Lesson 4 — Newton polygon method

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Support types

Definition

*A **support type** for a monomial monoid* $\mathfrak{M}$ *is a subset* $\mathscr{S}(\mathfrak{M}) \subseteq \mathscr{P}(\mathfrak{M})$ *such that*

T1. *Every* $\mathfrak{S} \in \mathscr{S}(\mathfrak{M})$ *is well-based.*

T2. *If* $\mathfrak{m} \in \mathfrak{M}$*, then* $\{\mathfrak{m}\} \in \mathscr{S}(\mathfrak{M})$.

T3. *If* $\mathfrak{S} \in \mathscr{S}(\mathfrak{M})$ *and* $\mathfrak{T} \subseteq \mathfrak{S}$*, then* $\mathfrak{T} \in \mathscr{S}(\mathfrak{M})$.

T4. *If* $\mathfrak{S}, \mathfrak{T} \in \mathscr{S}(\mathfrak{M})$*, then* $\mathfrak{S} \cup \mathfrak{T} \in \mathscr{S}(\mathfrak{M})$.

T5. *If* $\mathfrak{S}, \mathfrak{T} \in \mathscr{S}(\mathfrak{M})$*, then* $\mathfrak{S}\,\mathfrak{T} := \{\mathfrak{m}\,\mathfrak{n} : \mathfrak{m} \in \mathfrak{S}, \mathfrak{n} \in \mathfrak{T}\} \in \mathscr{S}(\mathfrak{M})$.

T6. *If* $\mathfrak{S} \in \mathscr{S}(\mathfrak{M})$ *and* $\mathscr{S} \prec 1$*, then* $\mathfrak{S}^* := \{\mathfrak{m}_1 \cdots \mathfrak{m}_n : \mathfrak{m}_1, \ldots, \mathfrak{m}_n \in \mathfrak{S}\} \in \mathscr{S}(\mathfrak{M})$.

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Let $\mathscr{S}$ *be a map that associates a support type* $\mathscr{S}(\mathfrak{M})$ *for* $\mathfrak{M}$ *to any monomial monoid* $\mathfrak{M}$.
We say that $\mathscr{S}$ *is a* ***support type*** *if:*

ST. *For every strictly increasing morphism* $\varphi : \mathfrak{M} \rightarrow \mathfrak{N}$ *and* $\mathfrak{S} \in \mathscr{S}(\mathfrak{M})$,
we have $\varphi(\mathfrak{S}) \in \mathscr{S}(\mathfrak{N})$.

$\mathscr{S}$-based series

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A series $f \in R[[\mathfrak{M}]]$ *is* ***$\mathscr{S}$-based*** *if* $\operatorname{supp} f \in \mathscr{S}(\mathfrak{M})$.

We denote by $R[[\mathfrak{M}]]_{\mathscr{S}}$ *the set of all such series.*

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Definition

A well-based family $(f_i)_{i \in \mathfrak{M}} \in R[[\mathfrak{M}]]_{\mathscr{S}}$ is ***$\mathscr{S}$-based*** *if* $\bigcup_{i \in I} \operatorname{supp} f_i \in \mathscr{S}(\mathfrak{M})$.

Then $\sum_{i \in I} f_i \in R[[\mathfrak{M}]]_{\mathscr{S}}$. *This defines "the natural" strong summation on* $R[[\mathfrak{M}]]_{\mathscr{S}}$.

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Proposition

a) $R[[\mathfrak{M}]]_{\mathscr{S}}$ *is a ring.*

b) If R *is a field and* $\mathfrak{M}$ *a totally ordered group, then* $R[[\mathfrak{M}]]_{\mathscr{S}}$ *is a field.*

Examples

Well-based supports.

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Finitely generated supports.

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$$\zeta(x) \;=\; 1 + 2^{-x} + 3^{-x} + \cdots \;=\; 1 + \mathrm{e}^{-(\log 2)x} + \mathrm{e}^{-(\log 3)x} + \cdots \;\notin\; \mathbb{R}[[\mathrm{e}^{-\mathbb{R}x}]]_{\mathscr{S}}.$$

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Intersections. Let $\mathscr{S}$ and $\mathscr{T}$ be support types.

$$(\mathscr{S} \cap \mathscr{T})(\mathfrak{M}) = \mathscr{S}(\mathfrak{M}) \cap \mathscr{T}(\mathfrak{M}).$$

Grid-based series

Definition

We say that $\mathfrak{S} \subseteq \mathfrak{M}$ *is* ***grid-based*** *if there exist finite sets* $\mathfrak{F} \subseteq \mathfrak{M}$ *and* $\mathfrak{E} \subseteq \mathfrak{M}^{\prec 1}$ *with*

$$\mathfrak{S} \subseteq \mathfrak{F}\mathfrak{E}^*.$$

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Note. If $\mathfrak{M}$ is a totally ordered group, then $\mathfrak{F}$ can be taken to be a singleton.

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The map $\mathscr{G} : \mathfrak{M} \longmapsto \{\mathfrak{S} \subseteq \mathfrak{M} : \mathfrak{S} \text{ is grid-based}\}$ *is a support type.*

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Lemma

If $\mathfrak{S} \subseteq \mathfrak{M}^{<1}$ is grid-based, then there is a finite $\mathfrak{E} \subseteq \mathfrak{M}^{<1}$ with $\mathfrak{S} \subseteq \mathfrak{E}^*$ (whence $\mathfrak{S}^* \subseteq \mathfrak{E}^*$).

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Proof. Let $\mathfrak{F} \subseteq \mathfrak{M}$ and $\mathfrak{G} \subseteq \mathfrak{M}^{\prec 1}$ be finite with $\mathfrak{S} \subseteq \mathfrak{F}\mathfrak{G}^*$.

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Proof. Let $\mathfrak{F} \subseteq \mathfrak{M}$ and $\mathfrak{G} \subseteq \mathfrak{M}^{\prec 1}$ be finite with $\mathfrak{S} \subseteq \mathfrak{F}\,\mathfrak{G}^*$.

Given $\mathfrak{f} \in \mathfrak{F}$, the set $(\mathfrak{f}\,\mathfrak{G}^*) \cap \mathfrak{M}^{\prec 1}$ is a final segment of $\mathfrak{f}\,\mathfrak{G}^*$ for $\succcurlyeq^!$.

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Let $\mathfrak{H}_\mathfrak{f} \subseteq \mathfrak{M}^{\prec 1}$ be a finite set of generators. Note that $(\mathfrak{f}\mathfrak{G}^*) \cap \mathfrak{M}^{\prec 1} \subseteq \mathfrak{H}_\mathfrak{f}\mathfrak{G}^*$.

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Now it suffices to take $\mathfrak{E} := \mathfrak{G} \cup \bigcup_{\mathfrak{f} \in \mathfrak{F}} \mathfrak{H}_{\mathfrak{f}}$. □

Grid-based series — continued

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Remark. For any other support type $\mathscr{S}$, we have $\mathscr{S}(\mathfrak{M})\supseteq\mathscr{G}(\mathfrak{M})$, for all $\mathfrak{M}$.

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Elements of $R[[\mathfrak{M}]]$ are called **grid-based series**.

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$\mathscr{G}$-based families are called **grid-based families**. Etc.

Cartesian representations

Proposition

For any $f \in R[[\mathfrak{M}]]$*, there exist power series* $\check{f}_1, \ldots, \check{f}_l \in R[[z_1, \ldots, z_k]]$*, monomials* $\mathfrak{f}_1, \ldots, \mathfrak{f}_l \in \mathfrak{M}$ *and* $\mathfrak{e}_1, \ldots, \mathfrak{e}_k \in \mathfrak{M}^{<1}$ *with*

$$f = \sum_{1 \leqslant i \leqslant l} (\check{f}_i \circ (\mathfrak{e}_1, \ldots, \mathfrak{e}_k)) \mathfrak{f}_i.$$

Proposition

Assume that $\mathfrak{M}$ *is a totally ordered group.*

For any $f \in R[[\mathfrak{M}]]$*, there exists a Laurent series* $\check{f} \in R((z_1, \ldots, z_k))$ *and* $\mathfrak{e}_1, \ldots, \mathfrak{e}_k \in \mathfrak{M}^{<1}$ *with*

$$f = \check{f} \circ (\mathfrak{e}_1, \ldots, \mathfrak{e}_k). \qquad (\star)$$

Here $(g z_1^{i_1} \cdots z_k^{i_k}) \circ (\mathfrak{e}_1, \ldots, \mathfrak{e}_k) := (g \circ (\mathfrak{e}_1, \ldots, \mathfrak{e}_k)) \mathfrak{e}_1^{i_1} \cdots \mathfrak{e}_k^{i_k}$ *for any* $g \in R[[z_1, \ldots, z_k]]$*,* $i_1, \ldots, i_k \in \mathbb{Z}$*.*

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For any $f \in R[[\mathfrak{M}]]$, there exist power series $\check{f}_1, \ldots, \check{f}_l \in R[[z_1, \ldots, z_k]]$, monomials $\mathfrak{f}_1, \ldots, \mathfrak{f}_l \in \mathfrak{M}$ and $\mathfrak{e}_1, \ldots, \mathfrak{e}_k \in \mathfrak{M}^{<1}$ with

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*We call $(\star)$ a **Cartesian representation** of f.*

Digression — local communities

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Let $\mathscr{L}$ *be a collection of subsets* $\mathscr{L}_k \subseteq R[[z_1,\ldots,z_k]]$ *for* $k \in \mathbb{N}$*, such that*

L1. $z_i \in \mathscr{L}_k$ *for* $i = 1,\ldots,k$*.*

L2. $\mathscr{L}_k$ *is an* R*-subalgebra of* $R[[z_1,\ldots,z_k]]$*.*

L3. *For any* $f \in \mathscr{L}_k$ *with* $z_1 \mid f$*, we have* $z^{-1} f \in \mathscr{L}_k$*.*

L4. *Given* $f \in \mathscr{L}_k$ *and* $g_1,\ldots,g_k \in \mathscr{L}_l^{<1}$*, we have* $f \circ (g_1,\ldots,g_k) \in \mathscr{L}_l$*.*

L5. *Given* $f \in \mathscr{L}_{k+1}$ *with* $f(0,\ldots,0) = 0$ *and* $(\partial f / \partial z_{k+1})(0,\ldots,0) = 1$*,*
the unique $\varphi \in R[[z_1,\ldots,z_k]]$ *with* $f \circ (z_1,\ldots,z_k,\varphi) = 0$ *is in* $\mathscr{L}_k$*.*

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- $\mathscr{L}_k = K[[z_1, \ldots, z_k]]^{\mathrm{alg}}$, algebraic power series, K any field.

