Toward a unification of infinities

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Mini-workshop on Transseries and Dynamical Systems
Fields institute, Toronto

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Outline

Part I — Regular growth rates at infinity

Regular growth rates at infinity

- **Dubois-Reymond (1870)**

  “calculus of infinities”

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  subfields of field of germs of continuous functions at infinity

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  Hardy fields

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- **Dahn–Göring (1984), Écalle (1992)**

  Transseries

Hausdorff fields

Subfields of the ring $\mathcal{G}^0$ of germs of continuous functions at infinity

What is regular growth?

Hausdorff fields

Subfields of the ring $\mathcal{G}^0$ of germs of continuous functions at infinity

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Differential fields of the ring $\mathcal{G}^1$ of germs of C^1 functions at infinity

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Hardy field shadows of *o*-minimal structures

Germs of definable functions at infinity in some *o*-minimal structure

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Germs of definable functions at infinity in some *o*-minimal structure

Formal germs constructed using primitives that preserve regular growth

Transseries: constructed from $x \succ 1$ and $\mathbb{R}$ using $\sum$, exp, log

The field $\mathbb{T}$ of transseries

$\mathbb{T} :=$ closure of $\mathbb{R} \cup \{x\}$ under exp, log and infinite summation

$$e^{e^{x}+\cdots} - 3\,e^{x^2} + 5\,(\log x)^{\pi} + 42 + x^{-1} + \cdots + e^{-x}$$

Here one should think of x as a positive infinite indeterminate.

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$$\sum_{\mathfrak{m}} f_{\mathfrak{m}} \mathfrak{m} \quad = \quad \mathrm{e}^{\mathrm{e}^x + \mathrm{e}^{x/2} + \cdots} - 3\,\mathrm{e}^{x^2} + 5\,(\log x)^{\pi} + 42 + x^{-1} + 2\,x^{-2} + 6\,x^{-3} + \cdots + \mathrm{e}^{-x}$$

x: positive infinite indeterminate

$f_{\mathfrak{m}}$: coefficent

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$$\begin{aligned} e^{e^x + e^{x/2} + \cdots} &\in \mathfrak{M} \\ e^x, e^{x/2}, \ldots &\succ 1 \end{aligned}$$

Missing growth rates

vdH (1997), vdDries–Macintyre–Marker (1997)

$(\log x \log \log x)^{\text{inv}}$ cannot be expanded w.r.t. the scale of exp-log functions

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duBois–Reymond, Hardy, Kneser, ...

There exist "regular" functions that grow faster than x, e^x, $\mathrm{e}^{\mathrm{e}^x}$, ...

$$E_\omega(x+1) \;=\; \mathrm{e}^{E_\omega(x)}$$

$\longrightarrow$ Écalle's "Grand Cantor"

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Part II — All numbers great and small

- **Cantor (1870)**

  ordinal calculus $0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \omega 2, \ldots, \omega^2, \ldots, \omega^\omega, \ldots$

Numbers beyond

- **Cantor (1870)**

  ordinal calculus $0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \omega 2, \ldots, \omega^2, \ldots, \omega^\omega, \ldots$

- **Robinson (1961)**

  non-standard analysis

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- **Robinson (1961)**

  non-standard analysis

- **Conway (1976)**

  surreal numbers

Conway's recursive definition

- Given sets $L, R \subseteq \mathbf{No}$ with $L < R$, there exists a $\{L|R\} \in \mathbf{No}$ with $L < \{L|R\} < R$
- All numbers in **No** can be obtained in this way

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Definition using sign sequences

- A surreal number x is a sequence $(x[\beta])_{\beta<\alpha} \in \{-,+\}^{\alpha}$ for some ordinal $\ell_x := \alpha \in \mathbf{On}$
- Lexicographical ordering on such sequences (modulo completion with zeros)

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Simplicity relation

$$x \sqsubseteq y \iff \ell_x \leqslant \ell_y \wedge (\forall \beta < \ell_a, a[\beta] = b[\beta])$$

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Equivalence between $(\mathbf{No}, \leqslant, \{\,|\,\})$ and $(\mathbf{No}, \leqslant, \sqsubseteq)$

$$\{L|R\} := \min_{\sqsubseteq} \{a \in \mathbf{No} : L < a < R\}$$

Operations on **No**

Ring structure: for $x = \{x_L | x_R\}$ and $y = \{y_L | y_R\}$, we define

$$
\begin{aligned}
0 &:= \{|\} \\
1 &:= \{0|\} \\
-x &:= \{-x_R | -x_L\} \\
x + y &:= \{x_L + y, x + y_L | x_R + y, x + y_R\} \\
xy &:= \{x'y + xy' - x'y', x''y + xy'' - x''y'' | x'y + xy'' - x'y'', x''y + xy' - x''y'\} \\
&\qquad (x' \in x_L, x'' \in x_R, y' \in y_L, y'' \in y_R).
\end{aligned}
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Conway's ω-map (generalizing Cantor's ordinal exponentiation)

