + \cdots$$
$$fg = 1 + \cdots$$

$$f = 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + 13z^6 + 21z^7 + \cdots$$
$$g = 1 + zfg$$

	$\uparrow$								
	g_6								
	g_5								
	g_4								
	g_3								
	g_2								
1	g_1	1	1	2					
1	g_0	1	1	2					
		f_0	f_1	f_2	f_3	f_4	f_5	f_6	$\rightarrow$
		1	1	2	3	5	8	13	

$$g = 1 + z + \cdots$$
$$fg = 1 + 2z + \cdots$$

Example

$$
\begin{aligned}
f &= 1 + z + 2\,z^2 + 3\,z^3 + 5\,z^4 + 8\,z^5 + 13\,z^6 + 21\,z^7 + \cdots \\
g &= 1 + z f g
\end{aligned}
$$

	$\uparrow$								
	g_6								
	g_5								
	g_4								
	g_3								
2	g_2	2							
1	g_1	1	1	2					
1	g_0	1	1	2					
		f_0	f_1	f_2	f_3	f_4	f_5	f_6	$\rightarrow$
		1	1	2	3	5	8	13	

$$
\begin{aligned}
g &= 1 + z + 2\,z^2 + \cdots \\
f g &= 1 + 2\,z + 5\,z^2 + \cdots
\end{aligned}
$$

$$
\begin{aligned}
f &= 1 + z + 2\,z^2 + 3\,z^3 + 5\,z^4 + 8\,z^5 + 13\,z^6 + 21\,z^7 + \cdots \\
g &= 1 + z\,f\,g
\end{aligned}
$$

	$\uparrow$								
	g_6								
	g_5								
	g_4								
5	g_3	5	5	10	15	25	40	65	
2	g_2	2	2	4	6	10	16	26	
1	g_1	1	1	2	3	5	8	13	
1	g_0	1	1	2	3	5	8	13	
		f_0	f_1	f_2	f_3	f_4	f_5	f_6	$\rightarrow$
		1	1	2	3	5	8	13	

$$
\begin{aligned}
g &= 1 + z + 2\,z^2 + 5\,z^3 + \cdots \\
f\,g &= 1 + 2\,z + 5\,z^2 + 12\,z^3 + \cdots
\end{aligned}
$$

$$f = 1 + z + 2\,z^2 + 3\,z^3 + 5\,z^4 + 8\,z^5 + 13\,z^6 + 21\,z^7 + \cdots$$
$$g = 1 + z\,f\,g$$

	$\uparrow$								
	g_6								
	g_5								
12	g_4	12							
5	g_3	5	5	10	15	25	40	65	
2	g_2	2	2	4	6	10	16	26	
1	g_1	1	1	2	3	5	8	13	
1	g_0	1	1	2	3	5	8	13	
		f_0	f_1	f_2	f_3	f_4	f_5	f_6	$\rightarrow$
		1	1	2	3	5	8	13	

$$g = 1 + z + 2\,z^2 + 5\,z^3 + 12\,z^4 + \cdots$$
$$f\,g = 1 + 2\,z + 5\,z^2 + 12\,z^3 + 29\,z^4 + \cdots$$

$$f = 1 + z + 2\,z^2 + 3\,z^3 + 5\,z^4 + 8\,z^5 + 13\,z^6 + 21\,z^7 + \cdots$$
$$g = 1 + z\,f\,g$$

	$\uparrow$								
	g_6								
29	g_5	29	29	58					
12	g_4	12	12	24					
5	g_3	5	5	10	15	25	40	65	
2	g_2	2	2	4	6	10	16	26	
1	g_1	1	1	2	3	5	8	13	
1	g_0	1	1	2	3	5	8	13	
		f_0	f_1	f_2	f_3	f_4	f_5	f_6	$\rightarrow$
		1	1	2	3	5	8	13	

$$g = 1 + z + 2\,z^2 + 5\,z^3 + 12\,z^4 + 29\,z^5 + \cdots$$
$$f\,g = 1 + 2\,z + 5\,z^2 + 12\,z^3 + 29\,z^4 + 70\,z^5 + \cdots$$

$$f = 1 + z + 2\,z^2 + 3\,z^3 + 5\,z^4 + 8\,z^5 + 13\,z^6 + 21\,z^7 + \cdots$$
$$g = 1 + z\,f\,g$$