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- $\mathscr{L}_k = K[[z_1,\dots,z_k]]^{\text{alg}}$, algebraic power series, K any field.
- $\mathscr{L}_k = K[[z_1,\dots,z_k]]^{\text{dalg}}$, d-algebraic power series, K any field with $\operatorname{char} K = 0$.

Digression — local communities

$\mathfrak{M}$ totally ordered monomial group
$\mathscr{L}$ local community

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Definition

We define $R[[\mathfrak{M}]]_{\mathscr{L}}$ *to be the set of* $f \in R[[\mathfrak{M}]]$ *with*

$$f = \check{f} \circ (\mathfrak{e}_1, \ldots, \mathfrak{e}_k),$$

for some $\check{f} \in \mathscr{L}_k z_1^{\mathbb{Z}} \cdots z_k^{\mathbb{Z}}$ *and* $\mathfrak{e}_1, \ldots, \mathfrak{e}_k \in \mathfrak{M}^{\prec 1}$.

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Theorem

If K *is a field, then so is* $K[[\mathfrak{M}]]_{\mathscr{L}}$ *is a field. Moreover, if* $\mathfrak{M}$ *has* $\mathbb{Q}$*-powers, then*

a) If K *is algebraically closed and of characteristic zero, then so is* $K[[\mathfrak{M}]]_{\mathscr{L}}$.

b) If K *is real closed, then so is* $K[[\mathfrak{M}]]_{\mathscr{L}}$.

Asymptotic algebraic equations

K algebraically closed field

Γ divisible totally ordered *abelian* group: $(\forall \gamma \in \Gamma)\,(\forall n \in \mathbb{N}^{>0})\,(\exists \alpha \in \Gamma)\, n\,\alpha = \gamma$

z^{Γ} corresponding monomial group, $z^{\alpha} \preccurlyeq z^{\beta} \Leftrightarrow \alpha \geqslant \beta$.

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Given $P \in K[[z^{\Gamma}]][Y] \setminus K[[z^{\Gamma}]]$, compute the solutions in $K[[z^{\Gamma}]]$ of

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We may replace $K[[z^{\Gamma}]]$ by $K[[z^{\Gamma}]]_{\mathscr{P}}$ or $K[\![z^{\Gamma}]\!]_{\mathscr{L}}$.

Starting monomials

$$P_d y^d + \cdots + P_0 \; = \; 0, \qquad (y \prec z^\gamma). \tag{$\star$}$$

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Setting $z^{\pi_k} := \mathfrak{d}_{P_k}$ for $k = 0, \ldots, d$, and $z^{\nu} := \mathfrak{d}_y$, this means that there exist $i \neq j$ with

$$\nu > \gamma, \qquad \pi_i + i\nu = \pi_j + j\nu \leqslant \pi_k + k\nu, \qquad \text{for all } k.$$

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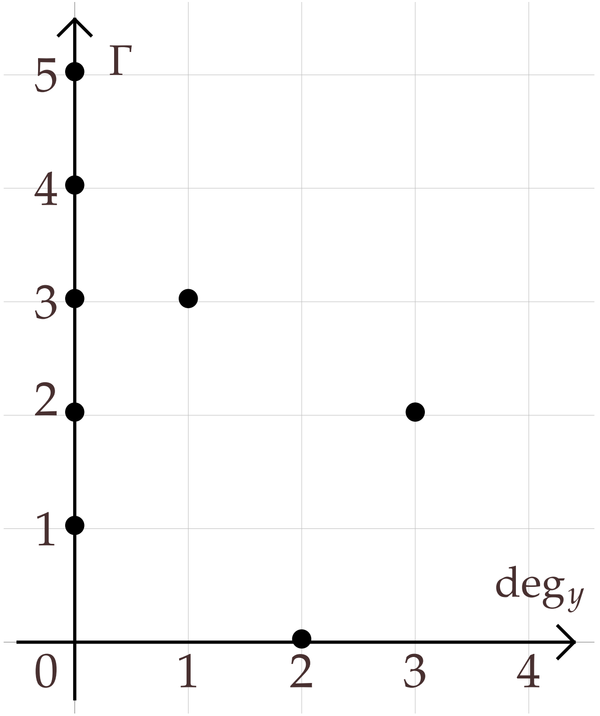
We call z^{ν} a **starting monomial** for the equation $(\star)$.

Newton polygons

$$5z^2y^3 + y^2 + 3z^3y - \frac{z}{1-z} = 0$$



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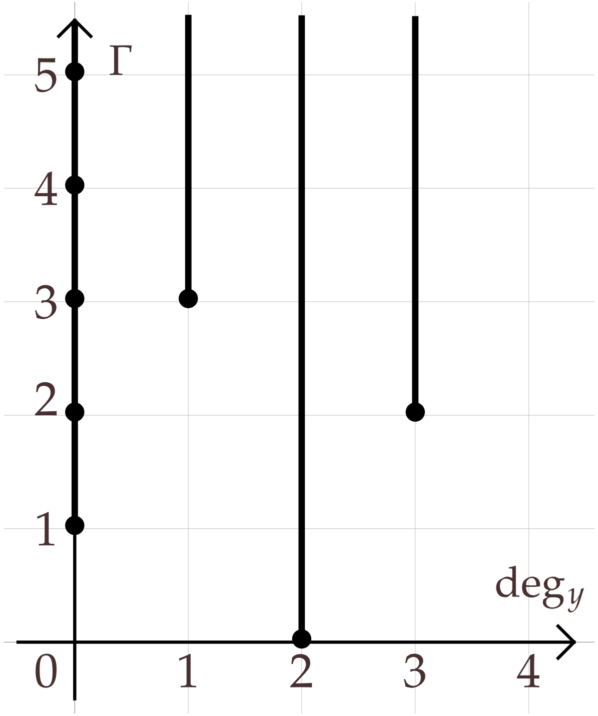

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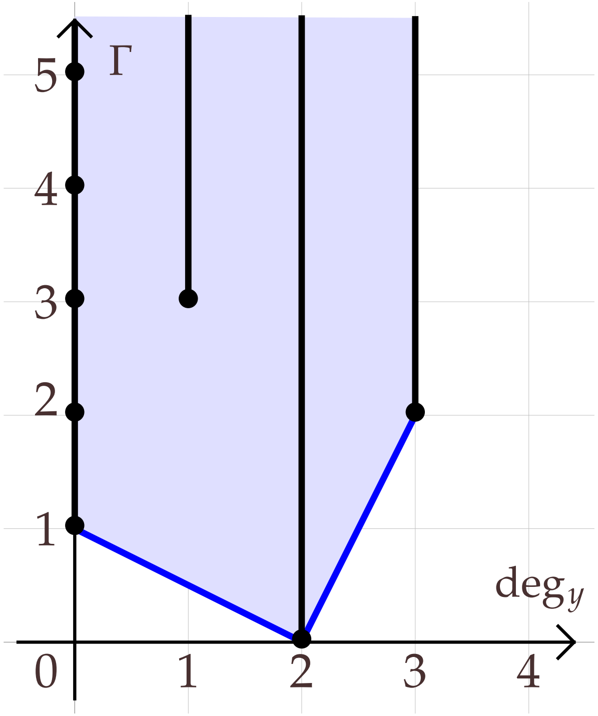

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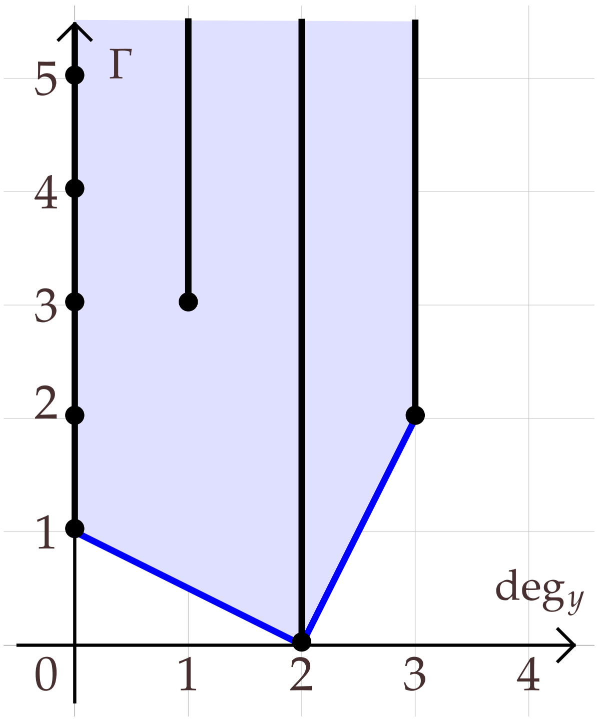

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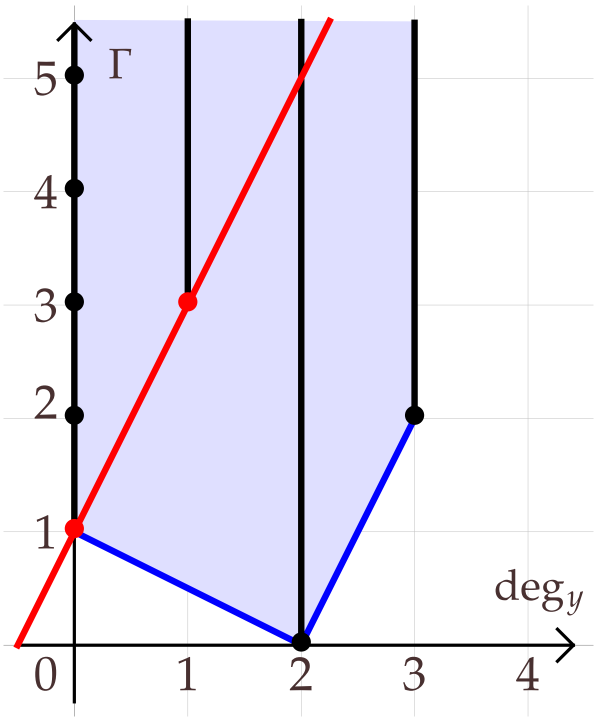

