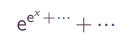
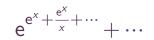
# Towards a model theory for transseries

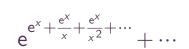
BY MATTHIAS ASCHENBRENNER (UCLA) LOU VAN DEN DRIES (Univ. of Illinois @ Urbana-Champaign) JORIS VAN DER HOEVEN (CNRS, École polytechnique)



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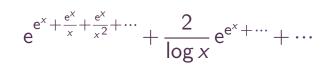


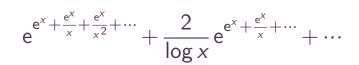


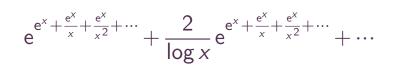


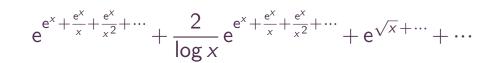
#### What is a transseries?

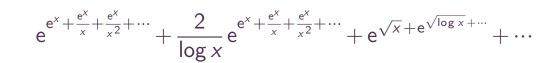
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22





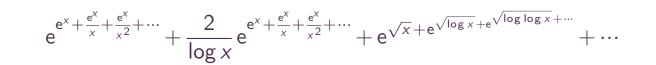


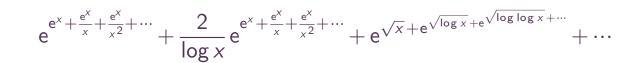




#### What is a transseries?

1 **2** 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22





- Dahn & Göring
- Écalle
- Detailed treatment in LNM 1888: "Transseries and Real Differential Algebra"

### More examples

1 2 **3** 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22

$$\frac{1}{1-x^{-1}-x^{-e}} = 1+x^{-1}+x^{-2}+x^{-e}+x^{-3}+x^{-e-1}+\cdots$$

$$\frac{1}{1-x^{-1}+e^{-x}} = 1+\frac{1}{x}+\frac{1}{x^2}+\cdots+e^{-x}+2\frac{e^{-x}}{x}+\cdots+e^{-2x}+\cdots$$

$$-e^x\int\frac{e^{-x}}{x} = \frac{1}{x}-\frac{1}{x^2}+\frac{2}{x^3}-\frac{6}{x^4}+\frac{24}{x^5}-\frac{120}{x^6}+\cdots$$

$$\Gamma(x) = \frac{\sqrt{2\pi}e^{x(\log x-1)}}{x^{1/2}}+\frac{\sqrt{2\pi}e^{x(\log x-1)}}{12x^{3/2}}+\frac{\sqrt{2\pi}e^{x(\log x-1)}}{288x^{5/2}}+\cdots$$

$$\zeta(x) = 1+2^{-x}+3^{-x}+4^{-x}+\cdots$$

$$\varphi(x) = \frac{1}{x}+\varphi(x^{\pi})=\frac{1}{x}+\frac{1}{x^{\pi}}+\frac{1}{x^{\pi^2}}+\frac{1}{x^{\pi^3}}+\frac{1}{e^{\log^8 x}}+\cdots$$

$$\psi(x) = \frac{1}{x}+\psi(e^{\log^2 x})=\frac{1}{x}+\frac{1}{e^{\log^2 x}}+\frac{1}{e^{\log^8 x}}+\frac{1}{e^{\log^8 x}}+\cdots$$

### Transseries are transfinite series

$$f = \sum_{m \in \mathfrak{T}} f_m \mathfrak{m} \qquad (\mathfrak{T}: \text{ set of transmonomials, coefficients } f_m \in \mathbb{R})$$

$$= -3 e^{e^x} + e^{\frac{e^x}{\log x} + \frac{e^x}{\log^2 x} + \frac{e^x}{\log^3 x} + \cdots} - x^{11} + 7 + \frac{\pi}{x} + \frac{1}{x \log x} + \frac{1}{x (\log x)^2} + \cdots$$

