Amortized bivariate multi-point evaluation

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CNRS, École polytechnique
Joint work with **Grégoire Lecerf**

July 21, 2021

Multi-point evaluation

Problem: multi-point evaluation

INPUT: $P \in \mathbb{K}[x_1, \ldots, x_D]$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{K}^D)^n$

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Theorem (Kedlaya–Umans 2008, 2011)

If $\mathbb{K} = \mathbb{F}_q$, then the bit-complexity of multi-point evaluation is $O((n \log q)^{1+o(1)})$.

If $\mathbb{K} = \mathbb{Z}$ or $\mathbb{K} = \mathbb{Q}$, then nearly optimal bound in terms of bit-size (CRT).

Link with polynomial system solving

System of equations

$$(\Sigma)\quad \begin{cases} P_1(x_1,\ldots,x_D) = 0 & \deg P_1 \leqslant d \\ \quad\vdots & \\ P_D(x_1,\ldots,x_D) = 0 & \deg P_D \leqslant d \end{cases}$$

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System solving $\longrightarrow$ Multi-point evaluation

Tentative solution $\boldsymbol{\alpha} \in (\mathbb{K}^D)^{d^D} \longrightarrow$ Proof that $P_1(\boldsymbol{\alpha}) = \cdots = P_D(\boldsymbol{\alpha}) = 0$

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Theorem (vdH–Lecerf, 2018)

If $\mathbb{K} = \mathbb{F}_q$, *then the Las Vegas bit-complexity to solve* (Σ) *is bounded by*

$$\tilde{O}(d^{(2+o(1))D-1} \log q).$$

If $\mathbb{K} = \mathbb{Z}$ *or* $\mathbb{K} = \mathbb{Q}$, *then the complexity is nearly optimal in the generic output size.*

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Assume that $\boldsymbol{\alpha}$ is in "general position".

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Theorem (this talk)

If $D = 2$ and $n = d^2$, then $\exists$ $\tilde{O}(n)$ algorithm for amortized multi-point evaluation.

k-degree and the admissible ordering $\prec_k$

Definition: $\deg_k$ and $\prec_k$ for $k \in \{1, 2, 3, \ldots\}$

$$\deg_k x^a y^b \;=\!=\; a + kb$$

$$x^a y^b \prec_k x^u y^v \iff \begin{cases} a + kb < u + kv & \text{or} \\ a + kb = u + kv \text{ and } b < v. \end{cases}$$

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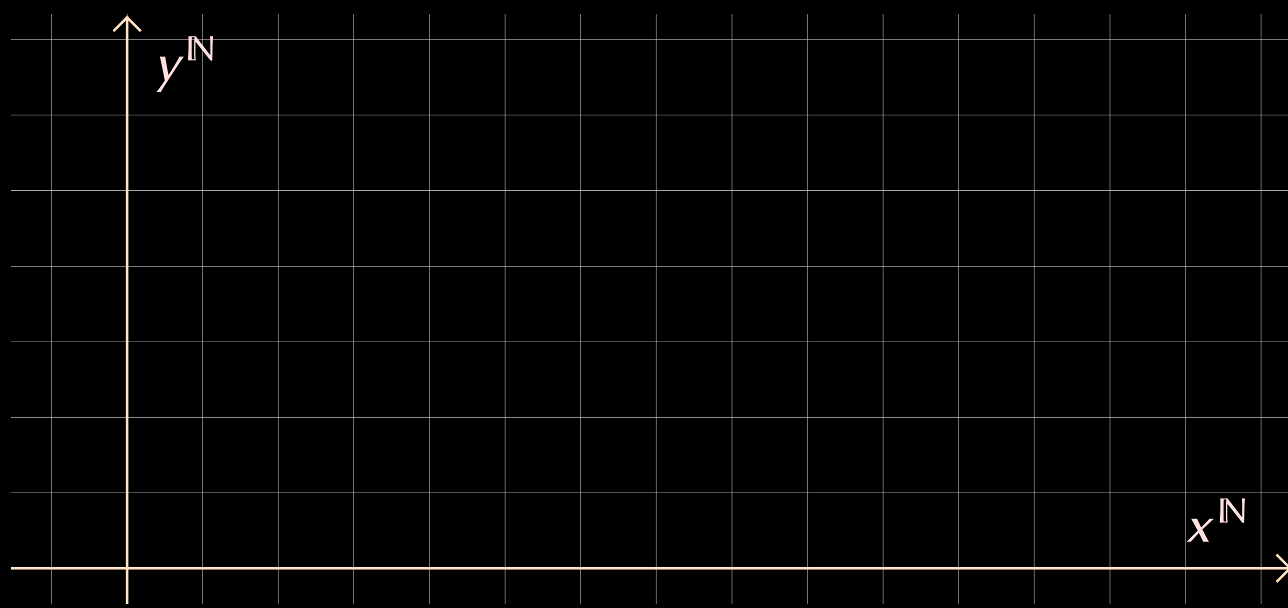
Illustration: sorting the monomomials $x^a y^b$ by $\prec_2$



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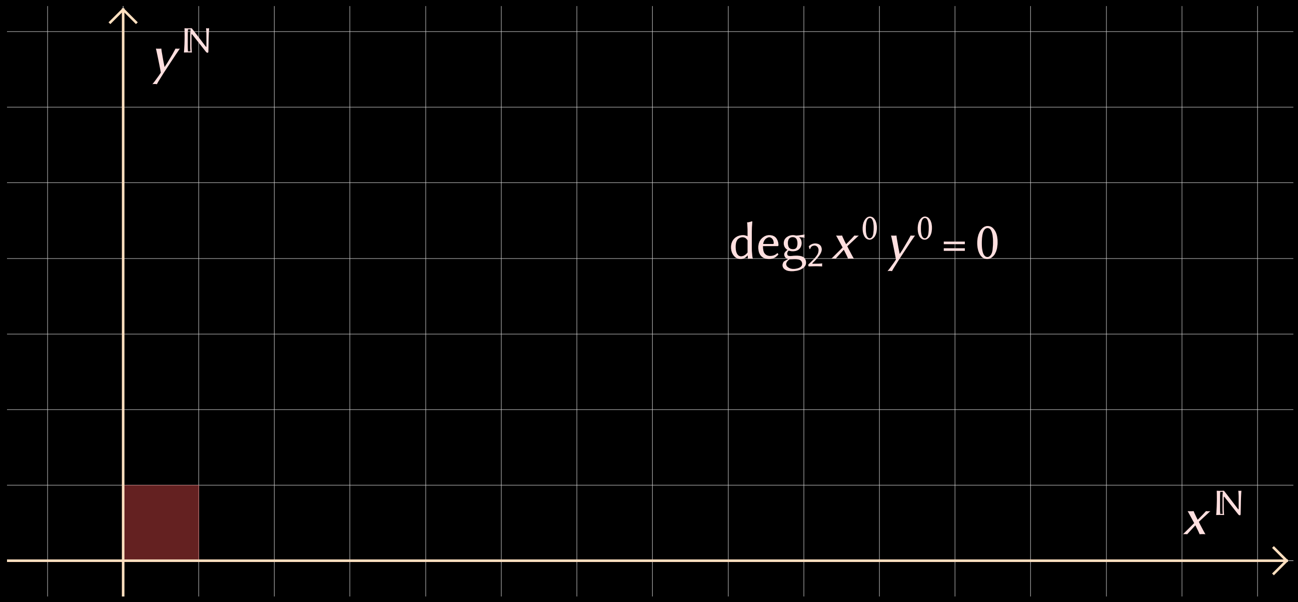

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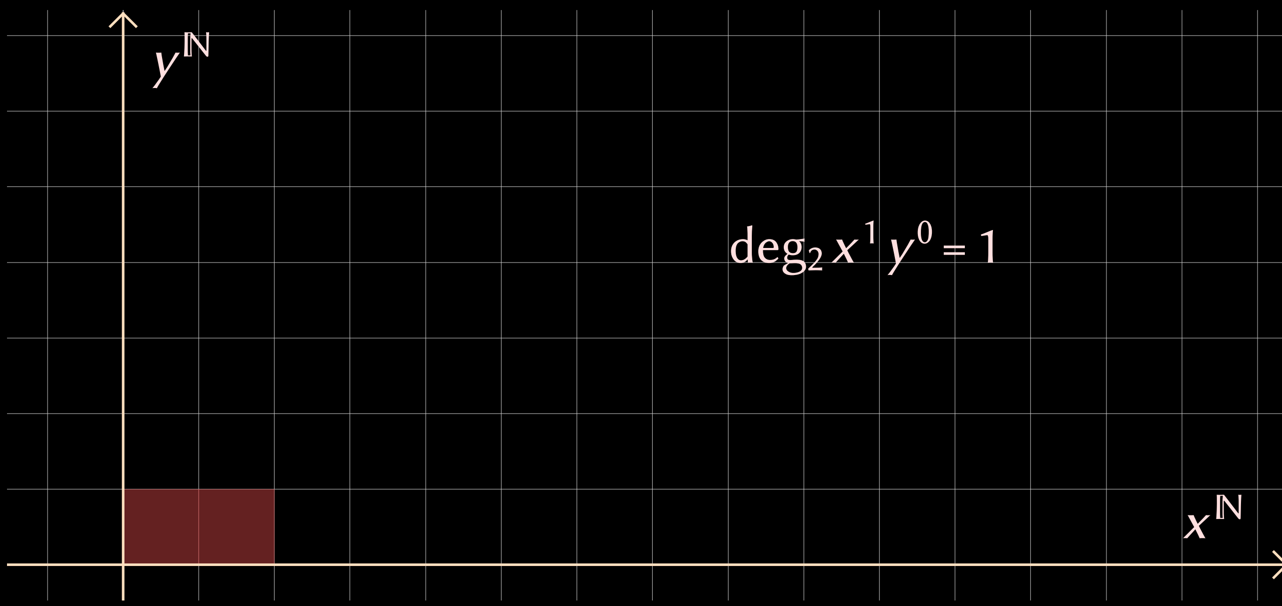

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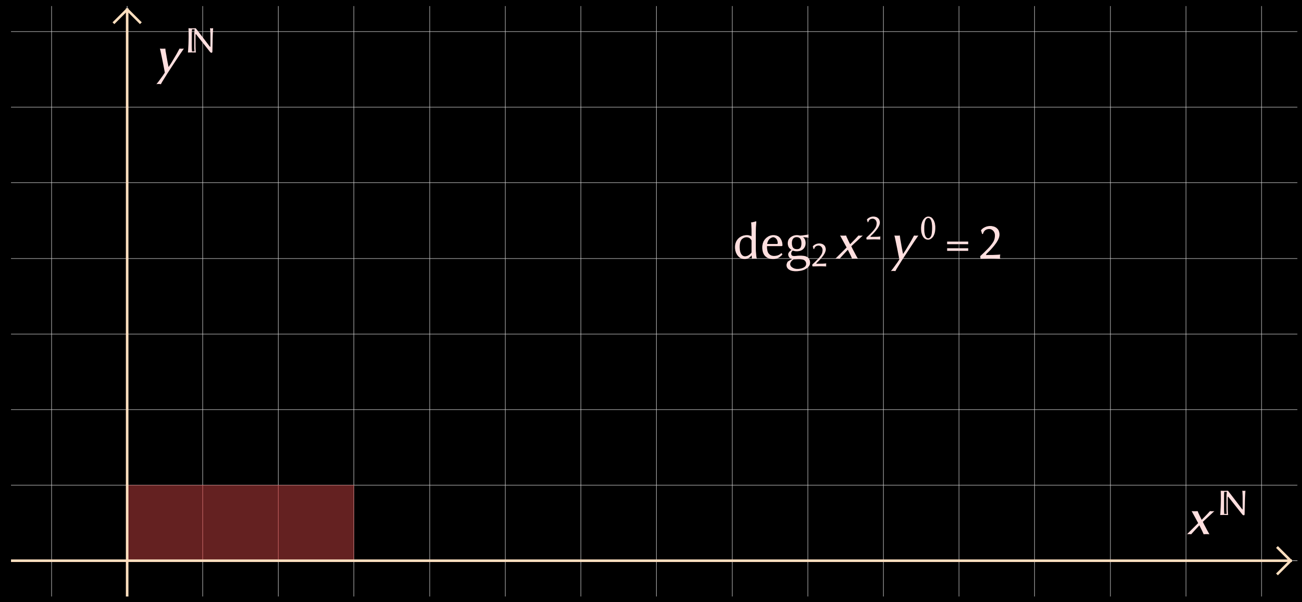

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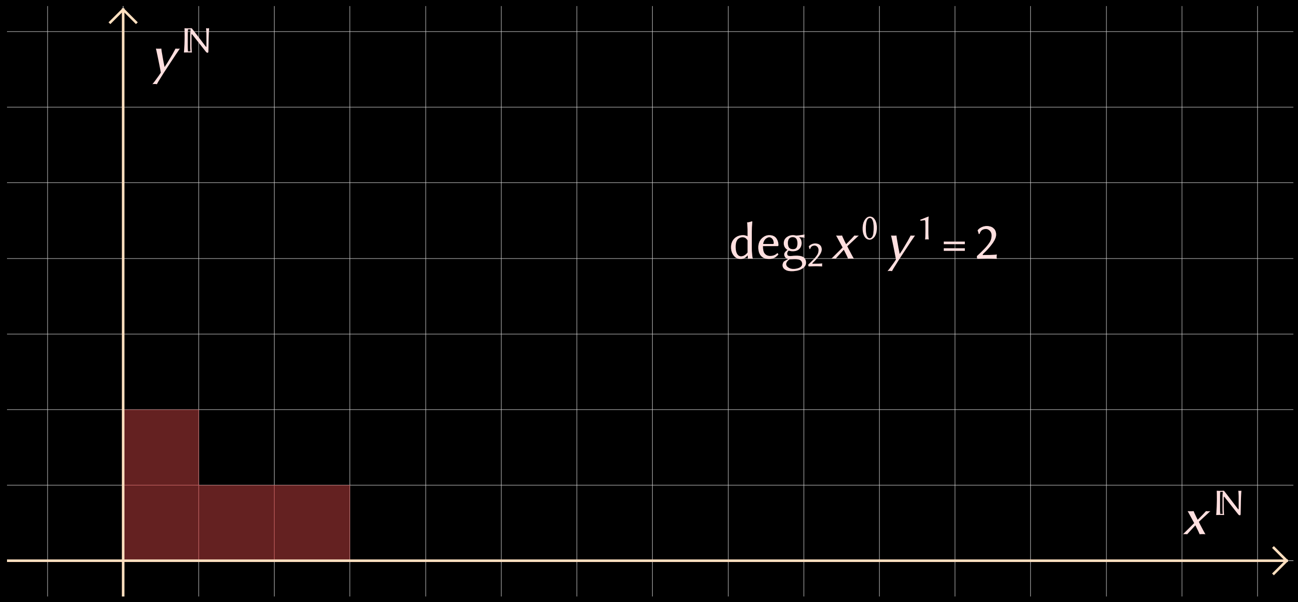

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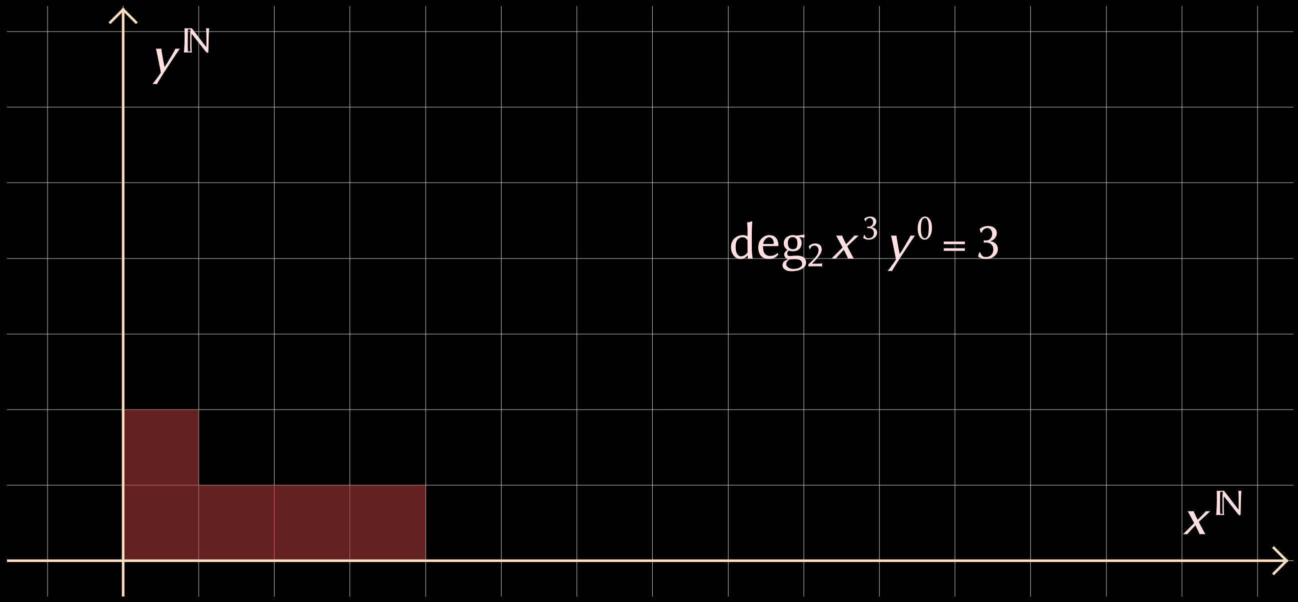

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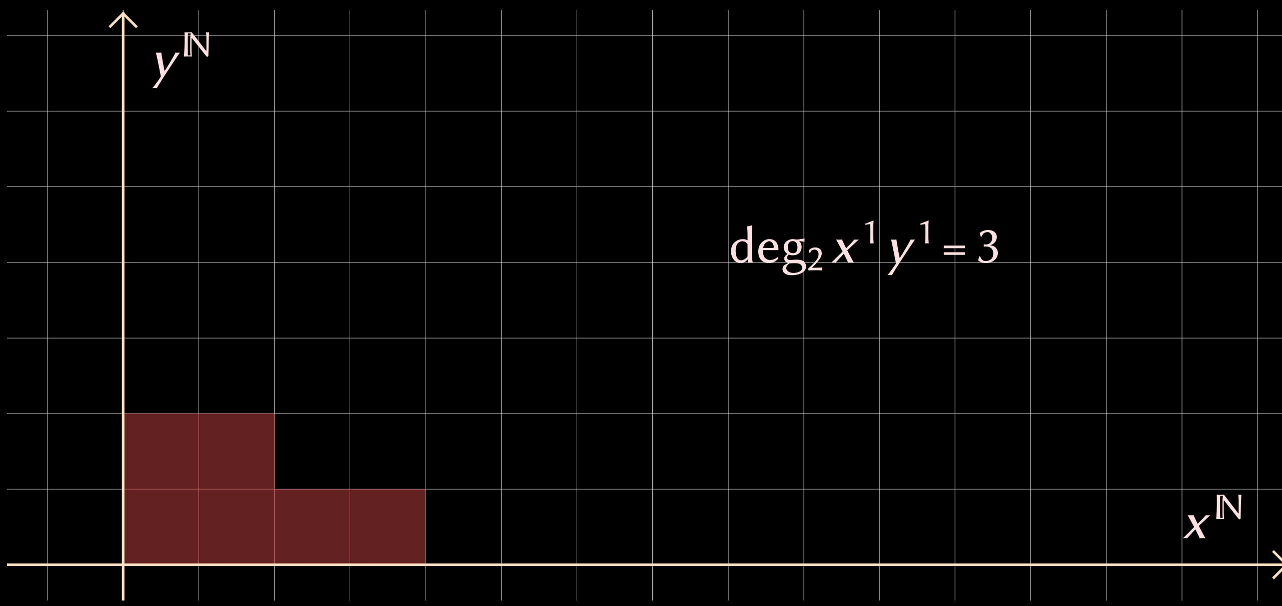

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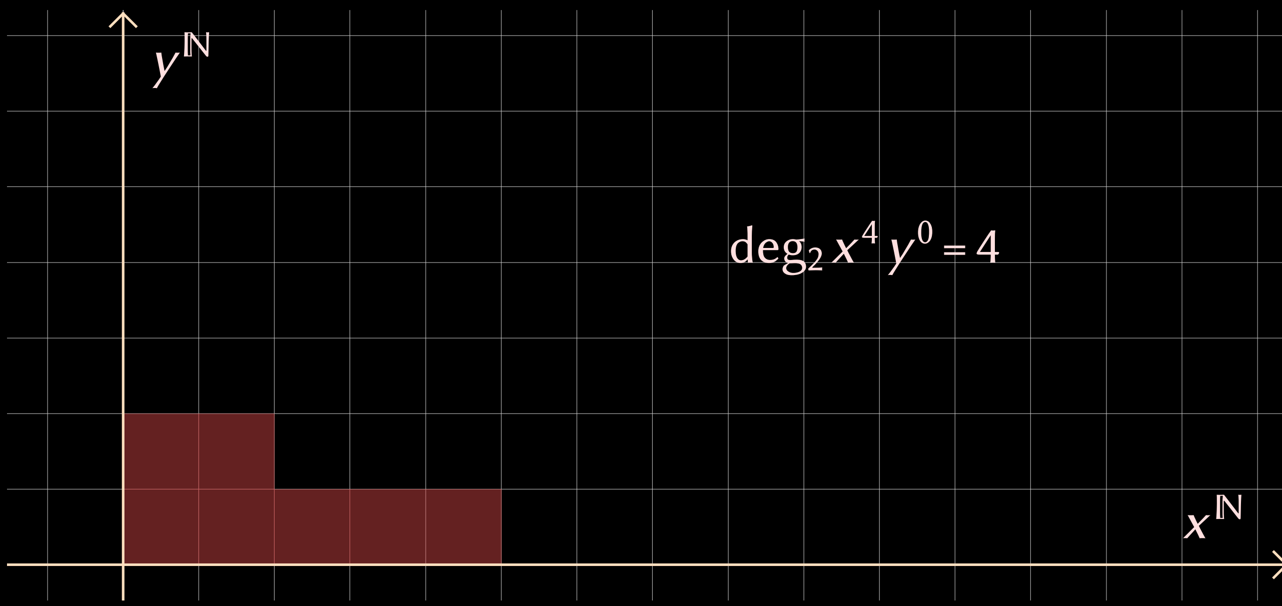

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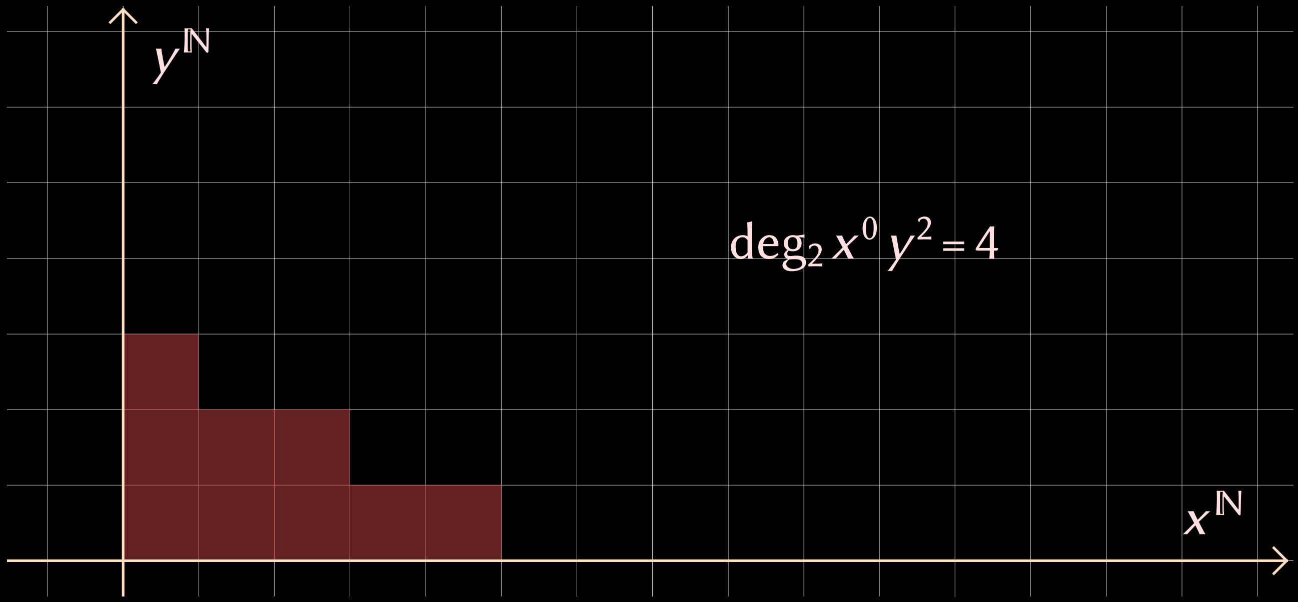

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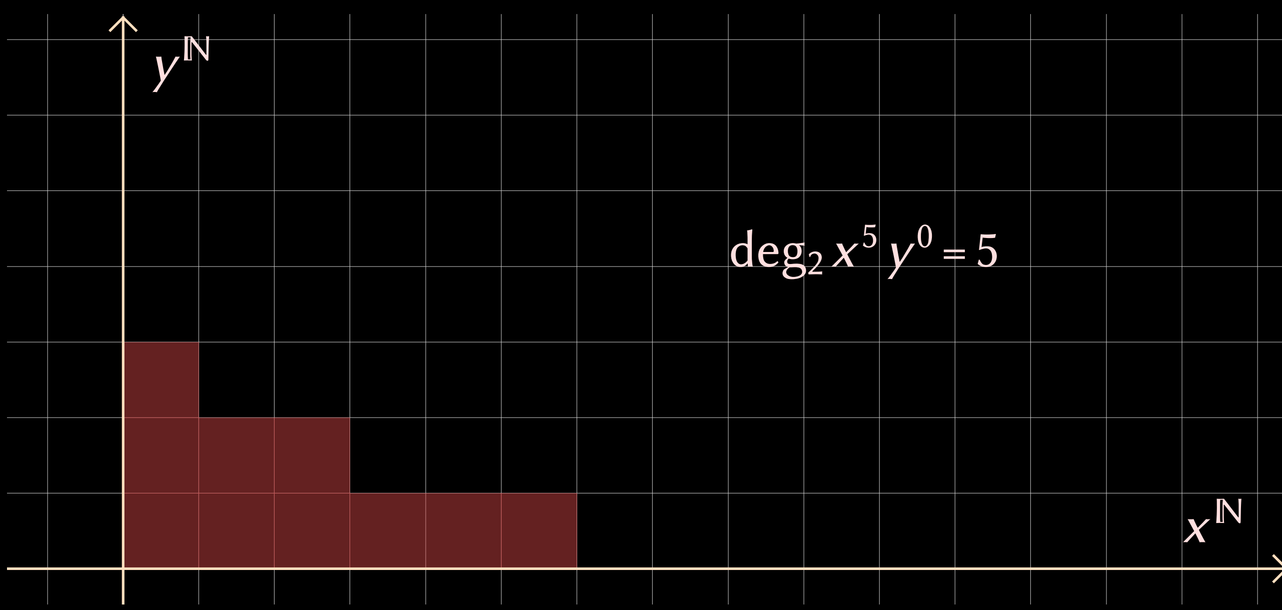