$$\omega^x := \{0, \mathbb{R}^> \omega^{x_L} | \mathbb{R}^> \omega^{x_R}\}$$

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Surreal numbers as Hahn series

$$\mathbf{No} \cong \mathbb{R}[[\mathbf{Mo}]], \qquad \mathbf{Mo} := \omega^{\mathbf{No}}$$

$$
\begin{aligned}
0 &:= \{\,|\,\} \\
1 &:= \{0\,|\,\} \\
2 &:= \{0,1\,|\,\} \\
&\vdots \\
-1 &:= \{\,|\,0\} \\
-2 &:= \{\,|\,-1,0\} \\
&\vdots \\
{}^1/_2 &:= \{0\,|\,1\} \\
{}^1/_4 &:= \{0\,|\,{}^1/_2,1\} \\
{}^3/_8 &:= \{0,{}^1/_4\,|\,{}^1/_2,1\} \\
&\vdots \\
{}^1/_3 &:= \{0,{}^1/_4,{}^5/_{16},\dots\,|\,\dots,{}^3/_8,{}^1/_2,1\} \\
\pi &:= \{0,1,2,3,3\,{}^1/_{16},\dots\,|\,\dots,3\,{}^1/_4,3\,{}^1/_2,4\} \\
&\vdots \\
\mathbb{R} &\subseteq \mathbf{No}
\end{aligned}
$$

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2 &:= \{0,1\,|\,\} \\
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\omega &:= \{0,1,2,\dots\,|\,\} \\
\omega+1 &:= \{0,1,2,\dots,\omega\,|\,\} \\
&\vdots \\
\omega 2 &:= \{0,1,2,\dots,\omega,\omega+1,\dots\,|\,\} \\
&\vdots \\
\omega^2 &:= \{0,1,2,\dots,\omega,\dots,\omega 2,\dots\,|\,\} \\
&\vdots \\
\mathbf{On} &\subseteq \mathbf{No} \\
\\
\omega^{-1} &:= \{0\,|\,\dots,{}^1/_4,{}^1/_2,1\} \\
\exp\omega &:= \{1,\omega,\omega^2,\omega^3,\dots\,|\,\}
\end{aligned}
$$

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Question: how to define a derivation with respect to ω on **No**?

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Question: how to define a composition on **No**?

Part III — The infinities conjecture

Summary so far

Transseries	**Surreal numbers**
Extension of $\mathbb{R}$ with infinite element x	Extension of $\mathbb{R}$ with infinite element ω
$\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$	$\mathbf{No} = \mathbb{R}[[\mathbf{Mo}]]$
Closed under $\exp$, $\log$	Closed under $\exp$, $\log$

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Closed under $\exp$, $\log$	Closed under $\exp$, $\log$
Closed under ∂	Closed under $\{\mid\}$
Closed under $\circ$	Simplicity relation $\sqsubseteq$

The infinities conjecture

Conjecture (vdH, 2006)

For a suitable generalization of transseries, called ***hyperseries****, the field* $\mathbb{H}$ *of hyperseries in* $x \succ 1$ *is naturally isomorphic to* **No***, via the map* $\mathbb{H} \longrightarrow \mathbf{No}; f \longmapsto f(\omega)$ *that evaluates a hyperseries* f *at* ω*.*

In particular, $\mathbb{H}$ *is closed under all* ***hyperexponentials*** E_α *and* ***hyperlogarithms*** L_α *for ordinal* α*, and* $\mathbb{H}$ *contains "nested hyperseries".*

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Transseries	**Surreal numbers**
Closed under $\{\,\mid\,\}$	Closed under ∂
Simplicity relation $\sqsubseteq$	Closed under $\circ$

Taming tameness

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- **Bagayoko, vdH 2022: No** $\cong \mathbb{H}$

Part IV — Hyperseries

Logarithmic hyperseries

$$\forall \alpha \in \mathbf{On}, \qquad \ell_\alpha := L_\alpha x$$

$$\mathfrak{L} := \left\{ \prod_{\beta<\alpha} \ell_\beta^{r_\beta} : \alpha \in \mathbf{On}, (r_\beta)_{\beta<\alpha} \in \mathbb{R}^\alpha \right\} \qquad \text{(lex monomial group)}$$

$$\mathbb{L} := \mathbb{R}[[\mathfrak{L}]]$$

Logarithmic hyperseries

$$\forall \alpha \in \mathbf{On}, \qquad \ell_\alpha := L_\alpha x$$

$$\mathfrak{L} := \left\{ \prod_{\beta<\alpha} \ell_\beta^{r_\beta} : \alpha \in \mathbf{On}, (r_\beta)_{\beta<\alpha} \in \mathbb{R}^\alpha \right\} \qquad \text{(lex monomial group)}$$