$$= f_{\succ} + f_{\asymp} + f_{\prec}$$

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$$f_{\preceq} = 7$$

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Transmonomials and exponential structure

$$\begin{aligned} \mathfrak{T} &= \exp \mathbb{T}_{\succ} \\ \mathbb{T}_{\succ} &= \{f \in \mathbb{T} \colon f_{\succ} = f\} \\ \exp f &= (\exp f_{\succ}) (\exp f_{\asymp}) (\exp f_{\prec}) \\ \exp f_{\prec} &= 1 + f_{\prec} + \frac{1}{2} f_{\prec}^2 + \frac{1}{6} f_{\prec}^3 + \cdots. \end{aligned}$$

### Differentiation

$$\begin{array}{rcl} x' &=& 1 \\ (\mathrm{e}^{f})' &=& f' \mathrm{e}^{f} \\ \left(\sum_{\mathfrak{m} \in \mathfrak{T}} f_{\mathfrak{m}} \mathfrak{m}\right)' &=& \sum_{\mathfrak{m} \in \mathfrak{T}} f_{\mathfrak{m}} \mathfrak{m}' \end{array}$$

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$$f \ge 0 \iff (\exists g \in \mathbb{T}, g^2 = f) \iff c(f) \ge 0$$
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Valuation

$$\begin{array}{lll} \mathcal{O} &=& \{f \in \mathbb{T} \colon \exists c \in \mathbb{R}^{>}, |f| \leqslant c \} \\ \\ \mathcal{O} &=& \{f \in \mathbb{T} \colon \forall c \in \mathbb{R}^{>}, |f| < c \} \\ \\ \mathcal{O} / \mathcal{O} &\cong& \mathbb{R} \end{array}$$

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$$\mathcal{O} / \mathcal{O} \cong \mathbb{R}$$

$$v(f) \ge v(g) \iff f \preccurlyeq g \iff f \in \mathcal{O} g$$
  
 $v(f) > v(g) \iff f \prec g \iff f \in \mathcal{O} g$ 

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Given a differential polynomial  $P \in \mathbb{T}\{Y\}$  (i.e.  $P = P(Y, Y', ..., Y^{(r)})$ ) and f < g in  $\mathbb{T}$  such that P(f) P(g) < 0, there exists an  $y \in \mathbb{T}$  with f < y < g and P(y) = 0.

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Any quasi-linear differential equation with asymptotic side-condition

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**Definition.** Let K be an ordered differential field and denote

$$C = \{c \in K : c' = 0\}$$
  

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K is an *H-field* if:

**H1.** For all  $a \in K$ , if a > C, then a' > 0.

**H2.**  $\mathcal{O} = \mathcal{C} + \mathcal{O}$ .

The derivation  $\partial$  is said to be *small*, if  $\partial \mathcal{O} \subseteq \mathcal{O}$ .

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### Differentially valued fields [Rosenlicht]

A *d-valued* field is a valued differential field such that

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### **Complex transseries**

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- Zeros of  $L \in \mathbb{T}[i][\partial]$  of order r form a subspace of  $\mathbb{C}[[e^{\mathbb{T} \succ [i]}]]$  of dimension r.
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### Differentially closed fields?

*K* is *d-closed* if for every  $P \in K\{Y\}$  of order *r* and  $Q \in K\{Y\}$  of order *s* < *r*, there exists an  $y \in K$  with P(y) = 0 and  $Q(y) \neq 0$ .

Unfortunately [Rosenlicht]: d-closed d-valued fields do not exist.

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- Generalize this theory to arbitrary H-fields.

• Following Robinson: systematically study the extension theory of H-fields K.

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- Generalize this theory to arbitrary H-fields.
- Main obstruction: problem with gaps

$$\gamma = \frac{1}{x \log x \log \log x \cdots}.$$

Indeed: should we have  $\int \gamma \succ 1$  or  $\int \gamma \prec 1$  in extensions?