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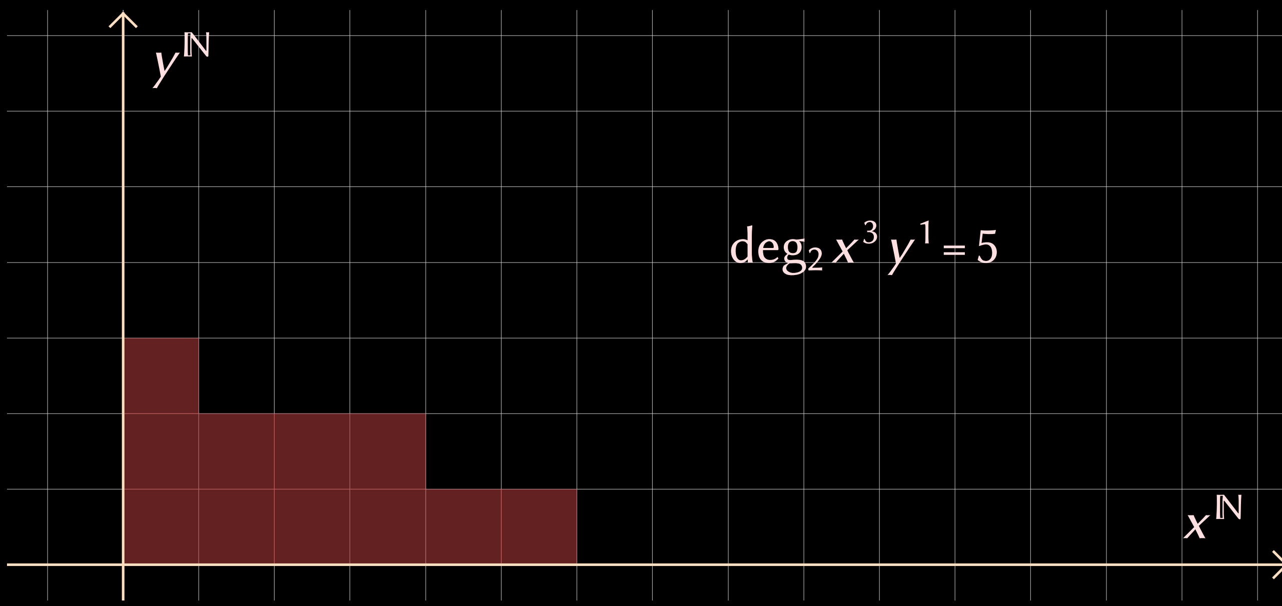

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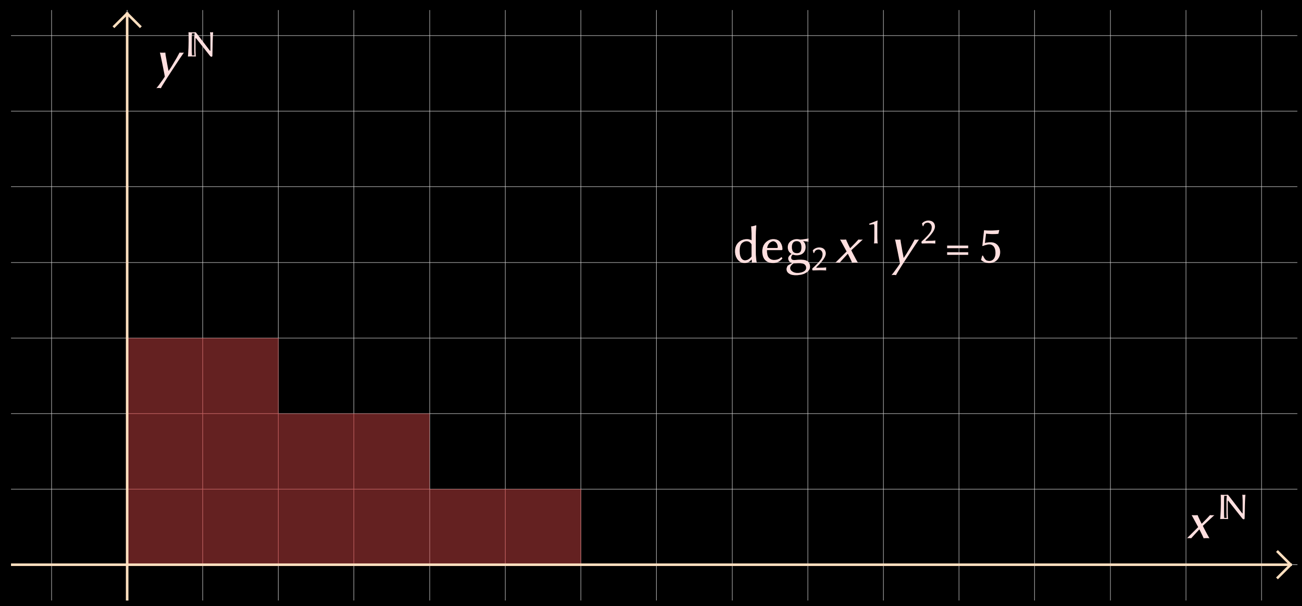

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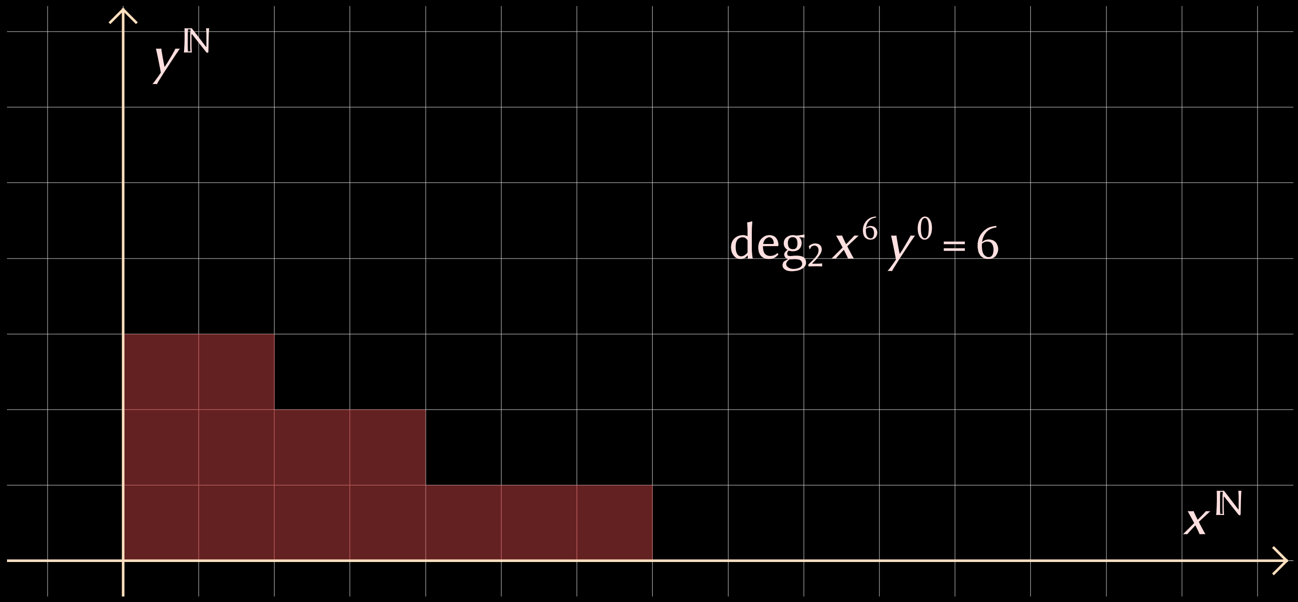

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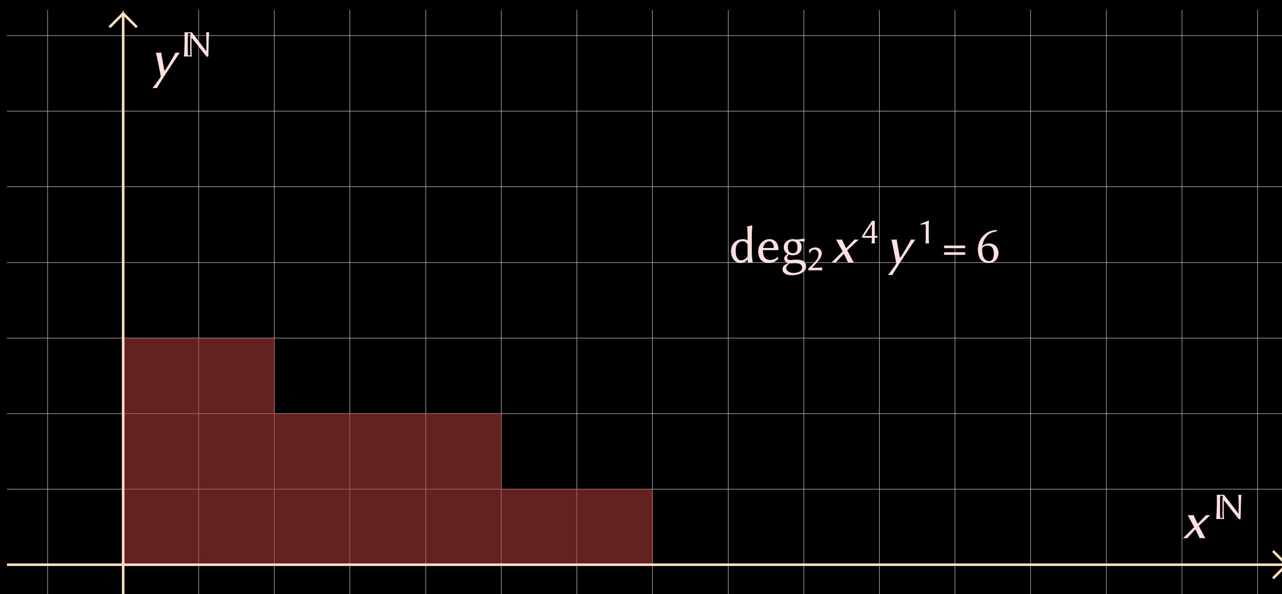

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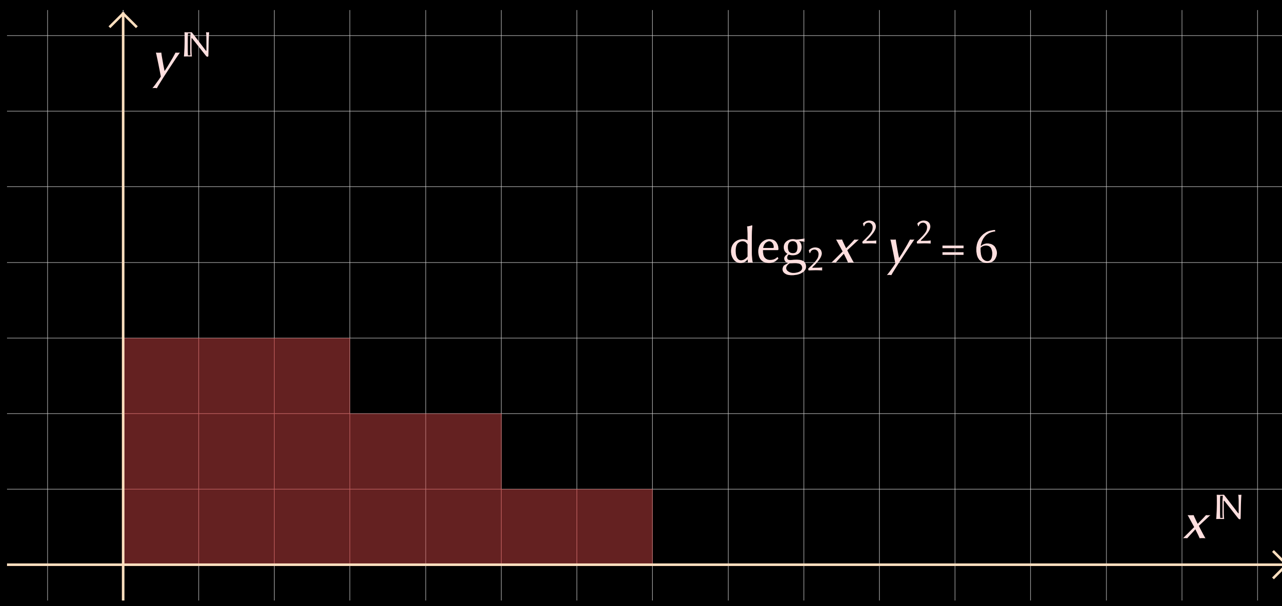

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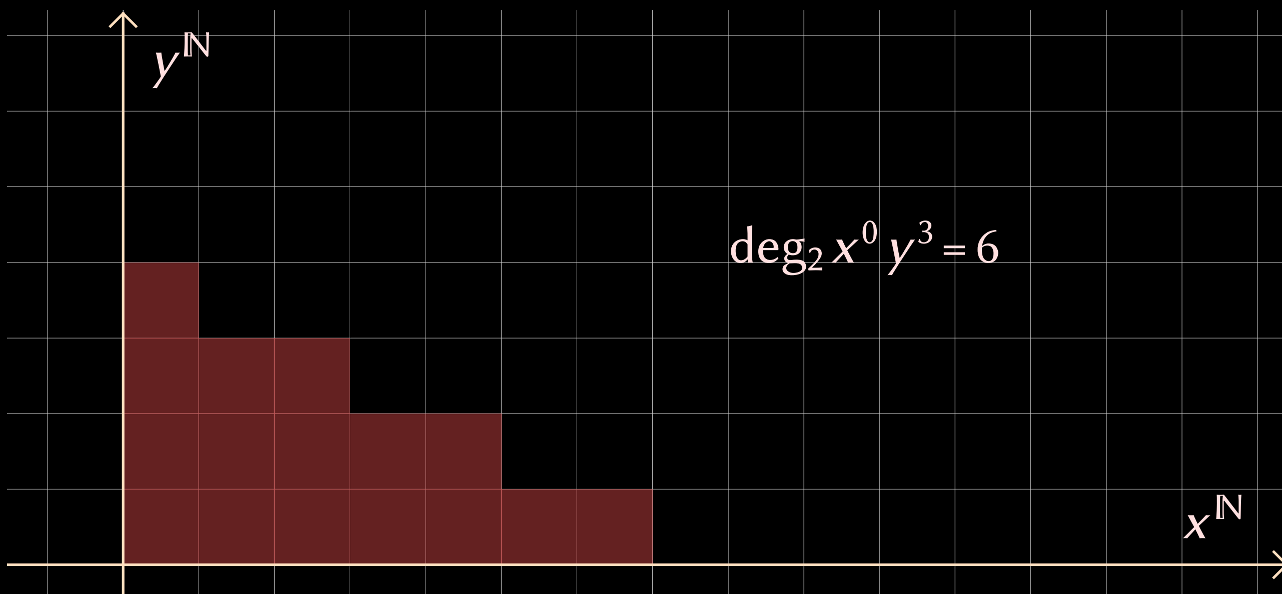

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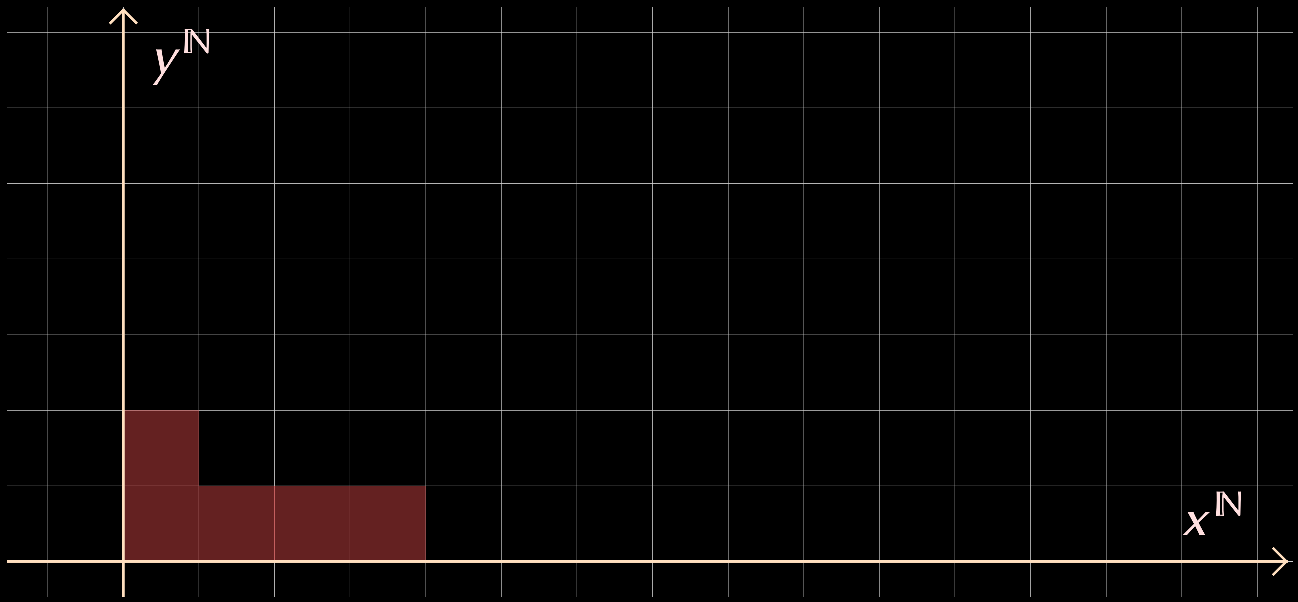

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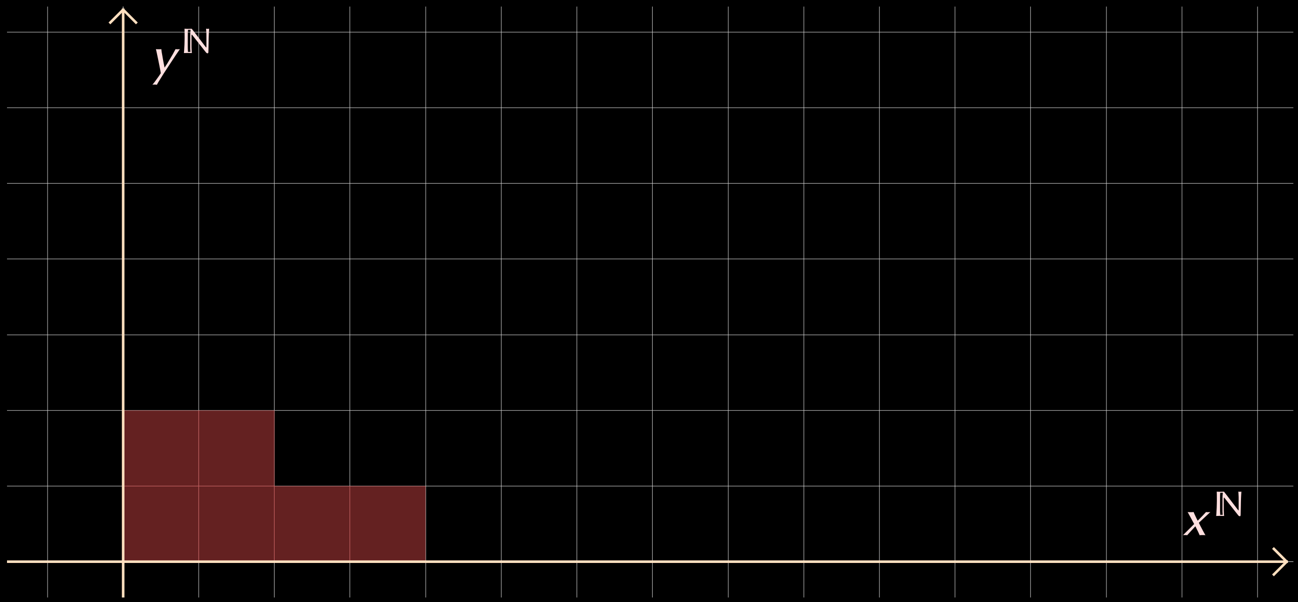

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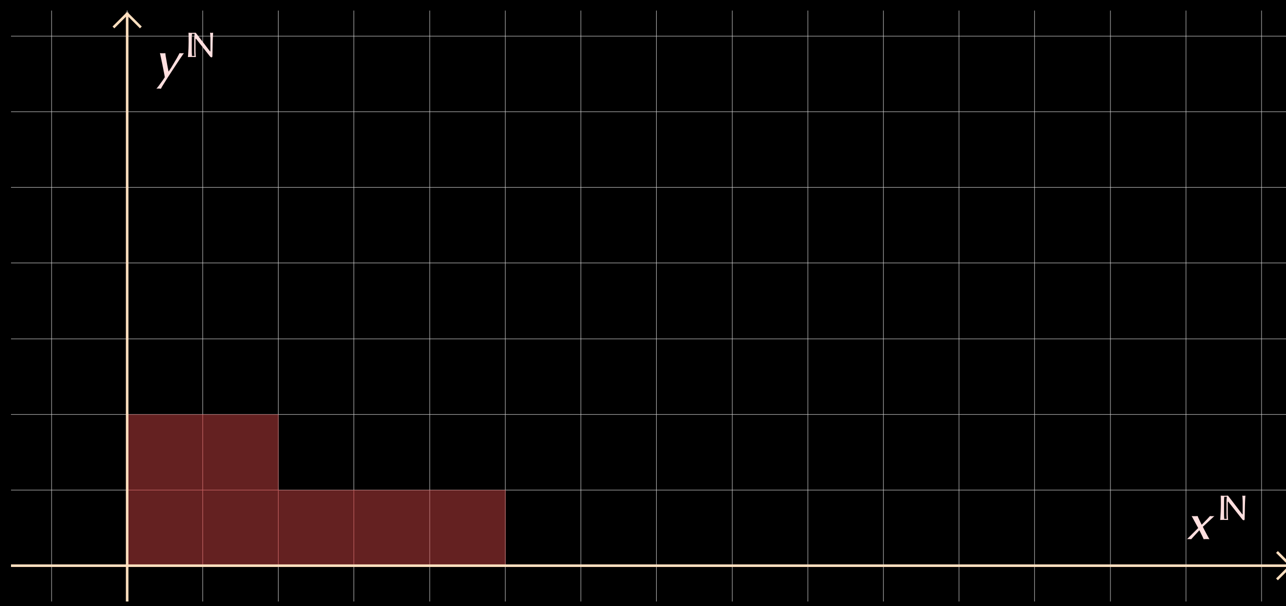

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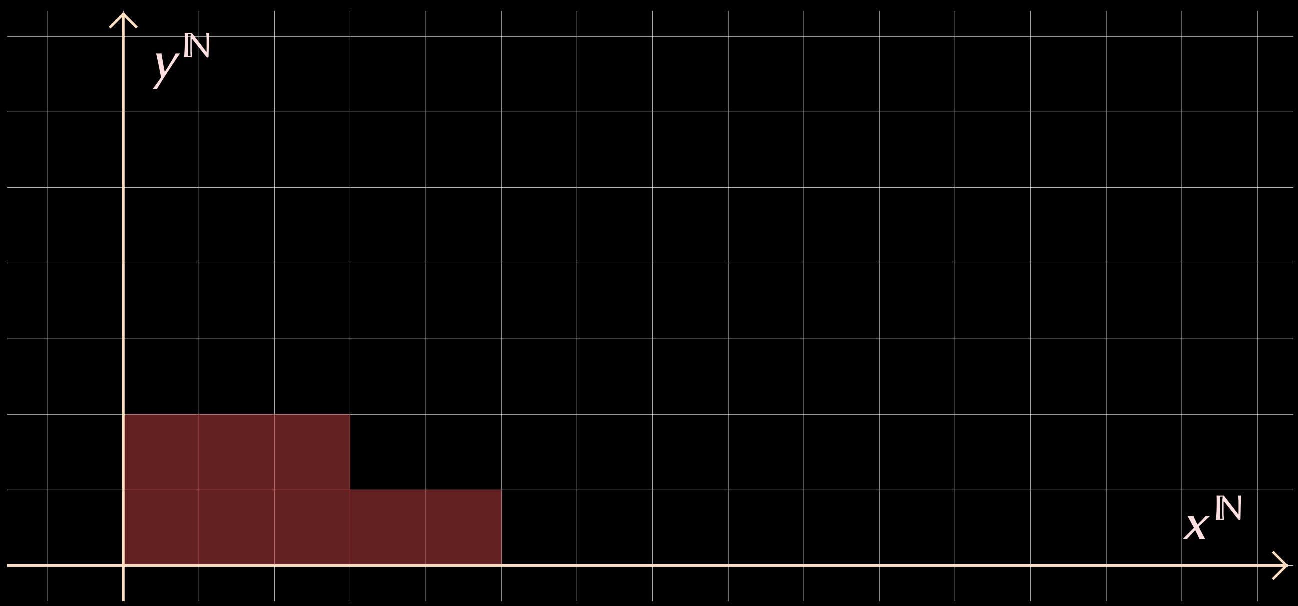

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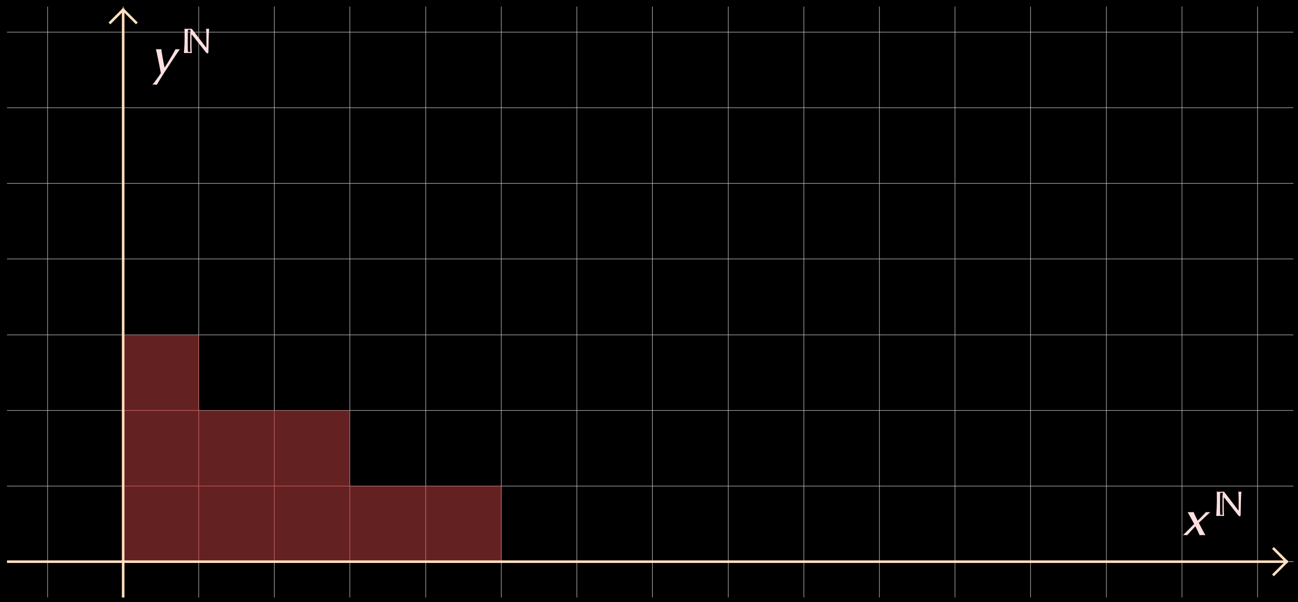