	$\uparrow$								
70	g_6	70							
29	g_5	29	29	58					
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5	g_3	5	5	10	15	25	40	65	
2	g_2	2	2	4	6	10	16	26	
1	g_1	1	1	2	3	5	8	13	
1	g_0	1	1	2	3	5	8	13	
		f_0	f_1	f_2	f_3	f_4	f_5	f_6	$\rightarrow$
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$$g = 1 + z + 2\,z^2 + 5\,z^3 + 12\,z^4 + 29\,z^5 + 70\,z^6 + \cdots$$
$$f\,g = 1 + 2\,z + 5\,z^2 + 12\,z^3 + 29\,z^4 + 70\,z^5 + 169\,z^6 + \cdots$$

$$f = 1 + z + 2\,z^2 + 3\,z^3 + 5\,z^4 + 8\,z^5 + 13\,z^6 + 21\,z^7 + \cdots$$
$$g = 1 + z\,f\,g$$

$\vdots$	$\uparrow$	$\vdots$							
70	g_6	70	$\vdots$	$\vdots$					
29	g_5	29	29	58					
12	g_4	12	12	24	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
5	g_3	5	5	10	15	25	40	65	
2	g_2	2	2	4	6	10	16	26	
1	g_1	1	1	2	3	5	8	13	
1	g_0	1	1	2	3	5	8	13	$\vdots$
		f_0	f_1	f_2	f_3	f_4	f_5	f_6	$\rightarrow$
		1	1	2	3	5	8	13	$\cdots$

$$g = 1 + z + 2\,z^2 + 5\,z^3 + 12\,z^4 + 29\,z^5 + 70\,z^6 + \cdots$$
$$f\,g = 1 + 2\,z + 5\,z^2 + 12\,z^3 + 29\,z^4 + 70\,z^5 + 169\,z^6 + \cdots$$

Classical reduction

$$
\begin{aligned}
A &= Q\,B + R \\
A(x) &= A_a\,x^a\,\tilde{A}(z) \qquad z = 1/x \\
B(x) &= B_b\,x^b\,\tilde{B}(z) \\
Q(x) &= (A_a/B_b)\,x^{a-b}\,(\tilde{A}/\tilde{B})(z) + \mathcal{O}(z^{a-b+1})
\end{aligned}
$$

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$$
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\end{aligned}
$$

Computable Laurent series

$\mathbb{K}((z))^{\mathrm{com}} = \{f z^k : f \in \mathbb{K}[[z]]^{\mathrm{com}}, k \in \mathbb{Z}\}$

Classical reduction

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Computable Laurent series

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Tagging and untagging

$$\begin{aligned} \hat{P}(x,z) &= P(x/z) \in \mathbb{K}[x]((z))^{\text{com}} \\ \check{f}(x) &= f(x,1) \end{aligned}$$

Classical reduction

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Tagging and untagging

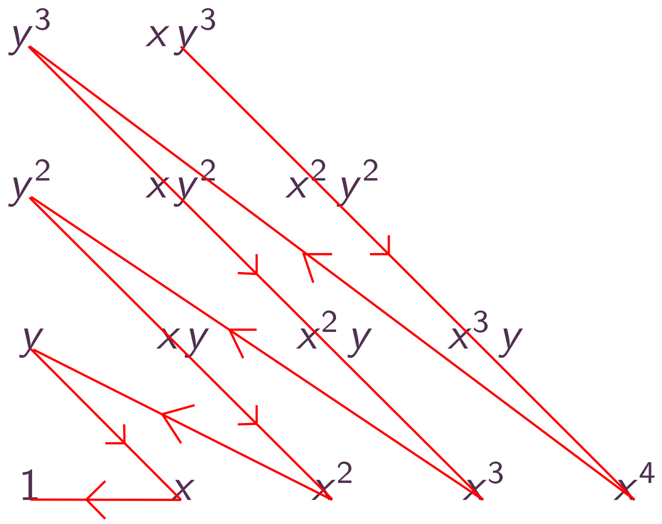
$$
\begin{aligned}
\hat{P}(x,z) &= P(x/z) \in \mathbb{K}[x]((z))^{\mathrm{com}} \\
\check{f}(x) &= f(x,1)
\end{aligned}
$$