$$\mathbb{L} := \mathbb{R}[[\mathfrak{L}]]$$

Semantics

$$\ell_n = L_1^{\circ n} x = \left(\log \circ \overset{n\times}{\cdots} \circ \log\right)(x)$$

$$\alpha = \omega^{\mu_1} n_1 + \cdots + \omega^{\mu_k} n_k, \quad \mu_1 > \cdots > \mu_k$$

$$\ell_\alpha = (L_{\omega^{\mu_k}}^{\circ n_k} \circ \cdots \circ L_{\omega^{\mu_1}}^{\circ n_1})(x)$$

$$\ell_{\omega^2+1} = \log L_{\omega^2} x$$

Calculus

$$\ell_{\omega^{\mu+1}} \circ \ell_{\omega^{\mu}} = \ell_{\omega^{\mu+1}} + 1$$

$$\ell'_{\alpha} = \prod_{\beta<\alpha} \ell_{\beta}^{-1}$$

$$\ell_{\omega^{\mu+1}} \circ \ell_{\omega^{\mu}} = \ell_{\omega^{\mu+1}} + 1$$

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$$\ell'_0 = x' = 1, \qquad \ell'_1 = \frac{1}{\ell_0}, \qquad \ell'_2 = \frac{1}{\ell_0\,\ell_1}, \qquad \ldots, \qquad \ell'_\omega = \frac{1}{\ell_0\,\ell_1\,\ell_2\cdots}, \qquad \ldots$$

Calculus

$$\ell_{\omega^{\mu+1}} \circ \ell_{\omega^\mu} = \ell_{\omega^{\mu+1}} + 1$$

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$$\ell'_0 = x' = 1, \qquad \ell'_1 = \frac{1}{\ell_0}, \qquad \ell'_2 = \frac{1}{\ell_0\,\ell_1}, \qquad \ldots, \qquad \ell'_\omega = \frac{1}{\ell_0\,\ell_1\,\ell_2\cdots}, \qquad \ldots$$

Theorem

- $\partial: \mathbb{L} \longrightarrow \mathbb{L}$, *strongly linear*
- $\circ: \mathbb{L} \times \mathbb{L}^{>,\succ} \longrightarrow \mathbb{L}$, *strongly linear in first argument*
- *For all* $f \in \mathbb{L}$, $g \in \mathbb{L}^{>,\succ}$ *and* $\delta \prec g$

$$f \circ (g + \delta) = f \circ g + (f' \circ g)\,\delta + \tfrac{1}{2}(f'' \circ g)\,\delta^2 + \cdots$$

What is a monomial?

$$\mathfrak{m} \in \mathfrak{M} \iff \log \mathfrak{m} \in \mathbb{T}_{\succ}$$

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Defining the exponential

$$\mathrm{e}^{\mathrm{e}^x + 3x^2 + \log x + \sqrt{2} + \frac{3}{x} + \frac{\pi}{x^2} + \cdots + \mathrm{e}^{-x}} = \mathrm{e}^{\mathrm{e}^x + 3x^2 + \log x}\, \mathrm{e}^{\sqrt{2}}\, \mathrm{e}^{\frac{3}{x} + \frac{\pi}{x^2} + \cdots + \mathrm{e}^{-x}}$$

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Defining the first hyperexponential

$$E_\omega(f+\delta) = E_\omega(f)+E'_\omega(f)\,\delta+\frac{1}{2}E''_\omega(f)\,\delta^2+\cdots$$

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Defining the first hyperexponential

$$E_\omega\left(x+\frac{1}{x}\right) \overset{?}{=} E_\omega(x)+E_\omega'(x)\frac{1}{x}+\frac{1}{2}E_\omega''(x)\frac{1}{x^2}+\cdots$$

Endorsing hypermonomials

What is a monomial?

$$\mathfrak{m} \in \mathfrak{M} \iff \log \mathfrak{m} \in \mathbb{T}_{\succ}$$

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$$\mathrm{e}^{\mathrm{e}^x + 3x^2 + \log x + \sqrt{2} + \frac{3}{x} + \frac{\pi}{x^2} + \cdots + \mathrm{e}^{-x}} = \mathrm{e}^{\mathrm{e}^x + 3x^2 + \log x} \, \mathrm{e}^{\sqrt{2}} \, \mathrm{e}^{\frac{3}{x} + \frac{\pi}{x^2} + \cdots + \mathrm{e}^{-x}}$$

Defining the first hyperexponential

$$E_\omega\left(x + \frac{1}{x}\right) \stackrel{?}{=} E_\omega(x) + (E_\omega(x)\, E_\omega(x-1)\, E_\omega(x-2) \cdots) \frac{1}{x} + \cdots$$