Systematic study of *asymptotic d-algebraic equations* 

$$P(y)=0, \quad y\prec v,$$

where  $P \in K\{Y\}$  and  $\varphi \in K$ . For example:

$$e^{-e^{x}}y^{2}y'' + y^{2} - 2xyy' - 7e^{-x}y' - 4 + \frac{1}{\log x} = 0, \quad y \prec x.$$

The slopes correspond to dominant monomials of candidate solutions. Two kinds:

Approximate solutions of algebraic type.  $y = \pm 2 + \cdots$ 

$$e^{-e^{x}}y^{2}y'' + y^{2} - 2xyy' - 7e^{-x}y' - 4 + \frac{1}{\log x} = 0, \quad y \prec x.$$

- Natural analogue of usual Newton polygon method.
- Slopes can be read off from the Newton diagram modulo "adjustments",
   e.g. y' = e<sup>e<sup>x</sup></sup> implies y ~ e<sup>e<sup>x</sup></sup>/e<sup>x</sup>. In general: equalizer theorem.

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Approximate solutions of differential type.  $y = c \sqrt{x} + \cdots$ 

$$e^{-e^{x}}y^{2}y'' + y^{2} - 2xyy' - 7e^{-x}y' - 4 + \frac{1}{\log x} = 0, \quad y \prec x.$$

Cancellation in homogeneous component  $y^2 - 2xyy' = (1 - 2x\frac{y'}{y})y^2 \rightsquigarrow$  Riccati equation

$$1-2xy^{\dagger}=0,$$

whence  $y^{\dagger} = \frac{1}{2}x$  and  $y \asymp \sqrt{x}$ .

**Refinements.** Given an approximate solution  $y \sim \varphi$ , performing the *refinement* 

$$y = \varphi + \tilde{y}, \quad \tilde{y} \prec \varphi$$

leads to a new asymptotic differential equation in  $\tilde{y}$ . Example:  $y = 2 + \tilde{y}, \tilde{y} \prec 1$  transforms

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**Newton degree.** [analogue of Weierstrass degree] Abscissa of highest slope of Newton diagram which satisfies the asymptotic side condition (e.g. two in our example).

- Newton degree can only decrease during refinements.
- If Newton degree is one, then the equation is said to be *quasi-linear*. In that case, it admits at least one transseries solution.
- Using a suitable generalization of Tschirnhausen transforms, one may reach a quasi-linear equation after only a finite number of refinements.

# Multiplicative conjugation

Reduce general asymptotic side condition  $y \prec v$  to the case when v = 1:

$$(P(y) = 0, y \prec v) \Longleftrightarrow (P_{\times v}(\tilde{y}) = P(\tilde{y}v) = 0, \tilde{y} \prec 1), \qquad \tilde{y} = y / v$$

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Consider  $P \in \mathbb{T}\{Y\}$  as a series  $P = \sum_{\mathfrak{m} \in \mathfrak{T}} P_{\mathfrak{m}}\mathfrak{m}$ , with  $P_{\mathfrak{m}} \in \mathbb{R}\{Y\}$ . Then  $D_P = D(P)$  is the "leading coefficient" of P.

$$D\left(\frac{3}{1-e^{-x}}Y^2Y'-\frac{1}{x}Y^2+(Y')^2+e^{-x}\right) = 3Y^2+(Y')^2.$$

Requires a cross section of the value group inside K for a general H-field.

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### **Dominant part**

Consider  $P \in \mathbb{T}\{Y\}$  as a series  $P = \sum_{\mathfrak{m} \in \mathfrak{T}} P_{\mathfrak{m}}\mathfrak{m}$ , with  $P_{\mathfrak{m}} \in \mathbb{R}\{Y\}$ . Then  $D_P = D(P)$  is the "leading coefficient" of P.

$$D\left(\frac{3}{1-e^{-x}}Y^2Y'-\frac{1}{x}Y^2+(Y')^2+e^{-x}\right) = 3Y^2+(Y')^2.$$

Requires a cross section of the value group inside K for a general H-field.