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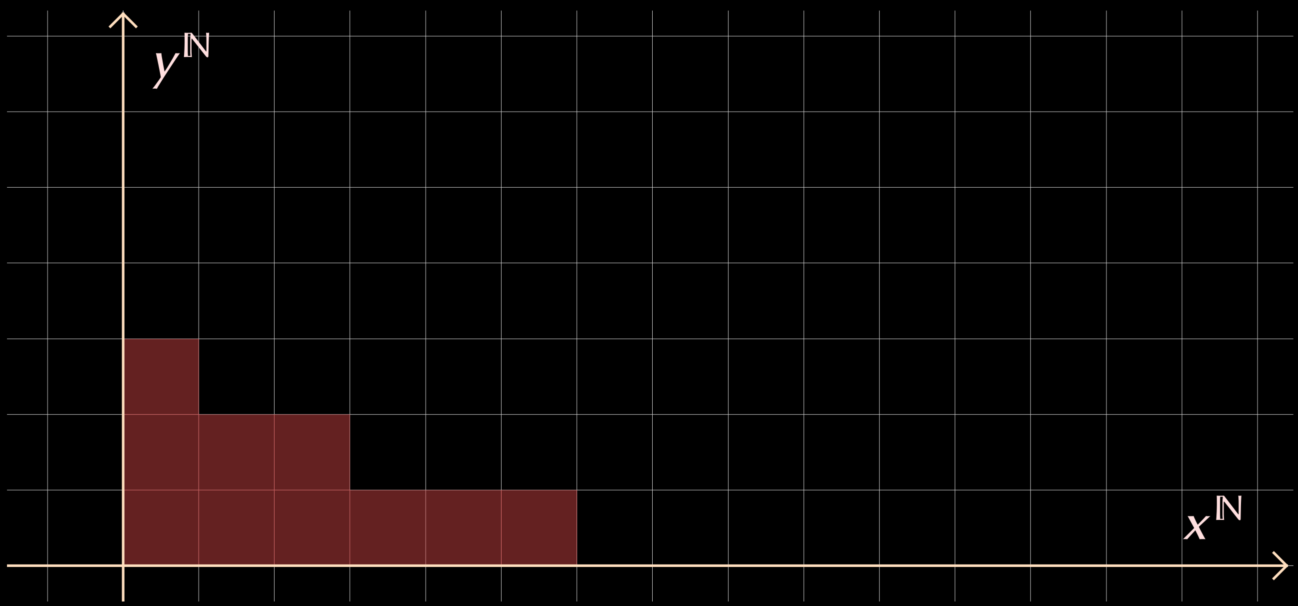

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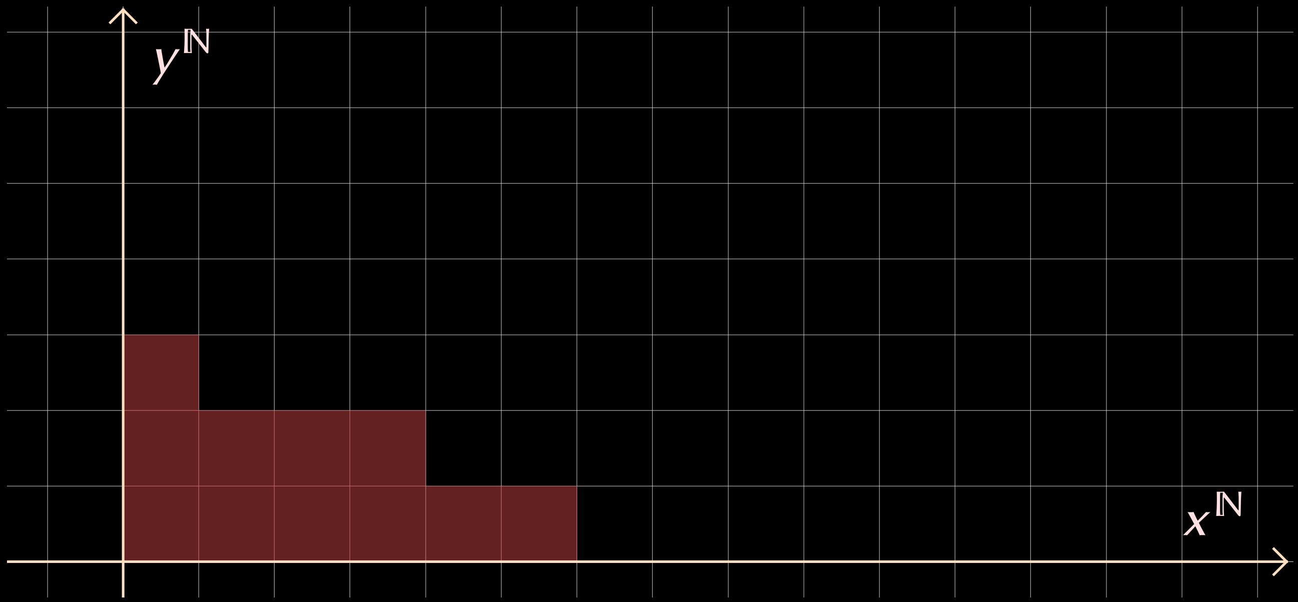
$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{K}^2)^n$$
$$I_{\boldsymbol{\alpha}} = \{P \in \mathbb{K}[x, y] : P(\boldsymbol{\alpha}) = 0\}$$

Gröbner basis for generic $\boldsymbol{\alpha}$



Gröbner basis of I_α for $<_k$

$$\boldsymbol{\alpha} \;==\; (\alpha_1, \ldots, \alpha_n) \;\in\; (\mathbb{K}^2)^n$$

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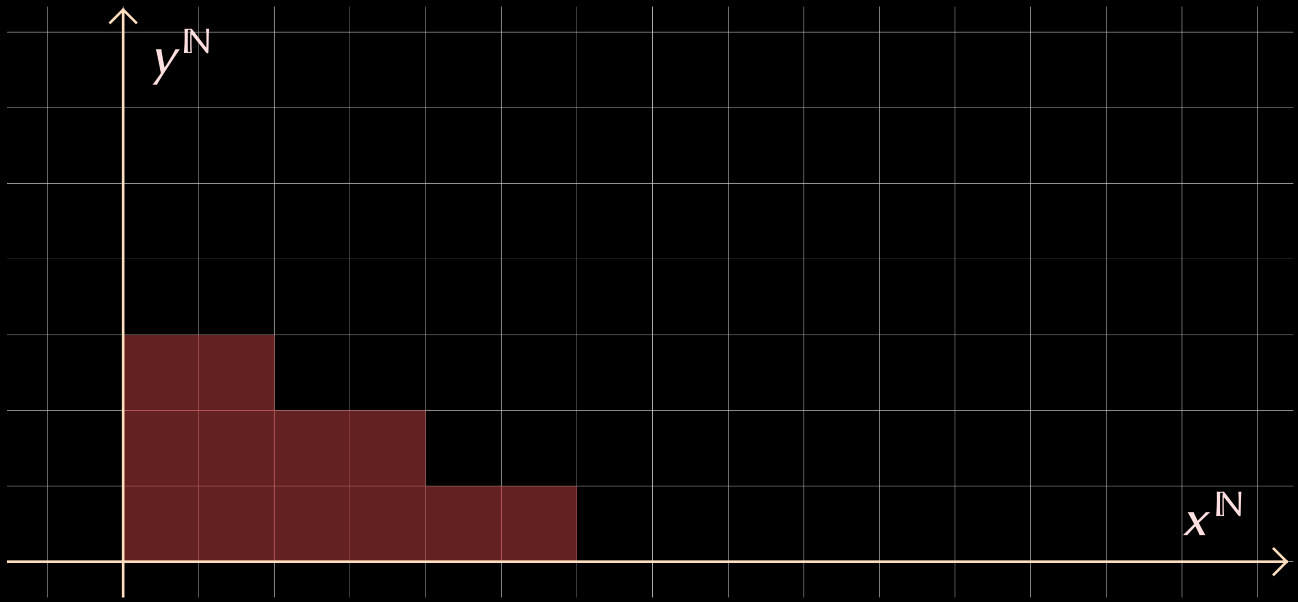

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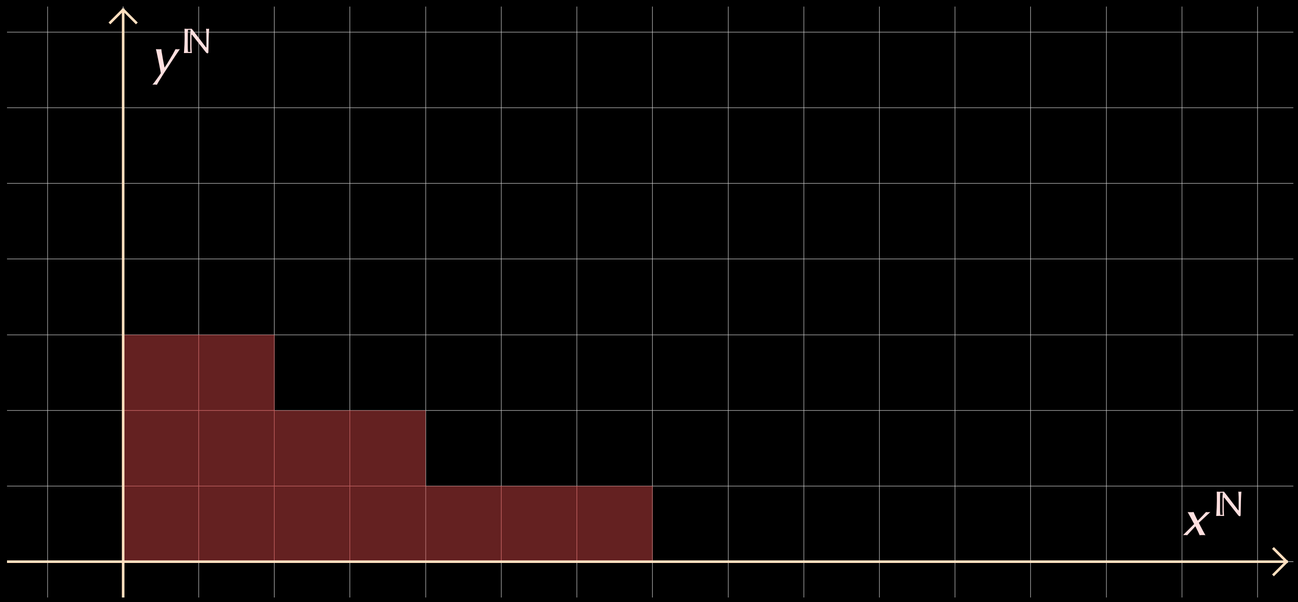

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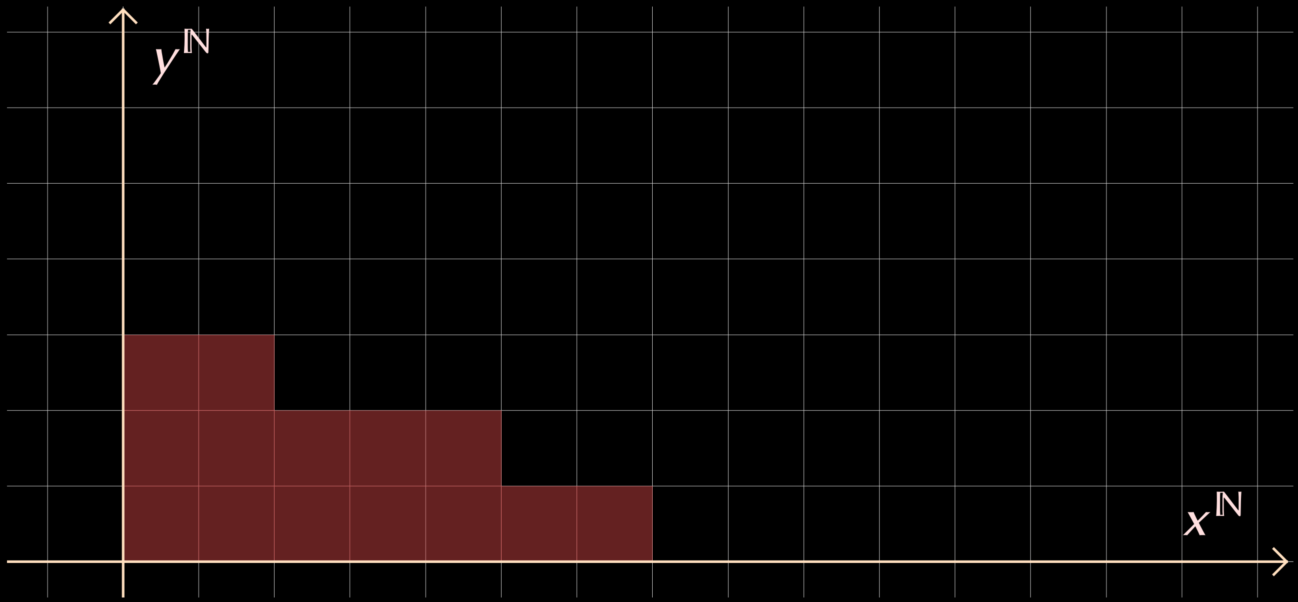

Gröbner basis of I_{α} for $<_k$

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{K}^2)^n$$
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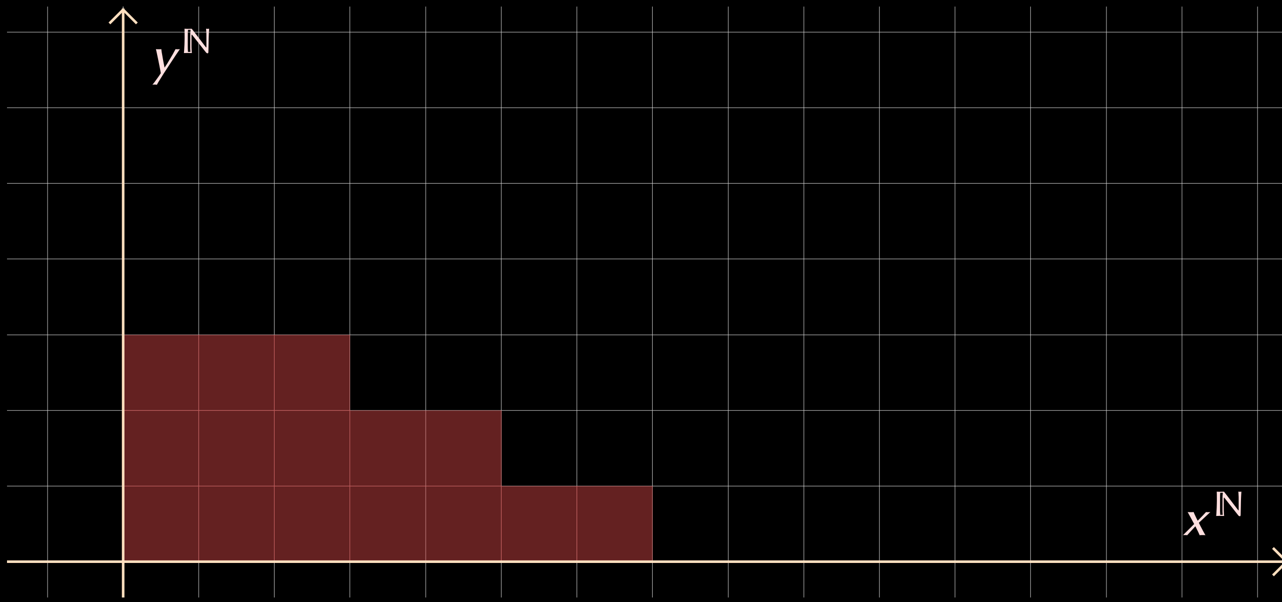
Gröbner basis for generic $\boldsymbol{\alpha}$



Gröbner basis of I_α for $<_k$

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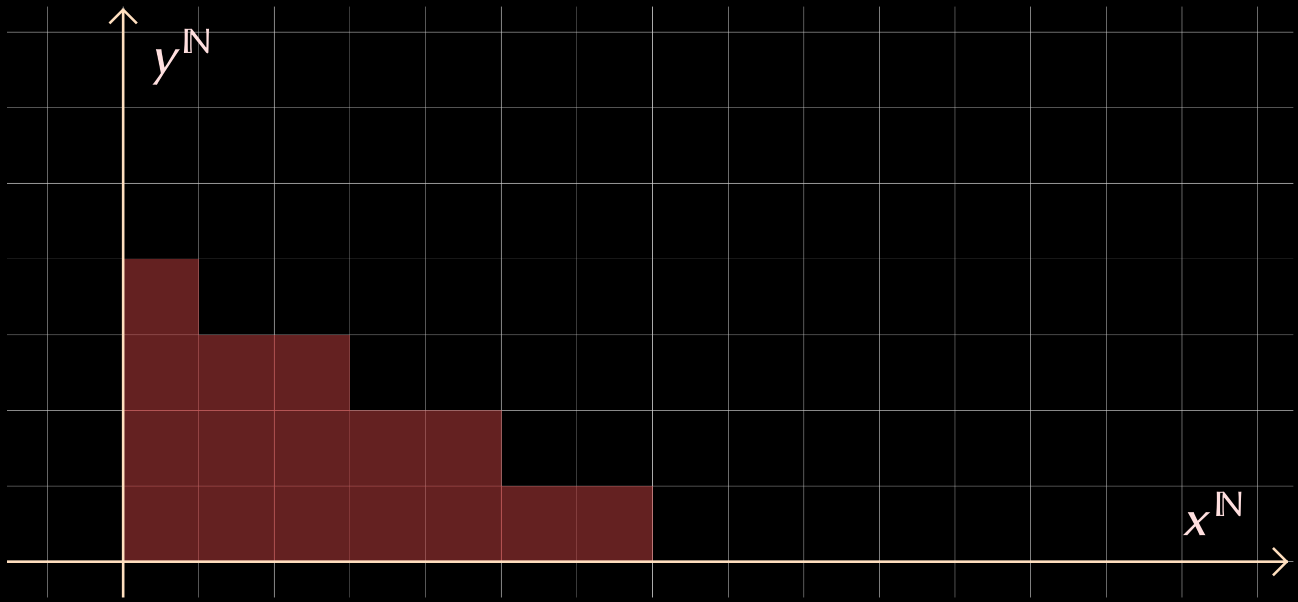
Gröbner basis for generic $\boldsymbol{\alpha}$



Gröbner basis of I_α for $<_k$

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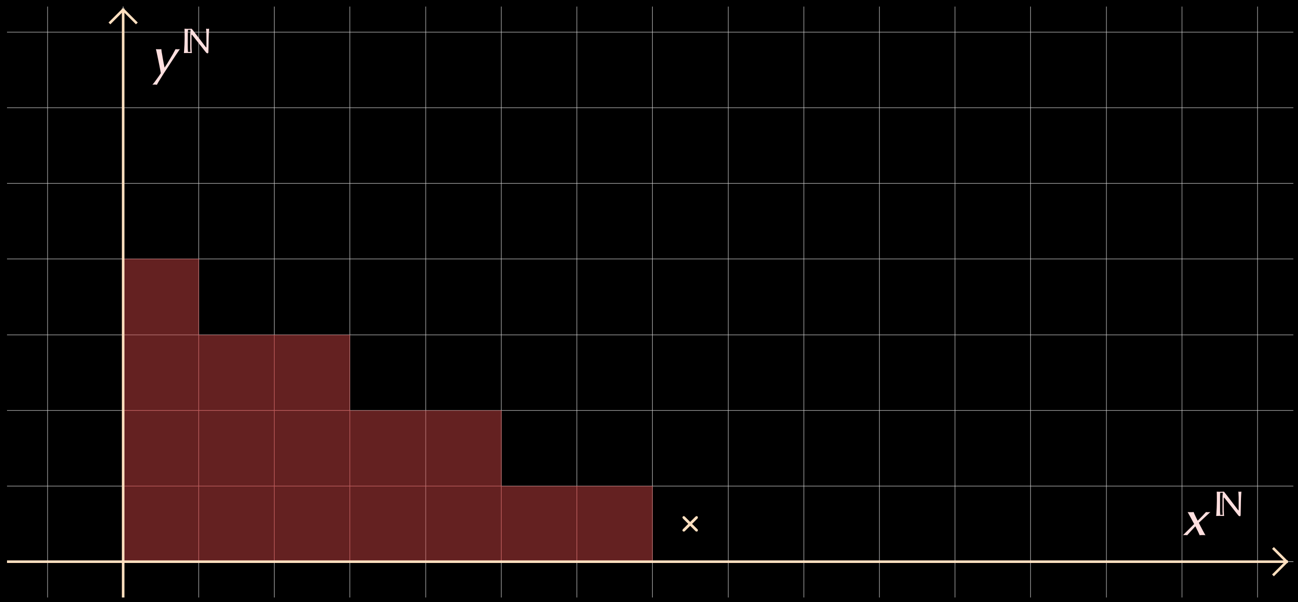

$$\boldsymbol{\alpha} \quad == \quad (\alpha_1, \ldots, \alpha_n) \ \in \ (\mathbb{K}^2)^n$$
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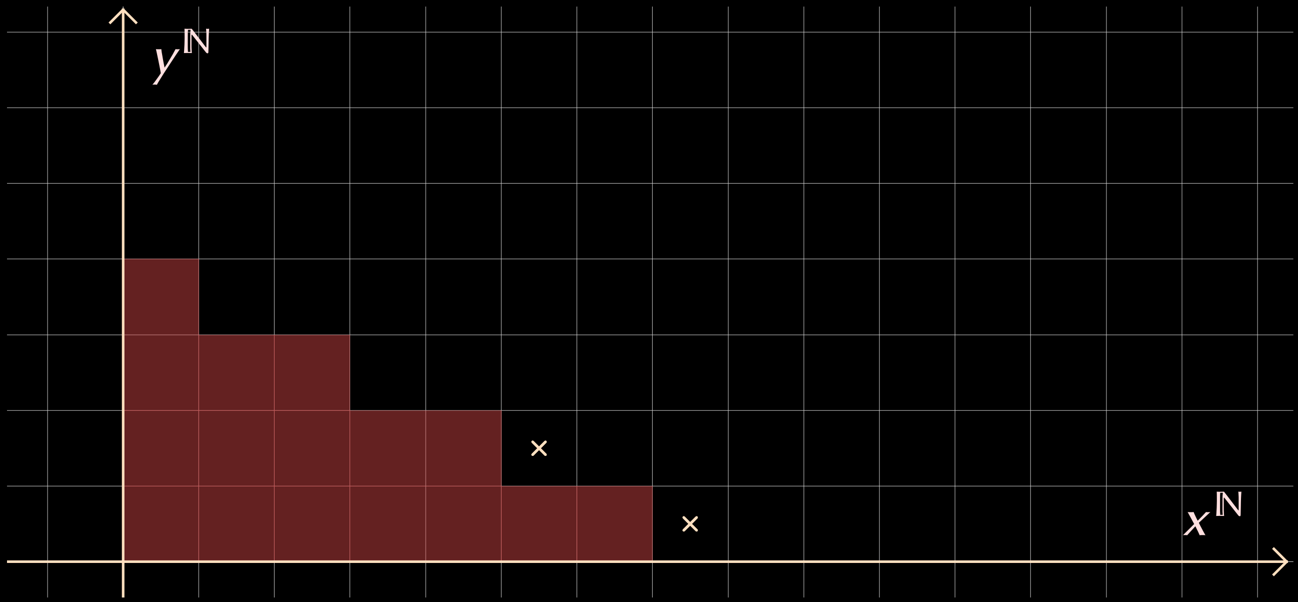

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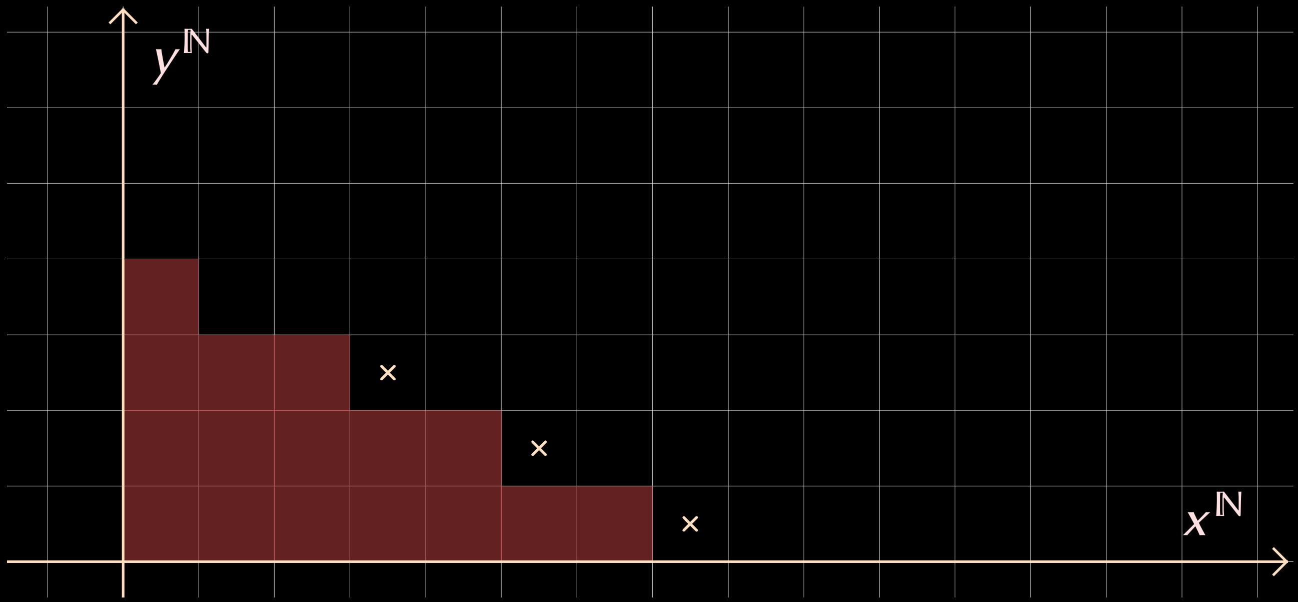