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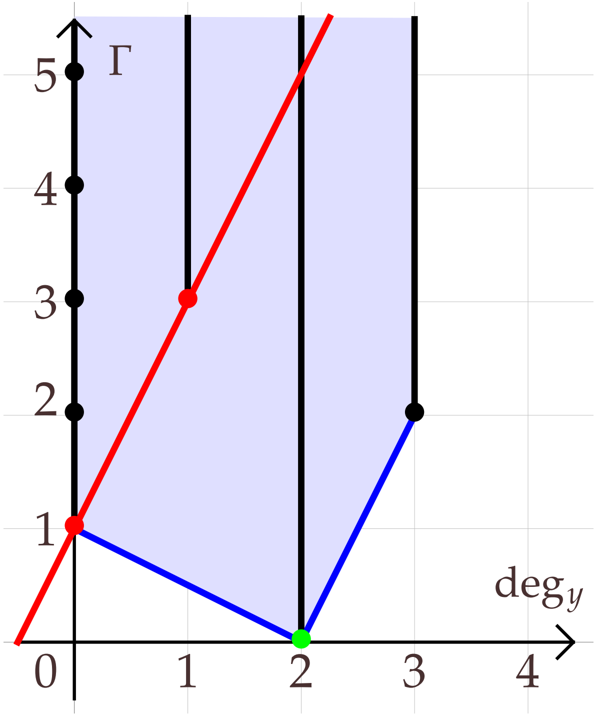
  But then $P_2 y^2 \asymp z^{0+2\nu} = z^{-4} \succ z \asymp P_0$



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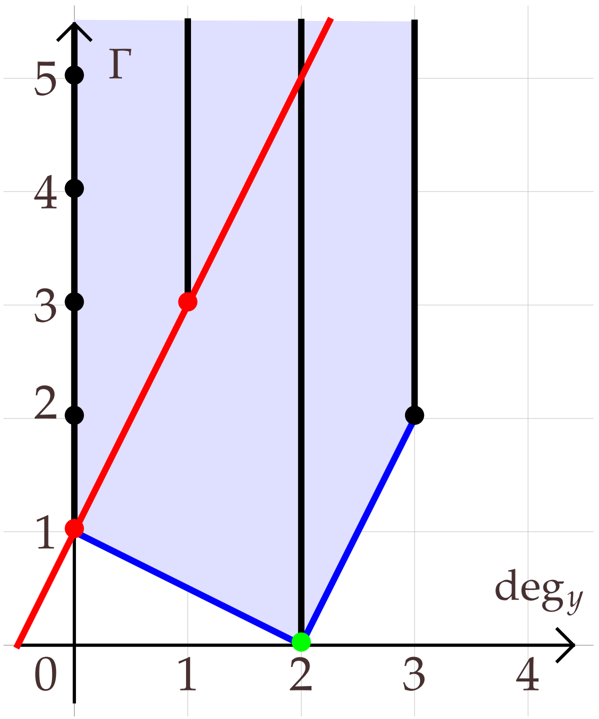

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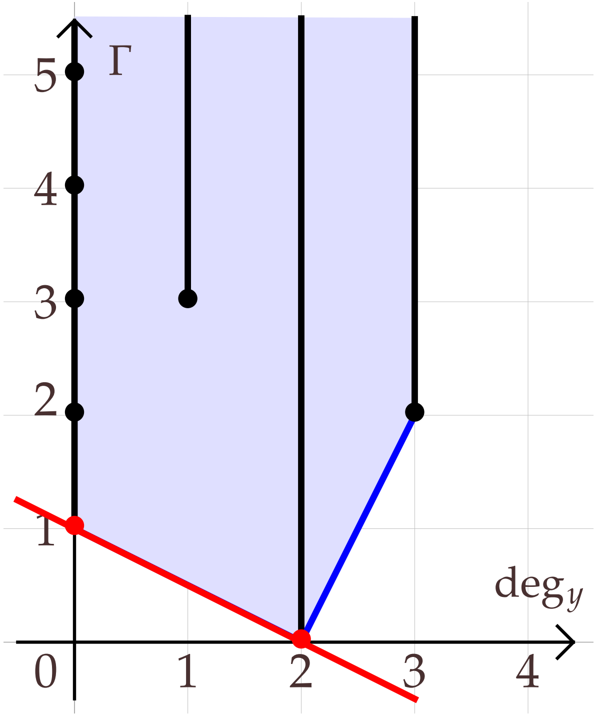

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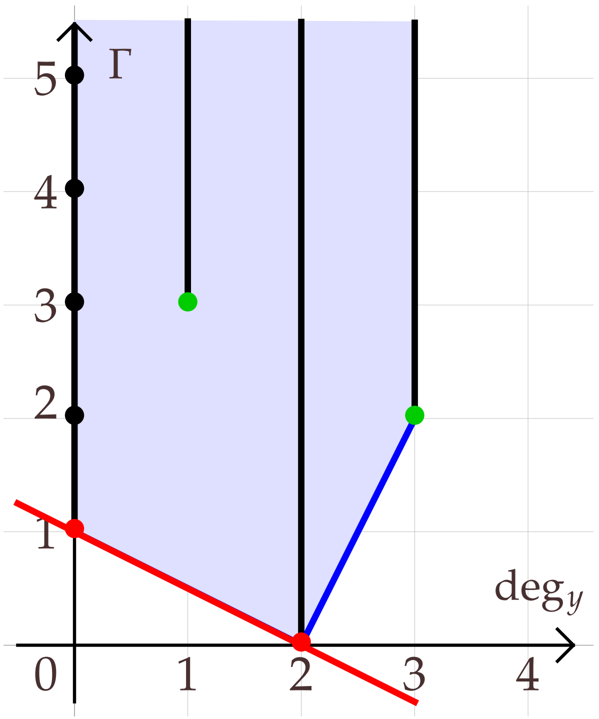
  OK, since $P_1 y \asymp z^{3+\nu} = z^{3 1/2} \preccurlyeq z \asymp P_0$

  $P_3 y^3 \asymp z^{2+3\nu} = z^{3 1/2} \preccurlyeq z \asymp P_0$



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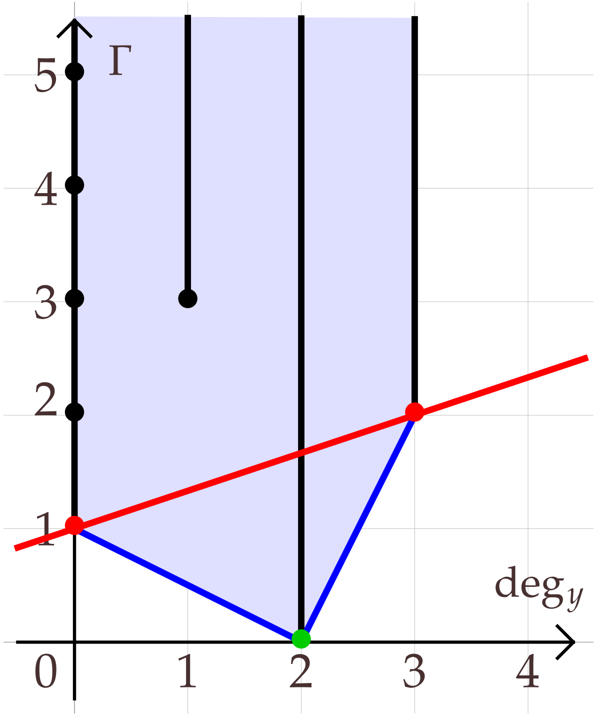

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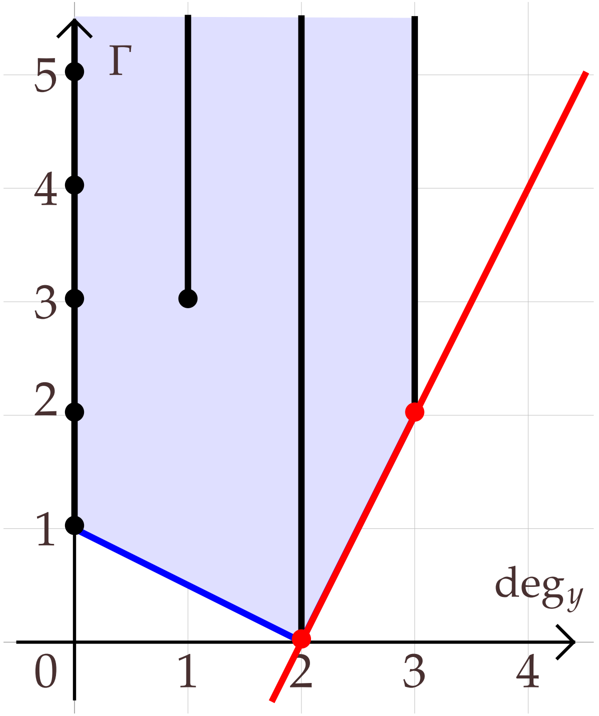
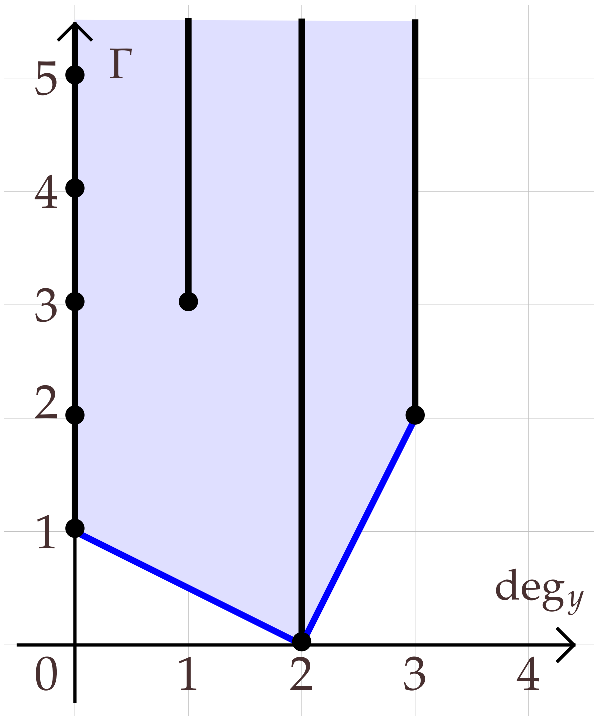


Newton polygons

$$5z^2y^3 + y^2 + 3z^3y - \frac{z}{1-z} = 0$$

Starting monomials $z^\nu \asymp y$?