$$
\begin{aligned}
P\,Q &= \widecheck{\hat{P}\,\hat{Q}} \qquad \text{(if we want to regard } P \text{ and } Q \text{ as series)} \\
f\,g &= \widehat{\check{f}\,\check{g}} \qquad \text{(for actual fast multiplications)}
\end{aligned}
$$

Monomial ordering

$$x^i y^j > x^{i'} y^{j'} \Leftrightarrow \begin{cases} i+j > i'+j' \\ i+j > i'+j' \wedge j > j' \end{cases}$$

or



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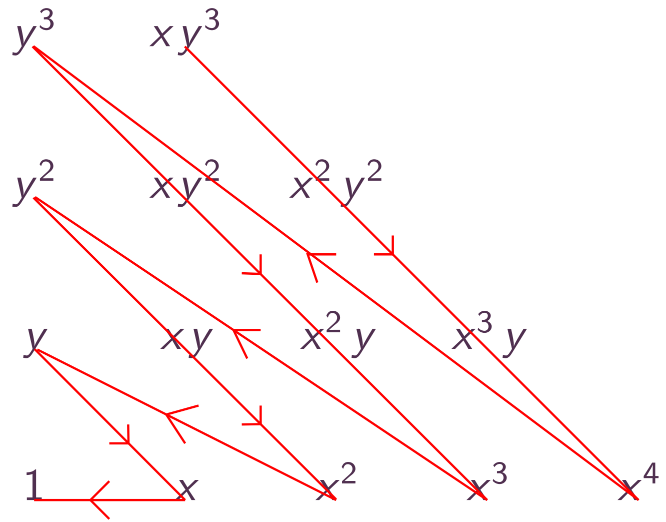

Tagging

$$x^i y^j \longrightarrow x^i y^j u^{-j} z^{-i-j}$$

$\rightsquigarrow$ Series in $\mathbb{K}[x, y]((u))((z))$

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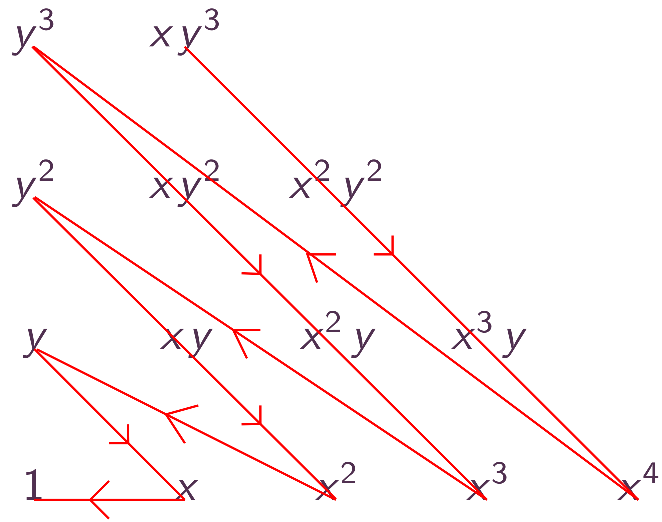

Tagging

$$x^i y^j \longrightarrow x^i y^j \, u^{-j} \, z^{-i-j}$$

$\rightsquigarrow$ Series in $\mathbb{K}[x,y]((u))((z))$

Example

$$\begin{array}{lcl}
P & = & x y^2 + 3 x^2 y + y^2 + 3 x^2 - 2 y + 5 \\
 & \Big\downarrow & x \to x z^{-1},\ y \to y u^{-1} z^{-1} \\
\hat{P} & = & x y^2 u^{-2} z^{-3} + 3 x^2 y u^{-1} z^{-3} + y^2 u^{-2} z^{-2} + 3 x^2 z^{-3} - 2 y u^{-1} z^{-1} + 5 \\
 & \Big\downarrow & u \to 1,\ z \to 1 \\
P & = & x y^2 + 3 x^2 y + y^2 + 3 x^2 - 2 y + 5
\end{array}$$

Monomial ordering

$$i \leqslant j \;\Leftrightarrow\; (\lambda_1 \cdot i, \ldots, \lambda_m \cdot i) \leqslant^{\mathrm{lex}} (\lambda_1 \cdot j, \ldots, \lambda_m \cdot j),$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{N}^n$ and $\gcd((\lambda_i)_1, \ldots, (\lambda_i)_n) = 1$ for all i.