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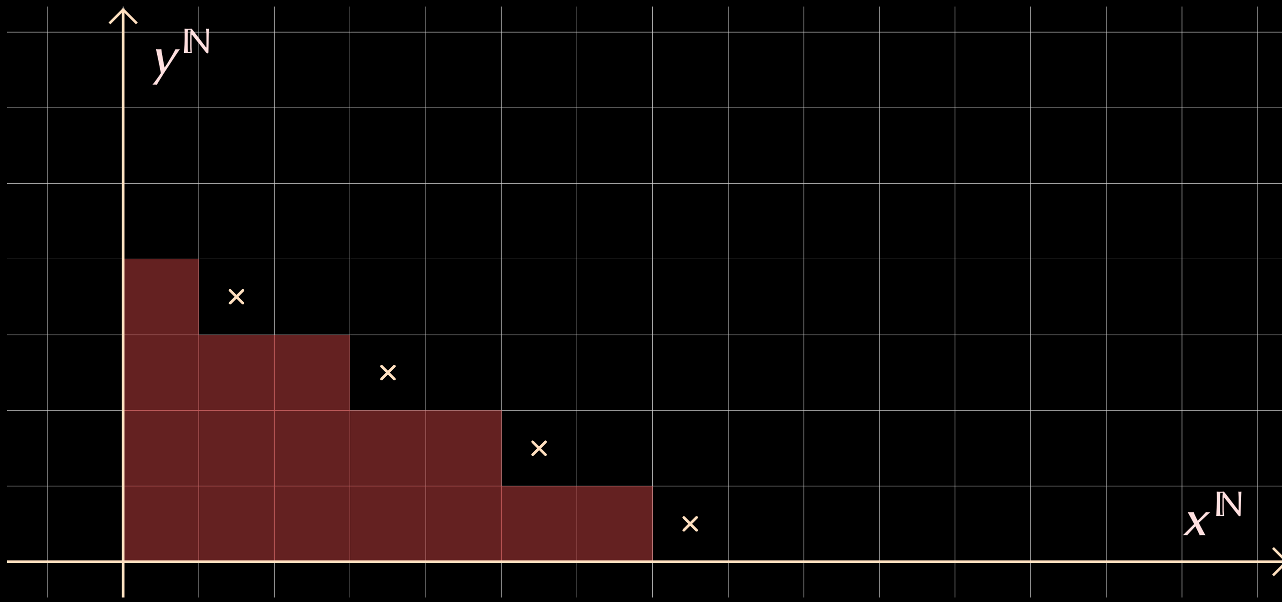

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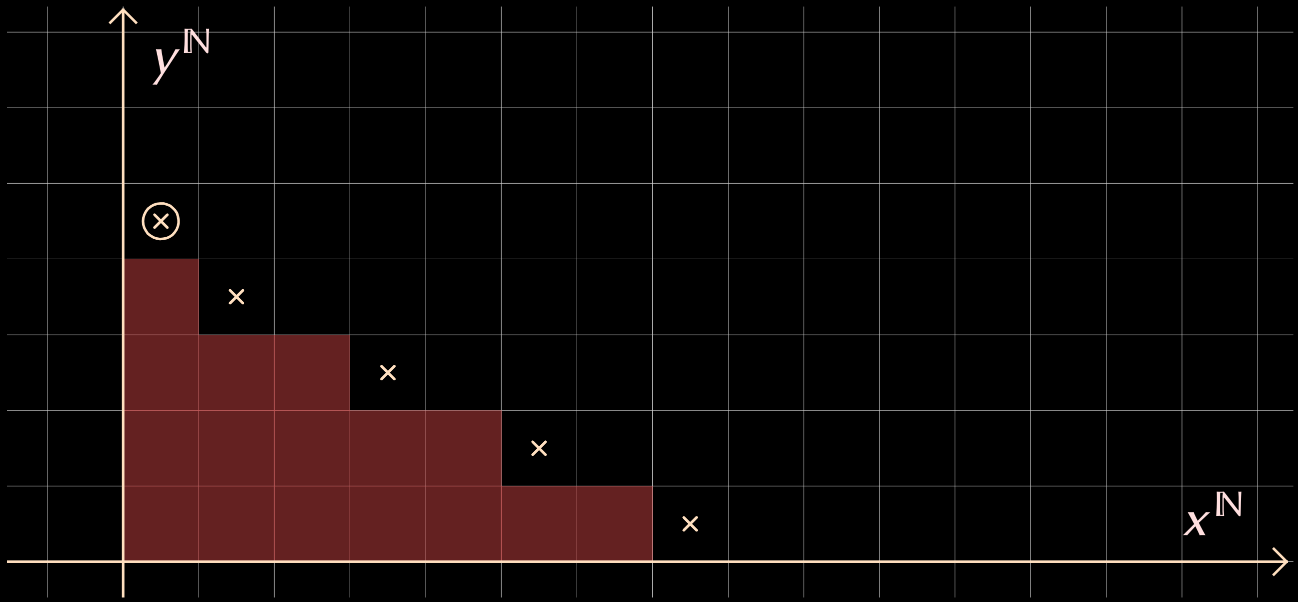

Gröbner basis of I_α for $<_k$

$$\boldsymbol{\alpha} \Longleftarrow (\alpha_1, \ldots, \alpha_n) \in (\mathbb{K}^2)^n$$
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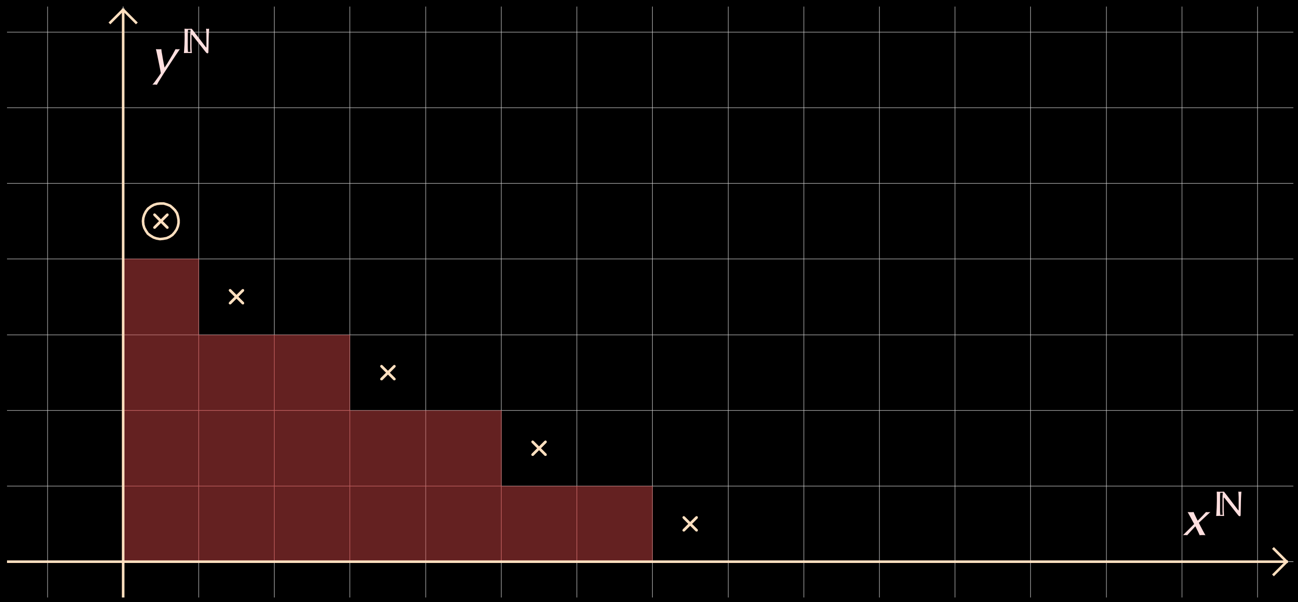
Gröbner basis for generic $\boldsymbol{\alpha}$



$\exists B_k \in I_\alpha, \quad \mathrm{LM}(B_k) = y^b, \quad b \leqslant \sqrt{\frac{2n}{k}}$

Amortized multi-point evaluation, generic case

vdH–Larrieu 2018: dichotomic Gröbner walk w.r.t. $\prec_1, \prec_2, \prec_4, \prec_8, \ldots$

Amortized multi-point evaluation, generic case

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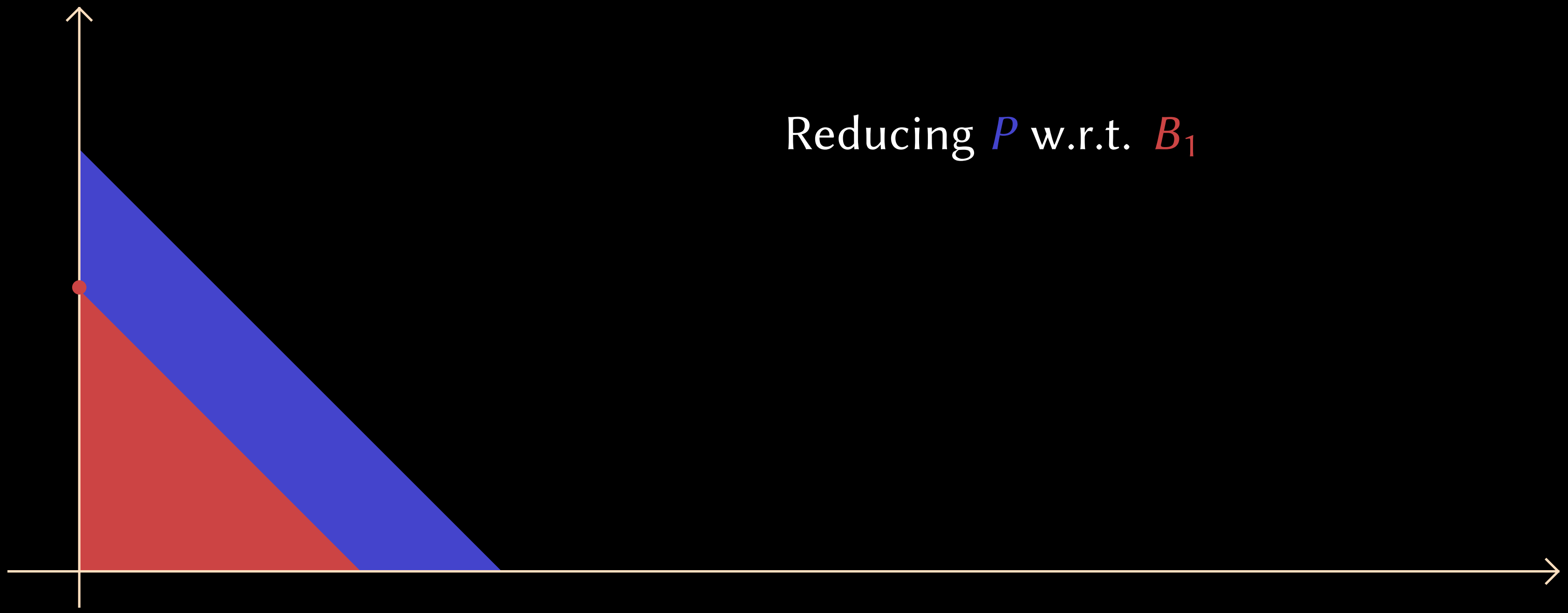

Reducing P w.r.t. B_1

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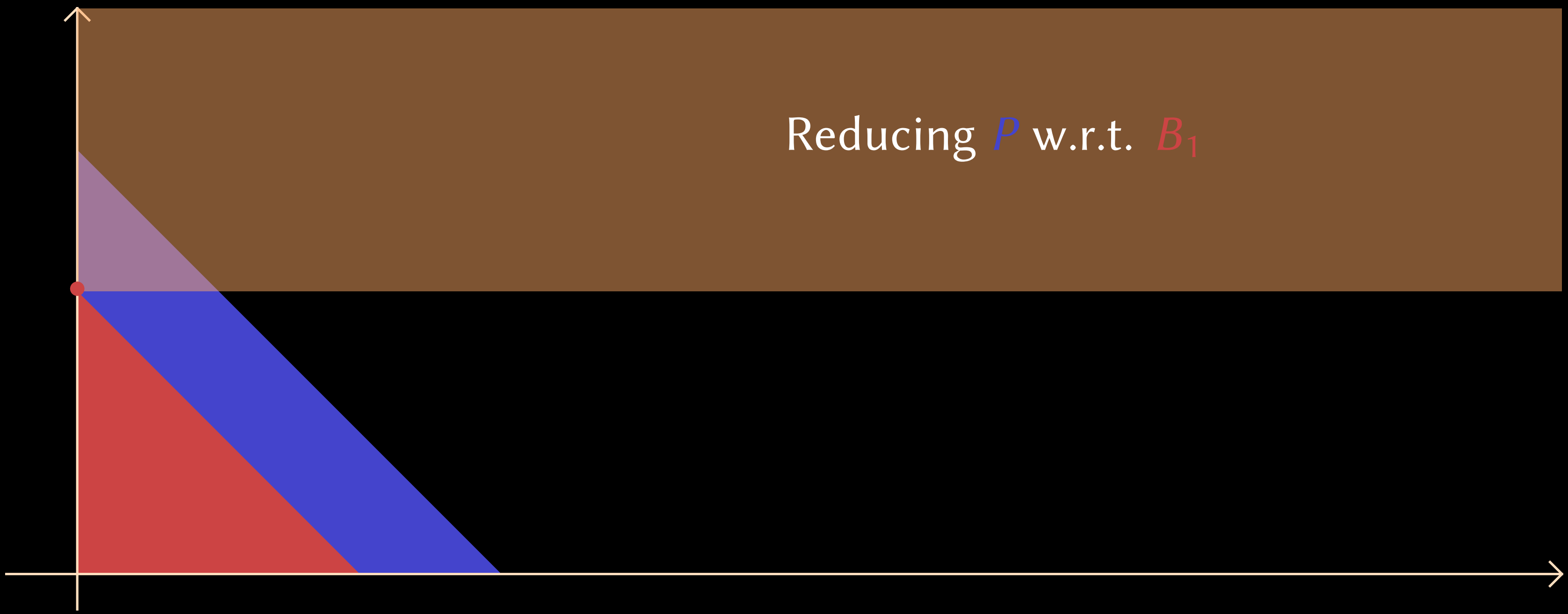

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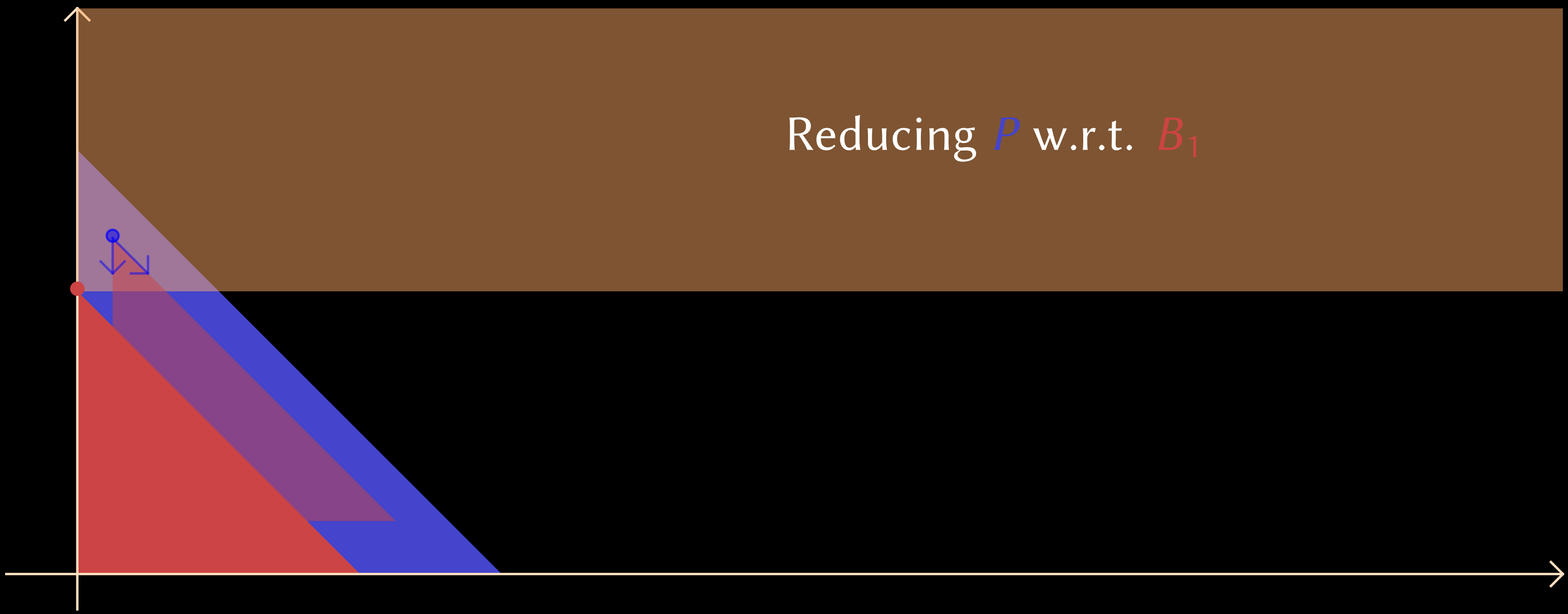

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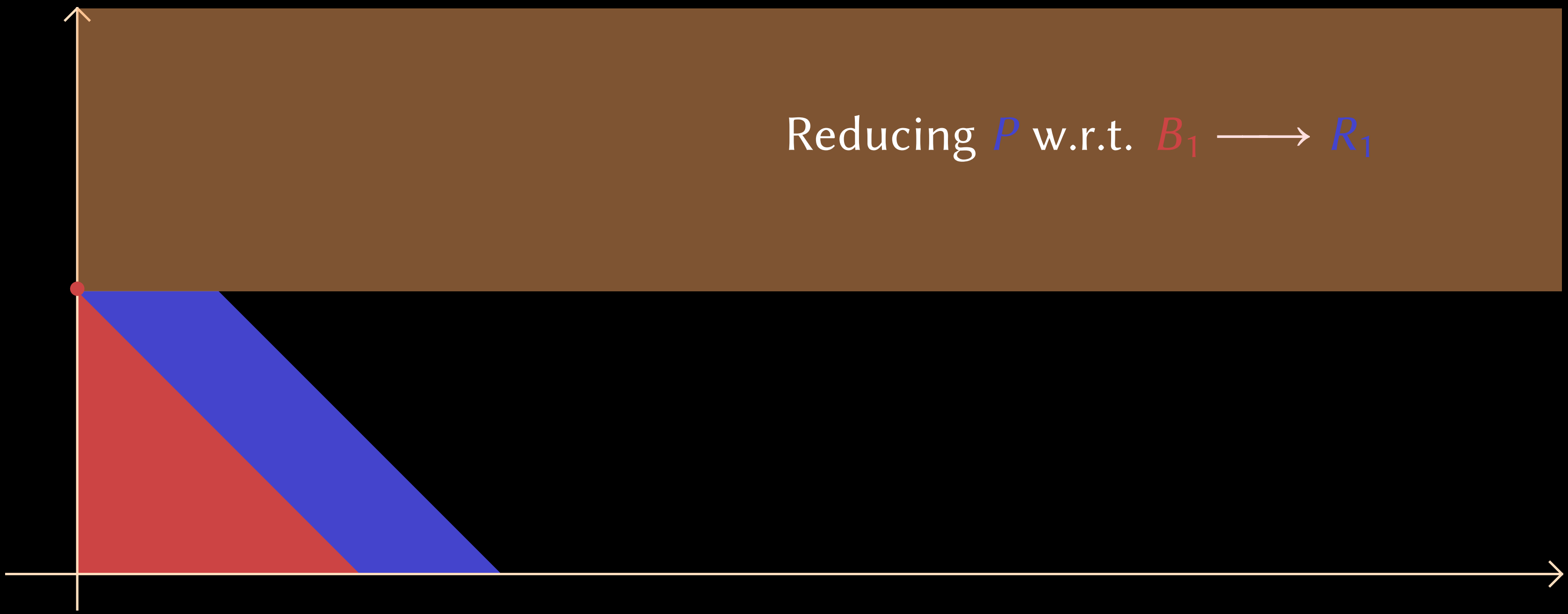

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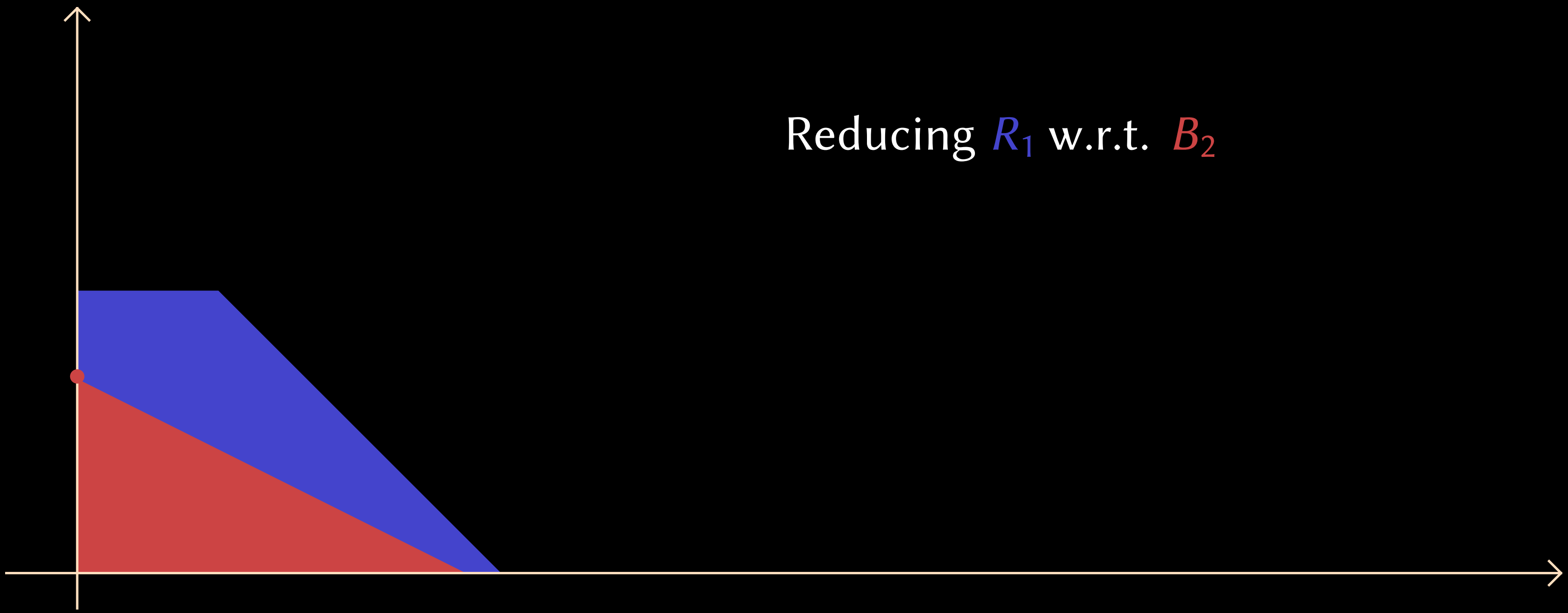

Reducing R_1 w.r.t. B_2

Amortized multi-point evaluation, generic case

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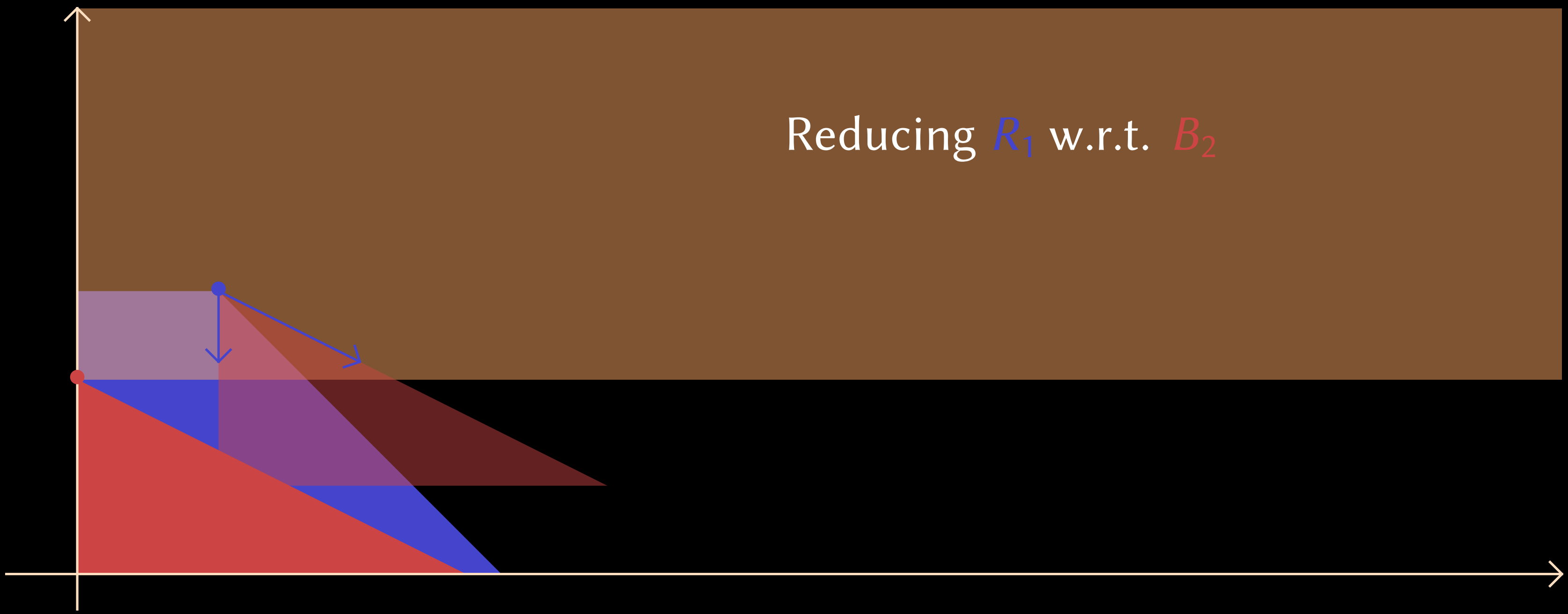