- $\nu = {}^1\!/_2$
- $\nu = -2$



Starting terms

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Consider the starting monomial $z^{1/2}$.

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If $y \sim c z^{1/2}$, then

$$\begin{aligned} 5z^2y^3 &\prec z \\ y^2 &\sim c^2 z \\ 3z^2 y &\prec z \\ -\frac{z}{1-z} &\sim -z \end{aligned}$$

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If the right-hand side vanishes, then

$$c^2 - 1 \;=\; 0,$$

whence

$$c = 1 \;\vee\; c = -1.$$

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We call $z^{1/2}$ and $-z^{1/2}$ **starting terms** for the equation.

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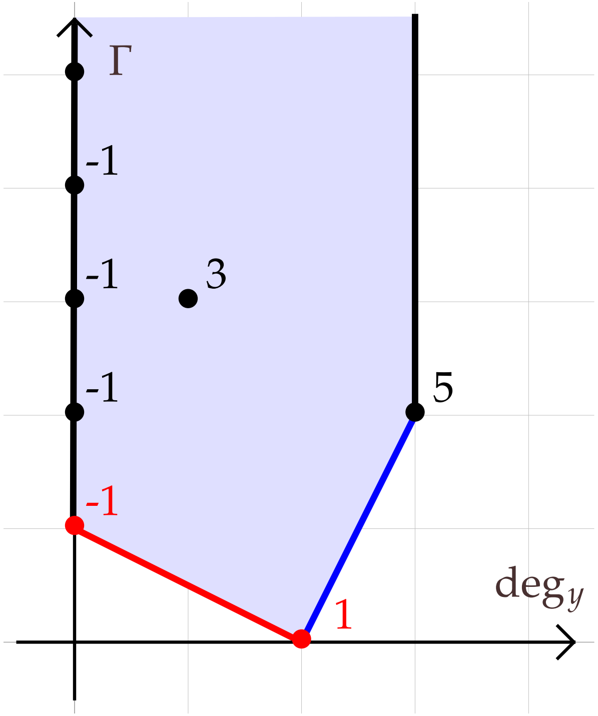

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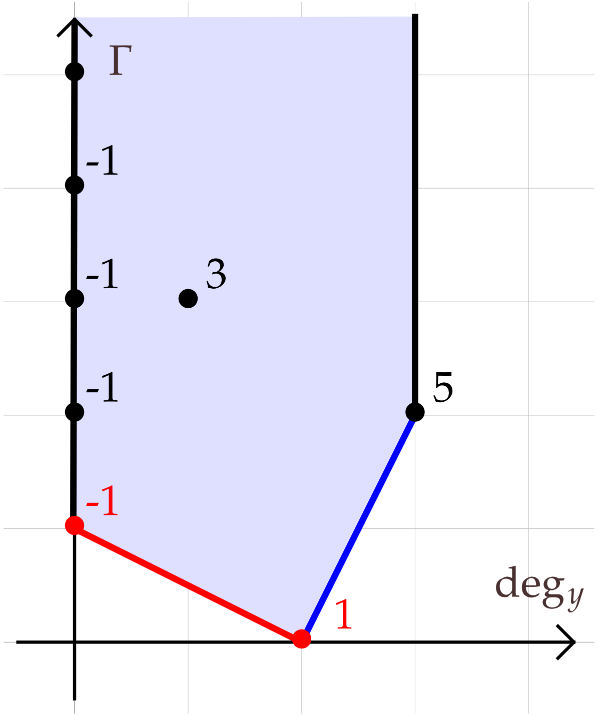
$N(c) = c^2 - 1$ is the **Newton polynomial** for $z^{1/2}$.



$$5z^2y^3 + y^2 + 3z^3y - \frac{z}{1-z} = 0$$

If $y \sim c z^{-2}$, then $c \neq 0$ and

$$5z^2y^3 + y^2 + 3z^3y - \frac{z}{1-z} = (5c^3 + c^2) z^{-4} + o(z^{-4}).$$

If the right-hand side vanishes, then

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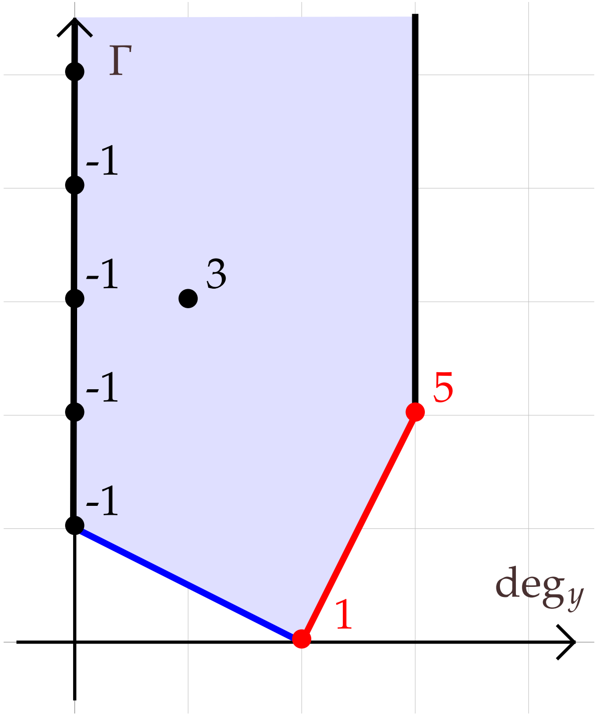

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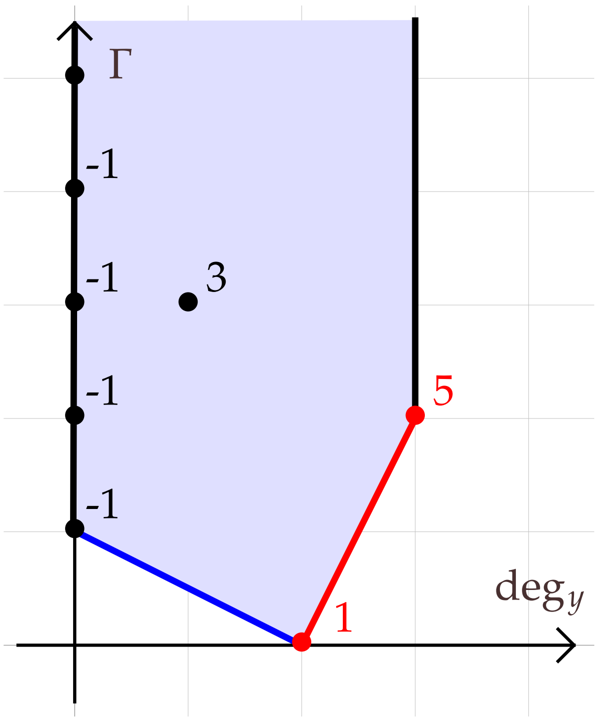
$-{}^1\!/_5 z^{-2}$ is a starting term for the equation.



Starting terms

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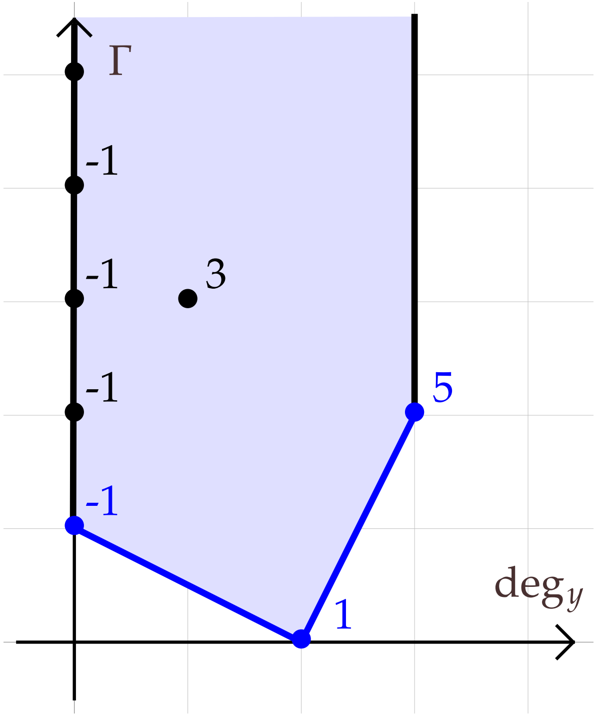
The starting terms for the equation are:

- $z^{1/2}$
- $-z^{1/2}$
- $-{}^1\!/_5\, z^{-2}$

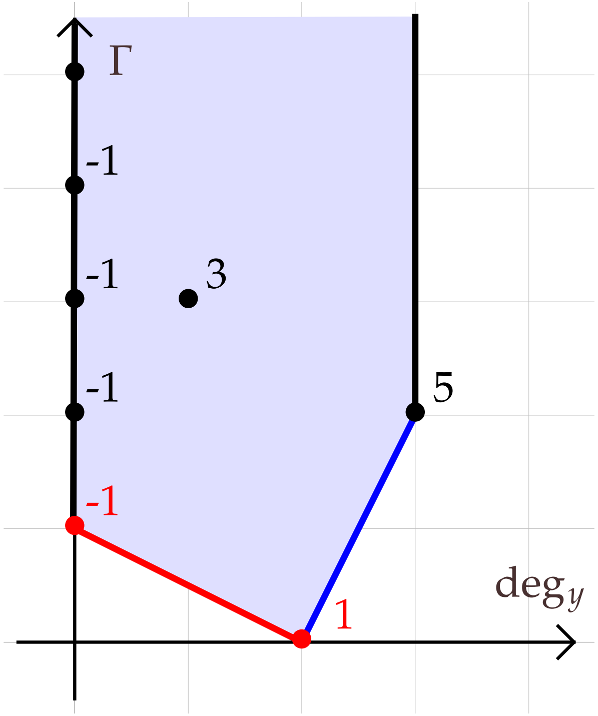


Refinements

$$5z^2y^3 + y^2 + 3z^3y - \frac{z}{1-z} = 0$$

Assume $y \sim z^{1/2}$ and peform the change of variables

$$y = z^{1/2} + \tilde{y}$$



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Assume $y \sim z^{1/2}$ and consider

