Lesson 3 — Generalized power series

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IMS summer school

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Quasi-orderings

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Trivial examples

- The finest ordering on any set E: the equality on E.
- The coarsest ordering on any set E: the relation $\approx$ with $a \approx b$ for all $a, b \in E$.

Operations on quasi-orderings

Let $(E, \leqslant_E)$ and $(F, \leqslant_F)$ be quasi-orderings.

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Disjoint union. $(E \sqcup F, \leqslant_{E \sqcup F})$

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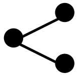
Anti-lexicographical product. $(E \dot{\times} F, \leqslant_{E \dot{\times} F})$

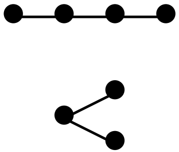
$$(a, a') \ \leqslant_{E \dot{\times} F} \ (b, b') \ :\Longleftrightarrow \ a' <_F b' \ \vee \ (a' \equiv_F b' \wedge a \leqslant_E b)$$

Operations on quasi-orderings — example

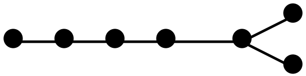
E



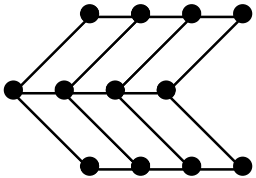
F



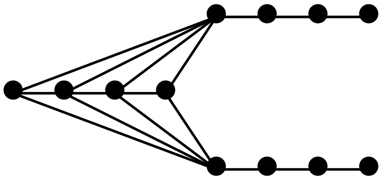
$E \sqcup F$



$E \dot{\sqcup} F$



$E \times F$



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*A total well-founded ordering is called a **well-ordering**.*

- A well-ordering is isomorphic to a unique **ordinal number**
  $0, 1, 2, \ldots, \omega, \omega+1, \ldots, \omega\cdot 2, \omega\cdot 2+1, \ldots, \omega\cdot 3, \ldots, \omega^2, \ldots, \omega^3, \ldots, \omega^\omega, \ldots$

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  $0,1,2,\ldots,\omega,\omega+1,\ldots,\omega\cdot 2,\omega\cdot 2+1,\ldots,\omega\cdot 3,\ldots,\omega^2,\ldots,\omega^3,\ldots,\omega^\omega,\ldots$
- The operations $\dot{\sqcup}$ and $\dot{\times}$ correspond to ordinal addition and multiplication.

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*A **well-quasi-ordering** is a well-founded quasi-ordering with no infinite anti-chains.*

Theorem

Let $(E, \leqslant)$ *be a quasi-ordering. The following conditions are equivalent:*

a) $(E, \leqslant)$ *is a well-quasi-ordering.*

b) Any final segment of E *is finitely generated.*

c) The final segments of E *satisfy the ascending chain condition.*

d) Each sequence $x_1, x_2, \ldots \in E$ *has an increasing subsequence.*

e) Any extension of $\leqslant$ *into a total quasi-ordering on* E *is well-founded.*

Well-quasi-orderings — continued

Proof $(a) \Rightarrow (b)$

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Repeating this argument yields a sequence $x_1 > x_2 > x_3 > \cdots$ in E: contradiction

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Let $H \subseteq G$ be such that H contains exactly one element in each class of $G/\equiv$.

Then $|H| = |G/\equiv|$ and H also generates F.

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Proof $(b) \Rightarrow (d)$

$b)$ *Any final segment of* E *is finitely generated.*

$d)$ *Each sequence* $x_1, x_2, \ldots \in E$ *has an increasing subsequence.*

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By induction on k, let us construct $x_{i_1} \leqslant x_{i_2} \leqslant \cdots \leqslant x_{i_k}$ such that the set $J_k := \{j > i_k : x_{i_k} \leqslant x_j\}$ is infinite.

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Take $i_{k+1} := x_{j_n}$, such that $\{j' \in J_k : x_{j_n} \leqslant x_{j'}\}$ is infinite.