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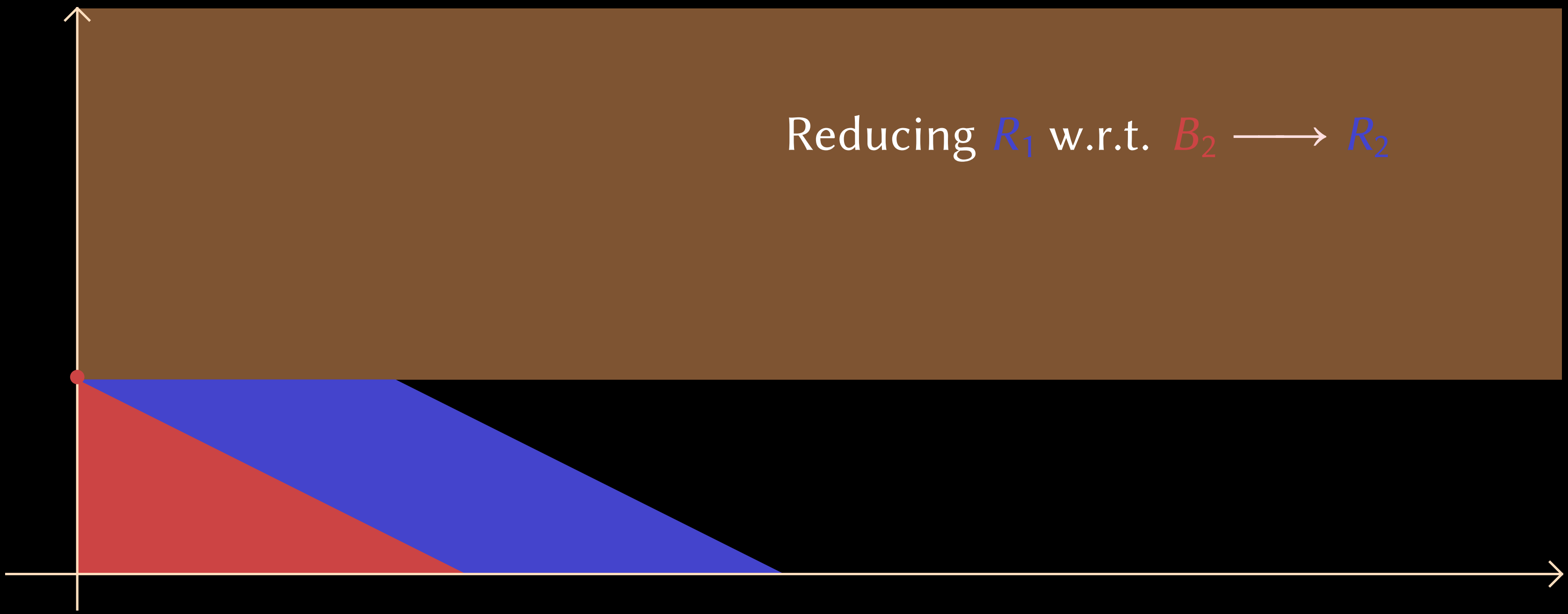

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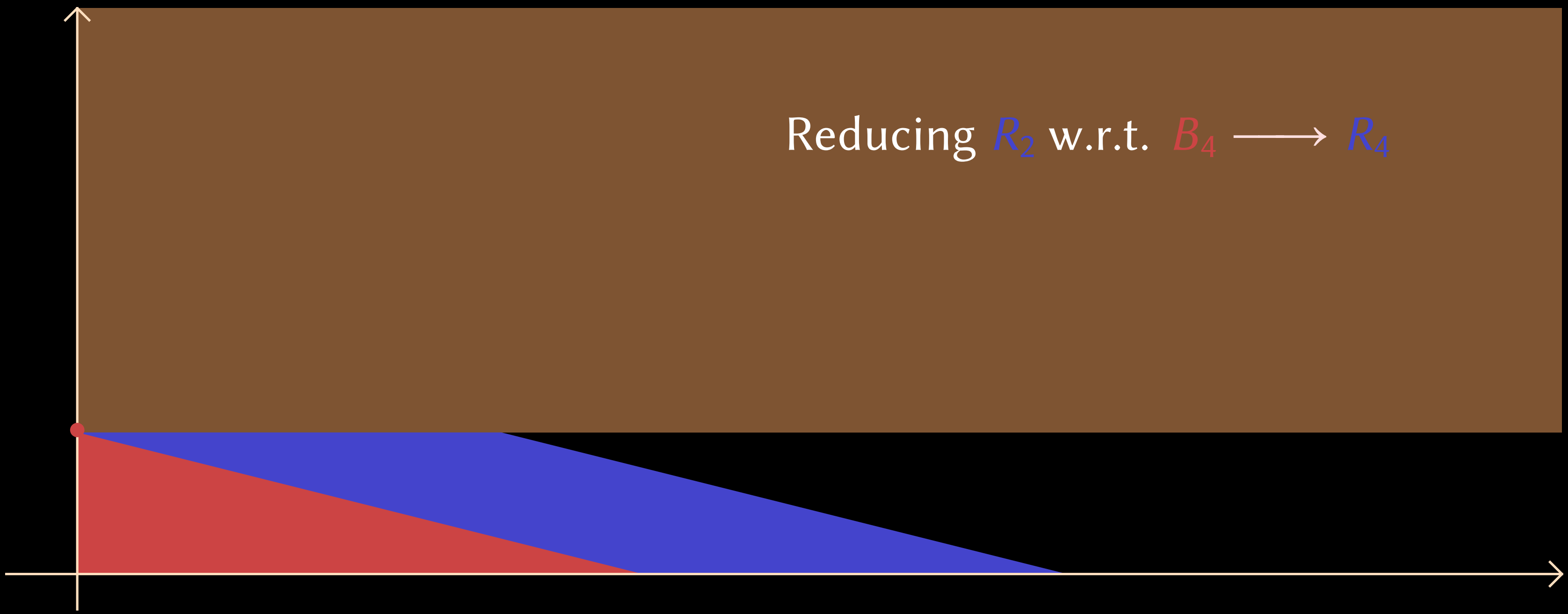

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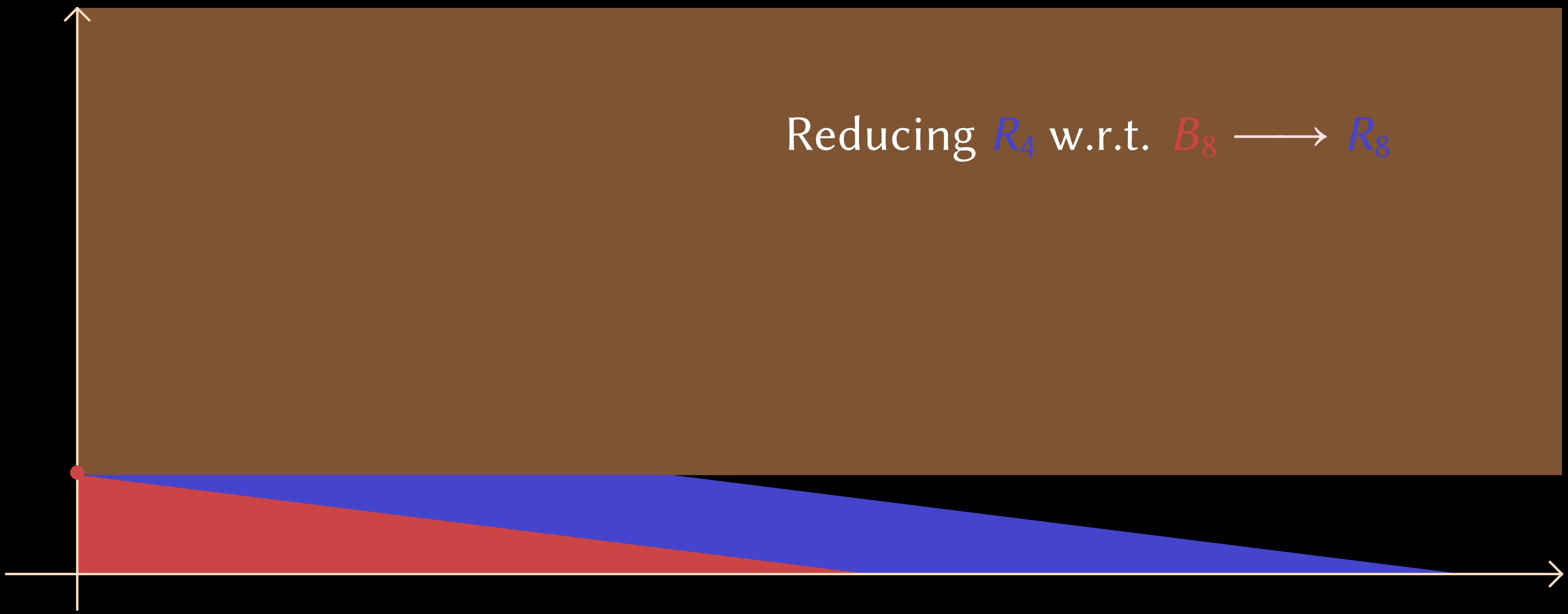

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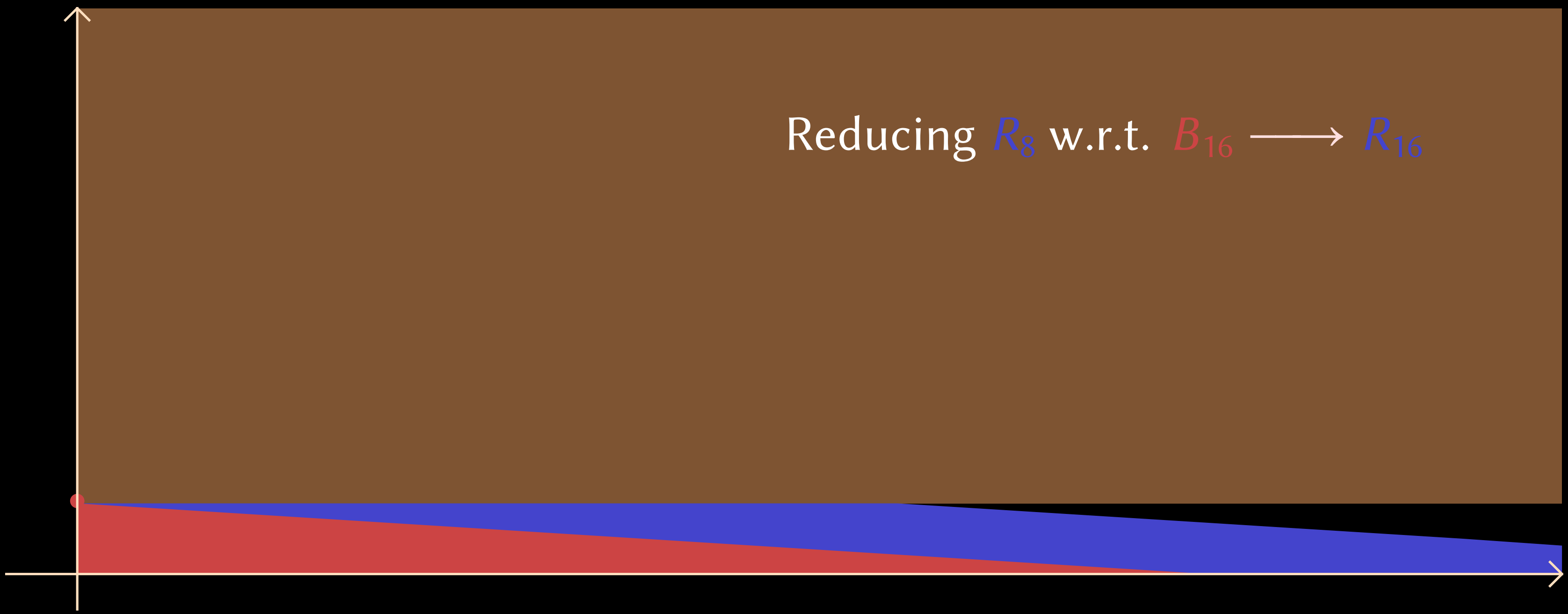

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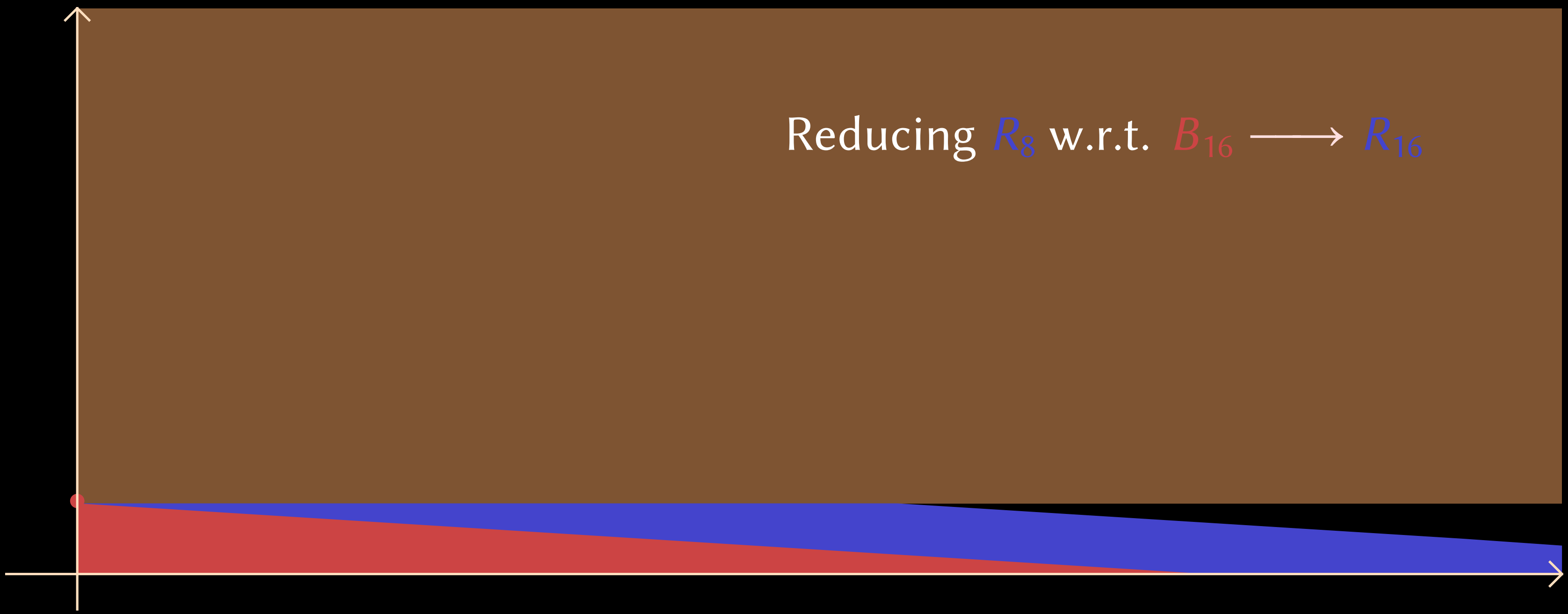

Result: $R = (((P \operatorname{rem} B_1) \operatorname{rem} B_2) \operatorname{rem} B_4) \operatorname{rem} \cdots \in \mathbb{K}[x]$ with $R - P \in I_\alpha$

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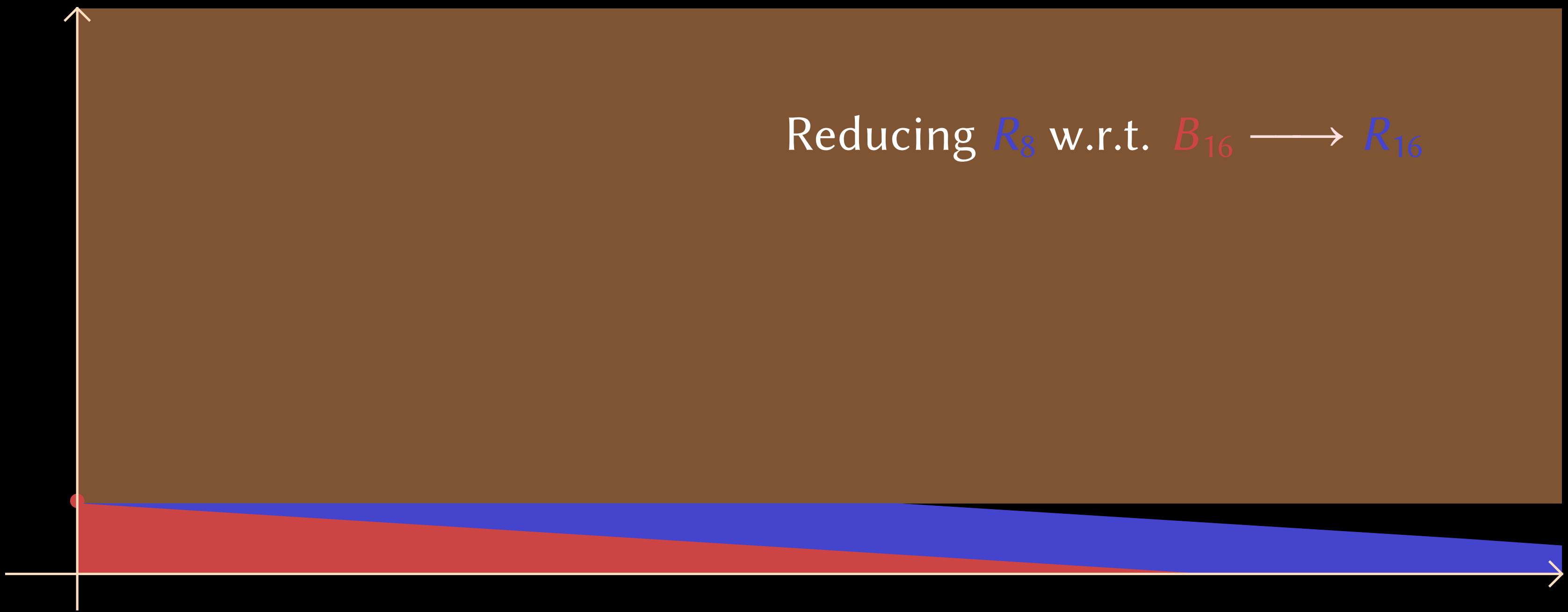

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Conclusion: reduction to univariate multi-point evaluation of R

Gröbner basis, non-generic case

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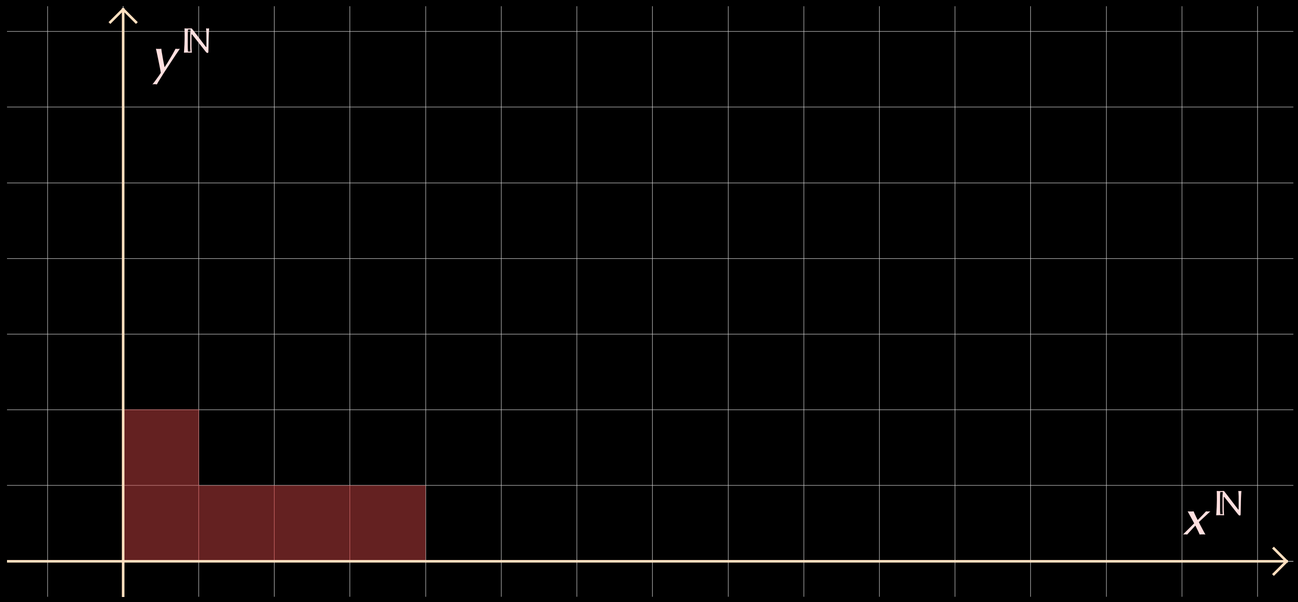

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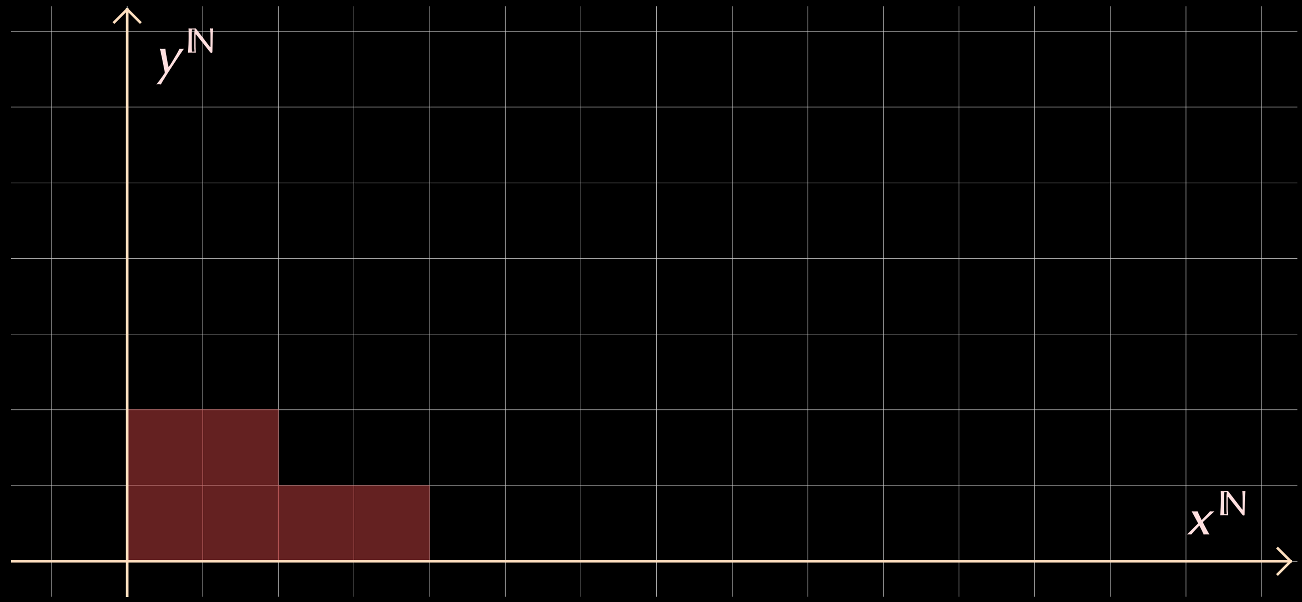

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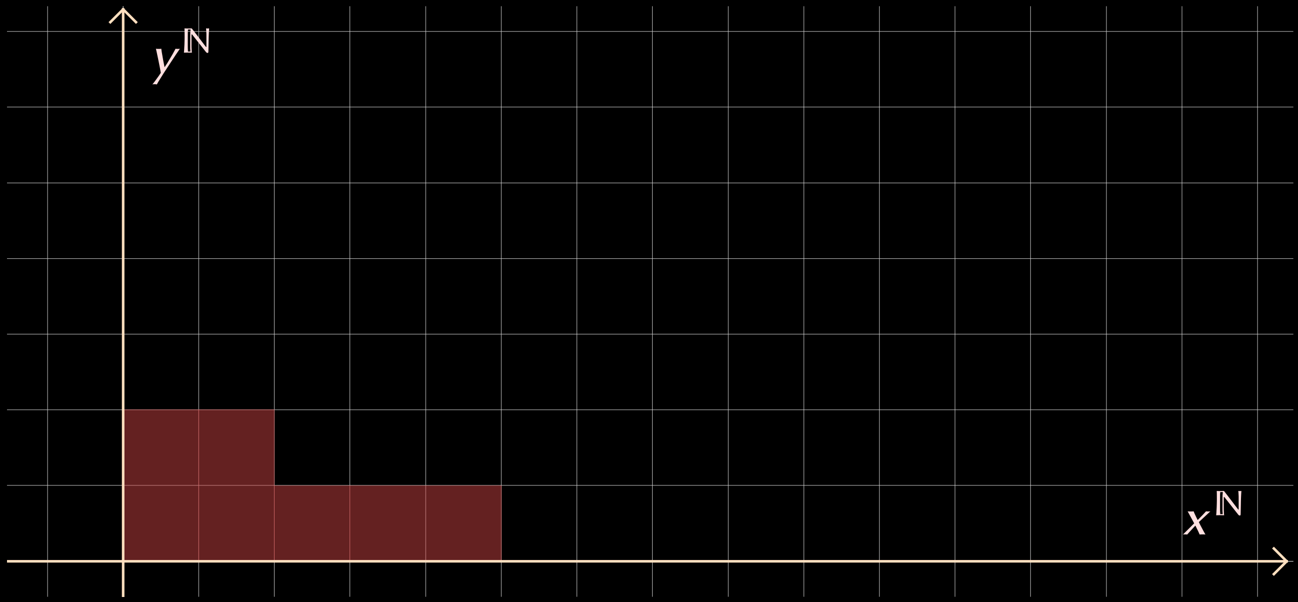