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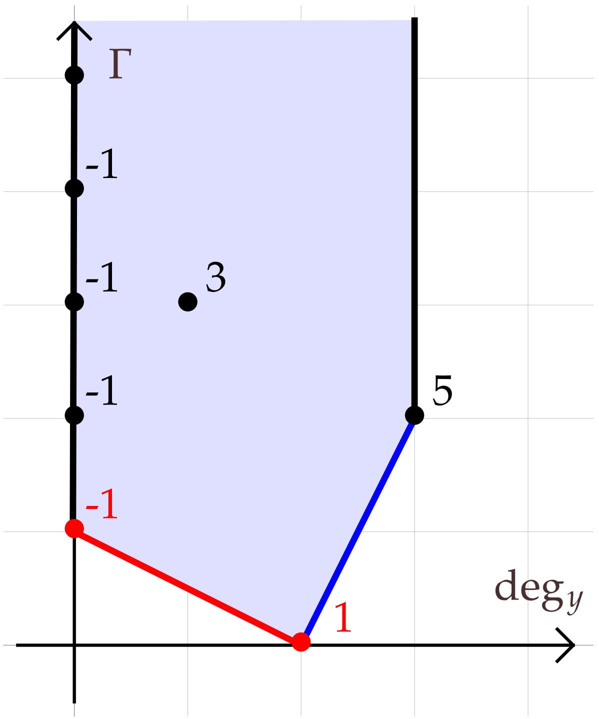
Refinement := change of variable
+
asymptotic constraint



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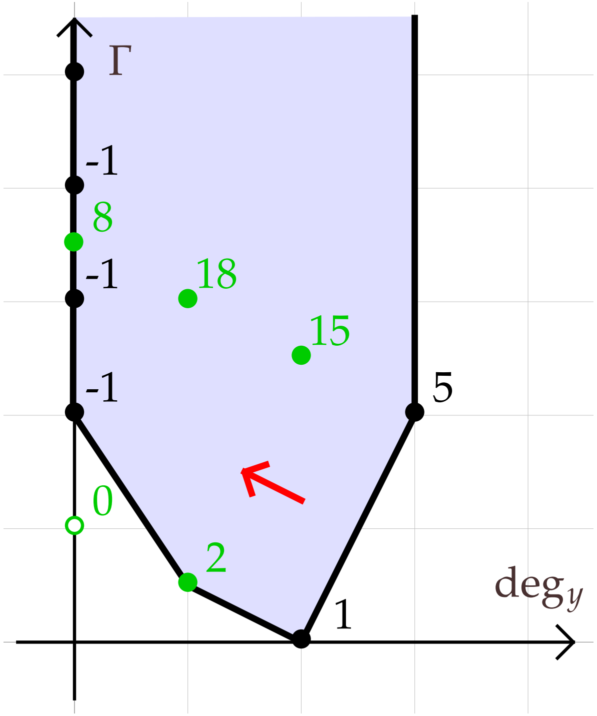
The refinement

$$y = z^{1/2} + \tilde{y} \qquad (\tilde{y} \prec z^{1/2}).$$

yields

$$\begin{aligned} & 5z^2\tilde{y}^3 \\ & +(1+15z^{2\frac{1}{2}})\tilde{y}^2 \\ & +(2z^{1/2}+18z^{2\frac{1}{2}})\tilde{y} \\ -z^2 - z^3 + 8z^{3\frac{1}{2}} - z^4 - \cdots & = 0, \quad (\tilde{y} \prec z^{1/2}). \end{aligned}$$



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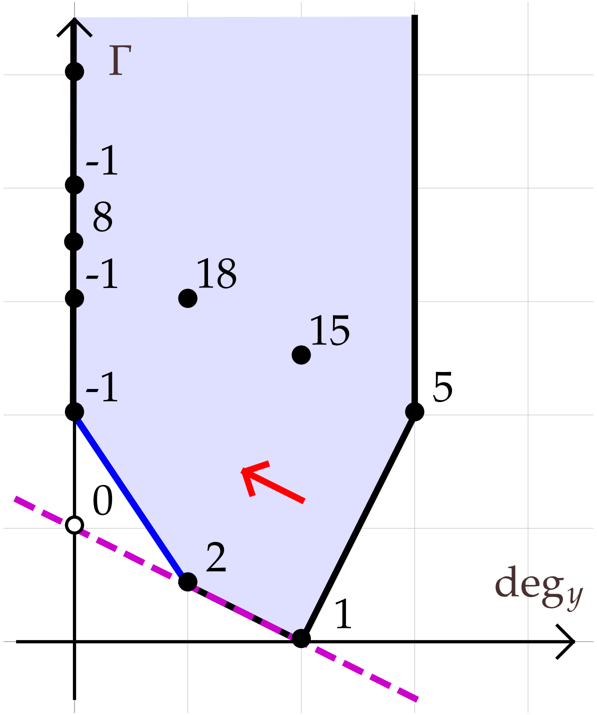

$$5z^2y^3 + y^2 + 3z^3y - \frac{z}{1-z} = 0$$

The refinement

$$y = z^{1/2} + \tilde{y} \qquad (\tilde{y} \prec z^{1/2}).$$

yields

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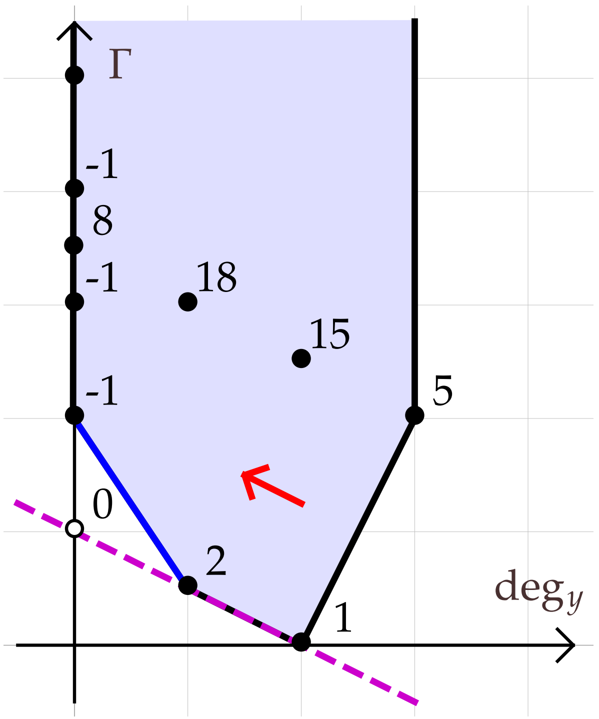
Only new starting monomial: $\tilde{y} \asymp z^{3/2}$.



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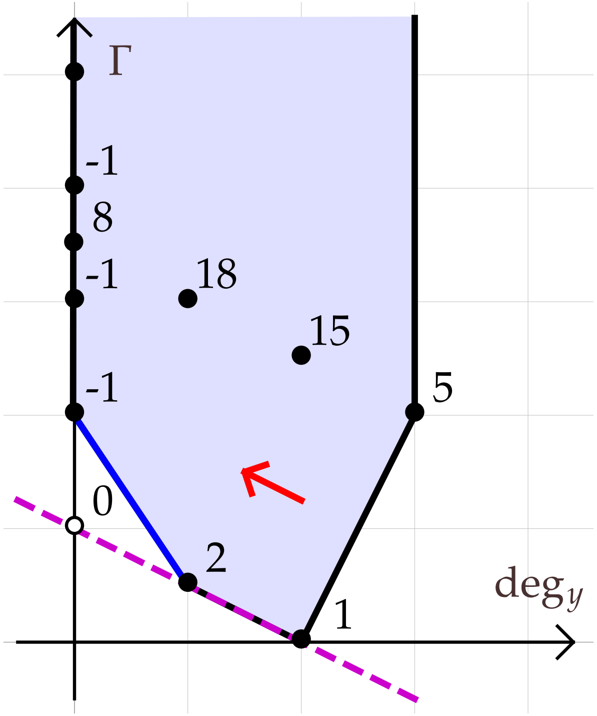

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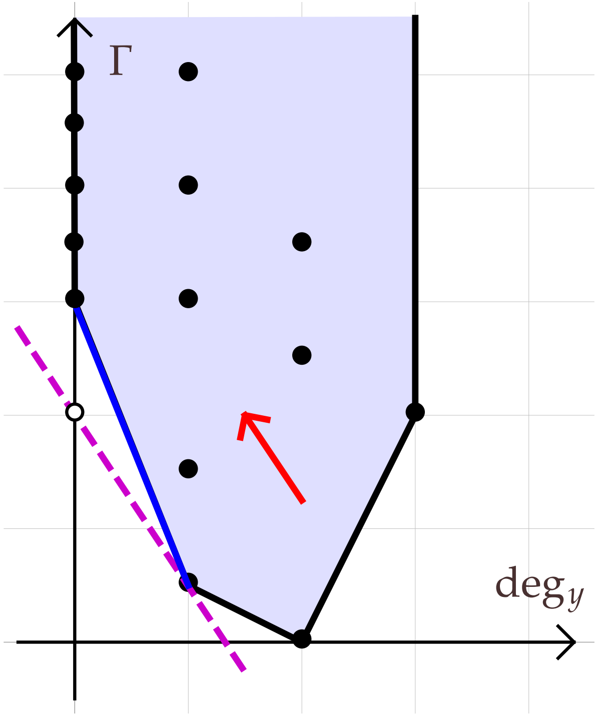
Continued refinement process

$$\begin{aligned} y &= z^{1/2}+\tilde{y} & (\tilde{y}\prec z^{1/2}) \\ \tilde{y} &= {}^1\!/_2\, z^{3/2}+\tilde{\tilde{y}} & (\tilde{\tilde{y}}\prec z^{3/2}) \\ &\vdots \end{aligned}$$