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where $\lambda_1, \ldots, \lambda_n \in \mathbb{N}^n$ and $\gcd((\lambda_i)_1, \ldots, (\lambda_i)_n) = 1$ for all i.

Example of reverse lexicographical ordering

$$\begin{cases} \lambda_1 &= (1, 1, \ldots, 1, 1) \\ \lambda_2 &= (0, 0, \ldots, 0, 1) \\ \lambda_3 &= (0, 0, \ldots, 1, 0) \\ & \vdots \\ \lambda_n &= (0, 1, \ldots, 0, 0) \end{cases}$$

Monomial ordering

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where $\lambda_1, \ldots, \lambda_n \in \mathbb{N}^n$ and $\gcd((\lambda_i)_1, \ldots, (\lambda_i)_n) = 1$ for all i.

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Tagging and untagging

$$\begin{aligned} x_i &\longrightarrow x_i\, z^{-\lambda_i} \\ \mathbb{K}[x_1, \ldots, x_n] &\longrightarrow \mathbb{K}[x_1, \ldots, x_n]((z_n)) \cdots ((z_1)) \end{aligned}$$

Monomial ordering

$$i \leqslant j \;\Leftrightarrow\; (\lambda_1 \cdot i, \ldots, \lambda_m \cdot i) \leqslant^{\mathrm{lex}} (\lambda_1 \cdot j, \ldots, \lambda_m \cdot j),$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{N}^n$ and $\gcd((\lambda_i)_1, \ldots, (\lambda_i)_n) = 1$ for all i.

Example of reverse lexicographical ordering

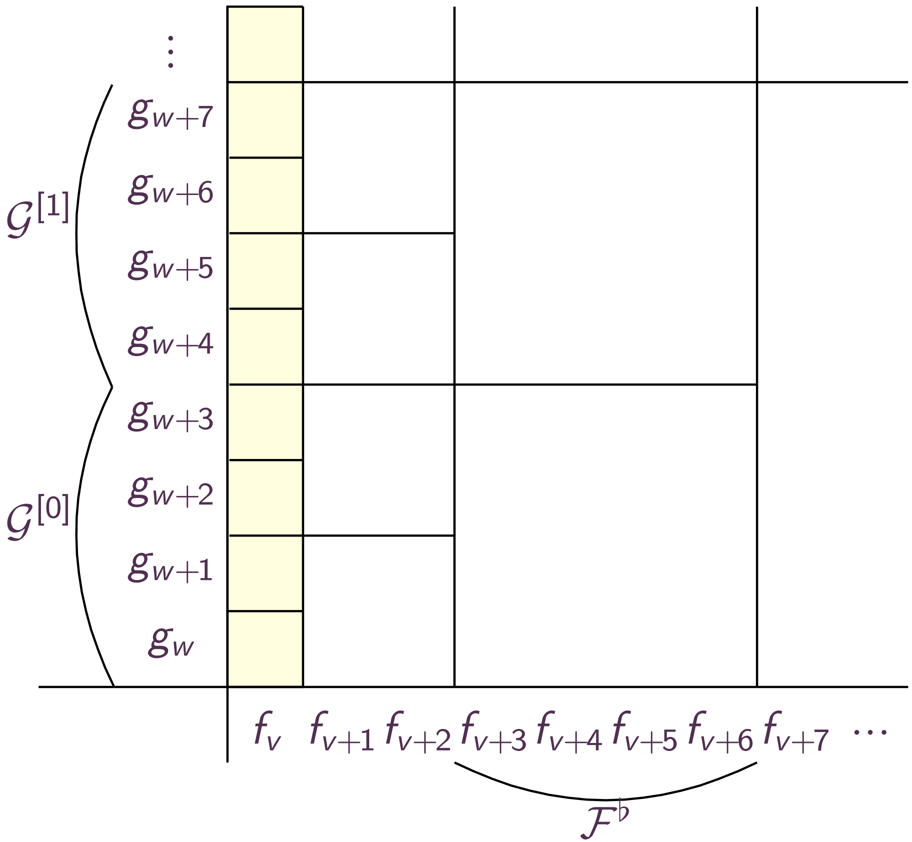
$$\begin{cases} \lambda_1 &= (1, 1, \ldots, 1, 1) \\ \lambda_2 &= (0, 0, \ldots, 0, 1) \\ \lambda_3 &= (0, 0, \ldots, 1, 0) \\ & \vdots \\ \lambda_n &= (0, 1, \ldots, 0, 0) \end{cases}$$