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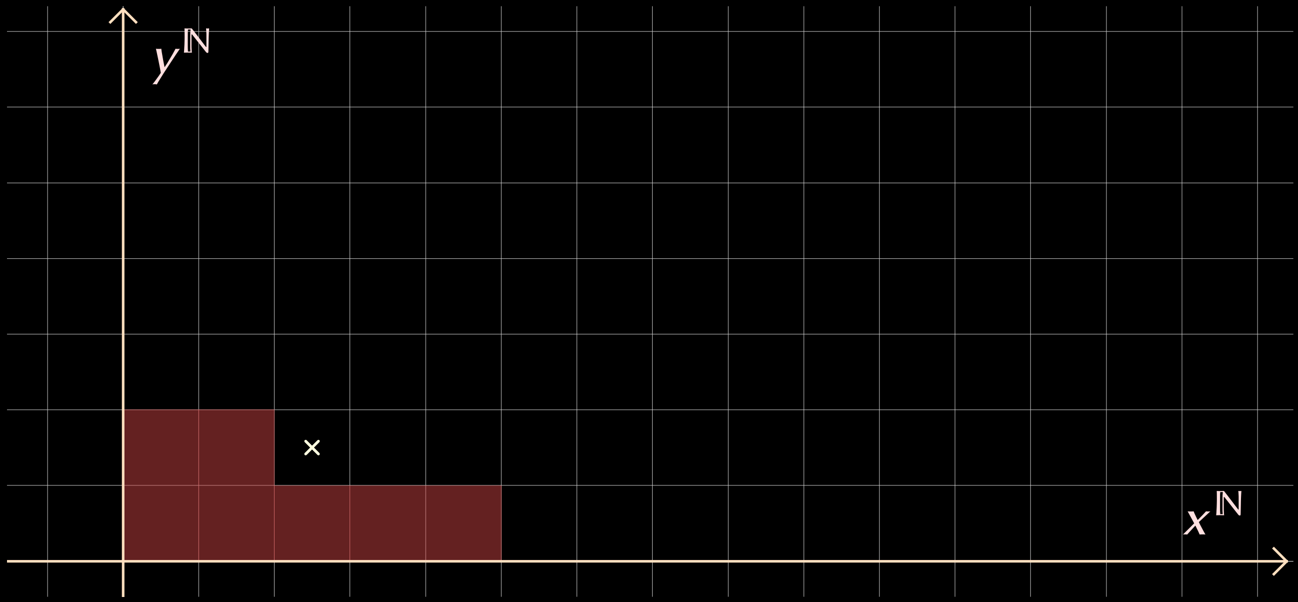

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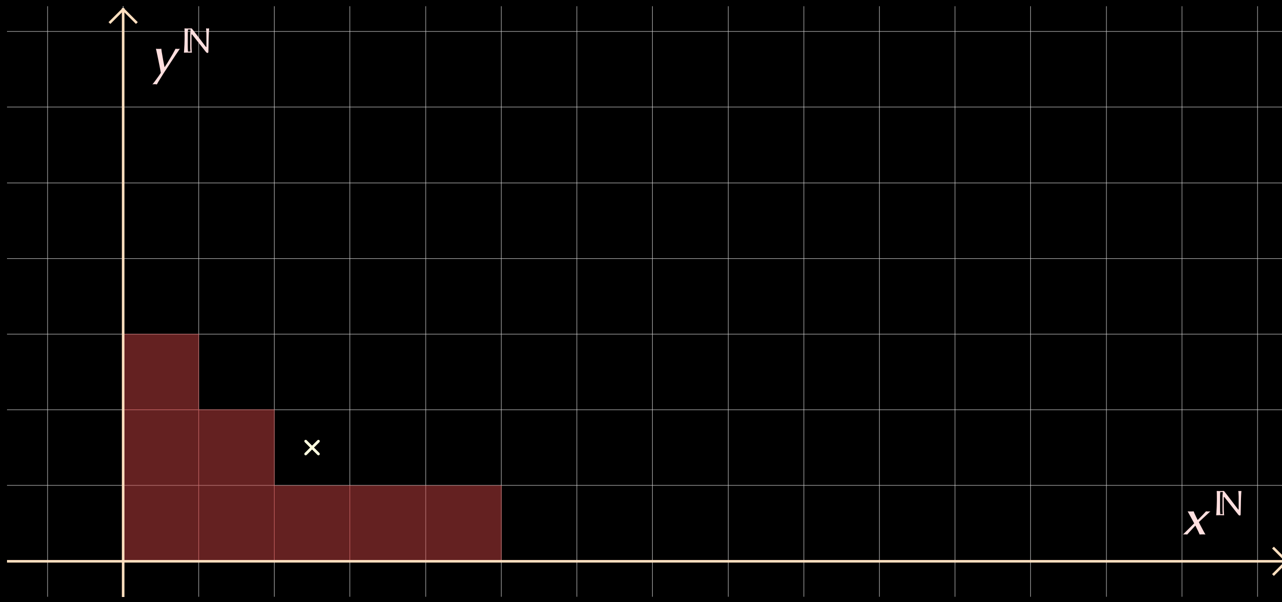

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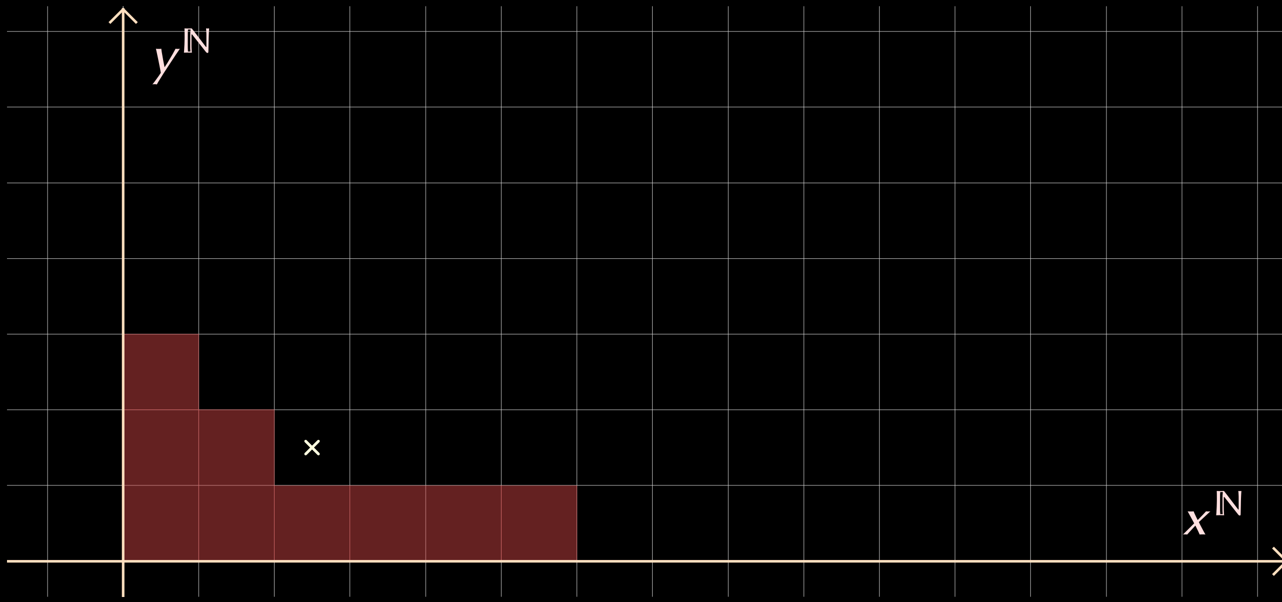

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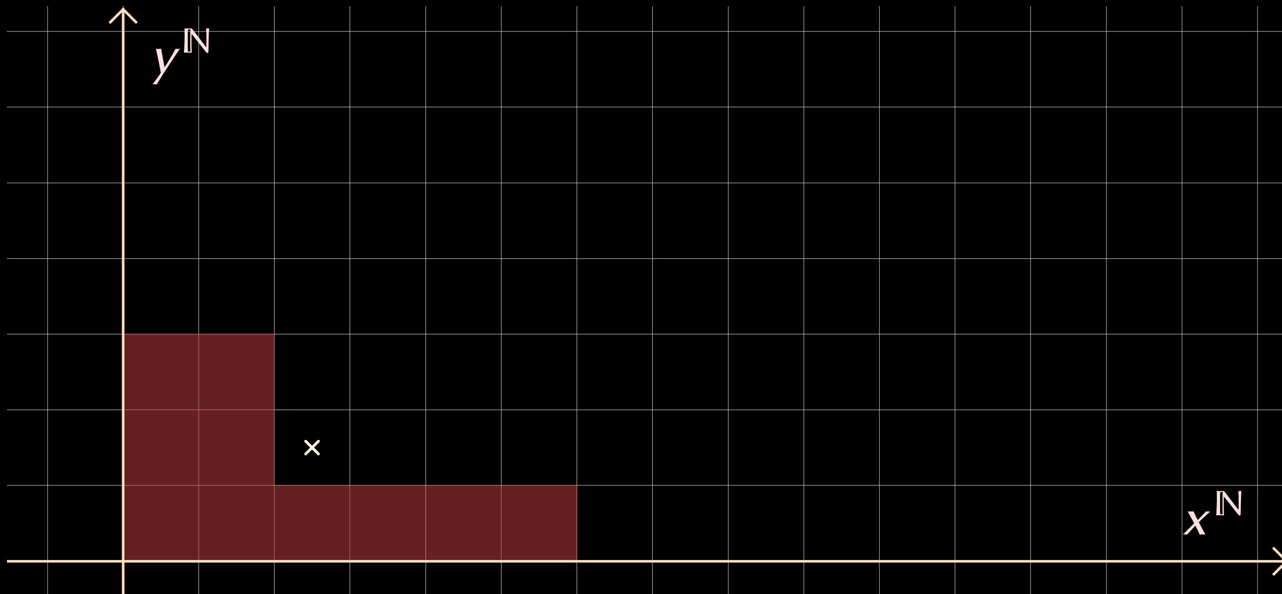

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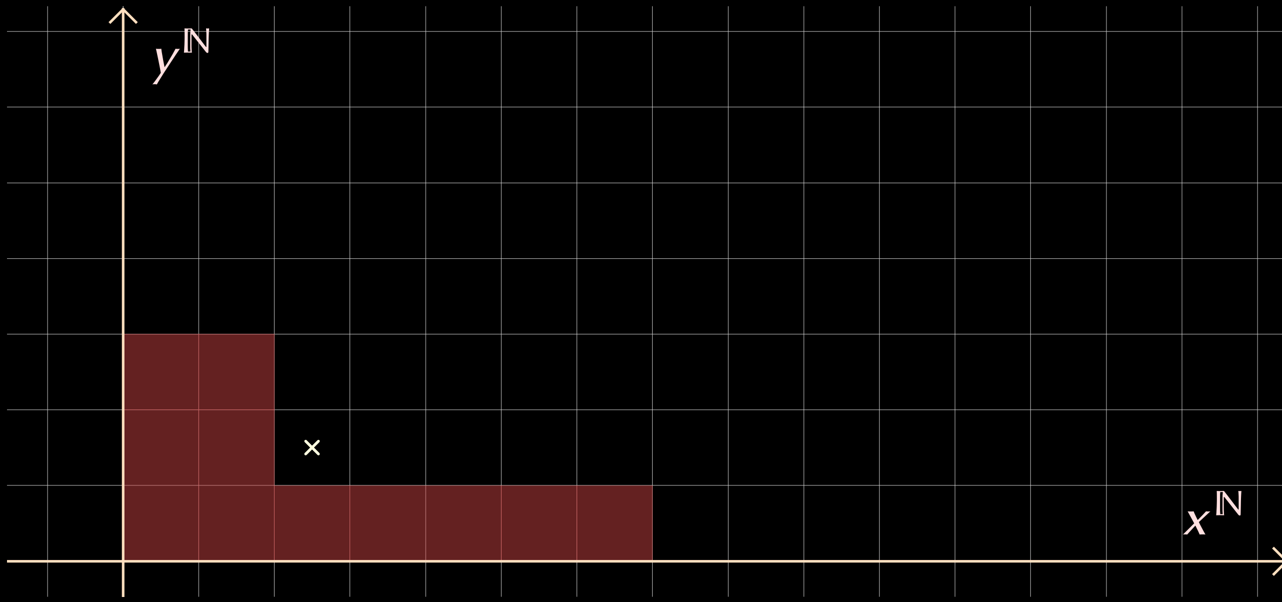

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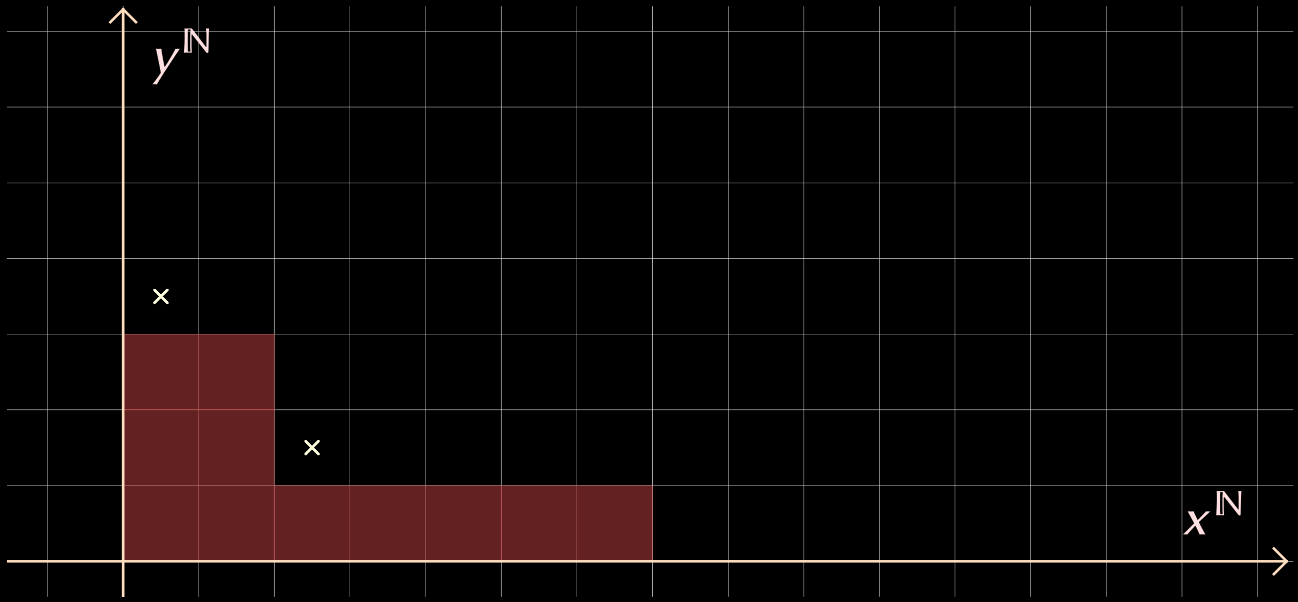

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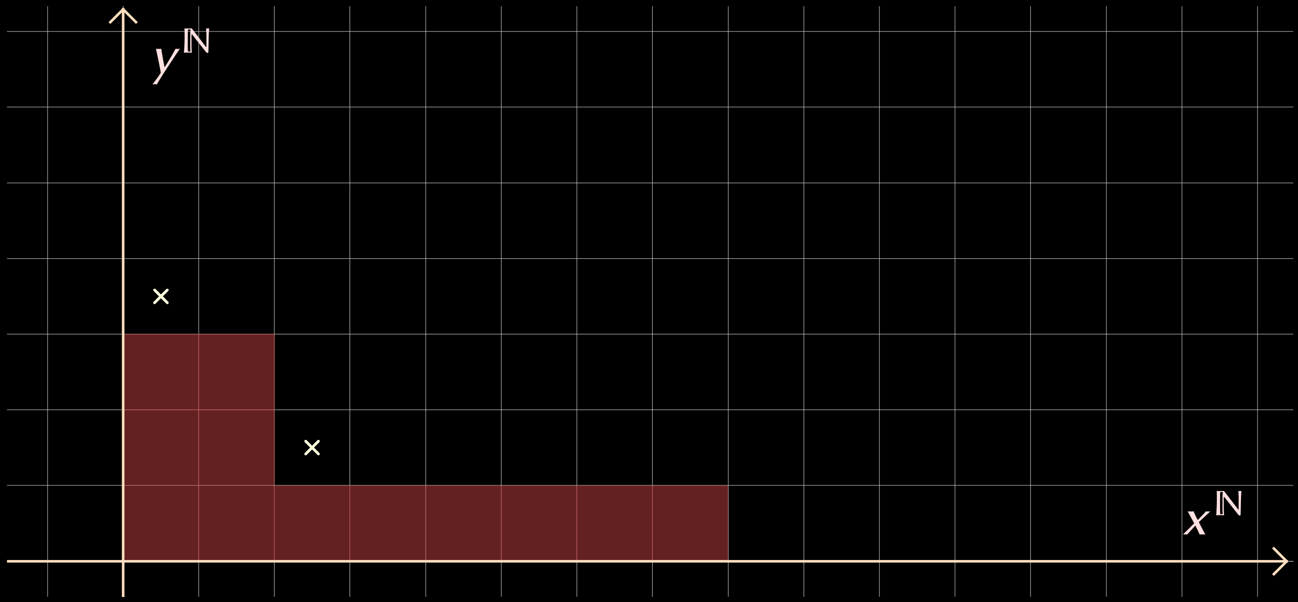

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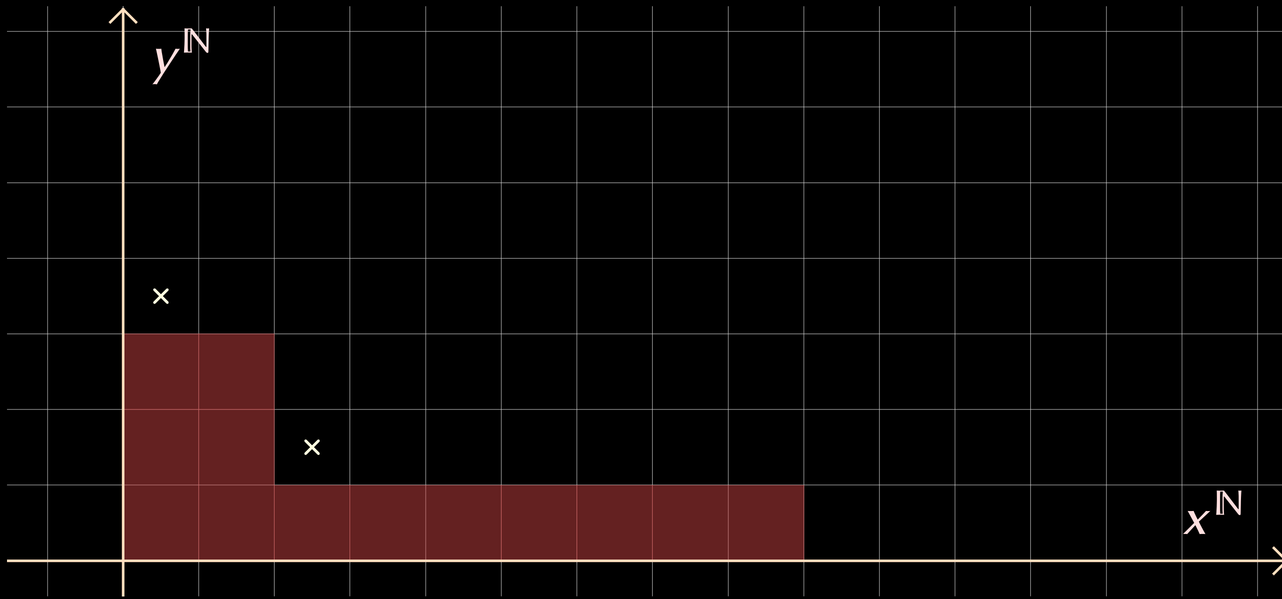

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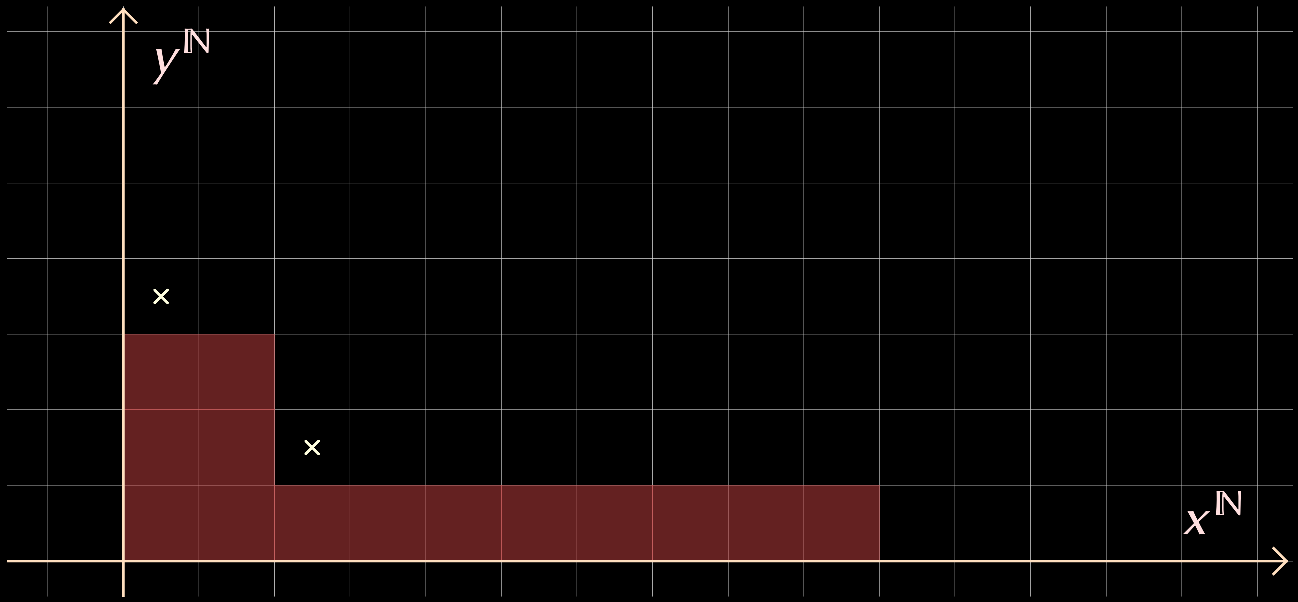

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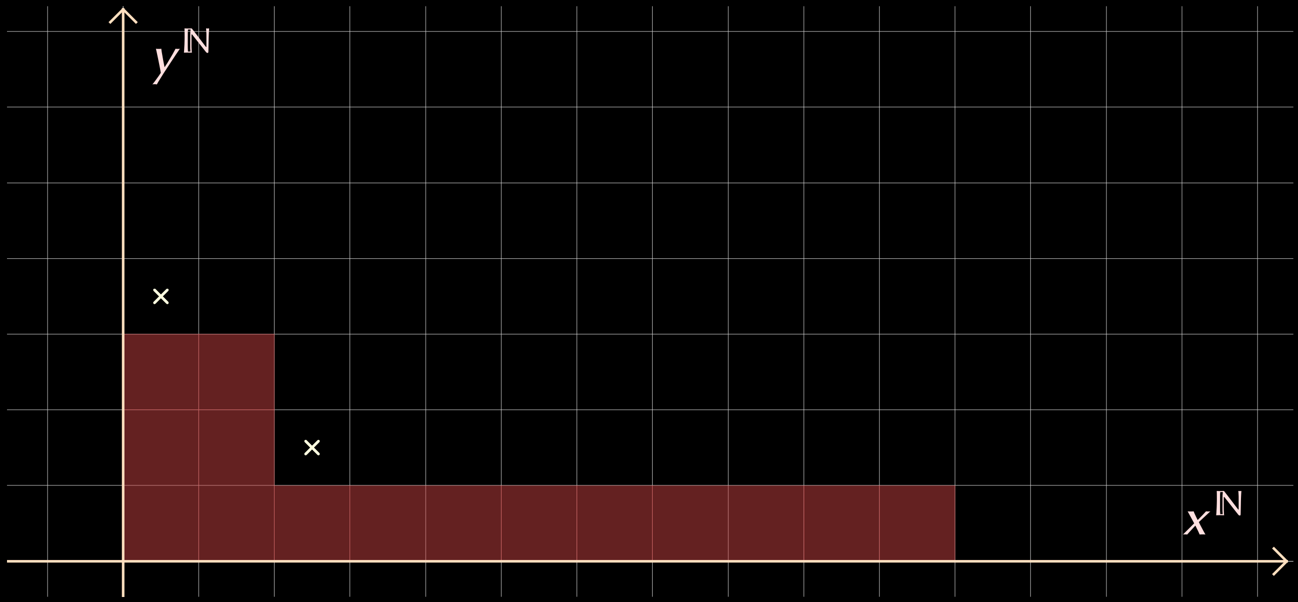

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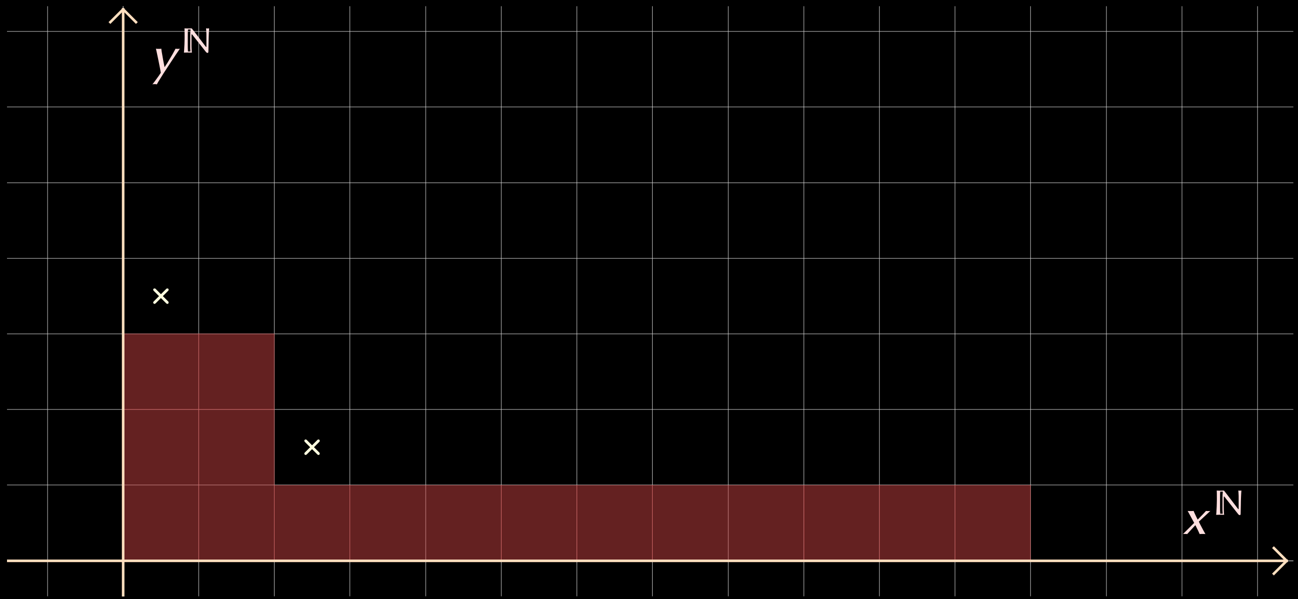