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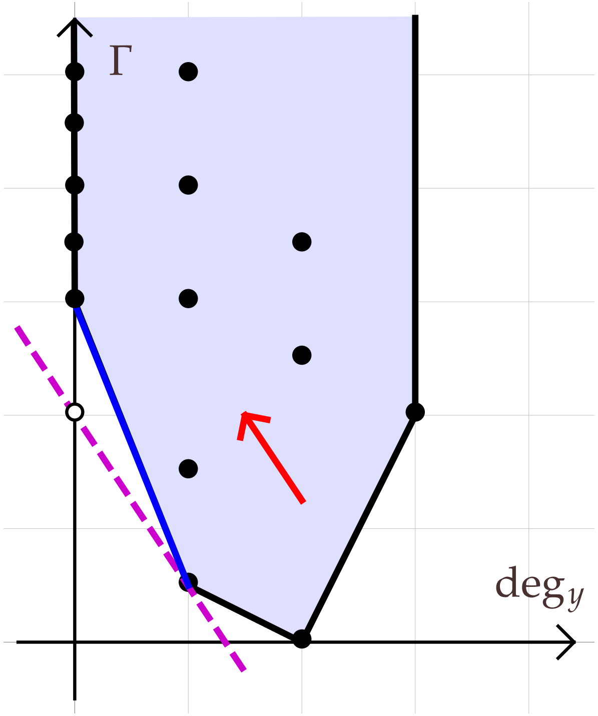
yields asymptotic expansion

$$y \approx z^{1/2} + {}^1\!/_2\, z^{3/2} + \cdots.$$



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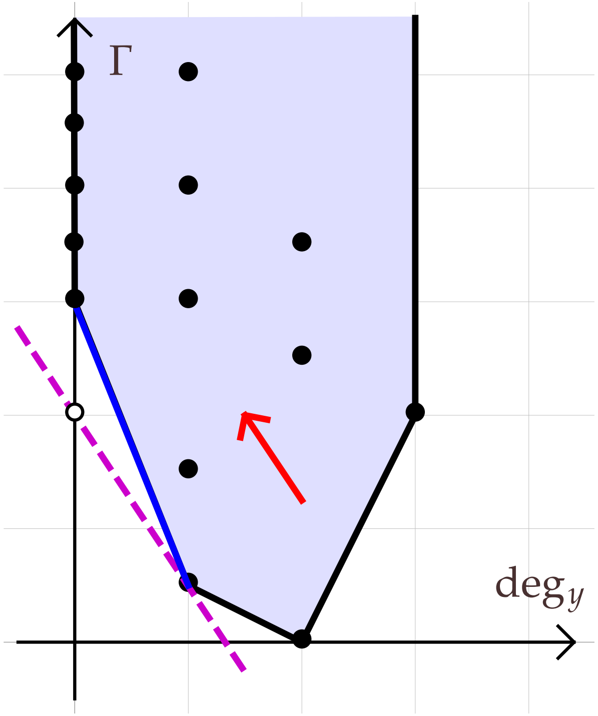
Continued refinement process

$$\begin{aligned} y &= z^{1/2} + \tilde{y} && (\tilde{y} \prec z^{1/2}) \\ \tilde{y} &= 1/2\, z^{3/2} + \tilde{\tilde{y}} && (\tilde{\tilde{y}} \prec z^{3/2}) \\ &\vdots \end{aligned}$$

yields asymptotic *solution*

$$y = z^{1/2} + 1/2\, z^{3/2} + \cdots ?$$



Multiplicative conjugations

$$P(y) = 5z^2y^3 + y^2 + 3z^3y - \frac{z}{1-z}$$

Multiplicative conjugate by $z^{1/2}$

$$P_{\times z^{1/2}}(y) := P(z^{1/2}y)$$
$$= 5z^{3 1/2}y^3 + zy^2 + 3z^{3 1/2}y - \frac{z}{1-z}$$

Multiplicative conjugations

$z^{1/2}$ is a starting monomial for

$$P(y) = 5z^2y^3 + y^2 + 3z^3y - \frac{z}{1-z} = 0$$

$\Longleftrightarrow$ 1 is a starting monomial for

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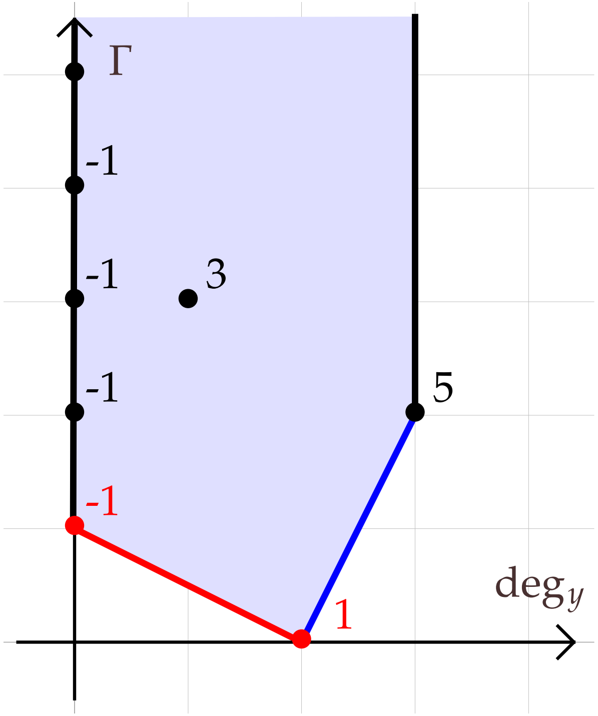

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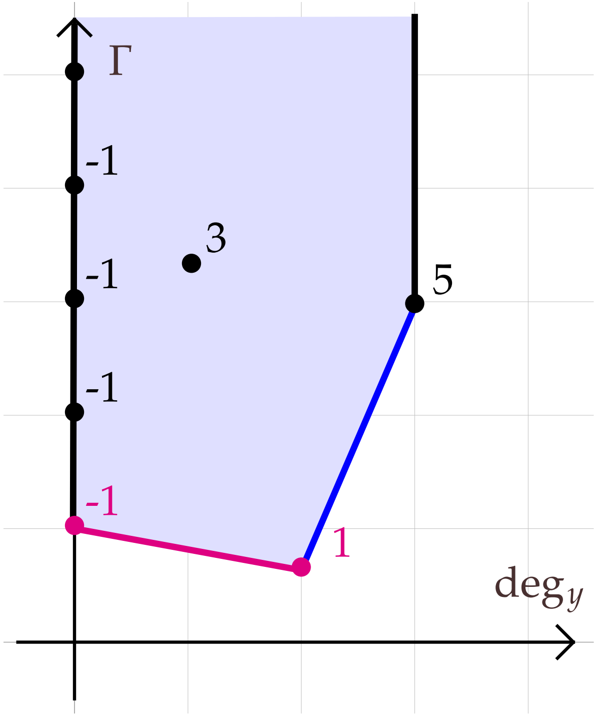

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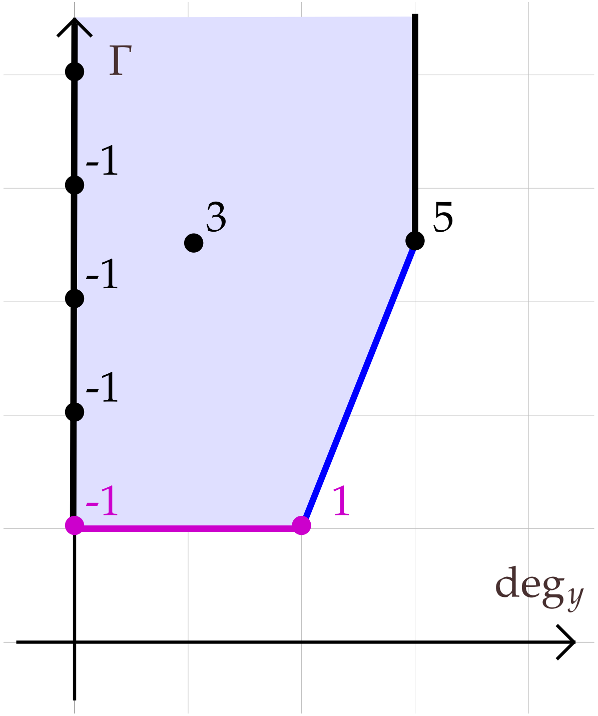

Starting terms revisited

$$K[[z^\Gamma]][Y] \subseteq K[Y][[z^\Gamma]]$$

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$$P = D_P \mathfrak{d}_P + o(\mathfrak{d}_P).$$

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Characterization of starting monomials

$$z^{\nu} \text{ is a starting monomial for } P(y) \;=\; 0 \quad\Longleftrightarrow\quad D_{P_{\times z^{\nu}}} \text{ is not homogeneous.}$$

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Characterization of starting terms

$$c z^{\nu} \text{ is a starting term for } P(y) = 0 \quad \Longleftrightarrow \quad N_{P_{\times z^{\nu}}}(c) = 0. \qquad (c \neq 0)$$