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Defining the first hyperexponential

$$E_\omega\left(x + \frac{1}{E_\omega(x)}\right) = E_\omega(x) + (E_\omega(x)\,E_\omega(x-1)\,E_\omega(x-2)\cdots)\frac{1}{E_\omega(x)} + \cdots$$

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What is a monomial?

$$\mathfrak{m} \in \mathfrak{M} \iff \log \mathfrak{m} \in \mathbb{T}_{\succ}$$

Defining the exponential

$$e^{e^x + 3x^2 + \log x + \sqrt{2} + \frac{3}{x} + \frac{\pi}{x^2} + \cdots + e^{-x}} = e^{e^x + 3x^2 + \log x} e^{\sqrt{2}} e^{\frac{3}{x} + \frac{\pi}{x^2} + \cdots + e^{-x}}$$

Defining the first hyperexponential

$$E_\omega\left(x + \frac{1}{E_\omega(x-1)}\right) \stackrel{?}{=} E_\omega(x) + (E_\omega(x)\, E_\omega(x-1)\, E_\omega(x-2) \cdots) \frac{1}{E_\omega(x-1)} + \cdots$$

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$$E_\omega\left(x + \frac{1}{E_\omega(x-1)}\right) = \exp E_\omega\left(x - 1 + \frac{1}{E_\omega(x-1)}\right)$$

Endorsing hypermonomials II

$$\delta \prec \frac{1}{L_n E_\omega f} \quad \Longrightarrow \quad E_\omega(f+\delta) = E_n E_\omega(f+\cdots)$$

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$$\delta \prec \frac{1}{L_n E_\omega f} \quad \Longrightarrow \quad E_\omega(f+\delta) = E_n E_\omega(f+\cdots)$$

$$\begin{aligned}\mathfrak{M}_\omega &:= \left\{\mathfrak{a} \in \mathfrak{M} : \forall n \in \mathbb{N}, \operatorname{supp} L_\omega \mathfrak{a} \succ \frac{1}{L_n \mathfrak{a}}\right\} \\ &= \{\mathfrak{a} \in \mathfrak{M} : \forall n \in \mathbb{N}, L_n \mathfrak{a} \in \mathfrak{M}\}\end{aligned}$$

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$$= \{ \mathfrak{a} \in \mathfrak{M} : \forall n \in \mathbb{N}, L_n \mathfrak{a} \in \mathfrak{M} \}$$

$$E_\omega\left(x+\tfrac{1}{x}\right), E_\omega\left(x-1+\tfrac{1}{x}\right), E_\omega\left(x-2+\tfrac{1}{x}\right), \ldots \in \mathfrak{M}_\omega \subseteq \mathfrak{M}$$

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$$E_\omega\!\left(x+\tfrac{1}{x}\right), E_\omega\!\left(x-1+\tfrac{1}{x}\right), E_\omega\!\left(x-2+\tfrac{1}{x}\right), \ldots \in \mathfrak{M}_\omega \subseteq \mathfrak{M}$$

L_ω on $\mathfrak{M}_\omega$ $\rightsquigarrow$ Definition of $E_\omega(L_\omega \mathfrak{a} + \delta)$ for any $\delta \prec \frac{1}{L_n \mathfrak{a}}$ for some $n \in \mathbb{N}$

Hyperserial fields

$\mathbb{H} = \mathbb{R}[[\mathfrak{M}]]$ with

HF1. $\circ \colon \mathbb{L} \times \mathbb{H}^{>,\succ} \longrightarrow \mathbb{H}$

HF2. Taylor expansions

HF3. $\operatorname{supp} \ell_{\omega^\mu} \circ \mathfrak{a} \succ \dfrac{1}{\ell_\gamma \circ \mathfrak{a}}$ for all $\mu \geqslant 1$, $\gamma < \omega^\mu$, and $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$

HF*. …

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HF*. ...

Theorem

$\exists$ *closure of* $\mathbb{H}$ *under all hyperexponentials* E_α *with* $\alpha \in \mathbf{On}$.

Part V — The surreals as a hyperserial field

Surreal substructures

Definition

$\mathbf{S} \subseteq \mathbf{No}$ *is a **surreal substructure** iff* $(\mathbf{S}, \leqslant_{\mathbf{S}}, \sqsubseteq_{\mathbf{S}}) \cong (\mathbf{No}, \leqslant_{\mathbf{No}}, \sqsubseteq_{\mathbf{No}})$

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Examples

- $\mathbf{No}^{>}$
- $\mathbf{No}^{>,\succ}$
- $\mathbf{Mo}$

Surreal substructures

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Proposition

The isomorphism $\Xi_{\mathbf{S}} \colon \mathbf{No} \longrightarrow \mathbf{S}$ *is unique and given by*

$$\Xi_{\mathbf{S}}\, x \;=\; \Xi_{\mathbf{S}}\, \{x_L \,|\, x_R\} \;=\; \{\Xi_{\mathbf{S}}\, x_L \,|\, \Xi_{\mathbf{S}}\, x_R\}_{\mathbf{S}}$$

