Ordering infinities

Joris van der Hoeven, CNRS, École polytechnique

Based on joint work with M. Aschenbrenner, L. van den Dries, V. Bagayoko, E. Kaplan

August 27, 2020

In honour of the 75th birthday of **Maurice Pouzet**

Georg Cantor

Georg Cantor

Cardinal numbers

Georg Cantor

Cardinal numbers

Ordinal numbers

$$0, 1, 2, \ldots$$

Georg Cantor

Cardinal numberſ

Ordinal numberſ

$$0, 1, 2, \ldots, \omega$$

Georg Cantor

Cardinal numberſ

Ordinal numberſ

$$0, 1, 2, \ldots, \omega, \omega + 1, \ldots$$

Georg Cantor

Cardinal numbers

Ordinal numbers

$$0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \omega \cdot 2, \omega \cdot 2 + 1, \ldots$$

Georg Cantor

Cardinal numberſ

Ordinal numberſ

$$0,1,2,\ldots,\omega,\omega+1,\ldots,\omega\cdot 2,\omega\cdot 2+1,\ldots,\omega^2$$

Georg Cantor

Cardinal numbers

Ordinal numbers

$$0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \omega \cdot 2, \omega \cdot 2 + 1, \ldots, \omega^2, \ldots, \omega^3$$

Georg Cantor

Cardinal numbers

Ordinal numbers

$$0, 1, 2, \ldots, \omega, \omega+1, \ldots, \omega\cdot 2, \omega\cdot 2+1, \ldots, \omega^2, \ldots, \omega^3, \ldots, \omega^\omega$$

Georg Cantor

Cardinal numbers

Ordinal numbers

$$0, 1, 2, \ldots, \omega, \omega+1, \ldots, \omega \cdot 2, \omega \cdot 2+1, \ldots, \omega^2, \ldots, \omega^3, \ldots, \omega^\omega, \ldots, \aleph_1, \ldots\ldots$$

Georg Cantor

Cardinal numbers

Ordinal numbers

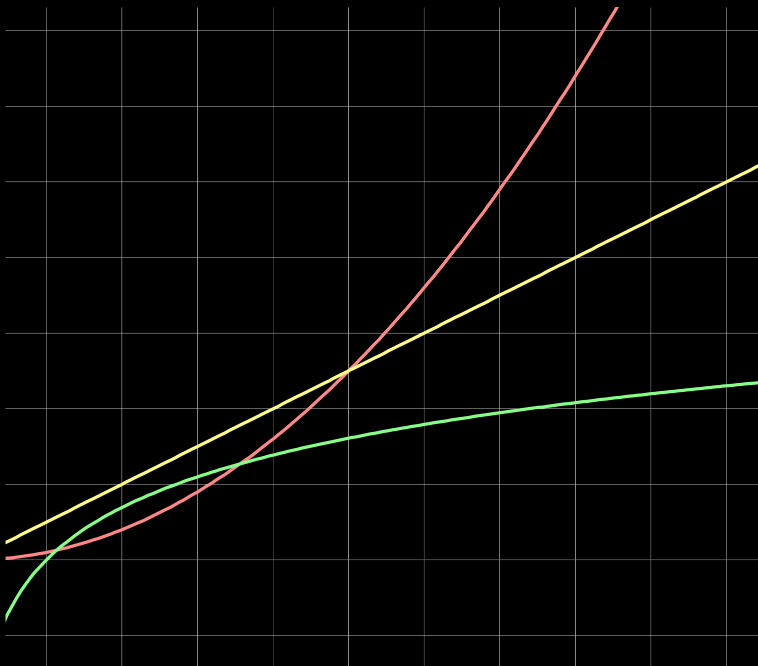
$$0, 1, 2, \ldots, \omega, \omega+1, \ldots, \omega\cdot 2, \omega\cdot 2+1, \ldots, \omega^2, \ldots, \omega^3, \ldots, \omega^\omega, \ldots, \aleph_1, \ldots\ldots$$

Cantor normal form

$$\omega^{\omega^{\omega+2}\cdot 3+\omega^8\cdot 7+\omega\cdot 3+2}\cdot 9+\omega^{\omega^{\omega+1}}\cdot 3+\omega^{\omega\cdot 7}\cdot 5+\omega^8+\omega^2\cdot 111+2020$$

Paul du Boiſ=Reymond

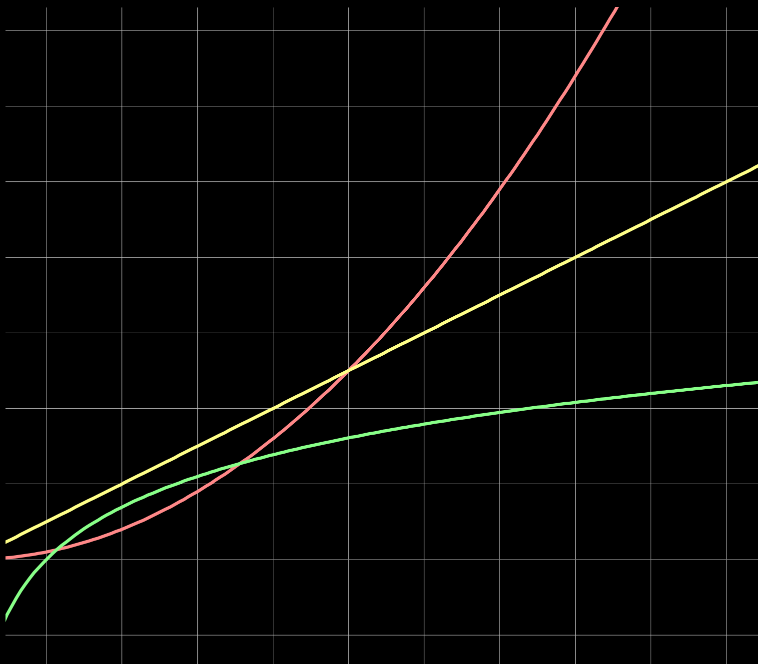
Paul du Bois-Reymond



Precursor of asymptotic calculus

$$\log x \prec \frac{x}{2} \prec \frac{x^2}{10} \qquad (x \longrightarrow \infty)$$

Paul du Bois-Reymond



Precursor of asymptotic calculus

$$\log x \prec \frac{x}{2} \prec \frac{x^2}{10} \qquad (x \longrightarrow \infty)$$

Diagonal argument

$$\exists f, \quad x \prec e^x \prec e^{e^x} \prec e^{e^{e^x}} \prec \cdots \prec f$$

Introduction

Three intimately related topics...

(surreal)
Numbers

Transseries

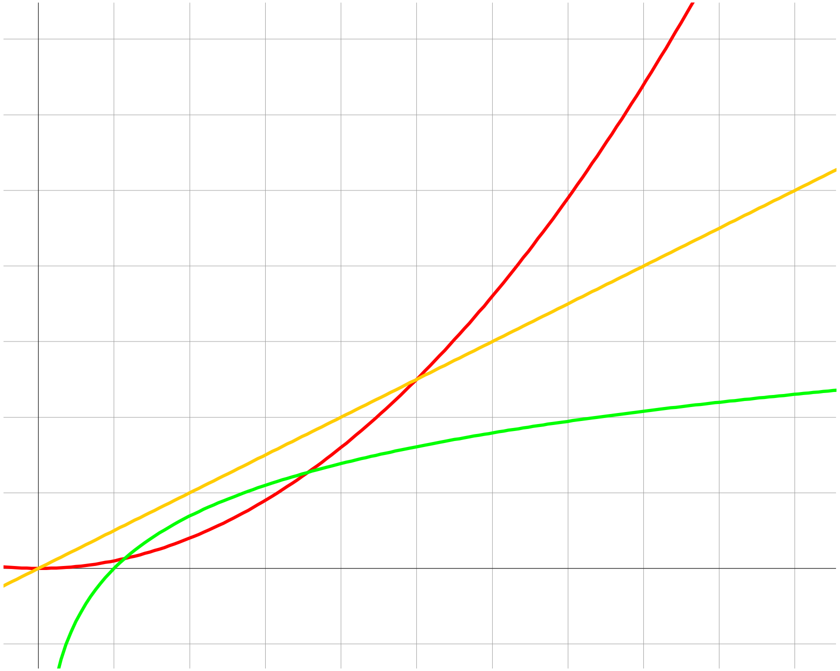
Germs
(in HARDY fields)

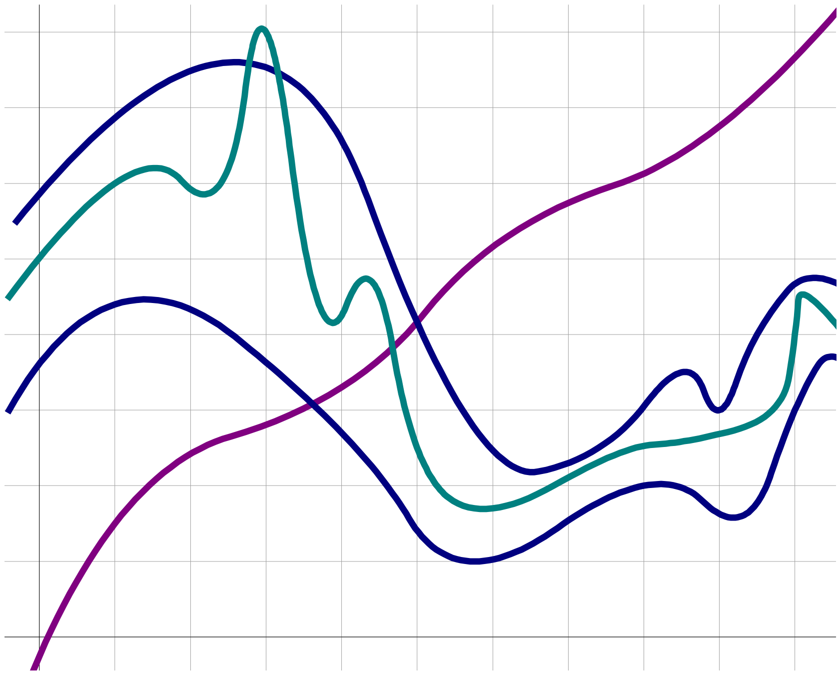
Numbers

Germs

(in HARDY fields)

Transseries





HARDY fields

Let $\mathscr{C}^1$ be the ring of germs at $+\infty$ of continuously differentiable functions $(a,\infty)\to\mathbb{R}$ ($a\in\mathbb{R}$).

We denote the germ at $+\infty$ of a function f also by f, relying on context.