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Corollary

Let $(E, \leqslant_E)$ *and* $(F, \leqslant_F)$ *be well-quasi-orderings.*

Then so are $E \sqcup F$, $E \dot{\sqcup} F$, $E \times F$, *and* $E \dot{\times} F$.

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Corollary (Dickson's lemma)

The set $\mathbb{N}^n = \mathbb{N} \times \overset{n\times}{\cdots} \times \mathbb{N}$ *is well-quasi-ordered.*

Higman's theorem

$(E, \leqslant)$: a quasi-ordering

E^{w}: the set of **words** $x_1 * \cdots * x_n$ with $x_1, \ldots, x_n \in E$

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$$H * O * T \ \leqslant\ T * E * M * P * E * R * A * T * E$$

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Theorem (Higman)

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Proof. (Following Nash-Williams.)

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If $w_k \leqslant w_l$, then $w_1, w_2, \ldots$ is a good sequence $\Longrightarrow$ contradiction.

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If $w_k \leqslant w'_{i_l}$, then $w_k \leqslant w'_{i_l} \leqslant x_{i_l} * w'_{i_l} = w_{i_l} \implies$ contradiction.

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Consider $w_1, \dots, w_{i_1 - 1}, w'_{i_1}, w'_{i_2}, \dots$
By the minimality of $w_1, w_2, \dots$, this sequence is good.
Contradiction. □

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R	the coefficient ring
$\mathfrak{M}$	the set (often a monoid or group) of monomials, quasi-ordered by $\preccurlyeq$

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*A **well-based series** in $\mathfrak{M}$ over R is a map $f : \mathfrak{M} \longrightarrow R$ such that*

$$\operatorname{supp} f \;:=\; \{\mathfrak{m} \in \mathfrak{M} : f(\mathfrak{m}) \neq 0\}$$

is well-based. We denote by $R[[\mathfrak{M}]]$ the set of all such series. Given $f \in R[[\mathfrak{M}]]$, we will also write $f_{\mathfrak{m}} := f(\mathfrak{m})$ for all $\mathfrak{m} \in \mathfrak{M}$ and

$$f \;=\; \sum_{\mathfrak{m}} f_{\mathfrak{m}} \mathfrak{m}.$$

Examples

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Infinite summation

Definition

A family $(f_i)_{i \in I} \in R[[\mathfrak{M}]]^I$ *is* ***well-based*** (*or* ***summable***) *if*

- $\bigcup_{i \in I} \operatorname{supp} f_i$ *is well-based.*
- $I_{\mathfrak{m}} := \{i \in I : f_{i,\mathfrak{m}} := (f_i)_{\mathfrak{m}} \neq 0\}$ *is finite, for every* $\mathfrak{m} \in \mathfrak{M}$.

In that case, we define $\sum_{i \in I} f_i \in R[[\mathfrak{M}]]$ *by*

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Basic arithmetic

Addition and subtraction. For $f, g \in R[[\mathfrak{M}]]$,

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Multiplication. Assuming that $\mathfrak{M}$ is a quasi-ordered monoid,

$$f g \ := \sum_{(\mathfrak{m}, \mathfrak{n}) \in \operatorname{supp} f \times \operatorname{supp} g} f_{\mathfrak{m}}\, g_{\mathfrak{n}}\, \mathfrak{m}\, \mathfrak{n}.$$

$\mathfrak{S} := \operatorname{supp} f \times \operatorname{supp} g, \quad c_{(\mathfrak{m}, \mathfrak{n})} := f_{\mathfrak{m}}\, g_{\mathfrak{n}}, \quad \varphi(\mathfrak{m}, \mathfrak{n}) = \mathfrak{m}\, \mathfrak{n}$

Ring structure

Proposition

Assume that R *is a ring and* $\mathfrak{M}$ *an ordered monoid. Then* $R[[\mathfrak{M}]]$ *is a ring.*