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$$\Xi_{\mathbf{S}}\, x \;=\; \Xi_{\mathbf{S}}\, \{x_L \mid x_R\} \;=\; \{\Xi_{\mathbf{S}}\, x_L \mid \Xi_{\mathbf{S}}\, x_R\}_{\mathbf{S}}$$

Examples

- $\mathbf{No}^{>}$ $\quad \Xi_{\mathbf{No}^{>}}\, x = 1 \dotplus x$
- $\mathbf{No}^{>,>}$ $\quad \Xi_{\mathbf{No}^{>,>}}\, x = \omega \dotplus x$
- $\mathbf{Mo}$ $\quad \Xi_{\mathbf{Mo}}\, x = \omega^{x}$

Simplest elements in halos under group actions

Theorem

Let $\mathcal{G} \times \mathbf{S} \longrightarrow \mathbf{S}$ *be a function group action on a surreal substructure* $\mathbf{S}$*.*
For any $x \in \mathbf{S}$*, the halo* $\mathcal{G}[x] := \mathbf{Hull}_{\mathbf{S}} \mathcal{G}x$ *admits a simplest element* $\pi_{\mathcal{G}}(x)$*.*
The class $\mathbf{Smp}_{\mathcal{G}} := \operatorname{im} \pi_{\mathcal{G}}$ *forms a surreal substructure.*

$\mathbf{Mo} \subseteq \mathbf{No}^{>} \quad x \longmapsto c\,x, \qquad c \in \mathbb{R}^{>}$

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$\mathbf{Mo} \subseteq \mathbf{No}^{>} \quad x \longmapsto c\, x, \qquad c \in \mathbb{R}^{>}$

$\mathbf{K} \subseteq \mathbf{No}^{>,>} \quad x \longmapsto \exp_n x, \qquad n \in \mathbb{Z}$

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$\mathbf{La} \subseteq \mathbf{No}^{>,>} \quad x \longmapsto \exp_n(c \log_n x), \qquad c \in \mathbb{R}^{>}, \quad n \in \mathbb{N}$

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$\mathbf{Ne} \subseteq \mathbf{Ad} \quad \sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega + \mathrm{e}^{\sqrt{\log \log \omega} + x}}} \longmapsto \sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega + \mathrm{e}^{\sqrt{\log \log \omega} + cx}}}, \quad c \in \mathbb{R}^{>}, \quad \text{height } n$

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Note: hyperserial solutions of $f(x) = \sqrt{x} + \mathrm{e}^{f(\log x)}$ parameterized by *surreal* constant

Defining E_ω on $\mathbf{No}^{\succ,>}$

Step 1: definition of $E_\omega\colon \mathbf{No}^{\gtrsim}_{\succ} \longrightarrow \mathbf{K}$, with $\mathbf{No}^{\gtrsim}_{\succ} := \mathbf{Smp}_{\mathcal{T}}$ and $x \overset{\mathcal{T}}{\longmapsto} x + \mathbb{R}$ on $\mathbf{No}^{\succ,>}$

$$E_\omega x \ := \ \left\{ E_{\mathbb{N}}\, x, E_{\mathbb{N}}\, E_\omega\, x_L^{\mathbf{No}^{\gtrsim}_{\succ}} \middle| L_{\mathbb{N}}\, E_\omega\, x_R^{\mathbf{No}^{\gtrsim}_{\succ}} \right\} \ \in \ \mathbf{K}$$

Defining E_ω on $\mathbf{No}^{\succ,>}$

Step 1: definition of $E_\omega \colon \mathbf{No}_{\succ}^{\gtrsim} \longrightarrow \mathbf{K}$, with $\mathbf{No}_{\succ}^{\gtrsim} := \mathbf{Smp}_{\mathcal{T}}$ and $x \overset{\mathcal{T}}{\longmapsto} x + \mathbb{R}$ on $\mathbf{No}^{\succ,>}$

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Step 2: define **Tr** as the surreal substructure of simplest elements of halos

$$\mathbf{\Pi}[a] \ := \ \left\{ b \in \mathbf{No}^{\succ,>} \ : \ \exists n \in \mathbb{N}, a - b \prec \frac{1}{L_{\mathbb{N}}\, E_\omega\, a_{\succ}} \right\}, \qquad a \ \in \ \mathbf{No}^{\succ,>}$$

Example: $\omega + \frac{1}{\omega} \in \mathbf{Tr} \setminus \mathbf{No}_{\succ}^{\gtrsim}$