Definition

A ***HARDY field*** *is a subring of $\mathscr{C}^1$ which is a field that contains with each germ of a function f also the germ of its derivative f' (where f' might be defined on a smaller interval than f).*

Examples. $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}(x)$, $\mathbb{R}(x,\mathrm{e}^x)$, $\mathbb{R}(x,\mathrm{e}^x,\log x)$, $\mathbb{R}(x,\mathrm{e}^{x^2},\operatorname{erf} x)$

HARDY fields

HARDY fields capture the somewhat vague notion of functions with "**regular growth**" at infinity (BOREL, DU BOIS-REYMOND, ...):

Let H be a HARDY field and $f \in H$. Then

$$f \neq 0 \implies \frac{1}{f} \in H \implies \begin{cases} f(x) > 0, \text{ eventually, or} \\ f(x) < 0, \text{ eventually.} \end{cases}$$

Consequently,

- H carries an ordering making H an **ordered field**:

$$f > 0 \iff f(x) > 0 \text{ eventually;}$$

- f is **eventually monotonic**, and

$$\lim_{x \to +\infty} f(x) \in \mathbb{R} \cup \{\pm\infty\}.$$

(surreal)
Numbers

Transseries

Germs
(in HARDY
fields)

(surreal)
Numbers

Germs
(in HARDY fields)

Transseries

The field $\mathbb{T}$ of transseries

$\mathbb{T} :=$ closure of $\mathbb{R} \cup \{x\}$ under exp, log and infinite summation

$$\mathrm{e}^{\mathrm{e}^{x}+\mathrm{e}^{x/2}+\mathrm{e}^{x/3}+\cdots} - 3\,\mathrm{e}^{x^2} + 5\,(\log x)^{\pi} + 42 + x^{-1} + 2\,x^{-2} + 6\,x^{-3} + \cdots + \mathrm{e}^{-x}$$

The field $\mathbb{T}$ of transseries

$\mathbb{T} = \mathbb{R}[[\mathfrak{M}]] :=$ closure of $\mathbb{R} \cup \{x\}$ under exp, log and infinite summation

$$\sum_{\mathfrak{m}} f_{\mathfrak{m}} \mathfrak{m} = \mathrm{e}^{\mathrm{e}^{x}+\mathrm{e}^{x/2}+\cdots} - 3\,\mathrm{e}^{x^2} + 5\,(\log x)^{\pi} + 42 + x^{-1} + 2\,x^{-2} + 6\,x^{-3} + \cdots + \mathrm{e}^{-x}$$

x: positive infinite indeterminate $\quad f_{\mathfrak{m}}$: coefficent $\quad \mathfrak{m}$: transmonomial

supp f: well-based subset of $\mathfrak{M}$

disallow $x + \log x + \log\log x + \cdots$ and $\mathrm{e}^{-x} + \mathrm{e}^{-\mathrm{e}^{x}} + \mathrm{e}^{-\mathrm{e}^{\mathrm{e}^{x}}} + \cdots$

$\mathbb{T}$ as an ordered differential field

- With the natural ordering of transseries (via the leading coefficient), $\mathbb{T}$ is a *real closed ordered field* extension of $\mathbb{R}$.
- Each $f \in \mathbb{T}$ can be *differentiated* term by term (with $x' = 1$):

$$\left(\sum_{n=0}^{\infty} n! \frac{\mathrm{e}^x}{x^n}\right)' = \sum_{n=0}^{\infty} n! \left(\frac{\mathrm{e}^x}{x^n}\right)' = \sum_{n=0}^{\infty} n! \left(\frac{\mathrm{e}^x}{x^n} - n \frac{\mathrm{e}^x}{x^{n+1}}\right) = \frac{\mathrm{e}^x}{x}$$

- This yields a *derivation* $f \mapsto f'$ on the field $\mathbb{T}$:

$$(f+g)' = f' + g', \qquad (f \cdot g)' = f' \cdot g + f \cdot g'$$

  Its constant field is $\{f \in \mathbb{T} : f' = 0\} = \mathbb{R}$.
- Given $f, g \in \mathbb{T}$, the equation $y' + fy = g$ admits a solution $y \neq 0$ in $\mathbb{T}$.

Surreal numbers

(surreal)
Numbers

Transseries

Germs
(in HARDY fields)

(surreal)
Numbers

Transseries

Germs

Class On of ordinal numbers

For any set L of ordinal numbers, there is a smallest ordinal number $\alpha > L$

Surreal numbers

Class On of ordinal numbers

For any set L of ordinal numbers, there is a smallest ordinal number $\alpha > L$

Class No of surreal numbers (CONWAY)

For any sets $L < R$ of surreal numbers, there is a **simplest** surreal number $\{L \mid R\}$ such that $L < \{L \mid R\} < R$.

Class On of ordinal numbers

For any set L of ordinal numbers, there is a smallest ordinal number $\alpha > L$

Class No of surreal numbers (CONWAY)

For any sets $L < R$ of surreal numbers, there is a **simplest** surreal number $\{L \mid R\}$ such that $L < \{L \mid R\} < R$.

We have **On** $\subseteq$ **No** by taking $R = \emptyset$:

$$
\begin{aligned}
0 &= \{|\} \\
1 &= \{0|\} \\
2 &= \{0,1|\} \\
\omega &= \{0,1,2,\ldots|\}
\end{aligned}
$$

$$0 = \{ | \}$$

Surreal numbers

$$0 = \{\,|\,\}$$

$$-1 = \{\,|0\} \qquad 1 = \{0|\,\}$$

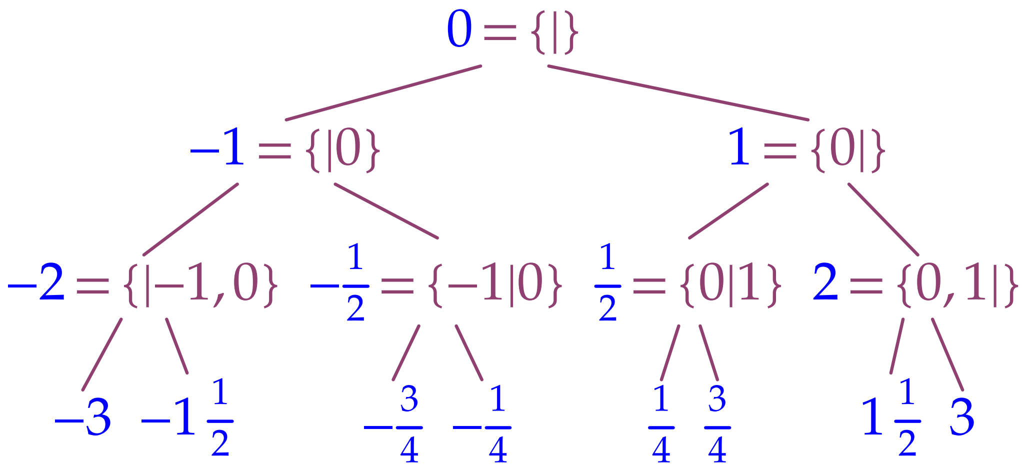
Surreal numbers

$$0 = \{ \mid \}$$

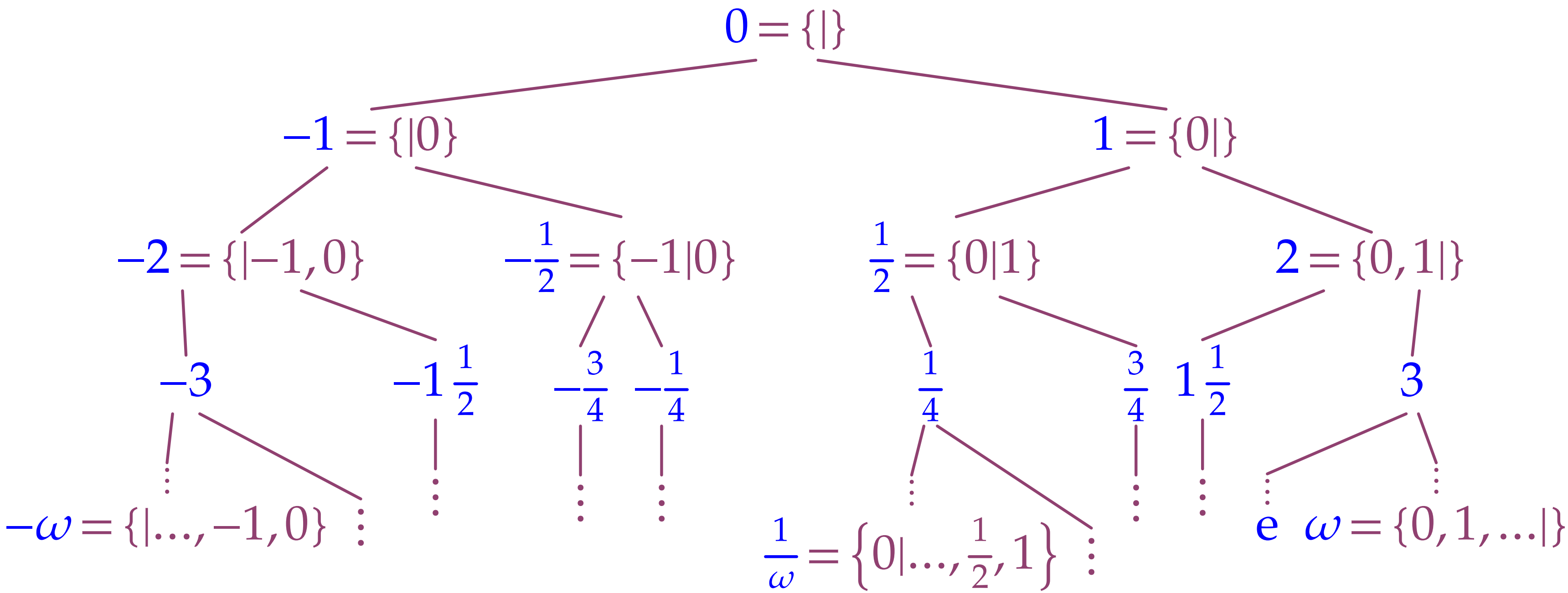
$$-1 = \{ \mid 0 \} \qquad 1 = \{ 0 \mid \}$$

$$-2 = \{ \mid -1, 0 \} \qquad -\frac{1}{2} = \{ -1 \mid 0 \} \qquad \frac{1}{2} = \{ 0 \mid 1 \} \qquad 2 = \{ 0, 1 \mid \}$$