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Associativity of multiplication. Let $f, g, h \in R[[\mathfrak{M}]]$ and $\mathfrak{u} \in \mathfrak{M}$. Then

$$((fg)h)_{\mathfrak{u}} = \sum_{\substack{(\mathfrak{v},\mathfrak{w}) \in \operatorname{supp}(fg) \times \operatorname{supp} h \\ \mathfrak{v}\mathfrak{w} = \mathfrak{u}}} (fg)_{\mathfrak{v}} h_{\mathfrak{w}}$$

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For any $(\mathfrak{m},\mathfrak{n})\in\operatorname{supp}\times\operatorname{supp}g$, there exists a unique $\mathfrak{v}\in\operatorname{supp}f\operatorname{supp}g$ with $\mathfrak{v}=\mathfrak{m}\mathfrak{n}$

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&= \sum_{\substack{(\mathfrak{v},\mathfrak{w})\in (\operatorname{supp} f\operatorname{supp} g)\times\operatorname{supp} h\\ \mathfrak{v}\mathfrak{w}=\mathfrak{u}\\ (\mathfrak{m},\mathfrak{n})\in\operatorname{supp} f\times\operatorname{supp} g\\ \mathfrak{m}\mathfrak{n}=\mathfrak{v}}} f_{\mathfrak{m}} g_{\mathfrak{n}} h_{\mathfrak{w}} = \sum_{\substack{(\mathfrak{m},\mathfrak{n},\mathfrak{w})\in \operatorname{supp} f\times\operatorname{supp} g\times\operatorname{supp} h\\ \mathfrak{m}\mathfrak{n}\mathfrak{w}=\mathfrak{u}}} f_{\mathfrak{m}} g_{\mathfrak{n}} h_{\mathfrak{w}}
\end{aligned}
$$

For any $(\mathfrak{m},\mathfrak{n})\in\operatorname{supp}\times\operatorname{supp} g$, there exists a unique $\mathfrak{v}\in\operatorname{supp} f\operatorname{supp} g$ with $\mathfrak{v}=\mathfrak{m}\mathfrak{n}$

Ring structure

Proposition

Assume that R is a ring and $\mathfrak{M}$ an ordered monoid. Then $R[[\mathfrak{M}]]$ is a ring.

Associativity of multiplication. Let $f, g, h \in R[[\mathfrak{M}]]$ and $\mathfrak{u} \in \mathfrak{M}$. Then

$$\begin{aligned}((fg)h)_{\mathfrak{u}} &= \sum_{\substack{(\mathfrak{v},\mathfrak{w}) \in \operatorname{supp}(fg) \times \operatorname{supp} h \\ \mathfrak{v}\mathfrak{w} = \mathfrak{u}}} (fg)_{\mathfrak{v}} h_{\mathfrak{w}} = \sum_{\substack{(\mathfrak{v},\mathfrak{w}) \in (\operatorname{supp} f \operatorname{supp} g) \times \operatorname{supp} h \\ \mathfrak{v}\mathfrak{w} = \mathfrak{u}}} (fg)_{\mathfrak{v}} h_{\mathfrak{w}} \\ &= \sum_{\substack{(\mathfrak{v},\mathfrak{w}) \in (\operatorname{supp} f \operatorname{supp} g) \times \operatorname{supp} h \\ \mathfrak{v}\mathfrak{w} = \mathfrak{u} \\ (\mathfrak{m},\mathfrak{n}) \in \operatorname{supp} f \times \operatorname{supp} g \\ \mathfrak{m}\mathfrak{n} = \mathfrak{v}}} f_{\mathfrak{m}} g_{\mathfrak{n}} h_{\mathfrak{w}} = \sum_{\substack{(\mathfrak{m},\mathfrak{n},\mathfrak{w}) \in \operatorname{supp} f \times \operatorname{supp} g \times \operatorname{supp} h \\ \mathfrak{m}\mathfrak{n}\mathfrak{w} = \mathfrak{u}}} f_{\mathfrak{m}} g_{\mathfrak{n}} h_{\mathfrak{w}}\end{aligned}$$

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&= \cdots = (f(gh))_{\mathfrak{u}},\end{aligned}$$

using a similar computation.