Defining E_ω on $\mathbf{No}^{\succ,>}$

Step 1: definition of $E_\omega \colon \mathbf{No}^{\gtrless}_{\succ} \longrightarrow \mathbf{K}$, with $\mathbf{No}^{\gtrless}_{\succ} := \mathbf{Smp}_{\mathcal{T}}$ and $x \overset{\mathcal{T}}{\longmapsto} x + \mathbb{R}$ on $\mathbf{No}^{\succ,>}$

$$E_\omega x \; := \; \{E_{\mathbb{N}}\, x, E_{\mathbb{N}}\, E_\omega\, x_L^{\mathbf{No}^{\gtrless}_{\succ}} | L_{\mathbb{N}}\, E_\omega\, x_R^{\mathbf{No}^{\gtrless}_{\succ}}\} \; \in \; \mathbf{K}$$

Step 2: define **Tr** as the surreal substructure of simplest elements of halos

$$\mathbf{\Pi}[a] \; := \; \left\{ b \in \mathbf{No}^{\succ,>} \; : \; \exists n \in \mathbb{N}, a - b < \frac{1}{L_{\mathbb{N}}\, E_\omega\, a_{\succ}} \right\}, \qquad a \; \in \; \mathbf{No}^{\succ,>}$$

Example: $\omega + \frac{1}{\omega} \in \mathbf{Tr} \setminus \mathbf{No}^{\gtrless}_{\succ}$

Step 3: extend $E_\omega \colon \mathbf{Tr} \longrightarrow \mathbf{La}$

$$E_\omega x \; := \; \{E_{\mathbb{N}}\, x, \mathcal{E} E_\omega\, x_L^{\mathbf{Tr}} | \mathcal{E} E_\omega\, x_L^{\mathbf{Tr}}\} \; \in \; \mathbf{La} \; = \; \mathbf{Smp}_{\mathcal{E}}$$

Defining E_ω on $\mathbf{No}^{\succ,>}$

Step 1: definition of $E_\omega\colon \mathbf{No}^{\gtrless}_{\succ} \longrightarrow \mathbf{K}$, with $\mathbf{No}^{\gtrless}_{\succ} := \mathbf{Smp}_{\mathcal{T}}$ and $x \overset{\mathcal{T}}{\longmapsto} x + \mathbb{R}$ on $\mathbf{No}^{\succ,>}$

$$E_\omega x \;:=\; \{E_{\mathbb{N}}\, x, E_{\mathbb{N}}\, E_\omega\, x_L^{\mathbf{No}^{\gtrless}_{\succ}} \,|\, L_{\mathbb{N}}\, E_\omega\, x_R^{\mathbf{No}^{\gtrless}_{\succ}}\} \;\in\; \mathbf{K}$$

Step 2: define $\mathbf{Tr}$ as the surreal substructure of simplest elements of halos

$$\mathbf{\Pi}[a] \;:=\; \left\{ b \in \mathbf{No}^{\succ,>} \;:\; \exists n \in \mathbb{N}, a - b < \frac{1}{L_{\mathbb{N}}\, E_\omega\, a_{\succ}} \right\}, \qquad a \;\in\; \mathbf{No}^{\succ,>}$$

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$$E_\omega x \;:=\; \{E_{\mathbb{N}}\, x, \mathcal{E} E_\omega\, x_L^{\mathbf{Tr}} \,|\, \mathcal{E} E_\omega\, x_L^{\mathbf{Tr}}\} \;\in\; \mathbf{La} \;=\; \mathbf{Smp}_{\mathcal{E}}$$

Step 4: extend $E_\omega\colon \mathbf{No}^{\succ,>} \longrightarrow \mathbf{No}^{\succ,>}$ using Taylor expansions

- The so-constructed function $E_\omega\colon \mathbf{No}^{>,>} \longrightarrow \mathbf{No}^{>,>}$ is an increasing bijection
- Similarly, we construct E_α for $\alpha \geqslant \omega^2$ in $\omega^{\mathbf{On}}$

Theorem

$(\mathbf{No}, (E_{\omega^\mu})_{\mu \in \mathbf{On}}, (L_{\omega^\mu})_{\mu \in \mathbf{On}})$ *is a hyperserial field.*

Part VI — Numbers as hyperseries

Well-nestedness

$$\sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} + \mathrm{e}^{\sqrt{\log\log \omega} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}$$

✓

$$\sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} + \mathrm{e}^{\sqrt{\log\log \omega} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}} \cdot \cdot + \log\log\log \omega} + \log\log \omega} + \log \omega$$

✗

Well-nestedness

$$\sqrt{\omega}+e^{\sqrt{\log\omega}+e^{\sqrt{\log\log\omega}+e^{\cdot^{\cdot^{\cdot}}}}} \quad ✓$$

$$\sqrt{\omega}+e^{\sqrt{\log\omega}+e^{\sqrt{\log\log\omega}+e^{\cdot^{\cdot^{\cdot}}}\cdot^{\cdot}\cdot+\log\log\log\omega}+\log\log\omega}+\log\omega \quad ✗$$