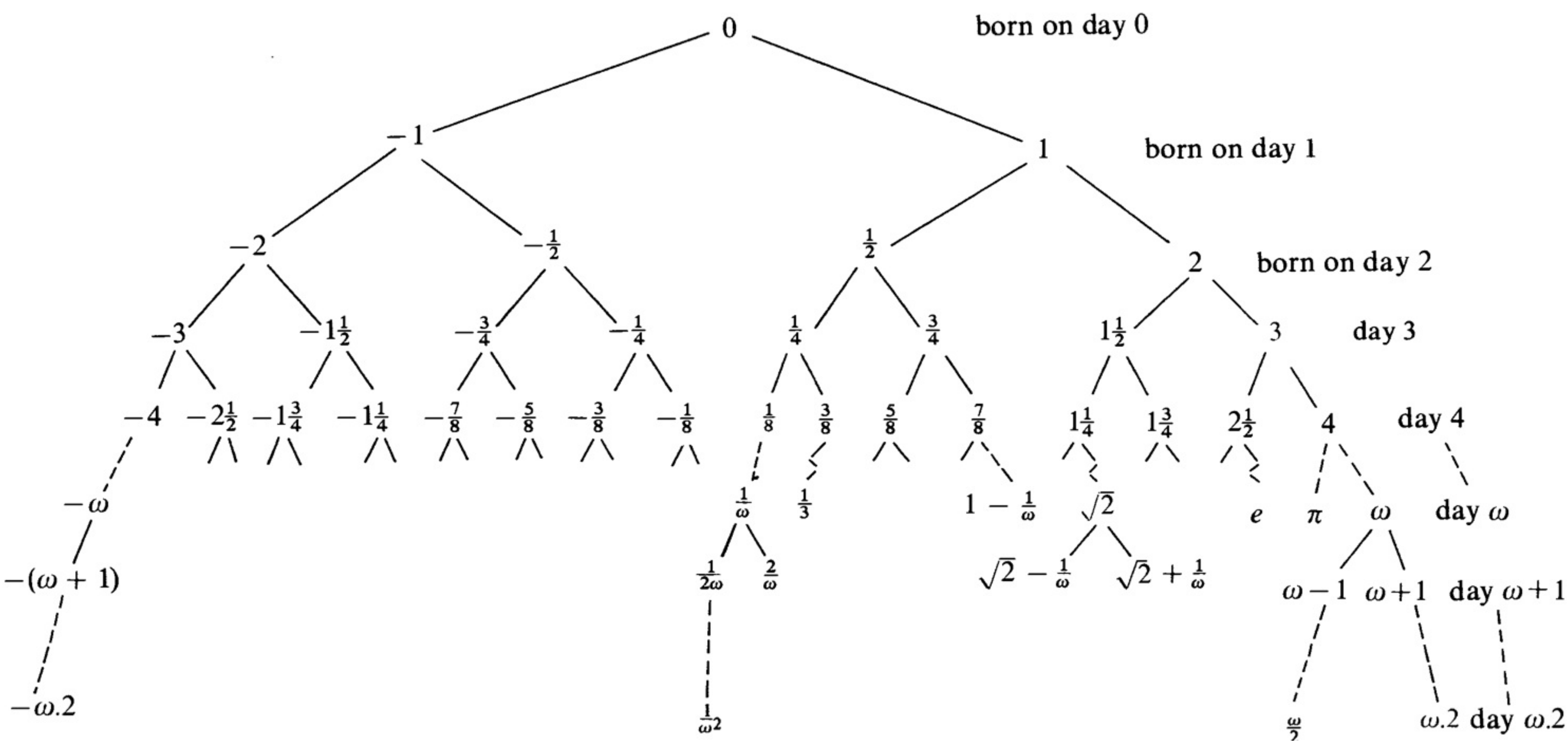
Surreal numbers



Surreal numbers



Surreal numbers



Arithmetic operations

Definition

If $x = \{x^L | x^R\}$ *and* $y = \{y^L | y^R\}$, *then*

$$x + y \; := \; \{x^L + y, x + y^L | x^R + y, x + y^R\}$$

(Idea: we want $x^L + y \; < \; x + y \; < \; x^R + y, \; \ldots$*)*

Arithmetic operations

Definition

If $x = \{x^L | x^R\}$ *and* $y = \{y^L | y^R\}$*, then*

$$x + y \;:=\; \{x^L + y, x + y^L | x^R + y, x + y^R\}$$

(Idea: we want $x^L + y \;<\; x + y \;<\; x^R + y$*, ...)*

Definition

If $x = \{x^L | x^R\}$ *and* $y = \{y^L | y^R\}$*, then*

$$xy \;:=\; \{\underline{x}y + x\underline{y} - \underline{x}\,\underline{y}, \bar{x}y + x\bar{y} - \bar{x}\bar{y} \,|\, \underline{x}y + x\bar{y} - \underline{x}\bar{y}, \bar{x}y + x\underline{y} - \bar{x}\underline{y}\},$$

where $x' \in x_L$*,* $x'' \in x_R$*,* $y' \in y_L$*,* $y'' \in y_R$

Arithmetic operations

Definition

If $x = \{x^L | x^R\}$ *and* $y = \{y^L | y^R\}$, *then*

$$x + y \ := \ \{x^L + y, x + y^L | x^R + y, x + y^R\}$$

(Idea: we want $x^L + y \ < \ x + y \ < \ x^R + y, \ \ldots$*)*

Definition

If $x = \{x^L | x^R\}$ *and* $y = \{y^L | y^R\}$, *then*

$$x\,y \ := \ \{\underline{x}\,y + x\,\underline{y} - \underline{x}\,\underline{y}, \bar{x}\,y + x\,\bar{y} - \bar{x}\,\bar{y} \,|\, \underline{x}\,y + x\,\bar{y} - \underline{x}\,\bar{y}, \bar{x}\,y + x\,\underline{y} - \bar{x}\,\underline{y}\},$$

where $x' \in x_L$, $x'' \in x_R$, $y' \in y_L$, $y'' \in y_R$

Theorem (Conway)

No *is a real closed field.*

Exponentiation and differentiation

- In the 1980s, GONSHOR (based on ideas of KRUSKAL) defined an exponential function $\exp\colon \mathbf{No} \to \mathbf{No}^{>0}$ that extends $x \mapsto e^x$ on $\mathbb{R}$.

- In 2006, BERARDUCCI and MANTOVA (using ideas of VDH and SCHMELING) defined a derivation ∂_{BM} on **No** with

  $$\ker \partial_{\mathrm{BM}} = \mathbb{R}, \quad \partial_{\mathrm{BM}}(\omega) = 1, \quad \partial_{\mathrm{BM}}(\exp(f)) = \partial_{\mathrm{BM}}(f) \cdot \exp(f) \text{ for } f \in \mathbf{No}.$$

  In a certain technical sense, it is the simplest such derivation that satisfies some natural further conditions.

- The BM-derivation on **No** behaves in many ways like the derivation on $\mathbb{T}$, with $\omega > \mathbb{R}$ in the role of $x > \mathbb{R}$. For instance, $\partial_{\mathrm{BM}}(\log \omega) = \frac{1}{\omega}$.

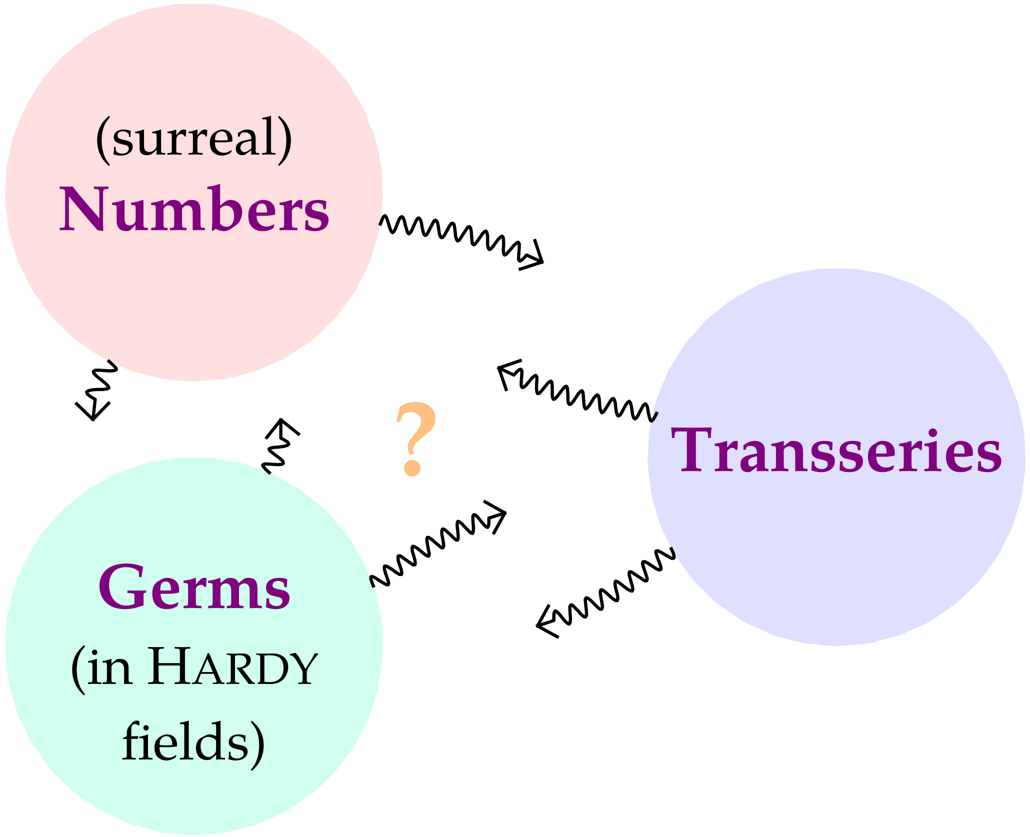
Towards a unified theory

(surreal)
Numbers

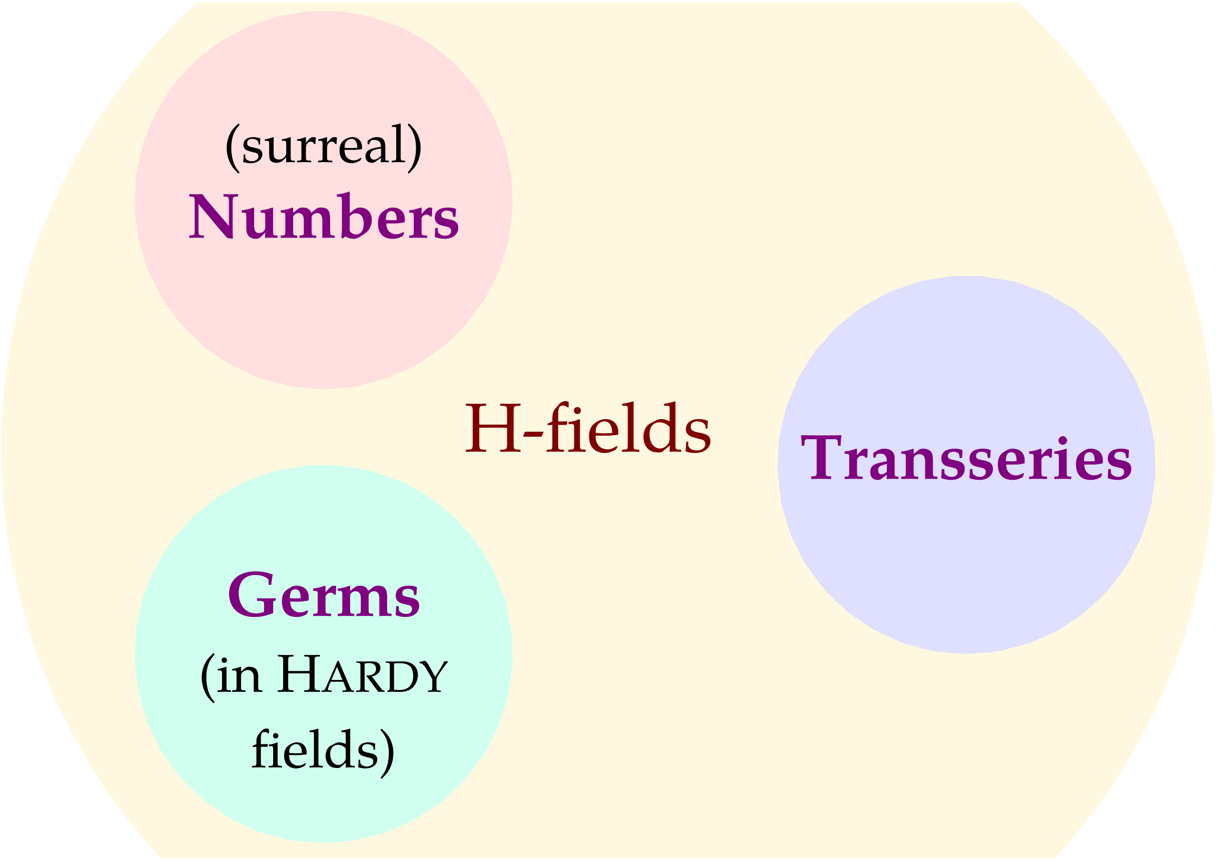
Transseries

Germs
(in HARDY fields)