Tagging and untagging

$$\begin{aligned} x_i &\longrightarrow x_i\, z^{-\lambda_i} \\ \mathbb{K}[x_1, \ldots, x_n] &\longrightarrow \mathbb{K}[x_1, \ldots, x_n]((z_n)) \cdots ((z_1)) \end{aligned}$$

Complexity measures for subsets $X \subseteq \mathbb{N}^n$

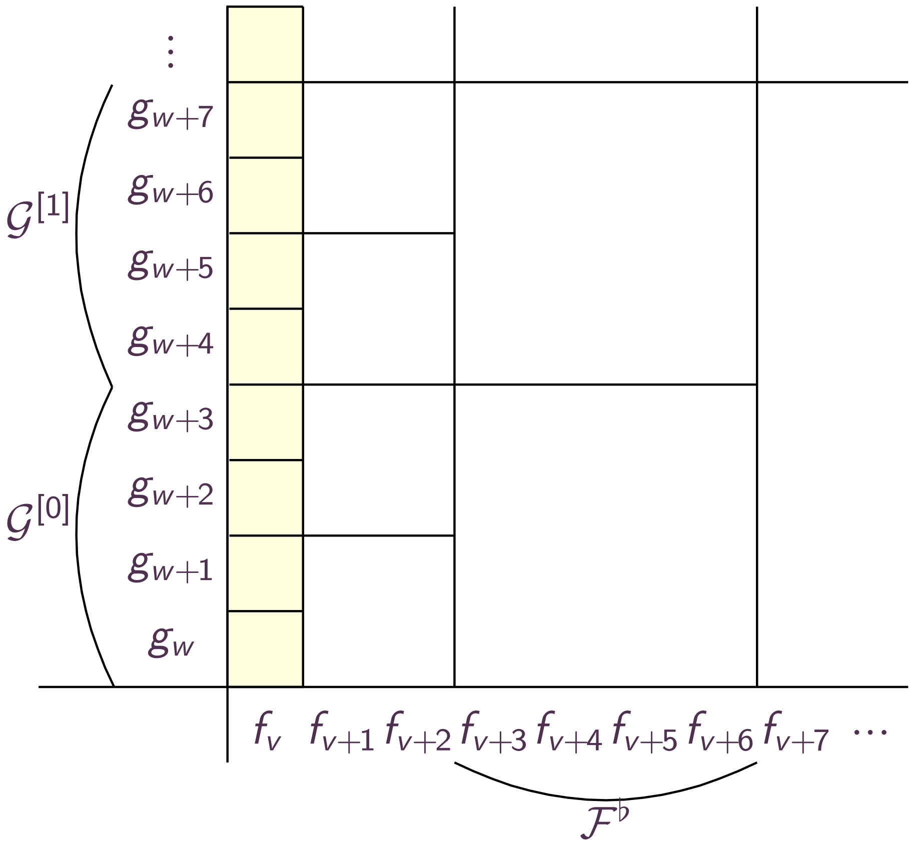
$$\begin{aligned} \delta_i(X) &= \max\{\lambda_i \cdot k : k \in X\} + 1 \\ \delta(X) &= \delta_1(X) \cdots \delta_n(X). \end{aligned}$$