Conceptually simpler proof

Fully expand and recombine in different ways.

$$\sum_{(\mathfrak{m},\mathfrak{n},\mathfrak{u})\in\operatorname{supp} f\times\operatorname{supp} g\times\operatorname{supp} h} f_{\mathfrak{m}} g_{\mathfrak{n}} h_{\mathfrak{u}}\,\mathfrak{m}\mathfrak{n}\mathfrak{u} = \left(\sum_{(\mathfrak{m},\mathfrak{n})\in\operatorname{supp} f\times\operatorname{supp} g} f_{\mathfrak{m}} g_{\mathfrak{n}}\,\mathfrak{m}\mathfrak{n}\right)\left(\sum_{\mathfrak{u}\in\operatorname{supp} h} h_{\mathfrak{u}}\,\mathfrak{u}\right) = (fg)h$$

$$= \left(\sum_{\mathfrak{m}\in\operatorname{supp} f} f_{\mathfrak{m}}\,\mathfrak{m}\right)\left(\sum_{(\mathfrak{n},\mathfrak{u})\in\operatorname{supp} g\times\operatorname{supp} h} g_{\mathfrak{n}} h_{\mathfrak{u}}\,\mathfrak{n}\mathfrak{u}\right) = f(gh)$$

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Strong distributivity.

If $(f_i)_{i\in I}$ and $(g_j)_{j\in J}$ are summable, then so is $(f_i g_j)_{(i,j)\in I\times J}$ and

$$\sum_{(i,j)\in I\times J} f_i g_j = \left(\sum_{i\in I} f_i\right)\left(\sum_{j\in J} g_j\right)$$

Properties of strong summation

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Strong associativity. Let $(f_i)_{i\in I}$ be summable and $I = \bigsqcup_{j\in J} I_j$.

Then $(f_i)_{i\in I_j}$ is summable for each j, the family $\left(\sum_{i\in I_j} f_i\right)_{j\in J}$ is summable, and

$$\sum_{i\in I} f_i = \sum_{j\in J} \sum_{i\in I_j} f_i.$$

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Termification. If $(f_i)_{i\in I}$ is summable, then so is $(f_{i,\mathfrak{m}}\,\mathfrak{m})_{i\in I,\mathfrak{m}\in\operatorname{supp} f_i}$ and

$$\sum_{i\in I} f_i = \sum_{i\in I,\mathfrak{m}\in\operatorname{supp} f_i} f_{i,\mathfrak{m}}\,\mathfrak{m}.$$

Asymptotic relations

Assume that $\mathfrak{M}$ is totally ordered.

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Dominant terms. Given $f \in R[[\mathfrak{M}]]^{\neq 0}$, we define

$\mathfrak{d}_f := \max_{\preccurlyeq} \operatorname{supp} f$	**dominant monomial** of f
$c_f := f_{\mathfrak{d}_f}$	**dominant coefficient** of f
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Ordering. If R is an ordered field and $f \in R[[\mathfrak{M}]]^{\neq 0}$, we may then define

$$f > 0 \iff c_f > 0.$$

Then $R[[\mathfrak{M}]]$ is an ordered ring (the ordering being total).