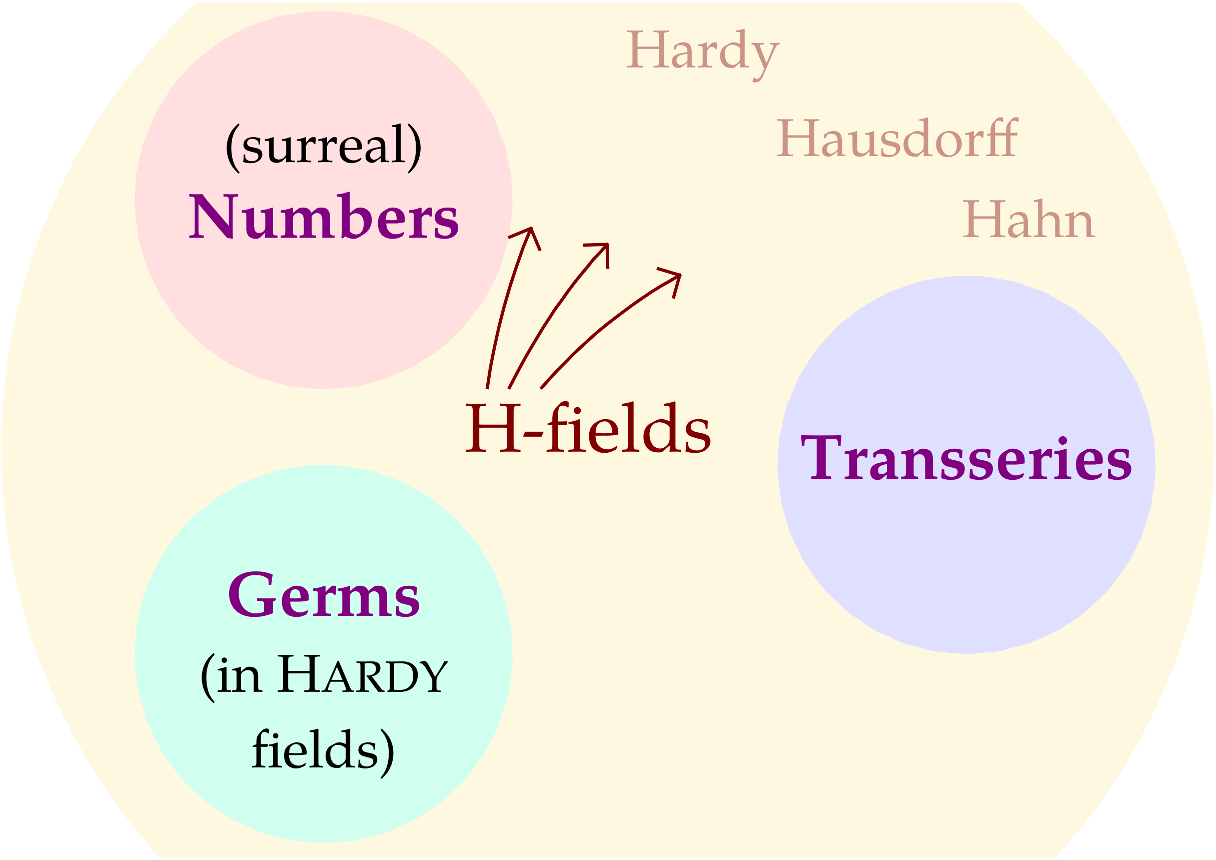
Towards a unified theory



Towards a unified theory



Towards a unified theory



Asymptotic relations

Let K be an ordered differential field with constant field

$$C = \{f \in K : f' = 0\}.$$

We define

$$f \preccurlyeq g \;:\Longleftrightarrow\; |f| \leqslant c\,|g| \text{ for some } c \in C^{>0} \qquad (f \text{ is } \textit{dominated} \text{ by } g)$$

$$f \prec g \;:\Longleftrightarrow\; |f| \leqslant c\,|g| \text{ for all } c \in C^{>0} \qquad (f \text{ is } \textit{negligible} \text{ w.r.t. } g)$$

$$f \asymp g \;:\Longleftrightarrow\; f \preccurlyeq g \preccurlyeq f \qquad (f \text{ is } \textit{asymptotic} \text{ to } g)$$

$$f \sim g \;:\Longleftrightarrow\; f - g \prec g \qquad (f \text{ is } \textit{equivalent} \text{ to } g)$$

Example. In $\mathbb{T}$: $0 \prec \mathrm{e}^{-x} \prec x^{-10} \prec 1 \asymp 100 \prec \log x \prec x^{1/10} \prec \mathrm{e}^{x} \sim \mathrm{e}^{x} + x \prec \mathrm{e}^{\mathrm{e}^{x}}$

Definition

*We call K an **H-field** if*

H1. $f > C \implies f' > 0$;

H2. $f \asymp 1 \implies f \sim c$ *for some* $c \in C$.

Examples. HARDY fields containing $\mathbb{R}$; ordered differential subfields of $\mathbb{T}$ or **No** that contain $\mathbb{R}$.

$\mathbb{T}$ admits further elementary properties in addition to being an H-field. It

- has **small derivation**, that is, $f \prec 1 \implies f' \prec 1$; and
- is **LIOUVILLE closed**, that is, it is real closed and for all f, g, there is some $y \neq 0$ with $y' + fy = g$.

One of our main results

We view $\mathbb{T}$ model-theoretically as a structure with the primitives

$$0, \quad 1, \quad +, \quad \times, \quad \partial \text{ (derivation)}, \quad \leqslant \text{ (ordering)}.$$

Theorem (Ann. of Math. Stud. vol. 195 + afterthought)

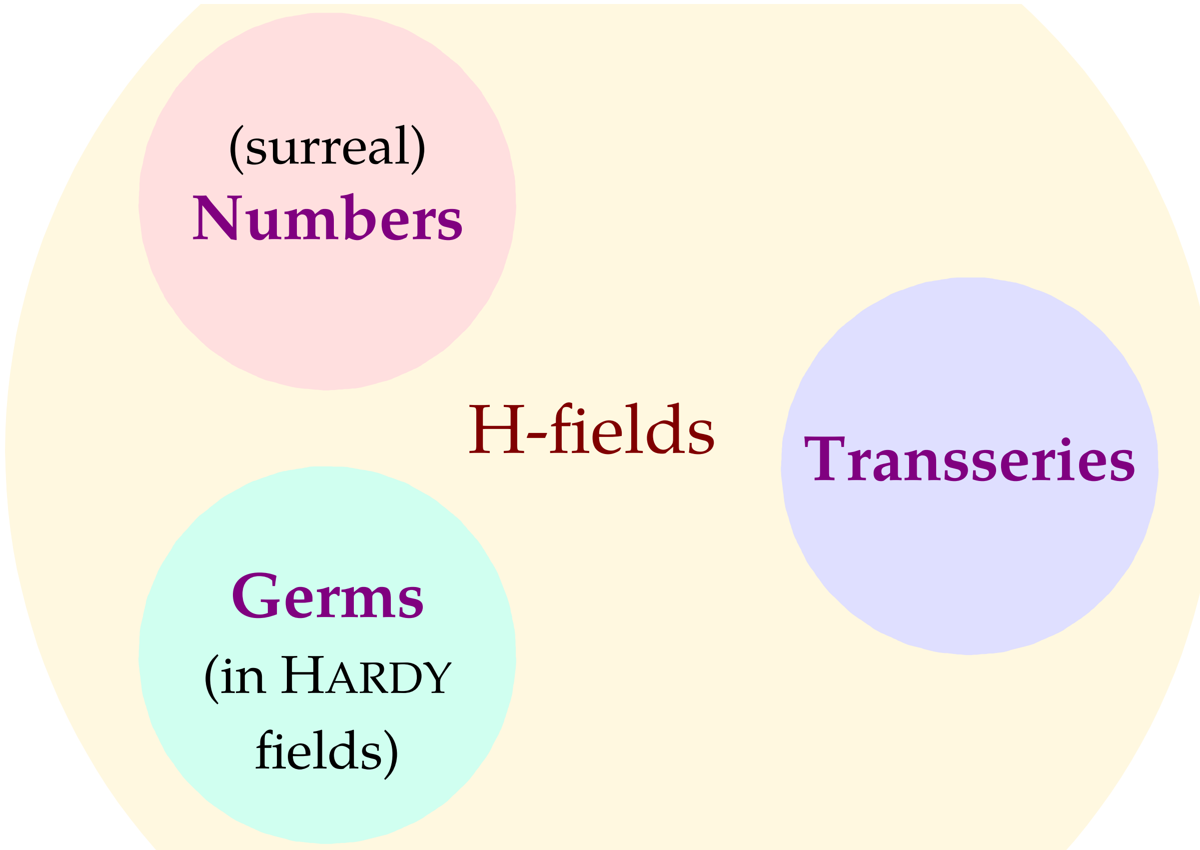
The elementary theory of $\mathbb{T}$ *is completely axiomatized by:*

1. $\mathbb{T}$ *is a* LIOUVILLE *closed H-field with small derivation;*
2. $\mathbb{T}$ *satisfies the intermediate value property for differential polynomials: Given* $P \in \mathbb{T}[Y, Y', \ldots, Y^{(r)}]$ *and* $u < v$ *in* $\mathbb{T}$ *with* $P(u)\,P(v) < 0$*, there exists a* $y \in \mathbb{T}$ *with* $u < y < v$ *and* $P(y) = 0$