### **Compositional conjugation**

Replacing the derivation  $\partial$  by a new derivation  $\tilde{\partial} = \phi^{-1} \partial$ .

Corresponds to a postcomposition  $\tilde{y} = y \circ u$ .

We typically want to take  $\phi$  as small as possible, while preserving the smallness of  $\tilde{\partial}$ . **Notation:**  $K^{\phi}$ : the field K with derivation  $\tilde{\partial}$ ,  $P^{\phi}$ : the counterpart of  $P \in K\{Y\}$  in  $K^{\phi}\{Y\}$ .

$$\frac{3}{1 - e^{-x}} Y^2 Y' - \frac{1}{x} Y^2 + (Y')^2 + e^{-x} \qquad \stackrel{D}{\longrightarrow} 3 Y^2 Y' + (Y')^2$$

$$\downarrow^{Y = \tilde{Y} \circ \log}$$

$$\frac{3}{(1 - e^{-x})x} (\tilde{Y}^2 \tilde{Y}') \circ \log - \frac{1}{x} \tilde{Y}^2 \circ \log + \frac{(\tilde{Y}')^2 \circ \log}{x^2} + e^{-x} \qquad \stackrel{D}{\longrightarrow} 3 \tilde{Y}^2 \tilde{Y}' - \tilde{Y}$$

$$\downarrow^{\tilde{Y} = \tilde{Y} \circ \log}$$

$$\frac{3}{(1 - e^{-x})x \log x} (\tilde{\tilde{Y}}^2 \tilde{\tilde{Y}}') \circ \log_2 - \frac{1}{x} \tilde{\tilde{Y}}^2 \circ \log_2 + \frac{(\tilde{\tilde{Y}}')^2 \circ \log_2}{x^2 \log^2 x} + e^{-x} \qquad \stackrel{D}{\longrightarrow} -\tilde{\tilde{Y}}$$

**Theorem.** For any  $P \in \mathbb{T}^{gb}\{Y\} \setminus \{0\}$ , there exists an  $N \in \mathbb{R}[Y](Y')^{\mathbb{N}}$  with

$$D_{P^{\phi}} = N,$$

for any  $\phi = \frac{1}{x \ell_1 \ell_2 \cdots \ell_{l-1} x} = \ell'_l$  with l sufficiently large, where  $\ell_k = \log \circ \overset{k \times}{\cdots} \circ \log$ .

The following special *cuts* will play a crucial role:

$$\begin{split} \gamma_{\mathbb{T}} &= \frac{1}{\ell_0 \,\ell_1 \,\ell_2 \cdots} \qquad (\forall n, (1/\ell_n)' \prec \gamma_{\mathbb{T}} \prec \ell'_n) \\ \lambda_{\mathbb{T}} &= \frac{1}{\ell_0} + \frac{1}{\ell_0 \,\ell_1} + \frac{1}{\ell_0 \,\ell_1 \,\ell_2} + \cdots = -\gamma_{\mathbb{T}}^{\dagger} \\ \omega_{\mathbb{T}} &= \frac{1}{\ell_0^2} + \frac{1}{\ell_0^2 \,\ell_1^2} + \frac{1}{\ell_0^2 \,\ell_1^2 \,\ell_2^2} + \cdots = -\lambda_{\mathbb{T}}^2 - 2\,\lambda'_{\mathbb{T}} \end{split}$$

Even though  $\gamma_{\mathbb{T}}$ ,  $\lambda_{\mathbb{T}}$  and  $\omega_{\mathbb{T}}$  are not in  $\mathbb{T}$ , the sets

$$\begin{split} &\Gamma(\mathbb{T}) = \{ a \in \mathbb{T} : a < \gamma_{\mathbb{T}} \} \\ &\Lambda(\mathbb{T}) = \{ a \in \mathbb{T} : a < \lambda_{\mathbb{T}} \} \\ &\Omega(\mathbb{T}) = \{ a \in \mathbb{T} : a < \omega_{\mathbb{T}} \} \end{split}$$

are definable subsets of  $\mathbb{T}$ . For instance,

$$\begin{split} \Gamma(\mathbb{T}) &= \{ a \in \mathbb{T} : \forall b \in \mathbb{T}, b \succ 1 \Rightarrow a \neq b^{\dagger} \} \\ &= \{ -a' : a \in \mathbb{T}, a \ge 0 \}. \end{split}$$