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Convention $\mathfrak{d}_0 \prec \mathfrak{M}$ and $f \sim 0 \Leftrightarrow 0 \sim f \Leftrightarrow f = 0$

Special inversion. Assume $\varepsilon \in R[[\mathfrak{M}]]$ with $\operatorname{supp} \varepsilon \prec 1$

$$\frac{1}{1-\varepsilon} := \sum_{\mathfrak{m}_1 * \cdots * \mathfrak{m}_k \in (\operatorname{supp} \varepsilon)^{\mathrm{w}}} \varepsilon_{\mathfrak{m}_1} \cdots \varepsilon_{\mathfrak{m}_k} \mathfrak{m}_1 \cdots \mathfrak{m}_k.$$

$$\mathfrak{S} := (\operatorname{supp} f)^{\mathrm{w}}, \quad c_{\mathfrak{m}_1 * \cdots * \mathfrak{m}_k} := \varepsilon_{\mathfrak{m}_1} \cdots \varepsilon_{\mathfrak{m}_k}, \quad \varphi(\mathfrak{m}_1 * \cdots * \mathfrak{m}_k) = \mathfrak{m}_1 \cdots \mathfrak{m}_k.$$

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General inversion. Given a field R, a totally ordered group $\mathfrak{M}$, and $f \in C[[\mathfrak{M}]]^{\neq 0}$,

$$f = c_f \mathfrak{d}_f (1-\varepsilon), \qquad \operatorname{supp} \varepsilon \prec 1,$$

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Proposition

Assume that R is a field and $\mathfrak{M}$ a totally ordered group. Then $R[[\mathfrak{M}]]$ is a field.

Division — proof inversion formula

$$\frac{1}{1-\varepsilon} := \sum_{\mathfrak{m}_1 * \cdots * \mathfrak{m}_k \in (\operatorname{supp} \varepsilon)^{\mathrm{w}}} \varepsilon_{\mathfrak{m}_1} \cdots \varepsilon_{\mathfrak{m}_k} \mathfrak{m}_1 \cdots \mathfrak{m}_k$$

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$$(1-\varepsilon) \cdot \frac{1}{1-\varepsilon} = 1$$

Strong linearity

Definition

Let $\mathfrak{M}$, $\mathfrak{N}$ be monomial sets and consider a linear map $\varphi\colon R[[\mathfrak{M}]] \to R[[\mathfrak{N}]]$. We say that φ is ***strongly linear*** *if $(\varphi(f_i))_{i\in I}$ is summable whenever $(f_i)_{i\in I} \in R[[\mathfrak{M}]]^I$ is, and*

$$\varphi\left(\sum_{i\in I} f_i\right) = \sum_{i\in I} \varphi(f_i).$$

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Definition

We say that $\varphi: \mathfrak{M} \to R[[\mathfrak{N}]]$ is ***well-based****, if, for any well-based subset $\mathfrak{S} \subseteq \mathfrak{M}$, the family $(\varphi(\mathfrak{m}))_{\mathfrak{m} \subseteq \mathfrak{S}}$ is well-based.*

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Theorem (extension by strong linearity)

Consider a well-based map $\varphi: \mathfrak{M} \rightarrow R[[\mathfrak{N}]]$. Then there exists a unique strongly linear map $\hat{\varphi}: R[[\mathfrak{M}]] \rightarrow R[[\mathfrak{N}]]$ that extends φ.

Extension by strong linearity — proof

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Uniqueness. Let $f \in R[[\mathfrak{M}]]$. Then $(\varphi(\mathfrak{m}))_{\mathfrak{m} \in \operatorname{supp} f}$ and $(f_{\mathfrak{m}} \varphi(\mathfrak{m}))_{\mathfrak{m} \in \operatorname{supp} f}$ are summable. Given a strongly linear extension $\hat{\varphi}$ of φ, we must have

$$\hat{\varphi}(f) = \hat{\varphi}\left(\sum_{\mathfrak{m} \in \operatorname{supp} f} f_{\mathfrak{m}} \mathfrak{m} \right) = \sum_{\mathfrak{m} \in \operatorname{supp} f} \hat{\varphi}(f_{\mathfrak{m}} \mathfrak{m}) = \sum_{\mathfrak{m} \in \operatorname{supp} f} f_{\mathfrak{m}} \hat{\varphi}(\mathfrak{m}) = \sum_{\mathfrak{m} \in \operatorname{supp} f} f_{\mathfrak{m}} \varphi(\mathfrak{m}).$$