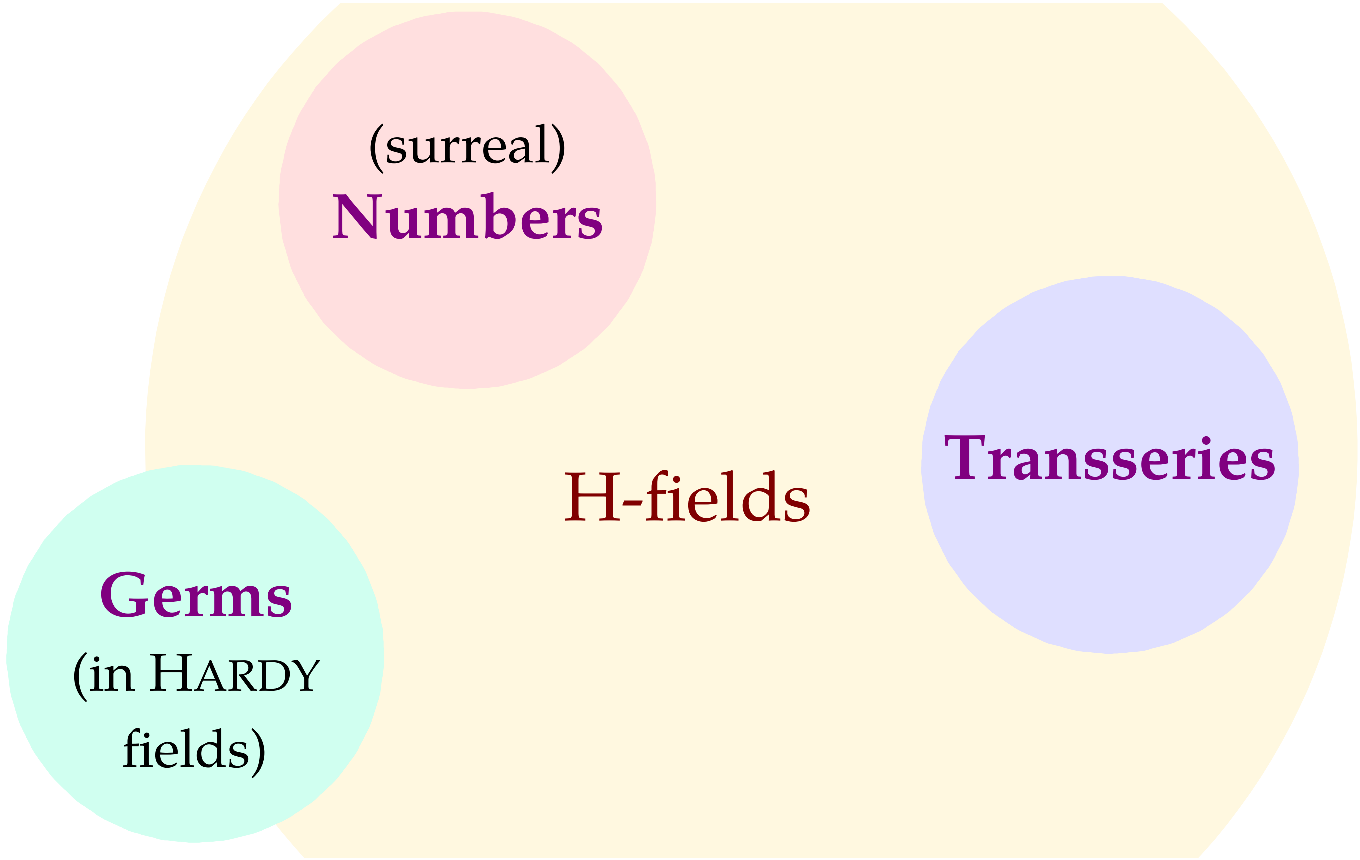
In particular: the theory of $\mathbb{T}$ *is decidable.*

We also prove a quantifier elimination result for $\mathbb{T}$ in a natural expansion of the above language.

H-field elements as germs



H-field elements as germs



Closure properties of Hardy fields

Theorem (Hardy 1910, Bourbaki 1951)

Any Hardy *field has a smallest* Liouville *closed* Hardy *field extension.*

Closure properties of HARDY fields

Theorem (HARDY 1910, BOURBAKI 1951)

Any HARDY *field has a smallest* LIOUVILLE *closed* HARDY *field extension.*

Conjecture

Let H be a maximal HARDY *field. Then*

- A H *satisfies the differential intermediate value property.*
- B *For countable subsets $L < R$ of H, there exists an $h \in H$ with $L < h < R$.*

Closure properties of HARDY fields

Theorem (HARDY 1910, BOURBAKI 1951)

Any HARDY *field has a smallest* LIOUVILLE *closed* HARDY *field extension.*

Conjecture

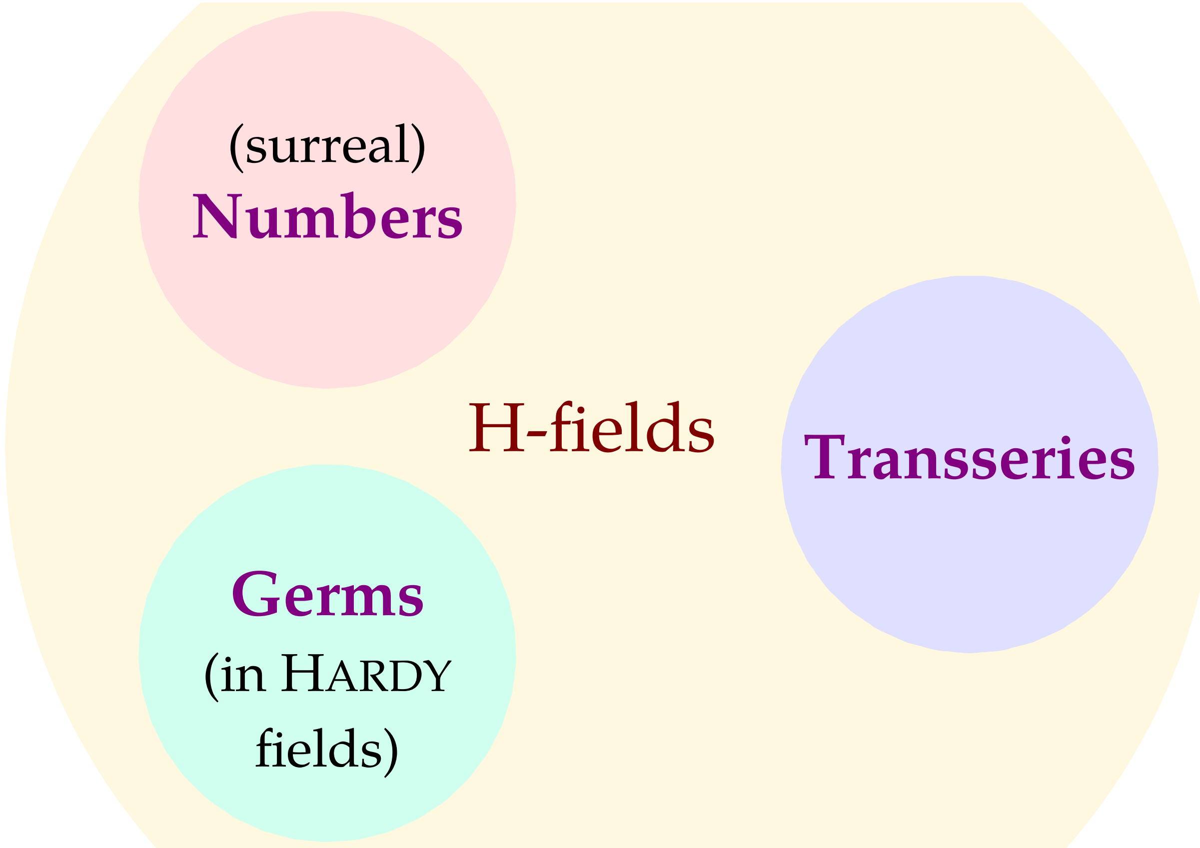
Let H be a maximal HARDY *field. Then*

- A H *satisfies the differential intermediate value property.*
- B *For countable subsets $L < R$ of H, there exists an $h \in H$ with $L < h < R$.*

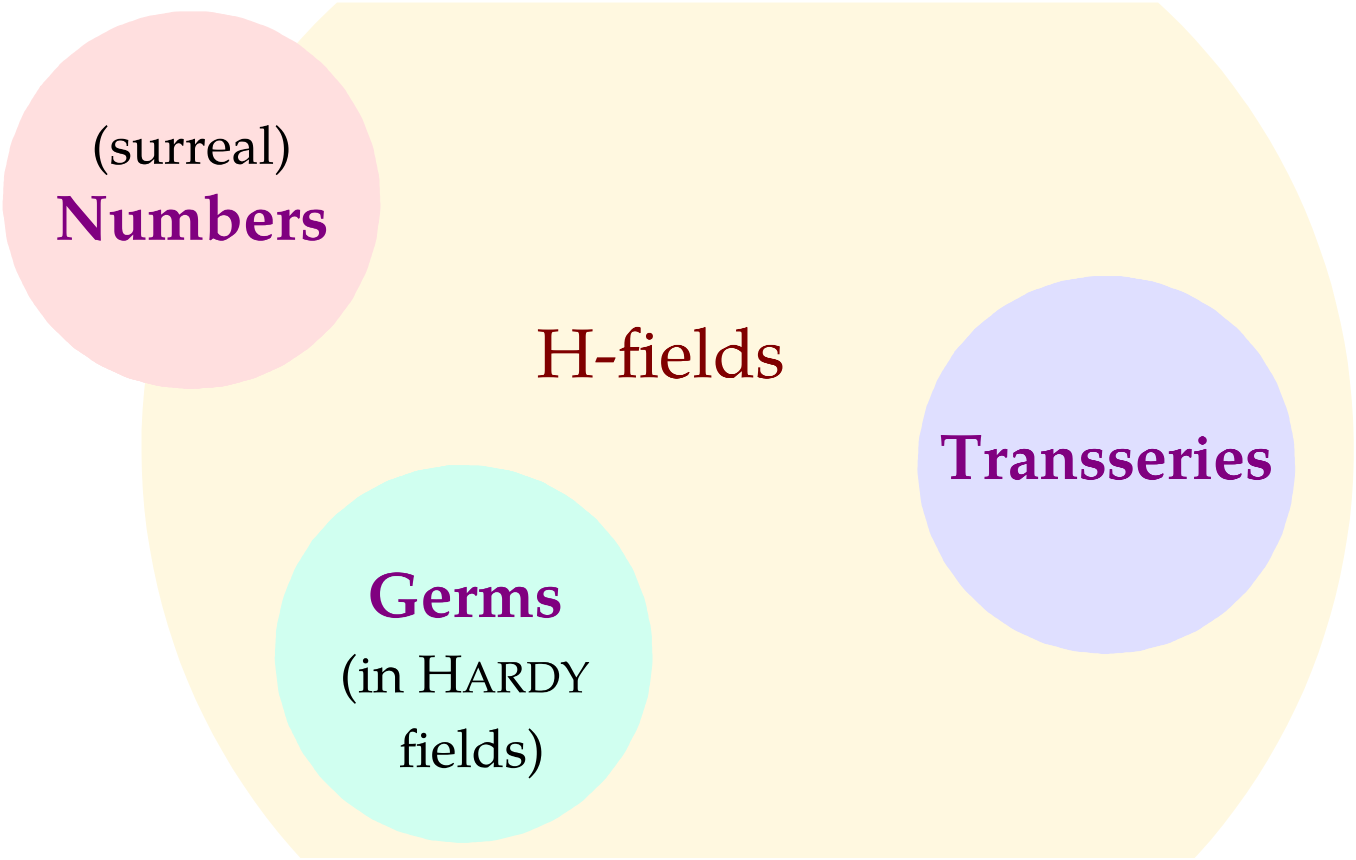
Corollary

- A H *is elementarily equivalent to $\mathbb{T}$ as an ordered differential field.*
- B *Under CH, all maximal* HARDY *fields are isomorphic.*

H-field elements as surreal numbers



H-field elements as surreal numbers



Embedding H-fields into the surreals

Theorem (JEMS 2019)

Every H-field with small derivation and constant field $\mathbb{R}$ can be embedded as an ordered differential field into $\mathbf{No}$.