$\mathcal{F} \supseteq \operatorname{supp} f$
$\mathcal{G} \supseteq \operatorname{supp} g$

$$\mathsf{SM}(|\mathcal{F}^\flat + \mathcal{G}^{[0]}|) + \mathsf{SM}(|\mathcal{F}^\flat + \mathcal{G}^{[1]}|) + \cdots + \mathsf{SM}(|\mathcal{F}^\flat + \mathcal{G}^{[\ell-1]}|) \leqslant 2\,\mathsf{SM}(|\mathcal{F} + \mathcal{G}|)$$

$$\mathsf{T}_{z_1} = \mathcal{O}(\mathsf{SM}(|\mathcal{F} + \mathcal{G}|) \log \delta_1(|\mathcal{F} + \mathcal{G}|)) + \mathsf{T}_{z_2,\ldots,z_n}$$



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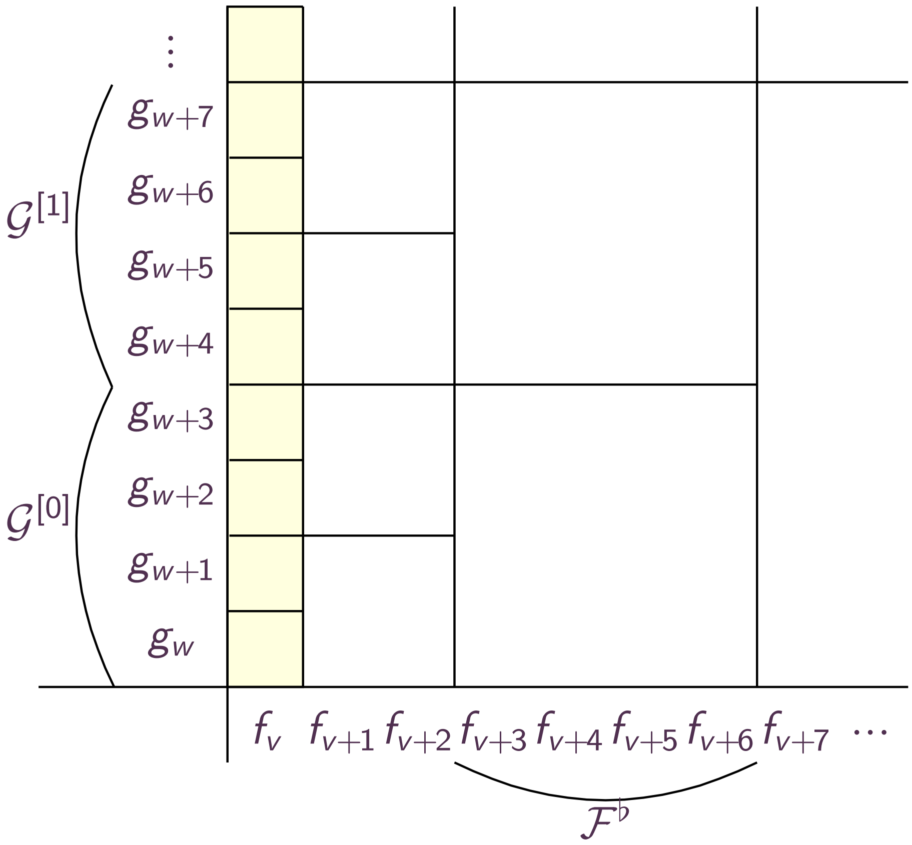