Extension by strong linearity — proof

Existence. Given $f \in R[[\mathfrak{M}]]$,

$$\hat{\varphi}(f) := \sum_{\mathfrak{m} \in \operatorname{supp} f} f_{\mathfrak{m}} \varphi(\mathfrak{m})$$

is well-defined, $\hat{\varphi}$ clearly extends φ and $\hat{\varphi}(c f) = c \hat{\varphi}(f)$ for any $c \in R$.

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Claim: $(f_{i,\mathfrak{m}} \varphi(\mathfrak{m}))_{(i,\mathfrak{m}) \in I \times \mathfrak{S}}$ is summable, where $\mathfrak{S} := \bigcup_{i \in I} \operatorname{supp} f_i$.

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The set $\bigcup_{(i,\mathfrak{m}) \in I \times \mathfrak{S}} \operatorname{supp} f_{i,\mathfrak{m}} \varphi(\mathfrak{m}) \subseteq \bigcup_{\mathfrak{m} \in \mathfrak{S}} \operatorname{supp} \varphi(\mathfrak{m})$ is well-based.

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Given $\mathfrak{n} \in \mathfrak{N}$, the set $\mathfrak{S}_{\mathfrak{n}} := \{\mathfrak{m} \in \mathfrak{S} : \mathfrak{n} \in \operatorname{supp} \varphi(\mathfrak{m})\}$ is finite.

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For every $\mathfrak{m} \in \mathfrak{S}_{\mathfrak{n}}$, the set $I_{\mathfrak{m},\mathfrak{n}} := \{i \in I : \mathfrak{m} \in \operatorname{supp} f_i\}$ is finite.

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Claim: $(f_{i,\mathfrak{m}} \varphi(\mathfrak{m}))_{(i,\mathfrak{m}) \in I \times \mathfrak{S}}$ is summable, where $\mathfrak{S} := \bigcup_{i \in I} \operatorname{supp} f_i$.

The set $\bigcup_{(i,\mathfrak{m}) \in I \times \mathfrak{S}} \operatorname{supp} f_{i,\mathfrak{m}} \varphi(\mathfrak{m}) \subseteq \bigcup_{\mathfrak{m} \in \mathfrak{S}} \operatorname{supp} \varphi(\mathfrak{m})$ is well-based.

Given $\mathfrak{n} \in \mathfrak{N}$, the set $\mathfrak{S}_{\mathfrak{n}} := \{\mathfrak{m} \in \mathfrak{S} : \mathfrak{n} \in \operatorname{supp} \varphi(\mathfrak{m})\}$ is finite.

For every $\mathfrak{m} \in \mathfrak{S}_{\mathfrak{n}}$, the set $I_{\mathfrak{m},\mathfrak{n}} := \{i \in I : \mathfrak{m} \in \operatorname{supp} f_i\}$ is finite.

Hence $\{(i,\mathfrak{m}) \in I \times \mathfrak{S} : \mathfrak{n} \in \operatorname{supp} f_{i,\mathfrak{m}} \varphi(\mathfrak{m})\}$ is finite.

Extension by strong linearity — proof

Existence. Given $f \in R[[\mathfrak{M}]]$, let

$$\hat{\varphi}(f) := \sum_{\mathfrak{m} \in \operatorname{supp} f} f_{\mathfrak{m}} \varphi(\mathfrak{m}).$$

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□

Application to power series substitution

$\mathfrak{M}$ monomial monoid, $\varepsilon \in R[[\mathfrak{M}]]$, $\operatorname{supp} \varepsilon \prec 1$.