Badly nested is bad: the following terms occur in the derivative

$$\frac{1}{\omega} \prec \frac{e^{\sqrt{\log\omega}+\cdots}}{\omega\log\omega} \prec \frac{e^{\sqrt{\log\omega}+\cdots}e^{\sqrt{\log\log\omega}+\cdots}}{\omega\log\omega\log\log\omega} \prec \cdots$$

Well-nestedness

$$\sqrt{\omega}+\mathrm{e}^{\sqrt{\log\omega+\mathrm{e}^{\sqrt{\log\log\omega+\mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}}} \quad ✓$$

$$\sqrt{\omega}+\mathrm{e}^{\sqrt{\log\omega+\mathrm{e}^{\sqrt{\log\log\omega+\mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}\cdot+\log\log\log\omega}+\log\log\omega}+\log\omega \quad ✗$$

Badly nested is bad: the following terms occur in the derivative

$$\frac{1}{\omega} \prec \frac{\mathrm{e}^{\sqrt{\log\omega}+\cdots}}{\omega\log\omega} \prec \frac{\mathrm{e}^{\sqrt{\log\omega}+\cdots}\mathrm{e}^{\sqrt{\log\log\omega}+\cdots}}{\omega\log\omega\log\log\omega} \prec \cdots$$

Can be avoided: solving $f(x)=\sqrt{x}+\mathrm{e}^{f(\log x)}+\log x \overset{\text{reduces}}{\rightsquigarrow}$ solving $g(x)=\sqrt{x}+\mathrm{e}^{g(\log x)}$

Well-nestedness

$$\sqrt{\omega}+\mathrm{e}^{\sqrt{\log\omega+\mathrm{e}^{\sqrt{\log\log\omega+\mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}}}\quad ✓ \qquad \sqrt{\omega}+\mathrm{e}^{\sqrt{\log\omega+\mathrm{e}^{\sqrt{\log\log\omega+\mathrm{e}^{\cdot^{\cdot^{\cdot}}}}+\log\log\log\omega}}+\log\log\omega}+\log\omega \quad ✗$$

Badly nested is bad: the following terms occur in the derivative

$$\frac{1}{\omega}\prec\frac{\mathrm{e}^{\sqrt{\log\omega+\cdots}}}{\omega\log\omega}\prec\frac{\mathrm{e}^{\sqrt{\log\omega+\cdots}}\,\mathrm{e}^{\sqrt{\log\log\omega+\cdots}}}{\omega\log\omega\log\log\omega}\prec\cdots$$

Can be avoided: solving $f(x)=\sqrt{x}+\mathrm{e}^{f(\log x)}+\log x \overset{\text{reduces}}{\rightsquigarrow}$ solving $g(x)=\sqrt{x}+\mathrm{e}^{g(\log x)}$

Theorem

The hyperserial field **No** *is well nested: it contains no badly nested elements.*

Nested numbers

Theorem

Let **Ad** *be the class of numbers* $x \approx \sqrt{\omega} + e^{\sqrt{\log \omega} + e^{\sqrt{\log\log \omega} + e^{\cdot^{\cdot^{\cdot}}}}}$ *that satisfy*

$$\begin{aligned} \sqrt{\omega} &< x < 2\sqrt{\omega} \\ \sqrt{\omega} + e^{\sqrt{\log \omega}} &< x < \sqrt{\omega} + e^{2\sqrt{\log \omega}} \\ &\vdots \end{aligned}$$

Let **Ne** *be the subclass of such numbers* x *for which the following are monomials:*

$$\begin{aligned} \mathfrak{m}_1 &:= x - \sqrt{\omega} = e^{\sqrt{\log \omega} + e^{\sqrt{\log\log \omega} + e^{\cdot^{\cdot^{\cdot}}}}} \\ \mathfrak{m}_2 &:= \log \mathfrak{m}_1 - \sqrt{\log \omega} = e^{\sqrt{\log\log \omega} + e^{\cdot^{\cdot^{\cdot}}}} \\ &\vdots \end{aligned}$$