Embedding H-fields into the surreals

Theorem (JEMS 2019)

Every H-field with small derivation and constant field $\mathbb{R}$ can be embedded as an ordered differential field into $\mathbf{No}$.

Theorem (JEMS 2019)

Let κ be an uncountable cardinal. The field $\mathbf{No}(\kappa)$ of surreal numbers of length $<\kappa$ is an elementary submodel of $\mathbf{No}$.

Embedding H-fields into the surreals

Theorem (JEMS 2019)

Every H-field with small derivation and constant field $\mathbb{R}$ *can be embedded as an ordered differential field into* $\mathbf{No}$.

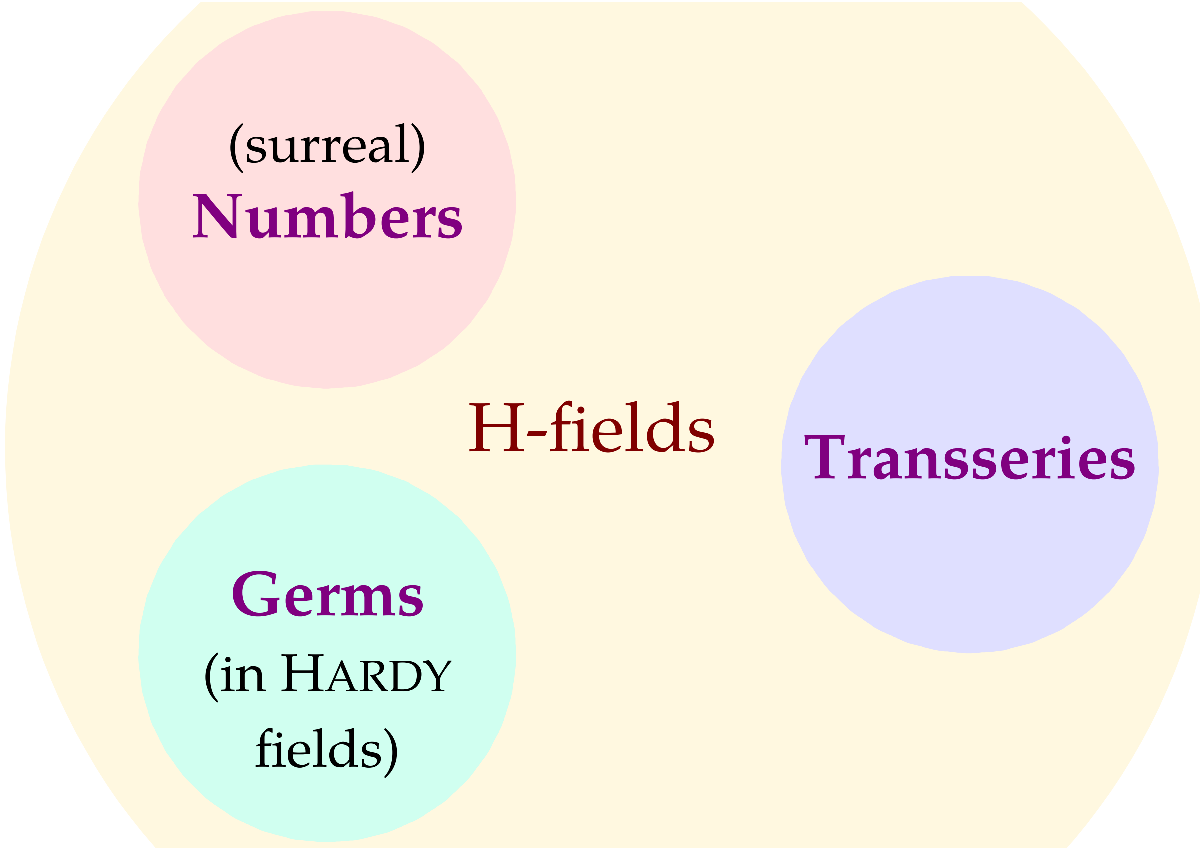
Theorem (JEMS 2019)

Let κ *be an uncountable cardinal. The field* $\mathbf{No}(\kappa)$ *of surreal numbers of length* $<\kappa$ *is an elementary submodel of* $\mathbf{No}$.

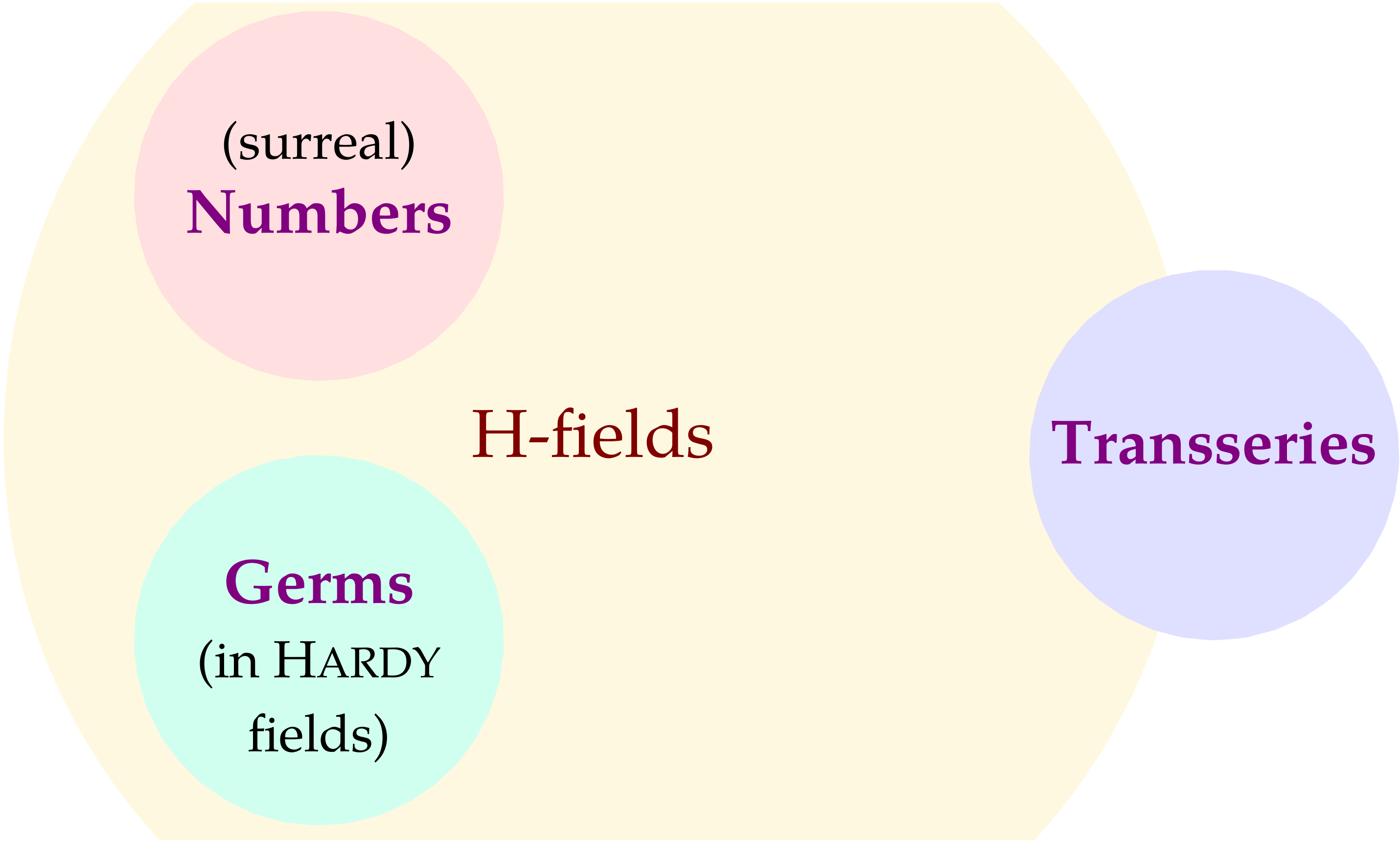
Corollary in progress

Under CH all maximal HARDY *fields are isomorphic to* $\mathbf{No}(\omega_1)$.