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$(\varepsilon^n)_{n \in \mathbb{N}}$ is summable, since $(\varepsilon_{\mathfrak{m}_1} \cdots \varepsilon_{\mathfrak{m}_n} \mathfrak{m}_1 \cdots \mathfrak{m}_n)_{\mathfrak{m}_1 * \cdots * \mathfrak{m}_n \in (\operatorname{supp} \varepsilon)^{\mathrm{w}}}$ is summable.

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a) For any $f, g \in R[[z]]$ with $g \prec 1$, we have $\operatorname{supp} g \circ \varepsilon \prec 1$ and $f \circ (g \circ \varepsilon) = (f \circ g) \circ \varepsilon$.

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Corollary

We have $(1-\varepsilon) \cdot \frac{1}{1-\varepsilon} = 1$.

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Proof. Follows from (b), since $(1 - z)\frac{1}{1 - z} = 1$ in $R[[z]]$. □

Associativity of composition

Proposition

et $\psi\colon \mathfrak{M} \to R[[\mathfrak{N}]]$ *and* $\varphi\colon \mathfrak{N} \to R[[\mathfrak{V}]]$ *be two well-based mappings. Then*

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Proof. The map $\hat{\varphi} \circ \psi: \mathfrak{M} \rightarrow R[[\mathfrak{V}]]$ is well-based.

The map $\hat{\varphi} \circ \hat{\psi}$ is the unique stongly linear map that extends $\hat{\varphi} \circ \psi$. □

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Composition as a ring homomorphism

Proposition

If $\varphi: \mathfrak{M} \rightarrow R[[\mathfrak{N}]]$ is a multiplicative well-based mapping, then $\hat{\varphi}$ is a ring homomorphism.

Claim 1: for any $\mathfrak{m} \in \mathfrak{M}$ and $g \in R[[\mathfrak{M}]]$, we have $\hat{\varphi}(\mathfrak{m}\, g) = \varphi(\mathfrak{m})\, \hat{\varphi}(g)$.

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Proof. The mapping $\chi_g\colon \mathfrak{m} \mapsto \hat{\varphi}(\mathfrak{m}\, g) = \varphi(\mathfrak{m})\, \hat{\varphi}(g)$ is well-based.

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$$\begin{aligned} \hat{\chi}_g(f) &= \sum_{\mathfrak{m} \in \operatorname{supp} f} f_{\mathfrak{m}}\, \hat{\varphi}(\mathfrak{m}\, g) = \hat{\varphi}\left(\sum_{\mathfrak{m} \in \operatorname{supp} f} f_{\mathfrak{m}}\, \mathfrak{m}\, g \right) = \hat{\varphi}(f g) \\ &= \sum_{\mathfrak{m} \in \operatorname{supp} f} f_{\mathfrak{m}}\, \varphi(\mathfrak{m})\, \hat{\varphi}(g) = \hat{\varphi}(g) \sum_{\mathfrak{m} \in \operatorname{supp} f} f_{\mathfrak{m}}\, \varphi(\mathfrak{m}) = \hat{\varphi}(f)\, \hat{\varphi}(g). \end{aligned}$$

□

Composition as a ring homomorphism

Proposition

If $\varphi: \mathfrak{M} \to R[[\mathfrak{N}]]$ is a multiplicative well-based mapping, then $\hat{\varphi}$ is a ring homomorphism.

Claim 2: for any $f, g \in R[[\mathfrak{M}]]$, we have $\hat{\varphi}(fg) = \hat{\varphi}(f)\,\hat{\varphi}(g)$.

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$$\begin{aligned}\hat{\chi}_g(f) &= \sum_{\mathfrak{m} \in \operatorname{supp} f} f_{\mathfrak{m}}\,\hat{\varphi}(\mathfrak{m}\,g) = \hat{\varphi}\left(\sum_{\mathfrak{m} \in \operatorname{supp} f} f_{\mathfrak{m}}\,\mathfrak{m}\,g\right) = \hat{\varphi}(fg) \\ &= \sum_{\mathfrak{m} \in \operatorname{supp} f} f_{\mathfrak{m}}\,\varphi(\mathfrak{m})\,\hat{\varphi}(g) = \hat{\varphi}(g) \sum_{\mathfrak{m} \in \operatorname{supp} f} f_{\mathfrak{m}}\,\varphi(\mathfrak{m}) = \hat{\varphi}(f)\,\hat{\varphi}(g).\end{aligned}$$

□

Remark. More elegant proof possible using "extension by strong bilinearity".