Then **Ad** *and* **Ne** *are surreal substructures.*

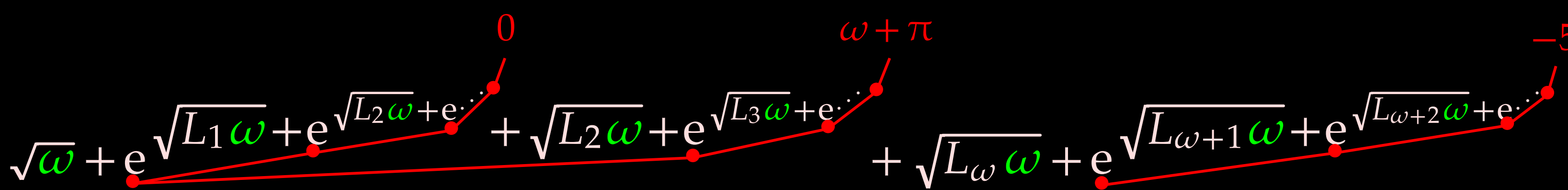
Numbers as hyperseries in ω

(tree) expression

$$\sqrt{\omega} + \mathrm{e}^{\sqrt{L_1 \omega} + \mathrm{e}^{\sqrt{L_2 \omega} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}} + \sqrt{L_2 \omega} + \mathrm{e}^{\sqrt{L_3 \omega} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}} + \sqrt{L_\omega \omega} + \mathrm{e}^{\sqrt{L_{\omega+1} \omega} + \mathrm{e}^{\sqrt{L_{\omega+2} \omega} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}}$$

(tree) description = expression + ranking

$$\sqrt{\omega}+\mathrm{e}^{\sqrt{L_1\omega}+\mathrm{e}^{\sqrt{L_2\omega}+\mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}+\sqrt{L_2\omega}+\mathrm{e}^{\sqrt{L_3\omega}+\mathrm{e}^{\cdot^{\cdot^{\cdot}}}}+\sqrt{L_\omega\,\omega}+\mathrm{e}^{\sqrt{L_{\omega+1}\omega}+\mathrm{e}^{\sqrt{L_{\omega+2}\omega}+\mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}$$



(tree) description = expression + ranking

$$\sqrt{\omega} + e^{\sqrt{L_1 \omega} + e^{\sqrt{L_2 \omega} + e^{\cdot^{\cdot^{\cdot}}}} + \sqrt{L_2 \omega} + e^{\sqrt{L_3 \omega} + e^{\cdot^{\cdot^{\cdot}}}}} + \sqrt{L_\omega \omega} + e^{\sqrt{L_{\omega+1} \omega} + e^{\sqrt{L_{\omega+2} \omega} + e^{\cdot^{\cdot^{\cdot}}}}}$$

Rankings: 0, $\omega + \pi$, -5

Theorem

Any surreal number has a unique hyperserial description in terms of ω.

Part VII — Bonus

Link with *o*-minimality

Conjecture

There exists an o-minimal structure that defines (an) E_ω.

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Question

Does any o-minimal structure with a transexponential function define (an) E_ω?

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***o*-maximus conjecture**

The germs at infinity of any o-minimal structure can be embedded in $(\mathbb{H}, +, \times, \partial, \circ)$.

Ultimate closure

Conjecture

Any functional defined using $+, \times, E_\alpha, L_\alpha, \partial, \circ$ *satisfies the intermediate value property.*

Ultimate closure

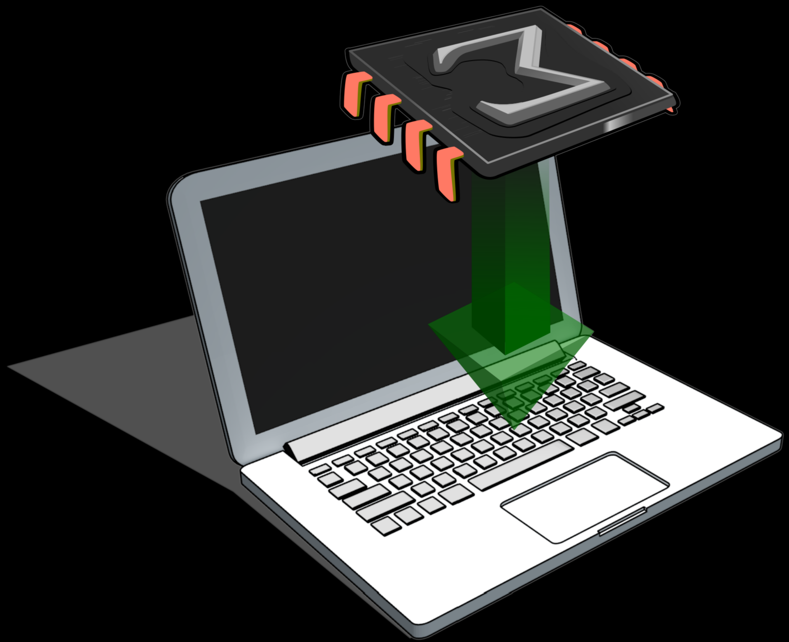
Conjecture

Any functional defined using $+$, $\times$, E_α, L_α, ∂, $\circ$ *satisfies the intermediate value property.*

Question

Let $A(n,k)$ *be the bivariate Ackermann function and take* $E_{\omega^\omega}(n) := A(n,n)$.
Compute the hyperserial expansion of $A(n,n^2)$.

Thank you !



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