Newton degree

$$P(y) = 0 \qquad (y \prec z^{\gamma}). \qquad (\star)$$

Newton degree of $(\star)$

$$\deg_{\prec z^{\gamma}} P := \operatorname{val} N_{P_{\times z^{\gamma}}}$$

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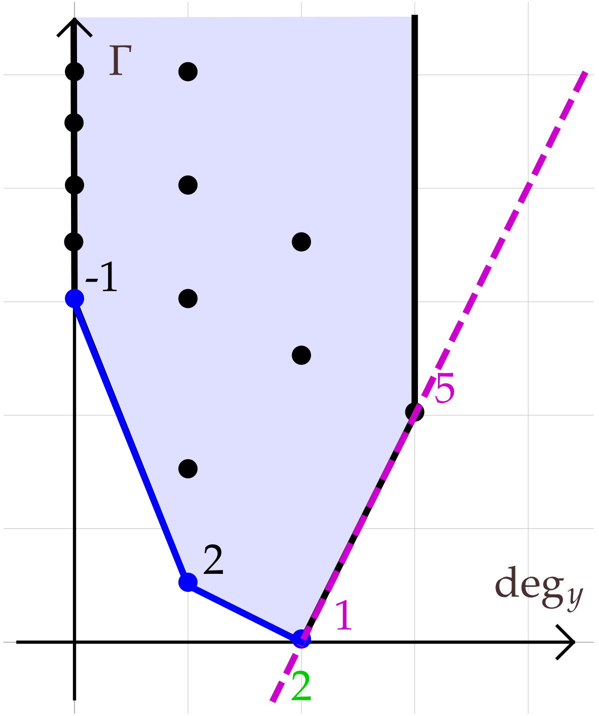
Example:

$$P(y) = 0 \qquad (y \prec z^{-2}).$$

$$\deg_{\prec z^{-2}} P = \operatorname{val}(5Y^3 + Y^2) = 2$$



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$$\deg_{\prec z^{\gamma}} P := \operatorname{val} N_{P_{\times z^{\gamma}}} = \max_{z^{\nu} \prec z^{\gamma}} \deg N_{P_{\times z^{\nu}}}$$

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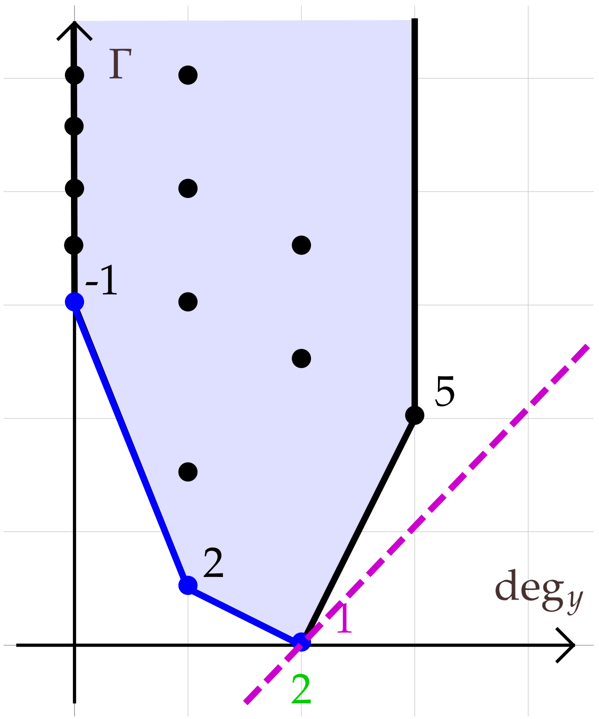

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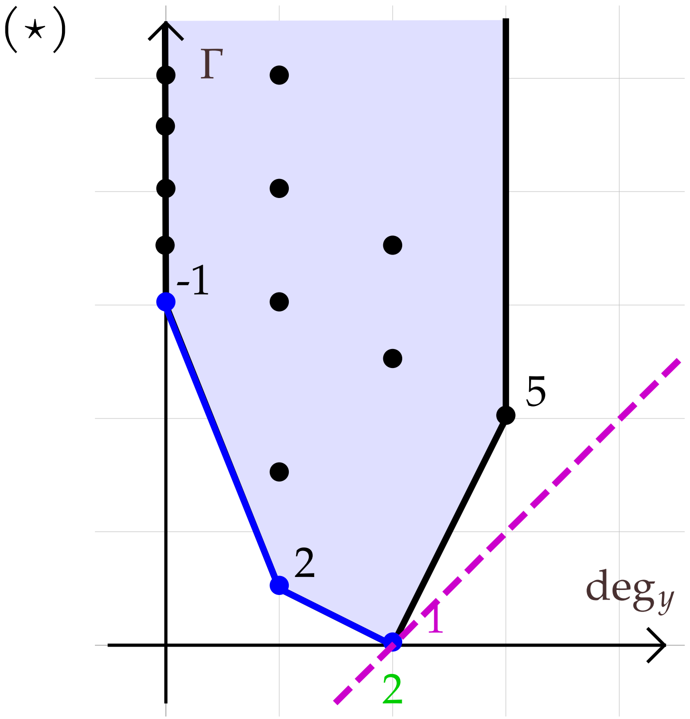

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Newton degree of $(\star)$ (if $\operatorname{val}_{\Upsilon} P < \deg_{\prec z^{\gamma}} P$)

$$\begin{aligned}\deg_{\prec z^{\gamma}} P &:= \operatorname{val} N_{P_{\times z^{\gamma}}}\\ &= \max_{z^{\nu} \prec z^{\gamma}} \deg N_{P_{\times z^{\nu}}}\\ &= \max_{\text{starting monomial } z^{\nu} \prec z^{\gamma}} \deg N_{P_{\times z^{\nu}}}\end{aligned}$$

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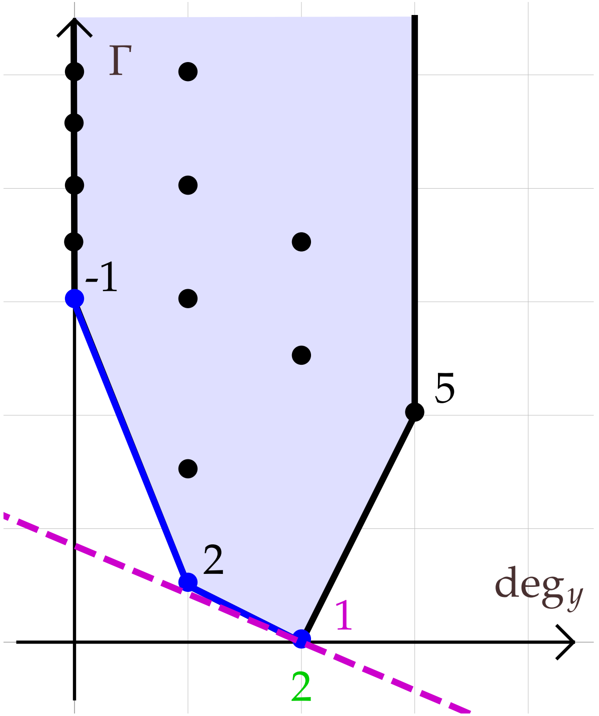
$$\begin{aligned}\deg_{\prec z^{-2}} P &= \operatorname{val}(5\Upsilon^3 + \Upsilon^2)\\ &= \deg \Upsilon^2\\ &= \deg(\Upsilon^2 + 2\Upsilon) = 2\end{aligned}$$



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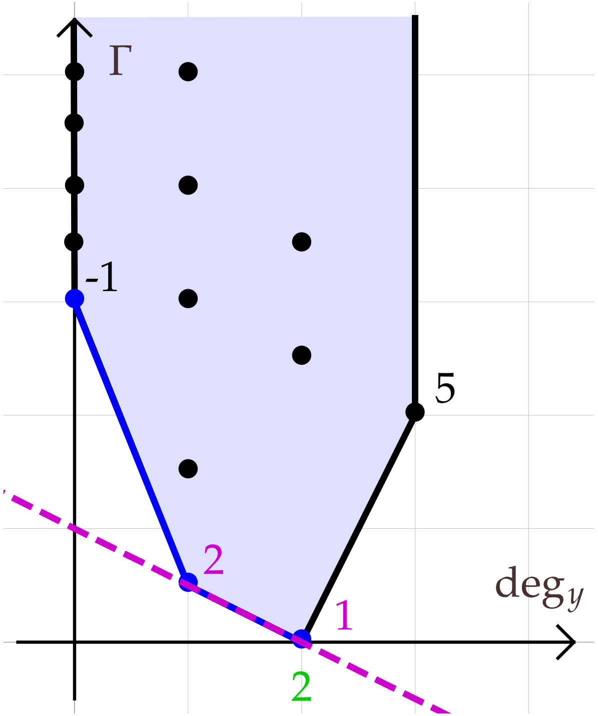

Additive conjugation

Given $P \in K[[z^\Gamma]][Y]$ and $\varphi \in K[[z^\Gamma]]$, the **additive conjugate** of P by φ is

$$P_{+\varphi}(y) \; := \; P(\varphi + y)$$

Let $N \in K[Y]^{\neq 0}$ and let $c \in K$. Then

$$\operatorname{val} N_{+c} \; = \; \text{multiplicity of } c \text{ as a root of } N$$

Let $c_1, \ldots, c_\ell \in K$ be the roots of N. Since K is algebraically closed, we have

$$\deg N \; = \; \operatorname{val} N_{+c_1} + \cdots + \operatorname{val} N_{+c_\ell}.$$

Refinements revisited

Consider an equation $P(y)=0,\ y\prec z^{\gamma}$ of Newton degree d:

$$d \;=\; \deg_{\prec z^{\gamma}} P.$$

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Consider an equation $P(y) = 0,\ y \prec z^\gamma$ of Newton degree d:

$$d = \deg_{\prec z^\gamma} P.$$

If $\operatorname{val}_Y P = d$, then $y = 0$ is a solution of multiplicity d.