In other words,  $\gamma_{\mathbb{T}}$ ,  $\lambda_{\mathbb{T}}$  and  $\omega_{\mathbb{T}}$  are *definable cuts* in  $\mathbb{T}$ .

## Gaps

 $\gamma = \gamma_K \in K$ , but  $\int \gamma \notin K$ . In other words, for all  $a \in K$  with  $a \succ 1$ , we have

 $a^{\dagger} \succ \gamma \succ (1/a)'.$ 

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### Indirect gaps

K admits no gap (i.e. K is  $\gamma$ -free), but  $\lambda \in K$  is such that for all  $a \in K$  with  $a \succ 1$ , we have

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### In general

Each of the following cases can occur:

$$\gamma \in K$$
  

$$\gamma \notin K \land \lambda \in K$$
  

$$\lambda \notin K \land \omega \in K$$
  

$$\omega \notin K$$

# Definition

$$\omega = \omega_{\mathcal{K}} \colon \mathcal{K} \to \mathcal{K}, \qquad \omega(z) := -2 \, z' - z^2,$$

 $\omega \in K$ , if for all  $a \succ 1$  in K, we have

$$\omega - \omega(a^{\dagger\dagger}) \prec (a^{\dagger})^2.$$

*K* is  $\omega$ -free if

$$\forall a, \exists b, [b \succ 1 \land a - \omega(b^{\dagger \dagger}) \succcurlyeq (b^{\dagger})^2].$$

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#### Examples

- $\mathbb{T}$  is  $\omega$ -free.
- If K has asymptotic integration and K is a union of H-subfields, each of which has a smallest comparability class, then K is  $\omega$ -free.
- There exist Liouville-closed H-fields that are not ω-free.

## **Differential Newton polynomials**

**Theorem.** If K is  $\omega$ -free, then we can define  $N_P$  for any  $P \in K\{Y\}$ , and  $N_P \in C[Y](Y')^{\mathbb{N}}$ .

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## Differentially algebraic extensions

**Theorem.** If L is a d-algebraic extension of an  $\omega$ -free H-field K, then L is  $\omega$ -free.

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## Relation with theorem of Écalle

Let  $\lambda = \frac{1}{x} + \frac{1}{x\ell_1} + \frac{1}{x\ell_1\ell_2} + \cdots$  and  $P \in \mathbb{R}\{Y\} \setminus \mathbb{R}$ . Then the first  $\omega$  terms of  $P(\lambda)$  either "behave" like  $\lambda$  or like  $\omega$ .

In particular, we cannot have  $P(\lambda) = \frac{1}{x^n} + \frac{1}{x^n \ell_1^n} + \frac{1}{x^n \ell_1^n} + \frac{1}{x^n \ell_1^n \ell_2^n} + \cdots$  for  $n \ge 3$ .

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#### Relation with second order linear differential equations

```
y'' = -y has no non-zero solution y \in \mathbb{T}.
```

y'' = xy has two  $\mathbb{R}$ -linearly independent solutions in  $\mathbb{T}$ .

In general, 4y'' + fy = 0 has a non-zero solution if and only if  $f < \omega$ .

**Definition** (for  $\omega$ -free H-field K)

Every  $P \in K\{Y\}$  with deg  $N_P = 1$  admits a zero in  $\mathcal{O}$ .

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**Theorem.** If K is a newtonian H-field with divisible value group, then K has no proper immediate d-algebraic H-field extension.