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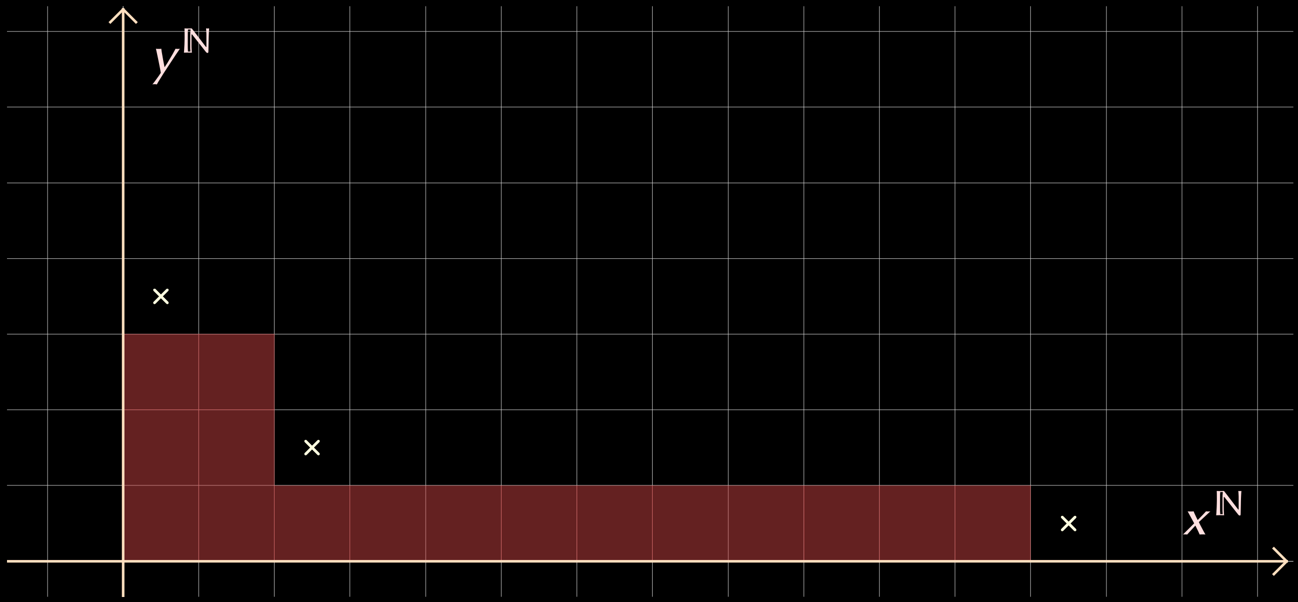

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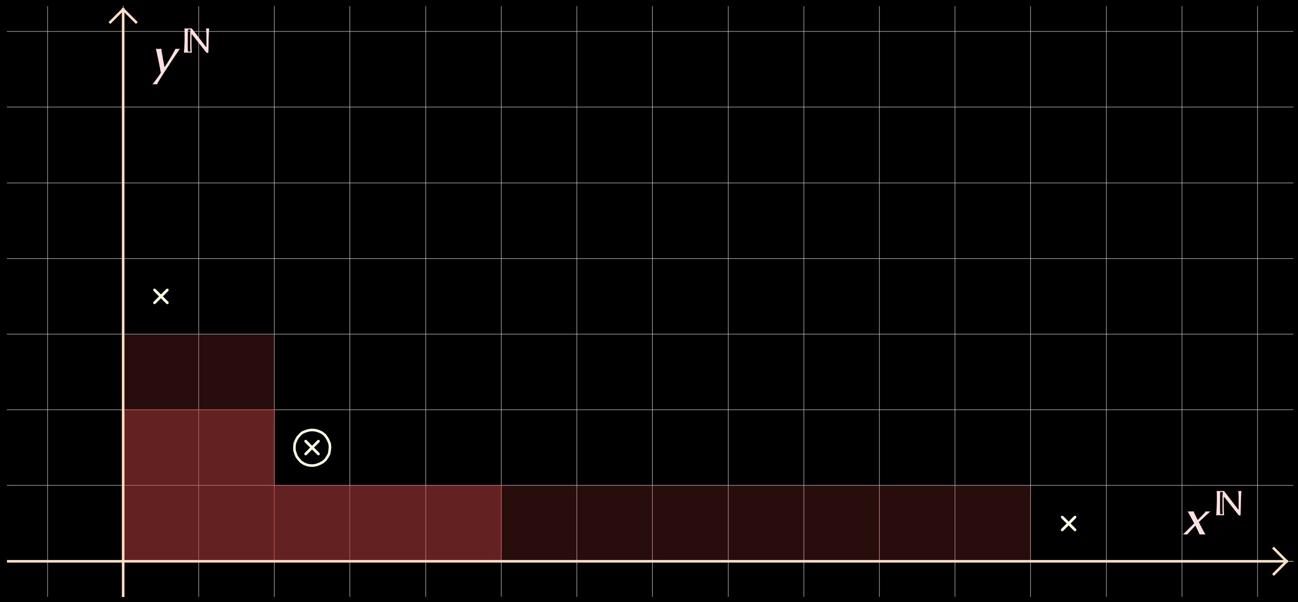

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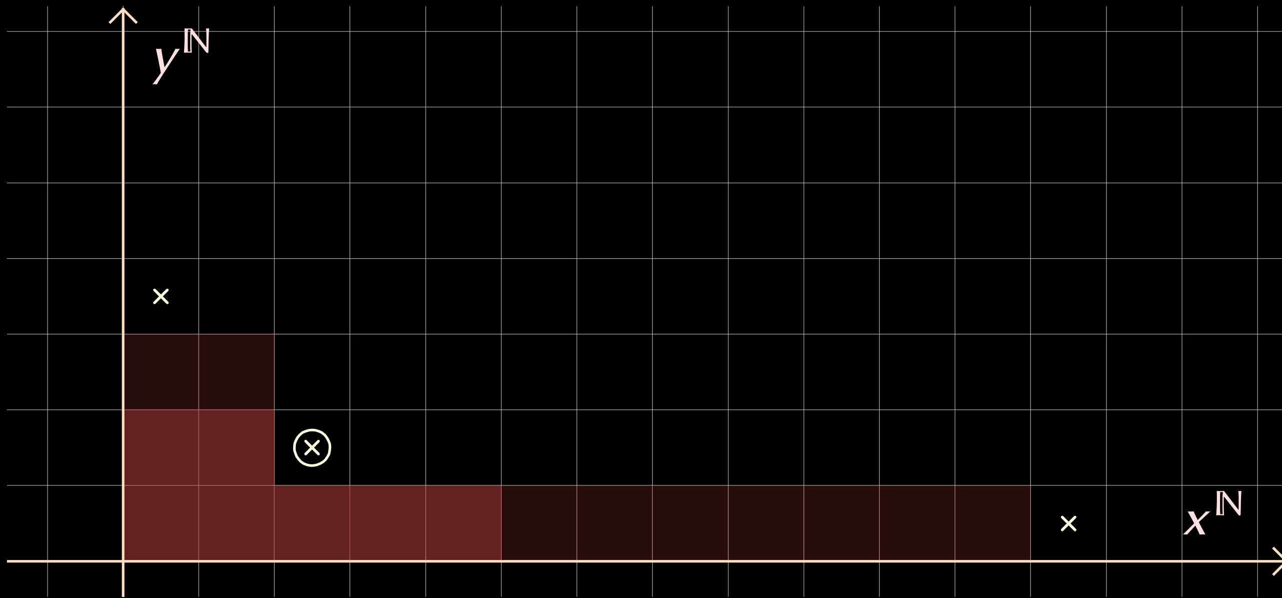
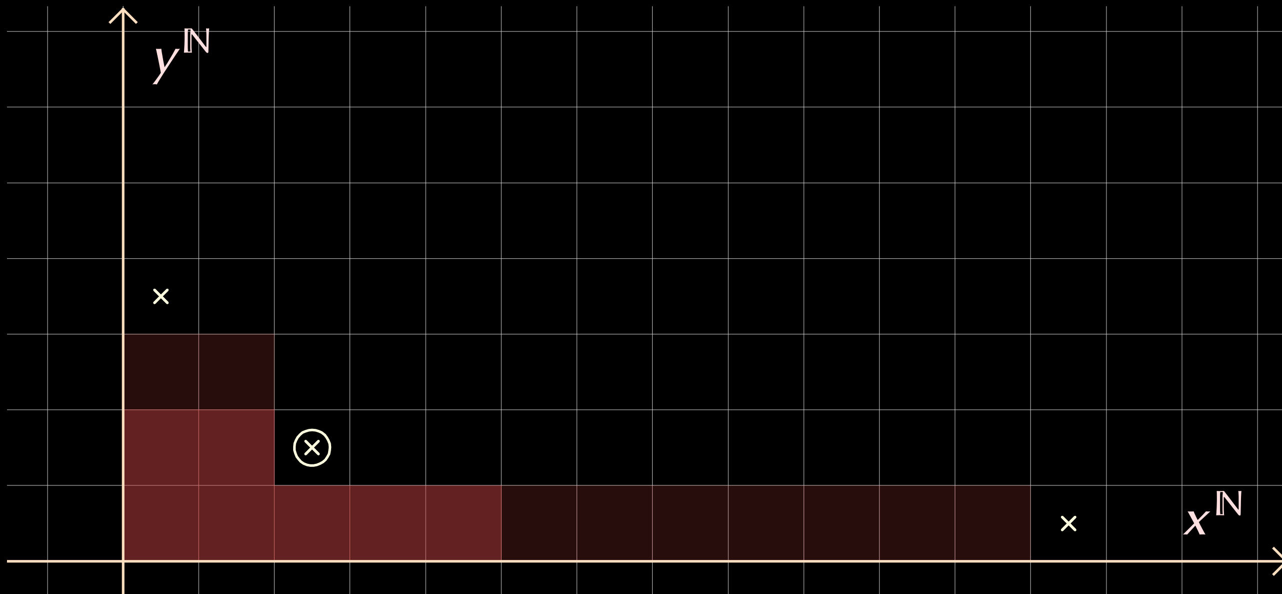


This time: $\exists B_k \in I_\alpha, \quad b \leq \sqrt{\frac{2n}{k}}$

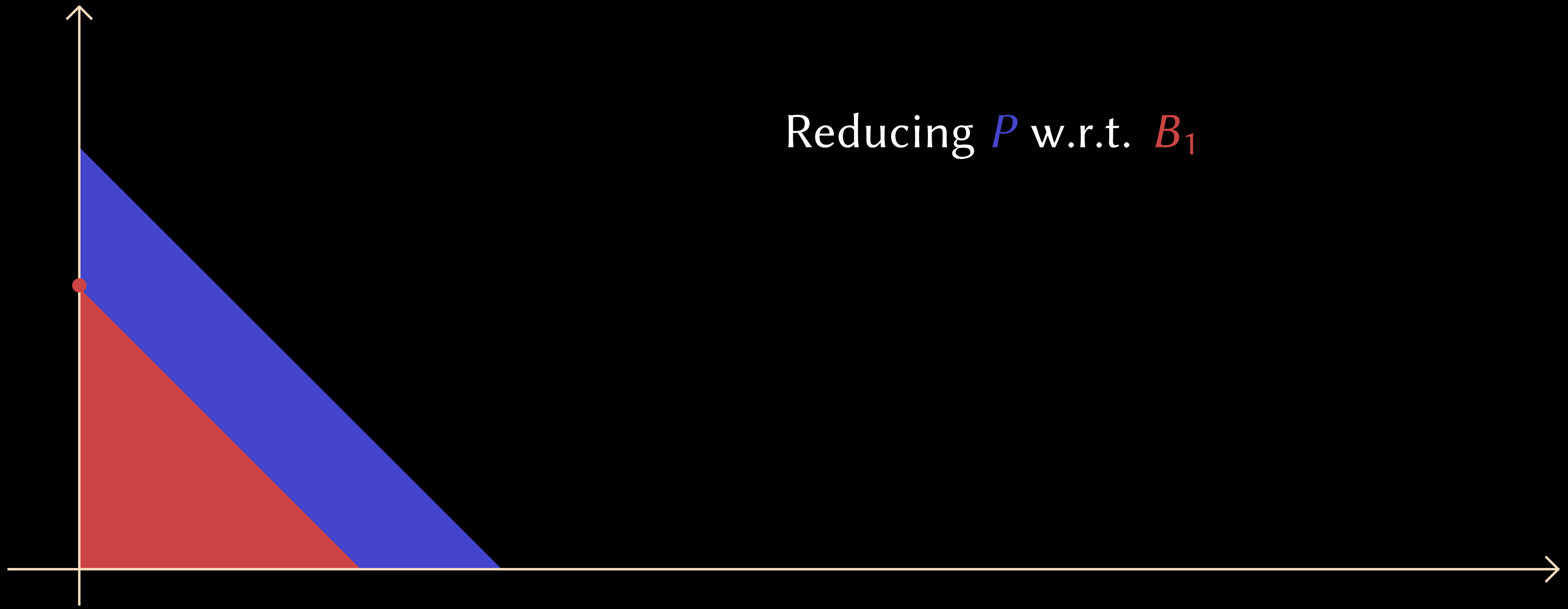
Gröbner basis, non-generic case



This time: $\exists B_k \in I_\alpha, \quad b \leqslant \sqrt{\frac{2n}{k}}$

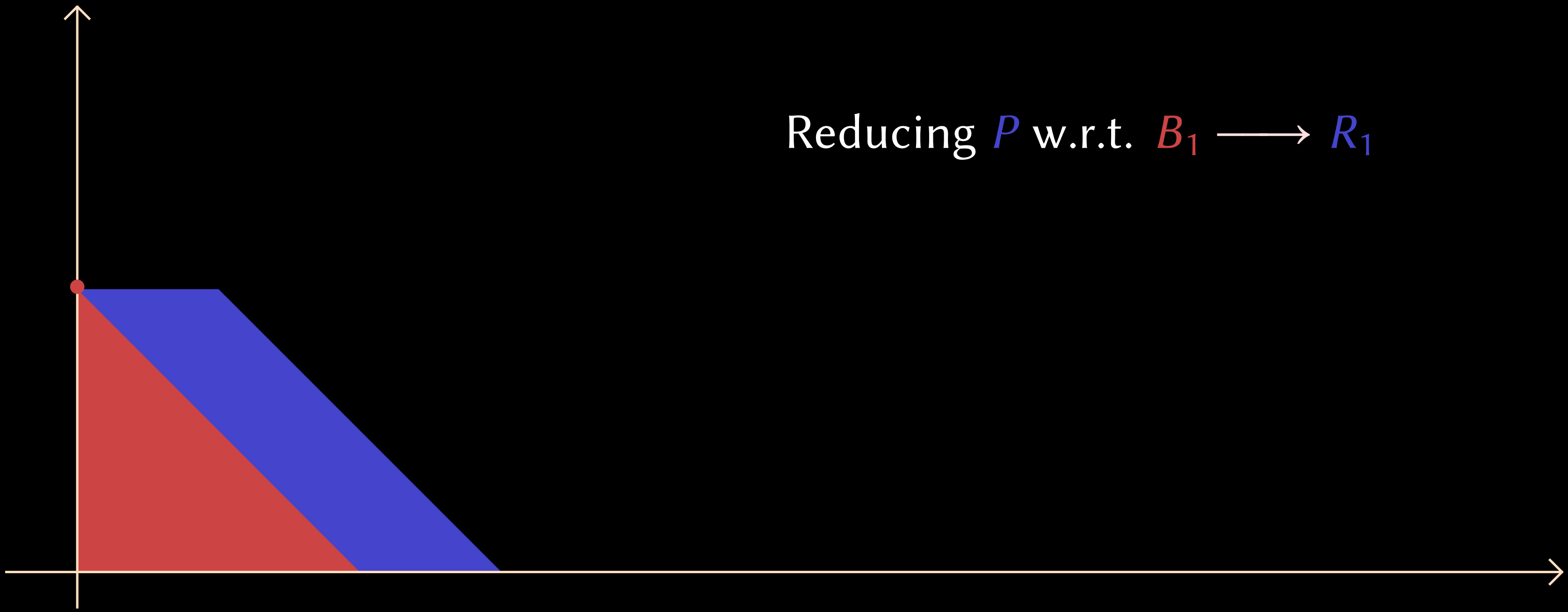
Also: modulo $x \longrightarrow \tilde{x} + \lambda y$, we may assume that $\mathrm{LM}(B_1) = y^b$