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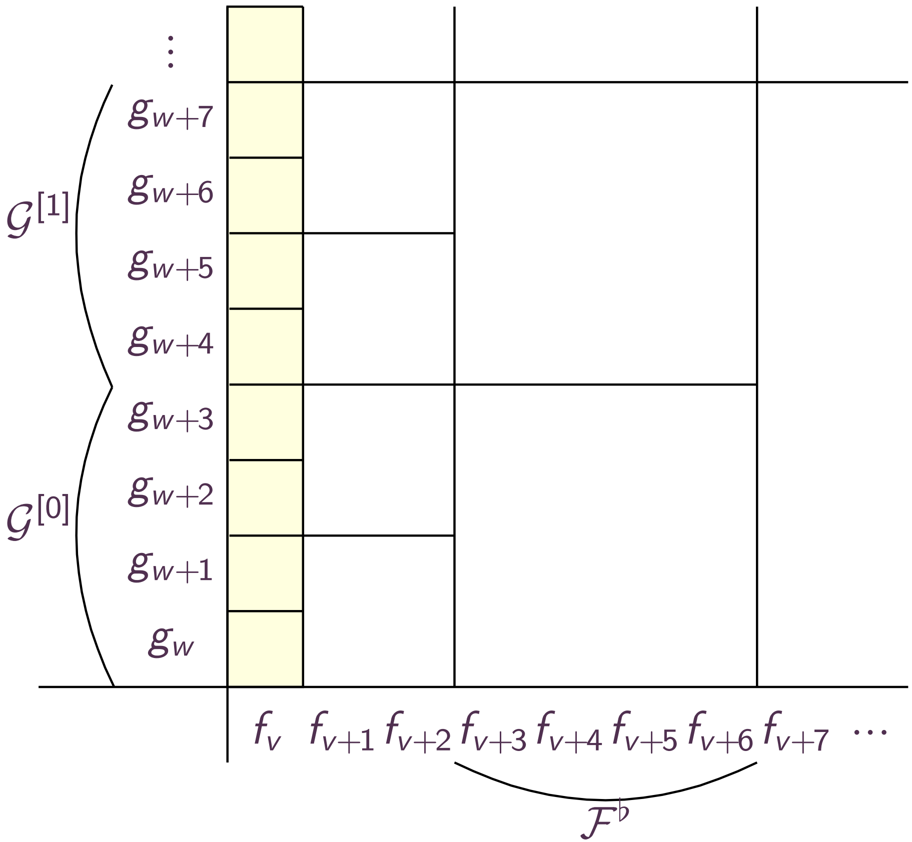
$$\mathsf{T} \leqslant \mathcal{O}(\mathsf{SM}(|\mathcal{F} + \mathcal{G}|)\,(\log \delta_1(|\mathcal{F} + \mathcal{G}|) + \cdots + \log \delta_n(|\mathcal{F} + \mathcal{G}|)))$$



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Constructing a "recursive" equation

$$g = \frac{1}{1 - zf}$$

$$\downarrow$$

$$g = 1 + zfg$$

Constructing a "recursive" equation

$$
\begin{array}{rcl}
A & = & Q_1 B_1 + \cdots + Q_b B_b + R \\
 & \big\downarrow & \\
(\hat{Q}_1, \ldots, \hat{Q}_b, \hat{R}) & = & \hat{\Phi}\big(\hat{A} - \hat{Q}_1 \hat{B}_1^* - \cdots - \hat{Q}_b \hat{B}_b^*\big)
\end{array}
$$

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$$\downarrow$$

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Cut the B_i in heads and tails

$$B_i = c_{B_i} x^{l_{B_i}} + B_i^*$$

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Dominant part of extended reduction

$$\Phi(x^k) = \begin{cases} c_{B_i}^{-1} x^{k - l_{B_i}} e_i & \text{if } k \in \mathrm{Fin}(\{l_{B_i}, \ldots, l_{B_b}\}) \text{ and} \\ & i \text{ is minimal with } l_{B_i} \preccurlyeq k \\ e_{b+1} x^k & \text{otherwise} \end{cases}$$

$$\Phi(P) = \sum_{i \in \operatorname{supp} P} P_i \, \Phi(P_i).$$

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$$\Phi(P) = \sum_{i \in \operatorname{supp} P} P_i \, \Phi(P_i).$$

Complexity bound

$$\mathrm{T} = \mathcal{O}(\mathrm{SM}(|\mathcal{B}_1 + \mathcal{Q}_1|) \log \delta(\mathcal{B}_1 + \mathcal{Q}_1) + \cdots + \mathrm{SM}(|\mathcal{B}_b + \mathcal{Q}_b|) \log \delta(\mathcal{B}_b + \mathcal{Q}_b) + |\mathcal{R}|).$$