**Corollary.** Let K be a real closed newtonian H-field. Then

- 1. Each d-polynomial in  $K[i]{Y}$  of positive degree has a zero in K[i].
- 2. Each linear differential operator in  $K[i][\partial]$  of positive order is a composition of such operators of order 1.
- 3. Each linear differential operator in  $K[\partial]$  of positive order is a composition of such operators of order 1 and order 2.

**Theorem.** If K is an  $\omega$ -free H-field with divisible value group, then K has an immediate d-algebraic newtonian H-field extension, and any such extension embeds over K into every  $\omega$ -free newtonian H-field extension of K.

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**Theorem.** If K is an  $\omega$ -free H-field, then K has a d-algebraic newtonian Liouville closed H-field extension that embeds over K into every  $\omega$ -free newtonian Liouville closed H-field extension of K.

# The theory

## The theory

$$\mathcal{L} = \{0, 1, +, -, \cdot, \partial, \leq, \preccurlyeq\}$$
  
$$T^{nI} = \text{theory of newtonian } (\omega\text{-free}) \text{ Liouville closed } H\text{-fields}$$

### Switchmen predicates

 $\mathcal{L}_{I,\Lambda,\Omega}^{\iota} \!=\! \mathcal{L} \cup \{\iota, I, \Lambda, \Omega\} \text{ and } \mathcal{T}_{I,\Lambda,\Omega}^{\mathrm{nl},\iota} \text{ is } \mathcal{T}^{\mathsf{nl}} \text{ with additional axioms}$ 

$$\begin{array}{ll} a \neq 0 \implies a \iota(a) = 1 \\ a = 0 \implies \iota(a) = 0 \\ \mathsf{I}(a) \iff [\exists y, (a \preccurlyeq y' \land y \preccurlyeq 1)] \iff [a = 0 \lor (a \neq 0 \land \neg \Lambda(-a^{\dagger}))] \\ \Lambda(a) \iff \exists y, (y \succ 1 \land a = -y^{\dagger \dagger}) \\ \Omega(a) \iff \exists y, (y \neq 0 \land 4 y'' + a y = 0) \end{array}$$

Assume that K contains a gap  $\gamma$  and that  $\Phi \in L \supseteq K$  such that  $\Phi' = \gamma$ .

Then we *must* have  $\Phi \preccurlyeq 1$  if  $I(\gamma)$  and  $\Phi \succ 1$  otherwise.

A and  $\Omega$  control what happens when adjoining  $\gamma$  and  $\lambda$  with  $\gamma^{\dagger} = -\lambda$  and  $\omega(\lambda) = \omega$ .

 $L_{\Lambda,\Omega}^{\iota} = \mathcal{L} \cup \{\iota, \Lambda, \Omega\}$  and  $T_{\Lambda,\Omega}^{\mathrm{nl},\iota}$  is  $T^{\mathrm{nl}}$  with above additional axioms for  $\iota, \Lambda$  and  $\Omega$ .

**Theorem.** The theory  $T_{\Lambda,\Omega}^{\mathrm{nl},\iota}$  admits elimination of quantifiers.

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**Theorem.** Let  $\mathcal{T}_{small}^{nl}$  be the L-theory whose models are the newtonian Liouville closed H-fields with small derivation. Then  $\mathcal{T}_{small}^{nl}$  is complete (and thus decidable) and model complete. Every H-field with small derivation can be embedded into some model of  $\mathcal{T}_{small}^{nl}$ .

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**Corollary.** Let K be a newtonian Liouville closed H-field. Then:

- 1. K is o-minimal at infinity: if  $X \subseteq K$  is definable in K, then for some  $a \in K$ , either  $(a, +\infty) \subseteq K$ , or  $(a, +\infty) \cap K = \emptyset$ .
- 2. If  $X \subseteq K^n$  is definable in K, then  $X \cap C^n$  is semialgebraic in the sense of the real closed constant field C of K.
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**Theorem.** If K is a newtonian Liouville closed H-field, then K has no proper d-algebraic H-field extension with the same constant field.