Reduction, non-generic case



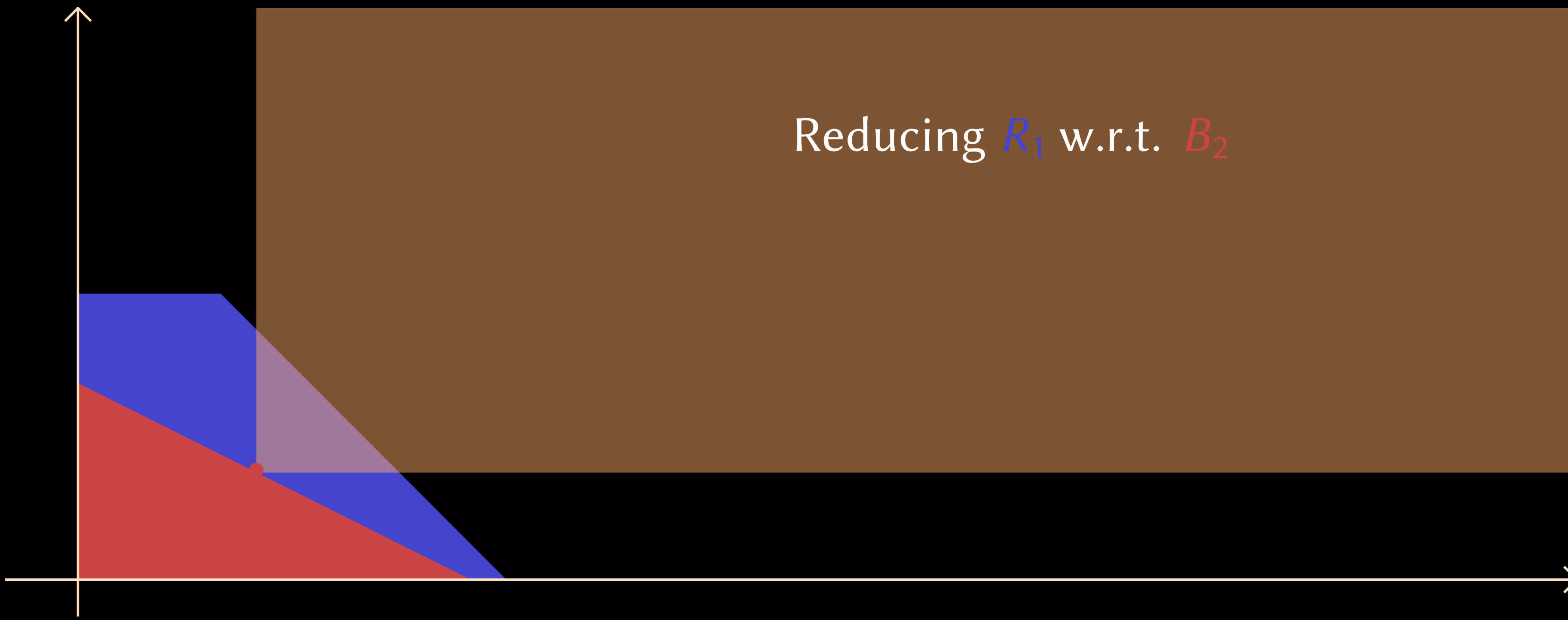
Reducing P w.r.t. B_1

Reduction, non-generic case

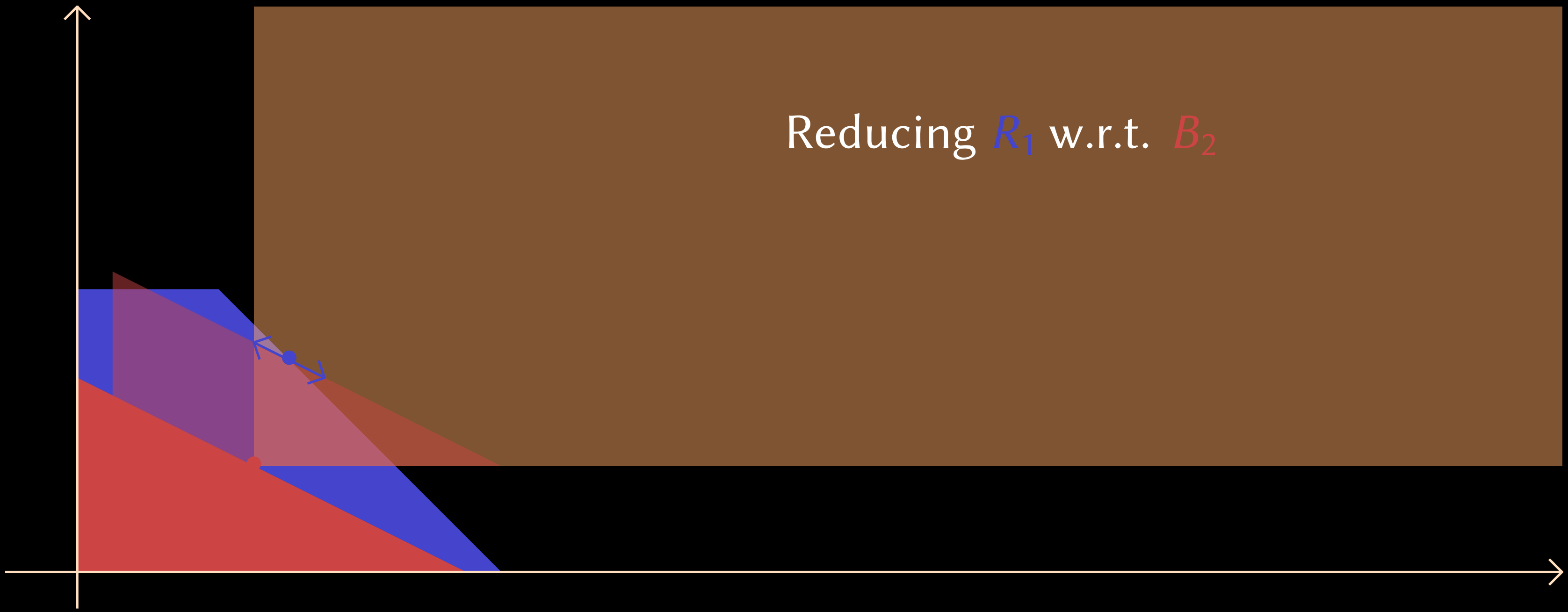


Reducing P w.r.t. $B_1 \longrightarrow R_1$

Reduction, non-generic case

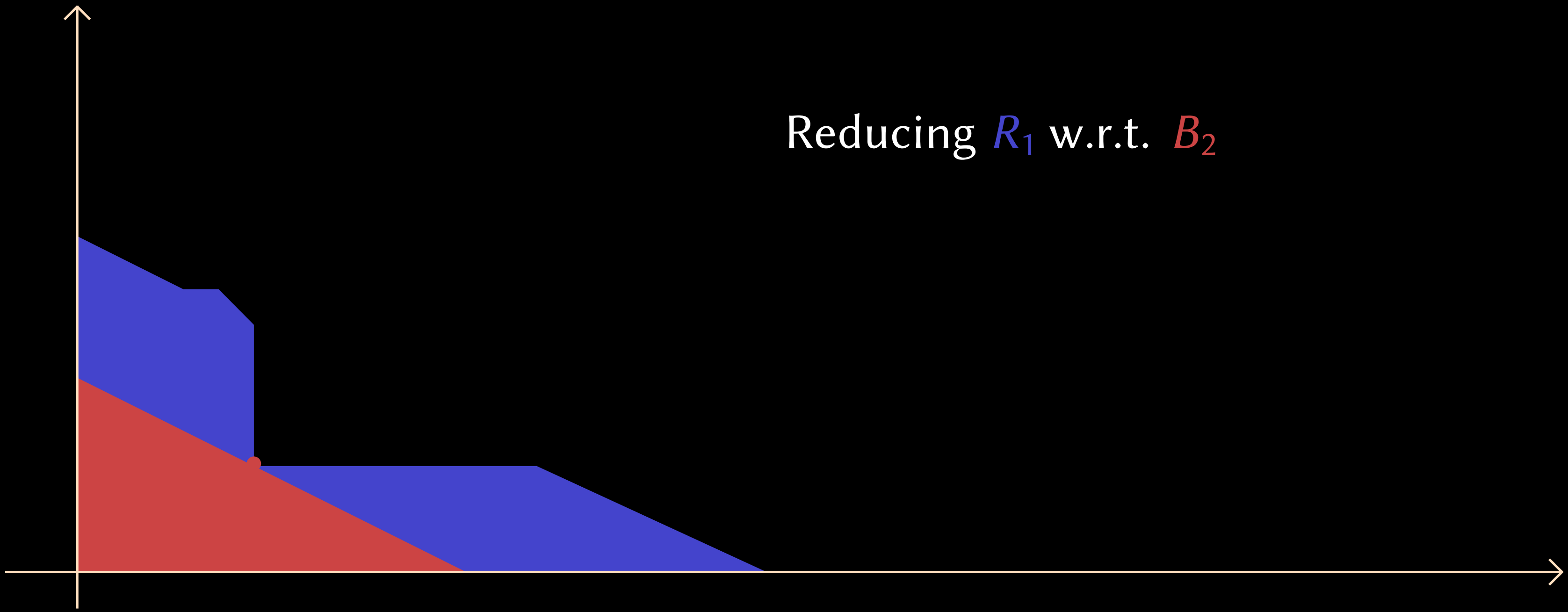


Reduction, non-generic case



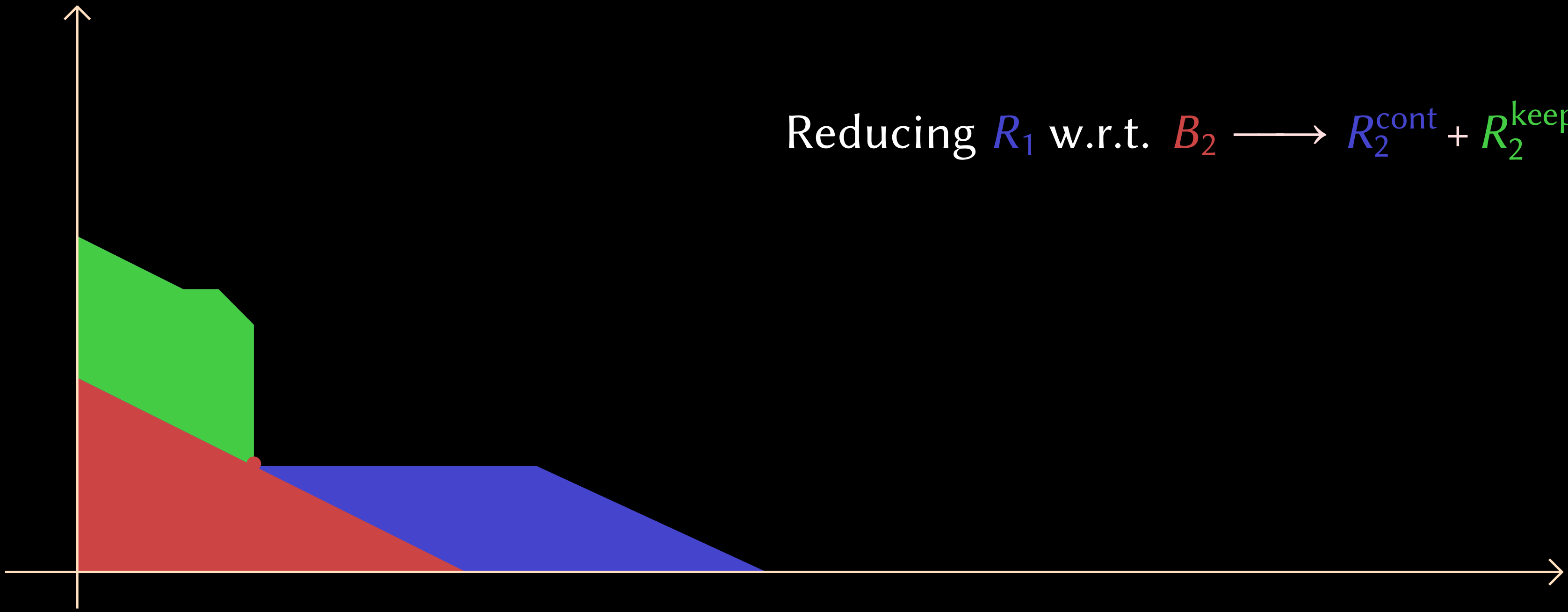
Reducing R_1 w.r.t. B_2

Reduction, non-generic case



Reducing R_1 w.r.t. B_2

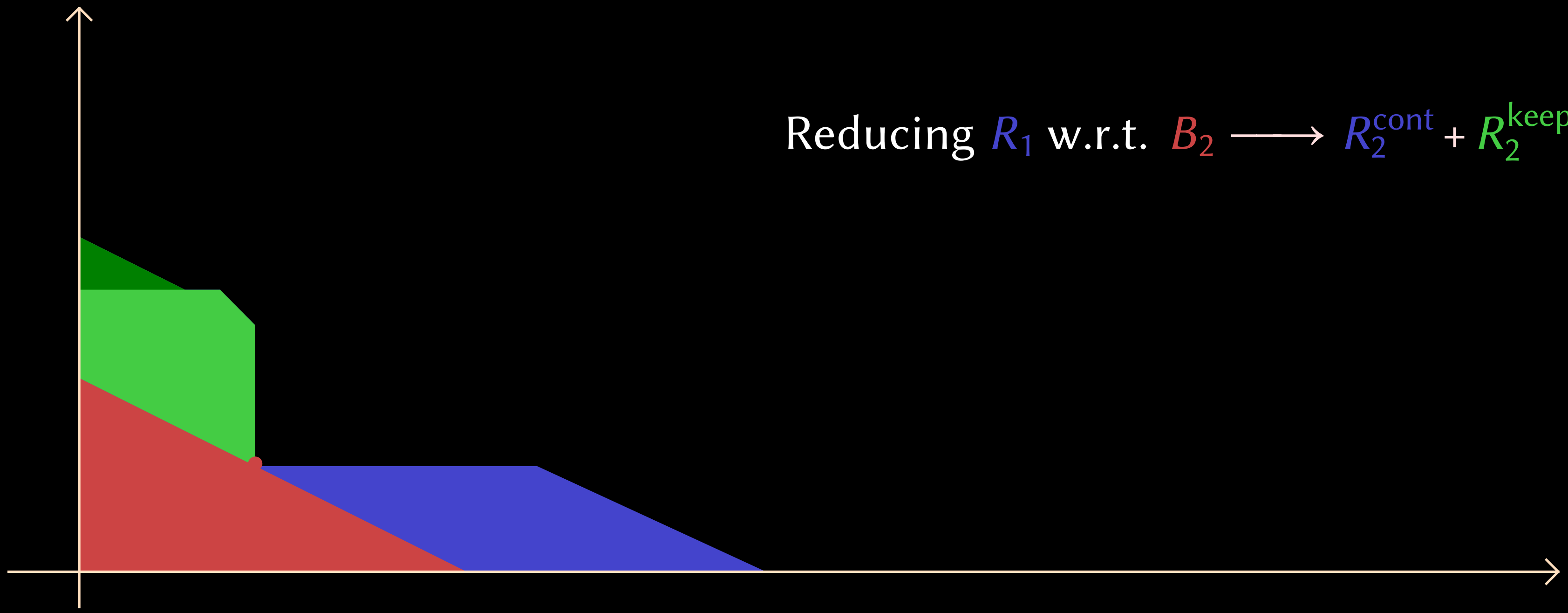
Reduction, non-generic case



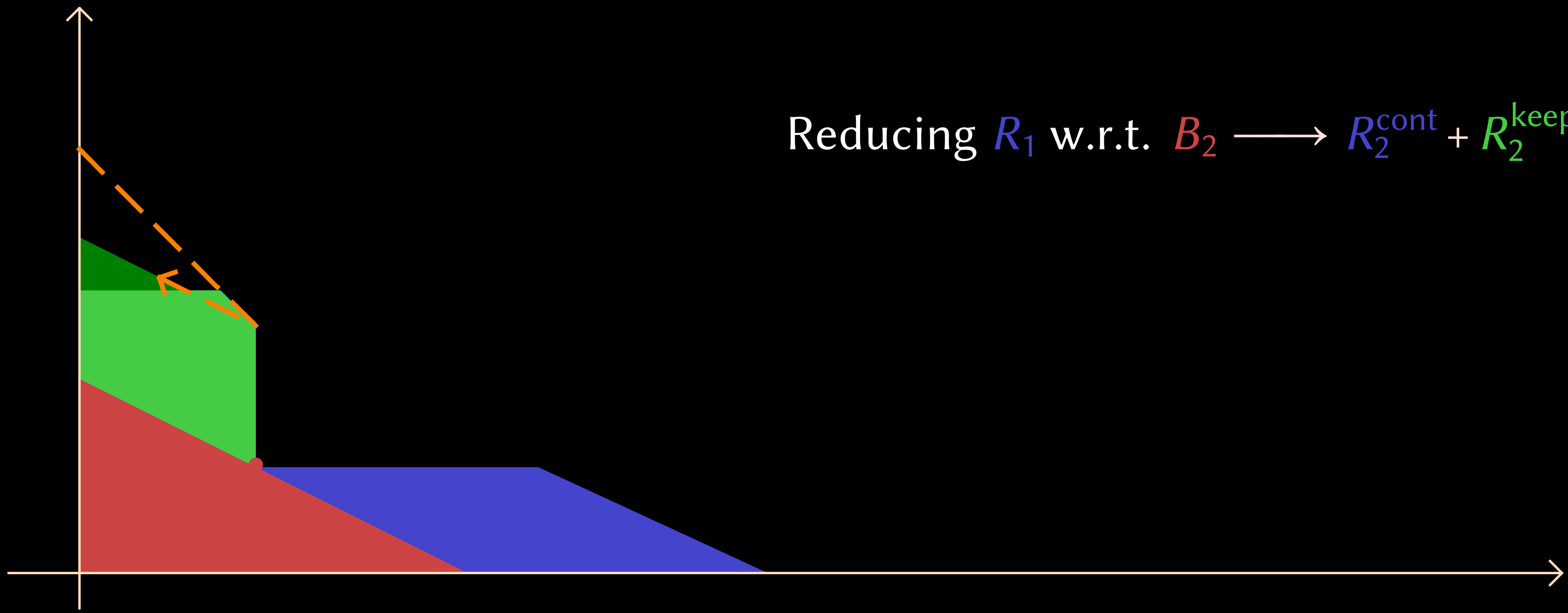
Reducing R_1 w.r.t. $B_2 \longrightarrow R_2^{\text{cont}} + R_2^{\text{keep}}$

Reduction, non-generic case

Reducing R_1 w.r.t. $B_2 \longrightarrow R_2^{\text{cont}} + R_2^{\text{keep}}$

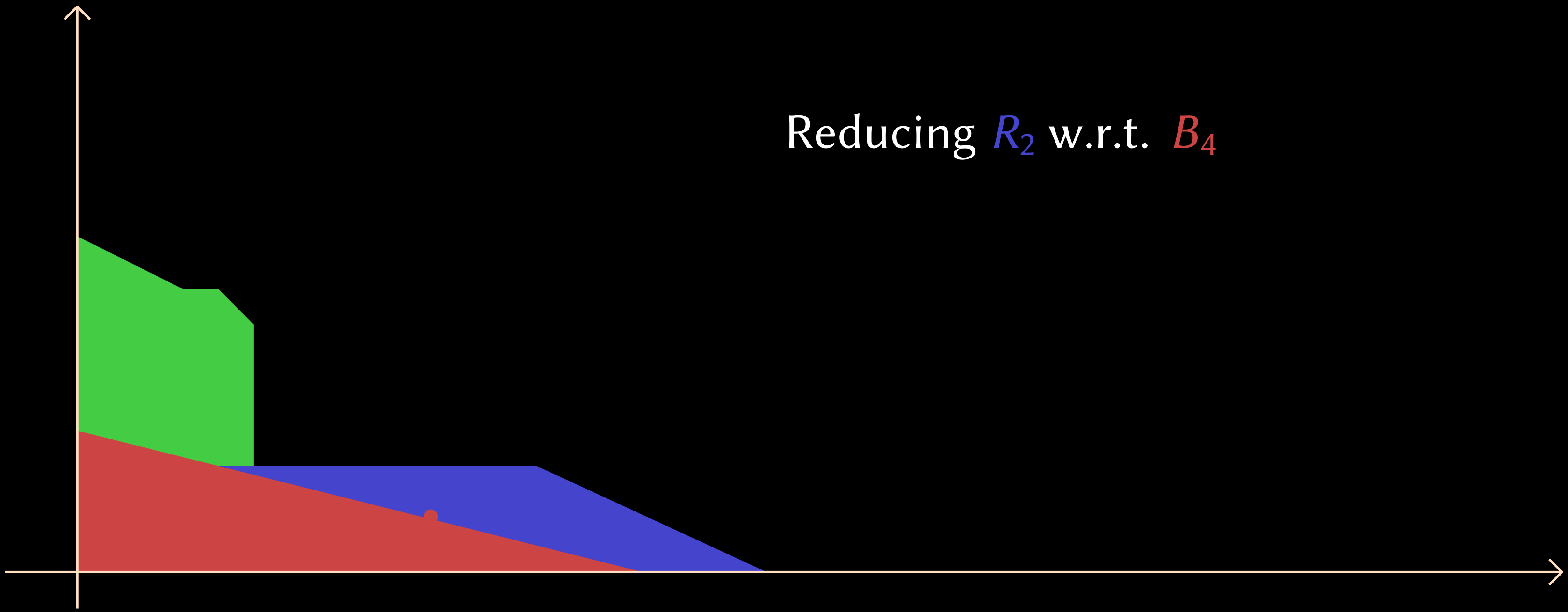


Reduction, non-generic case



Reducing R_1 w.r.t. B_2 $\longrightarrow$ $R_2^{\text{cont}} + R_2^{\text{keep}}$

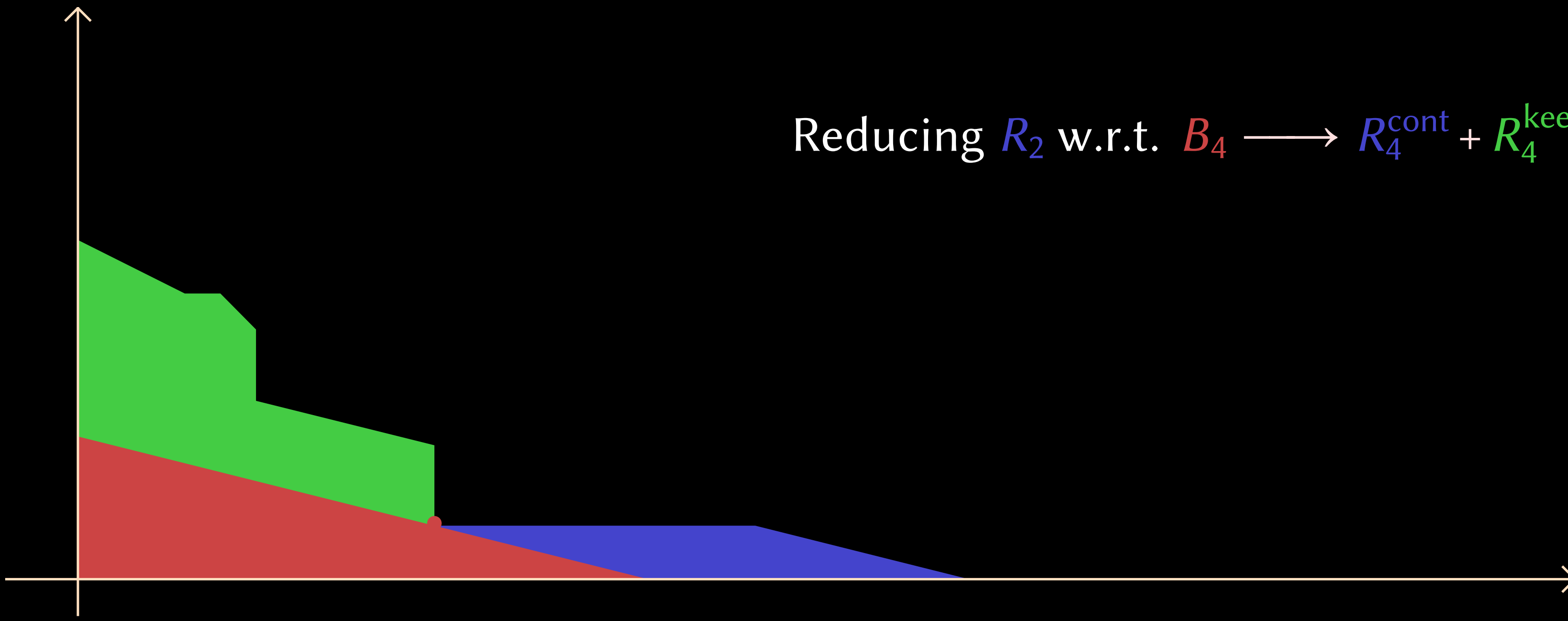
Reduction, non-generic case



Reducing R_2 w.r.t. B_4

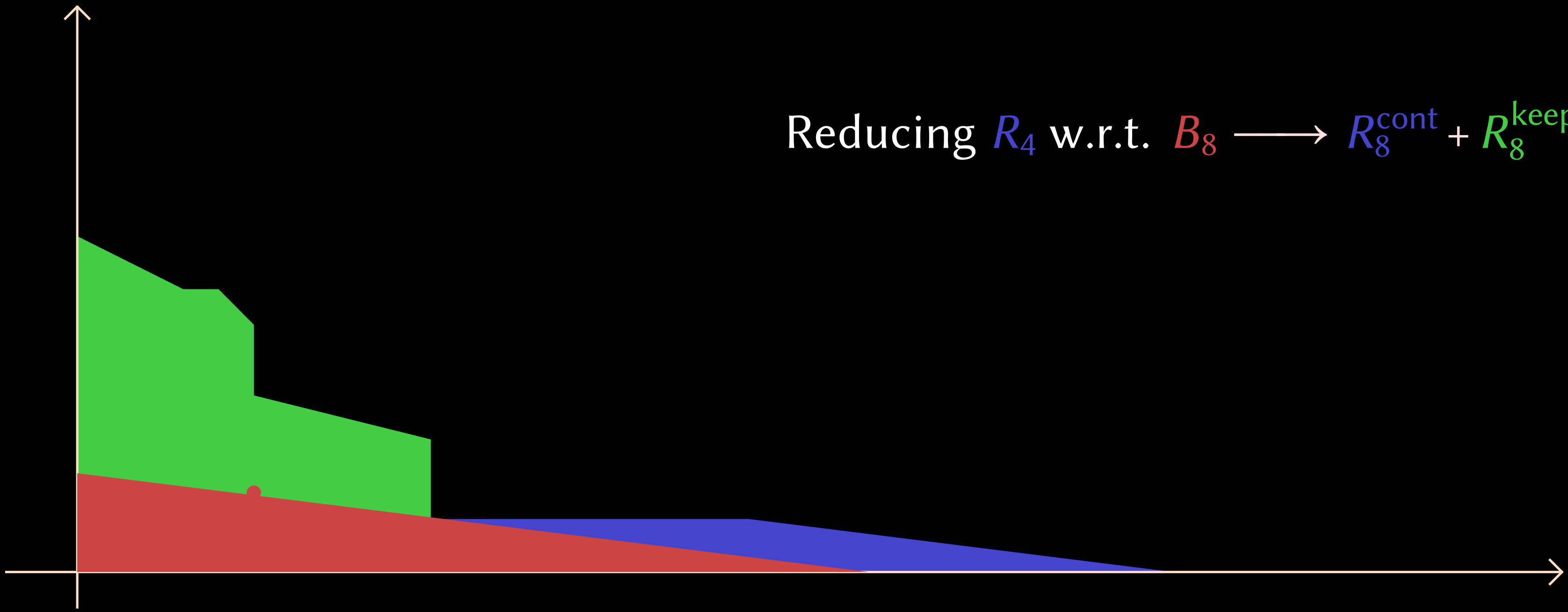
Reduction, non-generic case

Reducing R_2 w.r.t. $B_4 \longrightarrow R_4^{\text{cont}} + R_4^{\text{keep}}$



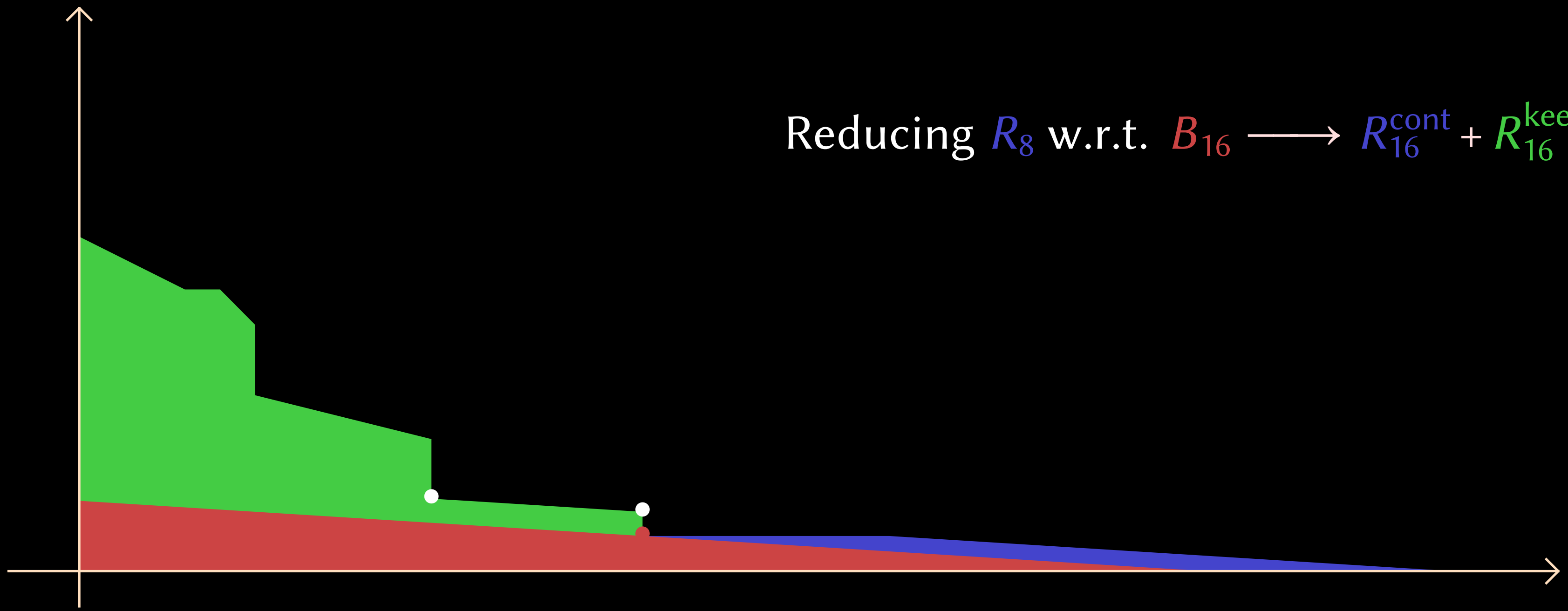
Reduction, non-generic case

Reducing R_4 w.r.t. $B_8 \longrightarrow R_8^{\text{cont}} + R_8^{\text{keep}}$

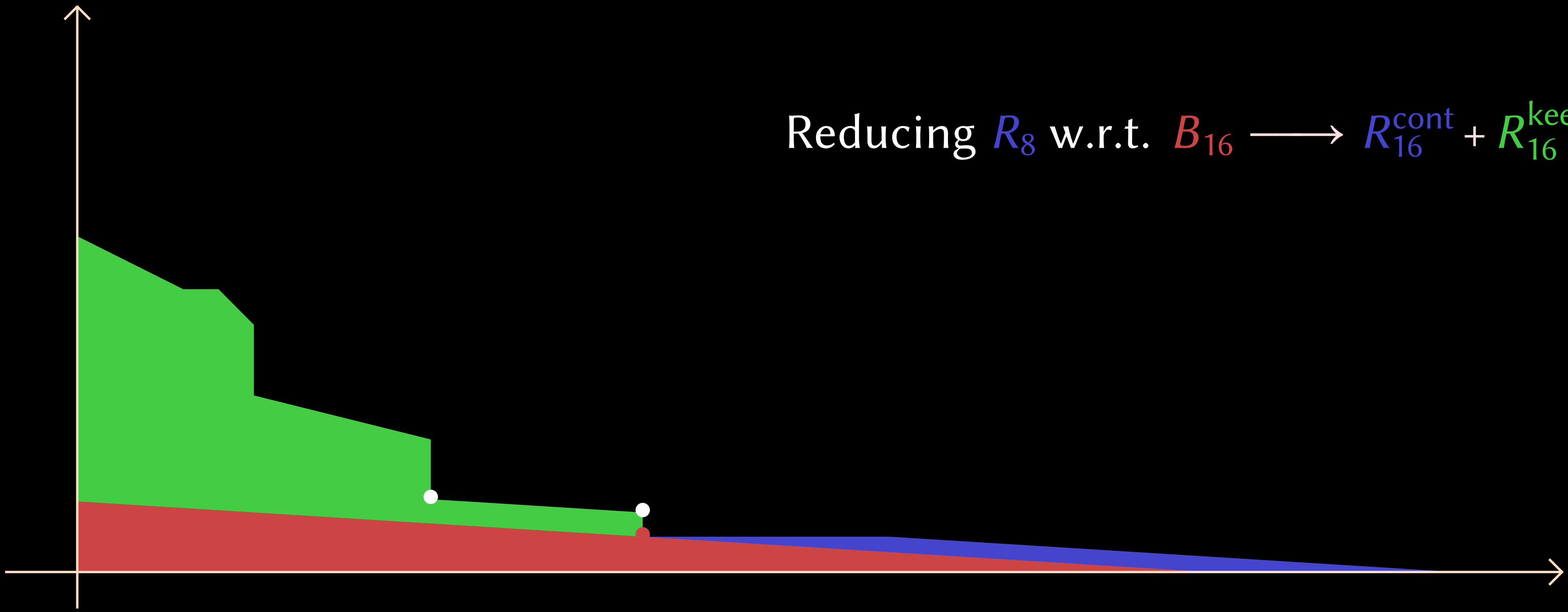


Reduction, non-generic case

Reducing R_8 w.r.t. B_{16} $\longrightarrow$ $R_{16}^{\text{cont}} + R_{16}^{\text{keep}}$



Reduction, non-generic case



Reducing R_8 w.r.t. B_{16} $\longrightarrow$ $R_{16}^{\text{cont}} + R_{16}^{\text{keep}}$

In general:

- $P = P_1 + P_2 + P_4 + \cdots \longrightarrow R = R_1 + R_2 + R_4 + \cdots, \qquad R - P \in I_\alpha$
- Controlled decrease of $\deg_k P_k$ during this reduction

Remainder trees for ideals

$$\boldsymbol{\alpha}_{\text{lo}} := (\alpha_1, \ldots, \alpha_{\lfloor n/2 \rfloor})$$
$$\boldsymbol{\alpha}_{\text{hi}} := (\alpha_{\lfloor n/2 \rfloor + 1}, \ldots, \alpha_n)$$

$$P$$

Remainder trees for ideals

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$$P \operatorname{rem} I_{\boldsymbol{\alpha}}$$

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$$(P \operatorname{rem} I_{\boldsymbol{\alpha}}) \operatorname{rem} I_{\boldsymbol{\alpha}_{\mathrm{lo}}} \qquad (P \operatorname{rem} I_{\boldsymbol{\alpha}}) \operatorname{rem} I_{\boldsymbol{\alpha}_{\mathrm{hi}}}$$

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$$P \operatorname{rem} I_{\boldsymbol{\alpha}_{\mathrm{lo,lo}}} \qquad P \operatorname{rem} I_{\boldsymbol{\alpha}_{\mathrm{lo,hi}}} \qquad P \operatorname{rem} I_{\boldsymbol{\alpha}_{\mathrm{hi,lo}}} \qquad P \operatorname{rem} I_{\boldsymbol{\alpha}_{\mathrm{hi,hi}}}$$

Remainder trees for ideals

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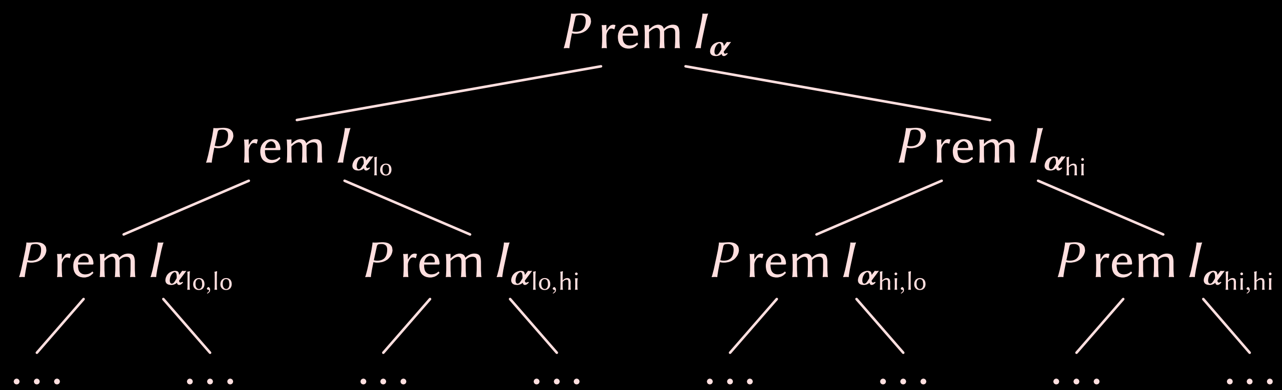

Thank you !

http://www.texmacs.org