Lesson 2 — Hardy fields

Joris van der Hoeven

IMS summer school

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- If K is a Hardy field and $g \in \mathscr{G}$ is such that $g' \circ g^{\mathrm{inv}} \in K$, then $K \circ g$ is a Hardy field.
- If K is a Hardy field, then its real closure K^{rc} is a Hardy field.
  Indeed, if $P(y) = 0$ for $P \in K[Y]$ and $y \in K^{\mathrm{rc}}$, then $y' = -\frac{\partial P}{\partial X}(y) / \frac{\partial P}{\partial Y}(y) \in K(y) \subseteq K^{\mathrm{rc}}$

Variants

Remark. If $K \subseteq \mathcal{G}^1$ is a Hardy field, then actually $K \subseteq \mathcal{G}^{<\infty} := \bigcap_{k \in \mathbb{N}} \mathcal{G}^k$.

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Remark. $y \in \mathcal{G}^{<\infty}$ is Hardian $\iff$

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Remark. Let $y = \frac{1}{x} + \frac{1}{e^x} + \frac{1}{e^{e^x}} + \cdots$.

Then $\mathbb{R}(y, y', \ldots)$ is a $\mathscr{C}^\infty$-Hardy field, but not a $\mathscr{C}^\omega$-Hardy field.

Initial value problems

Theorem (Cauchy, Lipschitz, Picard, Lindelöf, ...)

Let $U \subseteq \mathbb{R}^n$ *and open set, and* $f\colon U \to \mathbb{R}^n$ *a* $\mathscr{C}^1$ *function. Then the differential equation*

$$y'(x) \;=\; f(y(x))$$

with initial condition $y(0) = y_0 \in U$ *has a unique solution* $y\colon [-\varepsilon, \varepsilon] \to U$ *for some* $\varepsilon > 0$.

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Proof. Given $\varepsilon > 0$, let $\mathscr{F}$ be the Banach space of $\mathscr{C}^0$ functions $[-\varepsilon, \varepsilon] \to \mathbb{R}^n$.

Given $\delta > 0$, let $\mathscr{B} \subseteq \mathscr{F}$ be the ball with center y_0 and radius δ.

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Taking δ and ε sufficiently small, we have a contracting functional

$$\begin{aligned} \Phi\colon \mathscr{B} &\longrightarrow \mathscr{B} \\ y &\longmapsto y_0 + \int_0^x f(y(t))\,\mathrm{d}t. \end{aligned}$$

Its unique fixed point is the desired solution. □

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Indeed,

$$\begin{aligned} \|\Phi(y_2) - \Phi(y_1)\| &= \left\| \int_0^x (f(y_2(t)) - f(y_1(t)))\,\mathrm{d}t \right\| \\ &\leqslant \int_0^x \|f(y_2(t)) - f(y_1(t))\|\,\mathrm{d}t \\ &\leqslant \varepsilon\,\|f \circ y_2 - f \circ y_1\| \\ &\leqslant \varepsilon\,\|J_f\|_{\mathscr{B}}\,\|y_2 - y_1\|. \end{aligned}$$

Take δ, ε with $\varepsilon\,\|J_f\|_{\mathscr{B}} < 1$. [...] □

First order closure

Theorem (Hardy, Bourbaki, Singer, Rosenlicht)

Let K *be a Hardy field and let* $f \in K(Y)^{\neq 0}$*. Let* $y \in \mathscr{G}^1$ *be a solution of*

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Then $K(y)$ *is a Hardy field.*

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Then $P(y(x)) = (y(x) - g(x))^2 + h(x) > 0$, eventually.

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So $y(x_1) = y(x_2) = 0$, y has constant sign on (x_1, x_2), but $\operatorname{sign} y'(x_1) = \operatorname{sign} y'(x_2)$.

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Contradiction. $\square$

Liouville closure

Corollary

Let K be a Hardy field and let $\varphi \in K$. Then

- $K(\int \varphi)$ *is a Hardy field.*
- $K(\mathrm{e}^{\varphi})$ *is a Hardy field.*
- $K(\log \varphi)$ *is a Hardy field, whenever* $\varphi > 0$.

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Given a Hardy field K, *the smallest real closed field* $K^{\mathrm{lc}} \subseteq \mathcal{G}^{<\infty}$ *which contains* K *and which is closed under* $\int$ *and* $\exp$ *is a Hardy field, called the* ***Liouville closure*** *of* K.

Exp-log functions

Definition

*An **exp-log function** (or **L-function**) is any function constructed from the real numbers and an indeterminate x, using the field operations, exponention, and the logarithm.*

Corollary

Let $\mathscr{E} \subseteq \mathscr{G}^{<\infty}$ be the set of germs of exp-log functions that are eventually defined.

Then $\mathscr{E}$ is a Hardy field.

Maximal and perfect Hardy fields

Definition

A Hardy field K *is* ***maximal*** *if there is no Hardy field* L *with* $L \supsetneq K$*. We define*

$$\mathrm{E}(K) := \bigcap_{L \supseteq K, L \text{ is maximal}} L.$$

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Questions

- *First order axiomatization of the theory of maximal Hardy fields?*
- *First order characterization of perfect hulls?*

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Theorem (Boshernitzan)

Any maximal Hardy field contains exactly one solution of $(\star)$.

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