A multivariate generalization

$\varepsilon_1, \ldots, \varepsilon_n \in R[[\mathfrak{M}]]^{\prec 1} := \{\varepsilon \in R[[\mathfrak{M}]] : \operatorname{supp} \varepsilon \prec 1\}$.

$\varphi: z_1^{\mathbb{N}} \times \cdots \times z_n^{\mathbb{N}} \to R[[\mathfrak{M}]]; (z_1^{k_1}, \ldots, z_n^{k_n}) \mapsto \varepsilon_1^{k_1} \cdots \varepsilon_n^{k_n}$ is well-based.

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$$f \circ (\varepsilon_1, \ldots, \varepsilon_n) \ := \ \hat{\varphi}(f), \qquad \text{for any } f \in R[[z_1, \ldots, z_n]].$$

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Proposition

a) $\hat{\varphi}$ *is a ring homomorphism.*

b) For any $f \in R[[u_1, \ldots, u_k]]$ *and* $g_1, \ldots, g_k \in R[[z_1, \ldots, z_n]]^{\prec 1}$, *we have*

$$f \circ (g_1 \circ (\varepsilon_1, \ldots, \varepsilon_n), \ldots, g_k \circ (\varepsilon_1, \ldots, \varepsilon_n)) \ = \ (f \circ (g_1, \ldots, g_k)) \circ (\varepsilon_1, \ldots, \varepsilon_n).$$

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Corollary

If $R \supseteq \mathbb{Q}$ *and* $\delta, \varepsilon \in R[[\mathfrak{M}]]^{\prec}$*, then* $\mathrm{e}^{\delta+\varepsilon} = \mathrm{e}^{\delta}\mathrm{e}^{\varepsilon}$*, where* $\mathrm{e}^{\delta} := \mathrm{e}^{z} \circ \delta$ *with* $\mathrm{e}^{z} \in R[[z]]$.

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$\varepsilon_1, \ldots, \varepsilon_n \in R[[\mathfrak{M}]]^{\prec 1} := \{\varepsilon \in R[[\mathfrak{M}]] : \operatorname{supp} \varepsilon \prec 1\}$.
$\varphi: z_1^{\mathbb{N}} \times \cdots \times z_n^{\mathbb{N}} \to R[[\mathfrak{M}]]; (z_1^{k_1}, \ldots, z_n^{k_n}) \mapsto \varepsilon_1^{k_1} \cdots \varepsilon_n^{k_n}$ is well-based.

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Corollary

If $R \supseteq \mathbb{Q}$ and $\delta, \varepsilon \in R[[\mathfrak{M}]]^{\prec}$, then $\mathrm{e}^{\delta+\varepsilon} = \mathrm{e}^{\delta}\mathrm{e}^{\varepsilon}$, where $\mathrm{e}^{\delta} := \mathrm{e}^{z} \circ \delta$ with $\mathrm{e}^{z} \in R[[z]]$.

Proof. $\mathrm{e}^{\delta+\varepsilon} = \mathrm{e}^{z_1+z_2} \circ (\delta, \varepsilon) = (\mathrm{e}^{z_1}\mathrm{e}^{z_2}) \circ (\delta, \varepsilon) = \mathrm{e}^{z_1}\mathrm{e}^{z_2}$, using $\mathrm{e}^{z_1+z_2} = \mathrm{e}^{z_1}\mathrm{e}^{z_2}$ in $R[[z_1, z_2]]$.