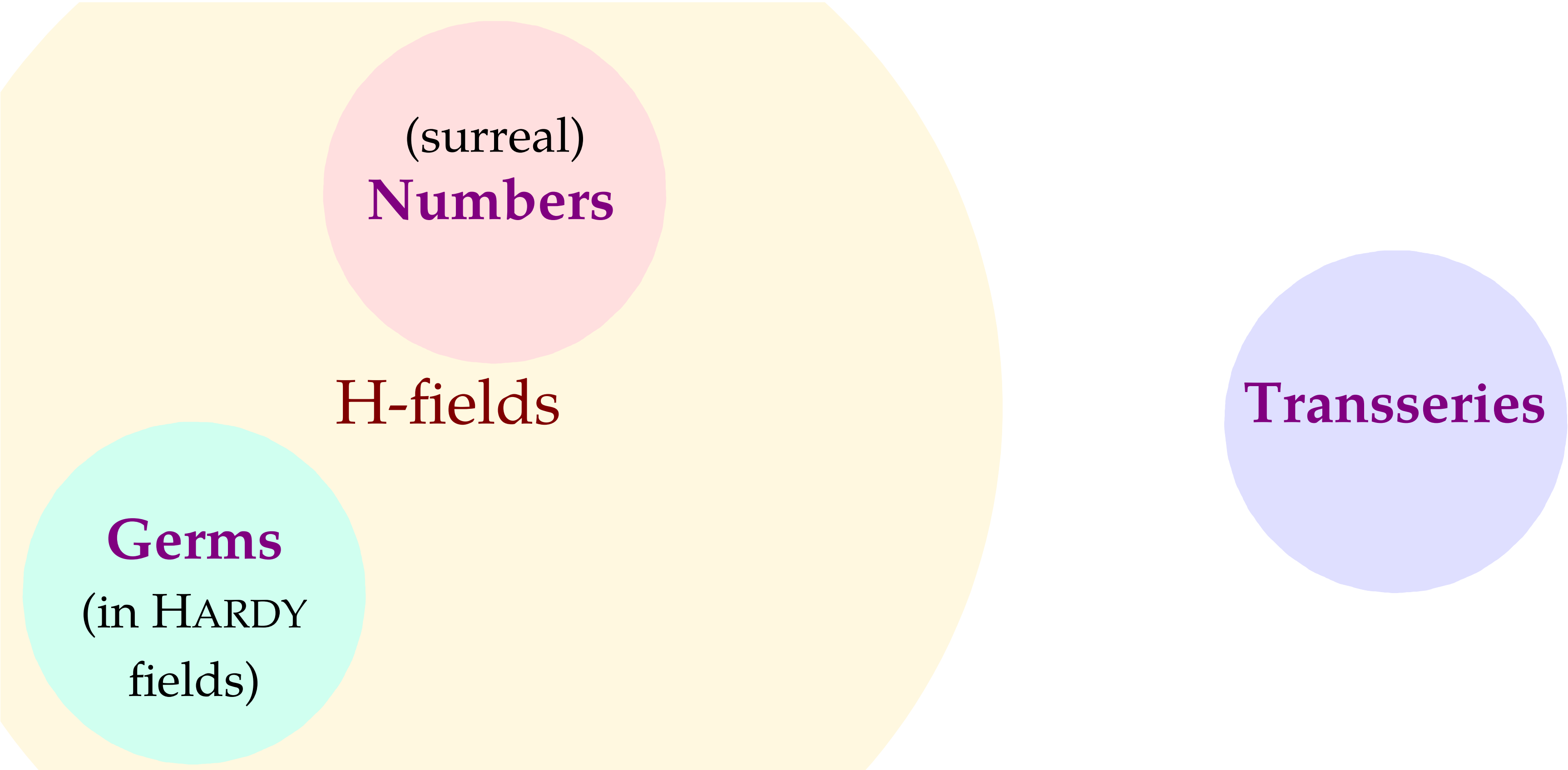
H-field elements as transseries



H-field elements as transseries



H-field elements as transseries



Surreal numbers as transseries

Definition (van der Hoeven 2000, Schmeling 2001)

A field $\mathbf{T} = \mathbb{R}[[\mathfrak{M}]]$ *with* $\log : \mathbf{T}^{>} \longrightarrow \mathbf{T}$ *is a* ***field of transseries*** *if …*

A ***transserial derivation*** *on* $\mathbf{T}$ *is a derivation* $\partial : \mathbf{T} \rightarrow \mathbf{T}$ *such that …*

Surreal numbers as transseries

Definition (van der Hoeven 2000, Schmeling 2001)

A field $\mathbf{T} = \mathbb{R}[[\mathfrak{M}]]$ *with* $\log : \mathbf{T}^{>} \longrightarrow \mathbf{T}$ *is a* ***field of transseries*** *if …*

A ***transserial derivation*** *on* $\mathbf{T}$ *is a derivation* $\partial : \mathbf{T} \rightarrow \mathbf{T}$ *such that …*

Theorem (Berarducci–Mantova, 2015)

No *is a field of transseries and* ∂_{BM} *is a transserial derivation.*

Surreal numbers as transseries

Definition (van der Hoeven 2000, Schmeling 2001)

A field $\mathbf{T} = \mathbb{R}[[\mathfrak{M}]]$ *with* $\log : \mathbf{T}^{>} \longrightarrow \mathbf{T}$ *is a* ***field of transseries*** *if* …

A ***transserial derivation*** *on* $\mathbf{T}$ *is a derivation* $\partial : \mathbf{T} \rightarrow \mathbf{T}$ *such that* …

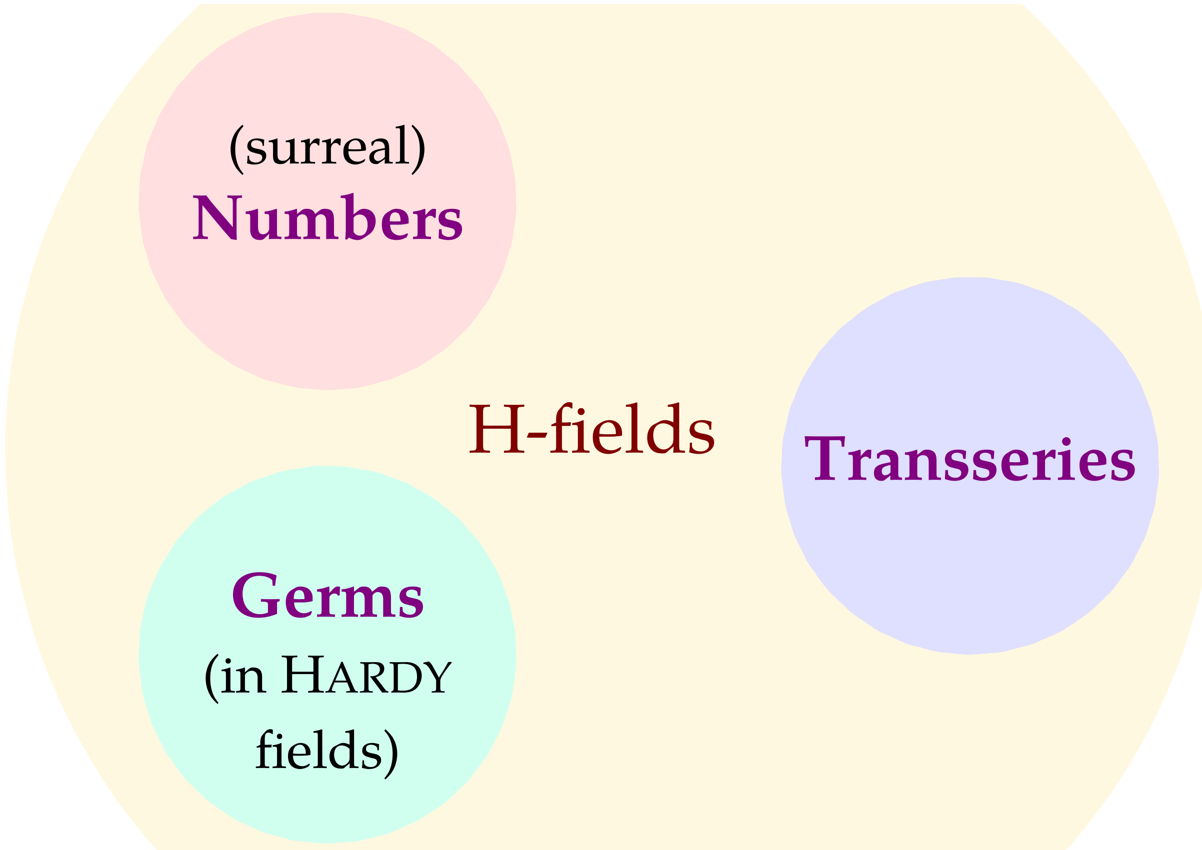
Theorem (Berarducci–Mantova, 2015)

$\mathbf{No}$ *is a field of transseries and* ∂_{BM} *is a transserial derivation.*

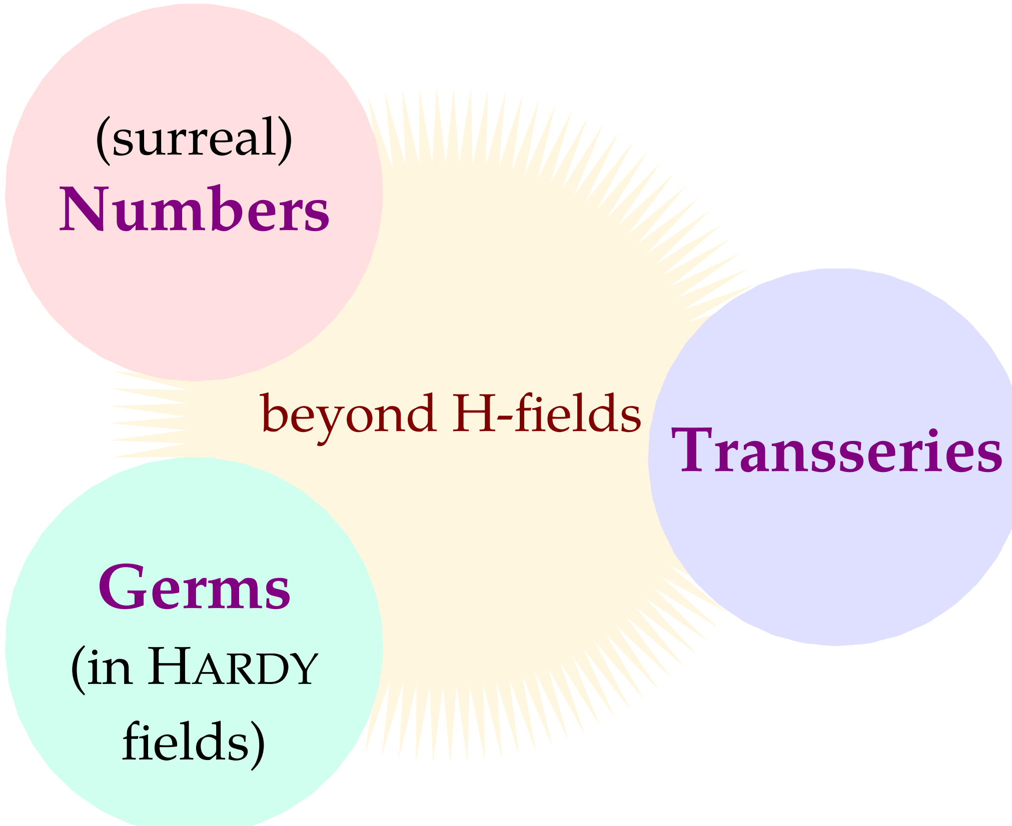
Corollary

Any H-field with constant field $\mathbb{R}$ *can be embedded in a field of transseries with a transserial derivation.*

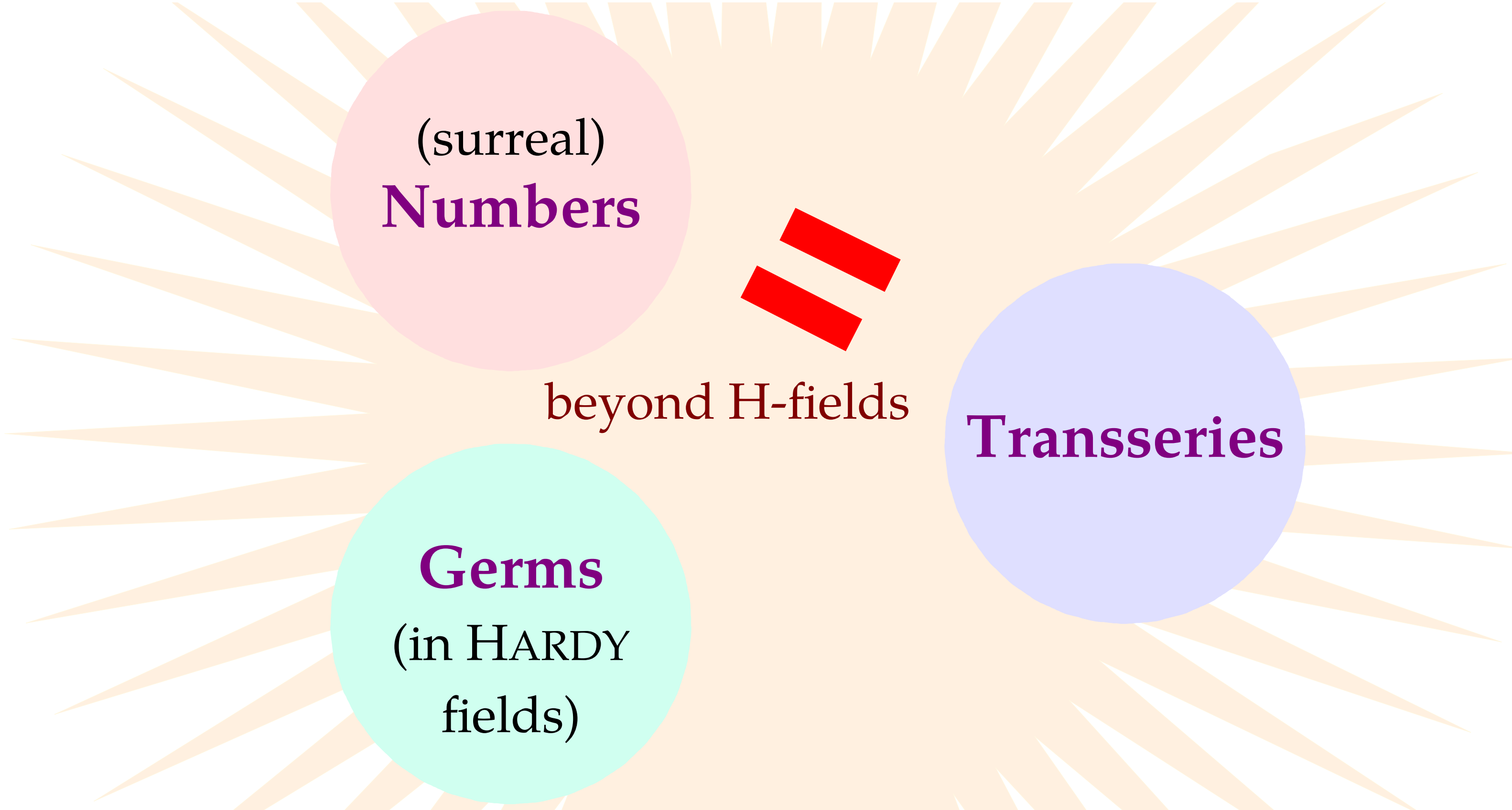
What next?


(surreal)
Numbers
H-fields
Transseries
Germs
(in HARDY
fields)

What next?


(surreal)
Numbers
beyond H-fields
Transseries
Germs
(in HARDY
fields)

What next?


(surreal)
Numbers
beyond H-fields
Transseries
Germs
(in HARDY
fields)

Iterated exponentials and logarithms

$$\begin{aligned}\exp_\omega(x+1) &= \exp \exp_\omega x \\ \exp_{\omega^2}(x+1) &= \exp_\omega \exp_{\omega^2} x \\ &\cdots\end{aligned}$$

→ stronger growth that $e^x, e^{e^x}, \ldots, \exp_\omega x, e^{\exp_\omega x}, \ldots, \exp_\omega \exp_\omega x, \ldots$

Iterated exponentials and logarithms

$$\begin{aligned}\exp_\omega(x+1) &= \exp\exp_\omega x\\ \exp_{\omega^2}(x+1) &= \exp_\omega\exp_{\omega^2} x\\ &\cdots\end{aligned}$$

→ stronger growth that $e^x, e^{e^x}, \ldots, \exp_\omega x, e^{\exp_\omega x}, \ldots, \exp_\omega\exp_\omega x, \ldots$

Functional equations

$$f(x) = \sqrt{x} + e^{f(\log x)} = \sqrt{x} + e^{\sqrt{\log x} + e^{\sqrt{\log\log x} + \cdot^{\cdot^{\cdot}}}}$$

Hyperlogarithms and hyperexponentials

$$\begin{aligned} \exp_\omega(x+1) &= \exp \exp_\omega x \\ \exp_{\omega^2}(x+1) &= \exp_\omega \exp_{\omega^2} x \\ &\vdots \end{aligned}$$

$$\begin{aligned} \log_\omega \log x &= \log_\omega x - 1 \\ \log_{\omega^2} \log_\omega x &= \log_{\omega^2} x - 1 \\ &\vdots \end{aligned}$$

Hyperlogarithms and hyperexponentials

$$
\begin{aligned}
\exp_\omega(x+1) &= \exp \exp_\omega x \\
\exp_{\omega^2}(x+1) &= \exp_\omega \exp_{\omega^2} x \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
\log_\omega \log x &= \log_\omega x - 1 \\
\log_{\omega^2} \log_\omega x &= \log_{\omega^2} x - 1 \\
&\vdots \\
\log_\omega x &= \int \frac{1}{x \log x \log \log x \cdots} \\
\log_\alpha x &= \int \prod_{\beta<\alpha} \frac{1}{\log_\beta x}
\end{aligned}
$$

Hyperseries

Hyperlogarithms and hyperexponentials

$$\begin{aligned} \exp_\omega(x+1) &= \exp \exp_\omega x \\ \exp_{\omega^2}(x+1) &= \exp_\omega \exp_{\omega^2} x \\ &\vdots \end{aligned}$$

$$\begin{aligned} \log_\omega \log x &= \log_\omega x - 1 \\ \log_{\omega^2} \log_\omega x &= \log_{\omega^2} x - 1 \\ &\vdots \\ \log_\omega x &= \int \frac{1}{x \log x \log \log x \cdots} \\ \log_\alpha x &= \int \prod_{\beta<\alpha} \frac{1}{\log_\beta x} \end{aligned}$$

Nested hyperseries

Solutions de $f(x) = \sqrt{x} + \mathrm{e}^{f(\log x)}$:

$$f_0(x)$$

Hyperlogarithms and hyperexponentials

$$\begin{aligned}\exp_\omega(x+1) &= \exp\exp_\omega x\\ \exp_{\omega^2}(x+1) &= \exp_\omega\exp_{\omega^2} x\\ &\vdots\end{aligned}$$

$$\begin{aligned}\log_\omega\log x &= \log_\omega x-1\\ \log_{\omega^2}\log_\omega x &= \log_{\omega^2} x-1\\ &\vdots\\ \log_\omega x &= \int\frac{1}{x\log x\log\log x\cdots}\\ \log_\alpha x &= \int\prod_{\beta<\alpha}\frac{1}{\log_\beta x}\end{aligned}$$

Nested hyperseries

Solutions de $f(x)=\sqrt{x}+\mathrm{e}^{f(\log x)}$:

$$f_{-1}(x) \quad < \quad f_0(x) \quad < \quad f_1(x)$$

Hyperlogarithms and hyperexponentials

$$\begin{aligned} \exp_\omega(x+1) &= \exp \exp_\omega x \\ \exp_{\omega^2}(x+1) &= \exp_\omega \exp_{\omega^2} x \\ &\vdots \end{aligned}$$

$$\begin{aligned} \log_\omega \log x &= \log_\omega x - 1 \\ \log_{\omega^2} \log_\omega x &= \log_{\omega^2} x - 1 \\ &\vdots \\ \log_\omega x &= \int \frac{1}{x \log x \log \log x \cdots} \\ \log_\alpha x &= \int \prod_{\beta<\alpha} \frac{1}{\log_\beta x} \end{aligned}$$

Nested hyperseries

Solutions de $f(x) = \sqrt{x} + \mathrm{e}^{f(\log x)}$:

$$f_{-2}(x) < f_{-1}(x) < f_{-1/2}(x) < f_0(x) < f_{1/2}(x) < f_1(x) < f_2(x)$$

Hyperlogarithms and hyperexponentials

$$\begin{aligned} \exp_\omega(x+1) &= \exp\exp_\omega x \\ \exp_{\omega^2}(x+1) &= \exp_\omega \exp_{\omega^2} x \\ &\vdots \end{aligned}$$

$$\begin{aligned} \log_\omega \log x &= \log_\omega x - 1 \\ \log_{\omega^2} \log_\omega x &= \log_{\omega^2} x - 1 \\ &\vdots \\ \log_\omega x &= \int \frac{1}{x \log x \log\log x \cdots} \\ \log_\alpha x &= \int \prod_{\beta<\alpha} \frac{1}{\log_\beta x} \end{aligned}$$

Nested hyperseries

Solutions de $f(x) = \sqrt{x} + e^{f(\log x)}$: $\qquad \longrightarrow \qquad f_{\mathbf{No}}(x)$

$$\cdots < f_{-2}(x) < \cdots < f_{-1}(x) < \cdots < f_0(x) < \cdots < f_{1/2}(x) < \cdots < f_1(x) < \cdots < f_2(x) < \cdots$$

Conjecture (vdH 2006)

For an appropriate definition of the class $\mathbf{Hy}$ *of hyperseries, we have* $\mathbf{No} \cong \mathbf{Hy}$ *for the map* $\phi: \mathbf{Hy} \longrightarrow \mathbf{No}; f \longmapsto f(\omega)$.

Hyperséries ≅ Nombres surréels

Conjecture (vdH 2006)

For an appropriate definition of the class $\mathbf{Hy}$ *of hyperseries, we have* $\mathbf{No} \cong \mathbf{Hy}$ *for the map* $\phi\colon \mathbf{Hy} \longrightarrow \mathbf{No}; f \longmapsto f(\omega)$.

Proof. By constructing a Conway bracket $\{|\}$ on $\mathbf{Hy}$.

Conjecture (vdH 2006)

For an appropriate definition of the class **Hy** *of hyperseries, we have* $\mathbf{No} \cong \mathbf{Hy}$ *for the map* $\phi: \mathbf{Hy} \longrightarrow \mathbf{No}; f \longmapsto f(\omega)$.

Proof. By constructing a Conway bracket $\{|\}$ on **Hy**.

Examples :

$$\{x, e^x, e^{e^x}, \ldots|\} = \exp_\omega x$$

$$\left\{\sqrt{x}, \sqrt{x} + e^{\sqrt{\log x}}, \ldots \middle| \ldots, \sqrt{x} + e^{2\sqrt{\log x}}, 2\sqrt{x}\right\} = f_0(x)$$

$$\left\{x^2, e^{\log^2 x}, e^{e^{\log^2 \log x}}, \ldots \middle| \ldots, e^{e^{e^{\sqrt{\log \log x}}}}, e^{e^{\sqrt{\log x}}}, e^{\sqrt{x}}\right\} = \exp_\omega\left(\log_\omega x + \tfrac{1}{2}\right)$$

Thank you!

http://www.TeXmacs.org