Refinements revisited

Consider an equation $P(y) = 0,\ y \prec z^\gamma$ of Newton degree d:

$$d = \deg_{\prec z^\gamma} P.$$

Assume that $\mathrm{val}_Y P < d$ and let z^ν be the largest starting monomial. We have

$$d = \deg N, \qquad N := N_{P_{\times z^\nu}}.$$

Refinements revisited

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For any $\alpha \in K$, we have $P_{\times z^{\nu}, +\alpha} = P_{+\alpha z^{\nu}, \times z^{\nu}}$ and $N_{P_{+\alpha}} = N_{P, +\alpha}$, whence

$$\operatorname{val} N_{+c_i} = \operatorname{val} N_{P_{+c_i z^{\nu}, \times z^{\nu}}} = \deg_{\prec z^{\nu}} P_{+c_i z^{\nu}}.$$

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Hence

$$d \;=\; \deg_{\prec z^{\nu}} P_{+c_1 z^{\nu}} + \cdots + \deg_{\prec z^{\nu}} P_{+c_{\ell} z^{\nu}}.$$

Refinements revisited — conclusion

Conservation of Newton degree

Consider an asymptotic algebraic equation

$$P(y) \; = \; 0 \qquad (y \prec z^{\gamma}),$$

with $\operatorname{val}_Y P < \deg_{\prec z^{\gamma}} P$ *and let* z^{ν} *be the largest starting monomial. Let* $c_1, \ldots, c_{\ell}$ *be the roots of* $N := N_{P_{\times z^{\nu}}}$*, so that each* c_i *determines a refined equation*

$$P_{+c_i z^{\nu}}(\tilde{y}) \; = \; 0 \qquad (\tilde{y} \prec z^{\nu}).$$

If K *is algebraically closed, then*

$$\deg_{\prec z^{\gamma}} P \; = \; \deg_{\prec z^{\nu}} P_{+c_1 z^{\nu}} + \cdots + \deg_{\prec z^{\nu}} P_{+c_{\ell} z^{\nu}}.$$

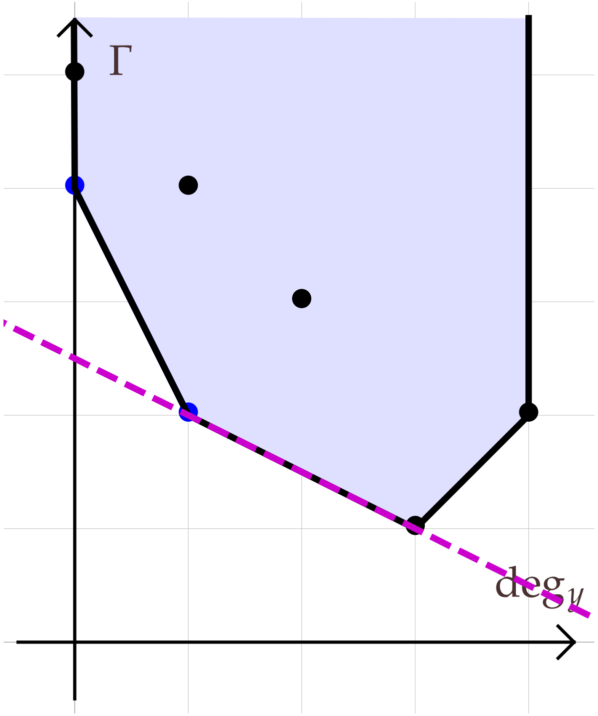
Quasi-linear equations

Definition

The equation

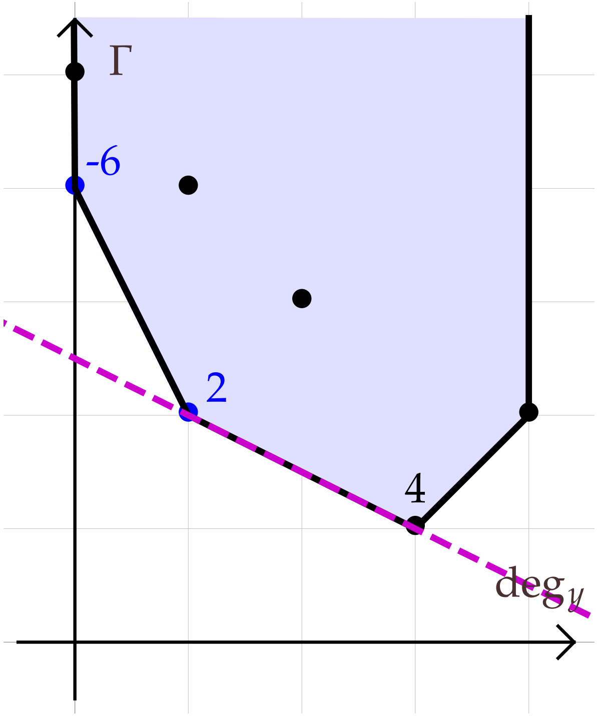
$$P(y) = 0 \qquad (y \prec z^{\gamma})$$

is ***quasi-linear*** *if* $\deg_{\prec z^{\gamma}} P = 1$.



Consider a quasi-linear equation

$$P(y) = 0 \qquad (y \prec z^{\gamma}) \qquad (\star)$$



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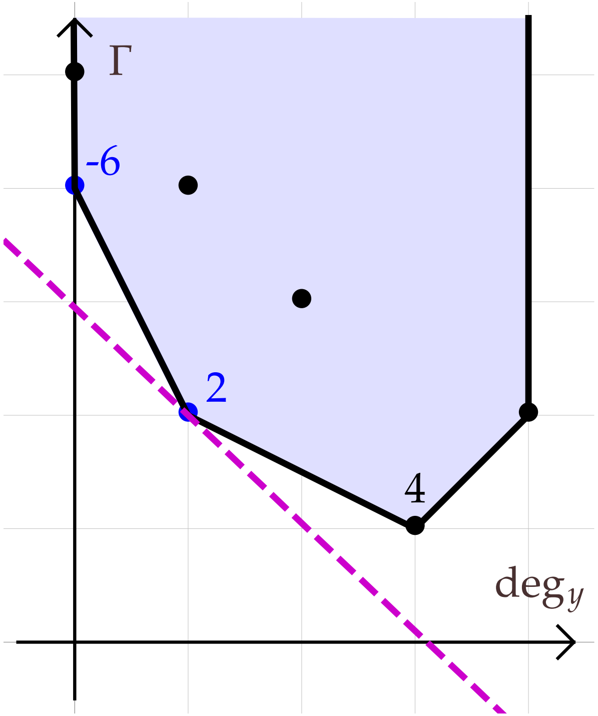
Without loss of generality, we may arrange that

$$\operatorname{val} N_{\times z^{\gamma}} = \deg N_{\times z^{\gamma}} = 1.$$



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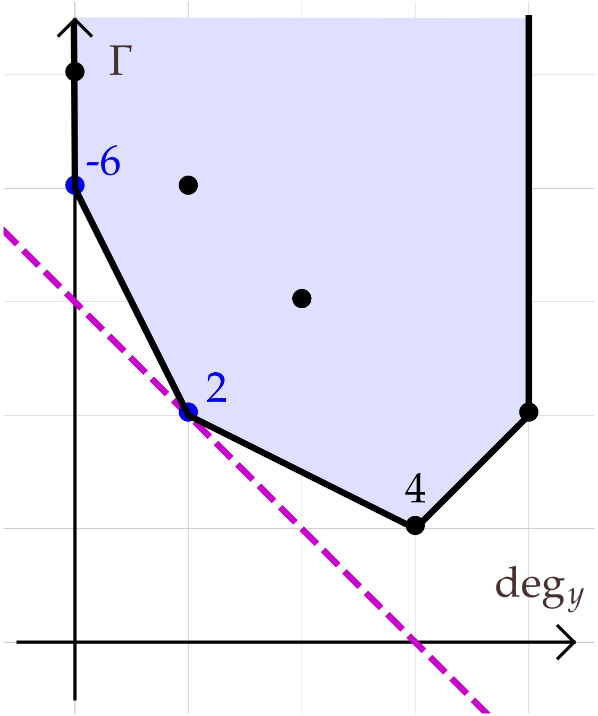


Quasi-linear equations

Consider a quasi-linear equation

$$P(y) = 0 \qquad (y \prec z^{\gamma}) \qquad (\star)$$

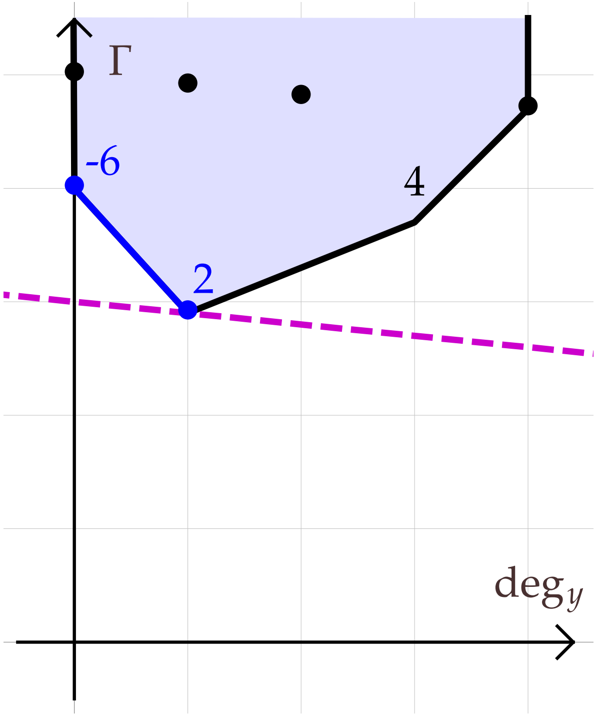
Let

$$y := z^{\gamma} u$$
$$Q := P_{\times z^{\gamma}}.$$

Then $(\star)$ is equivalent to

$$Q(u) = 0 \qquad (u \prec 1)$$

We have $\deg_{\prec 1} Q = \operatorname{val} N_Q = \deg N_Q = 1$.



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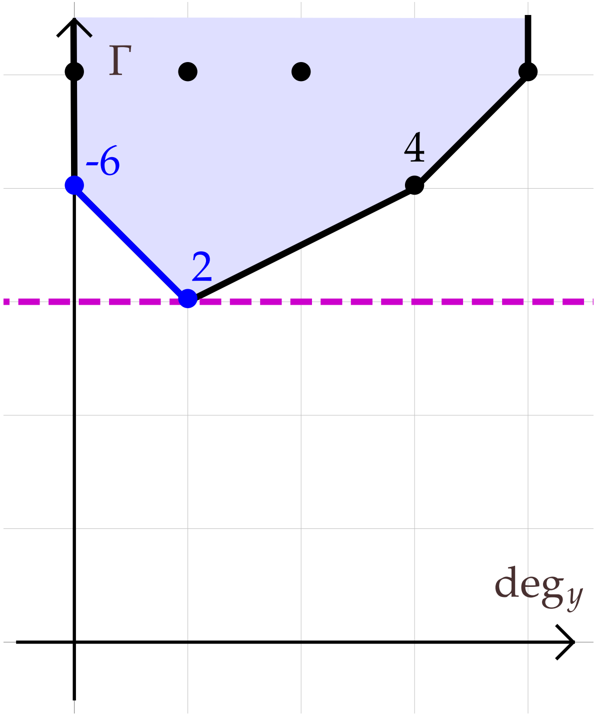

Quasi-linear equations

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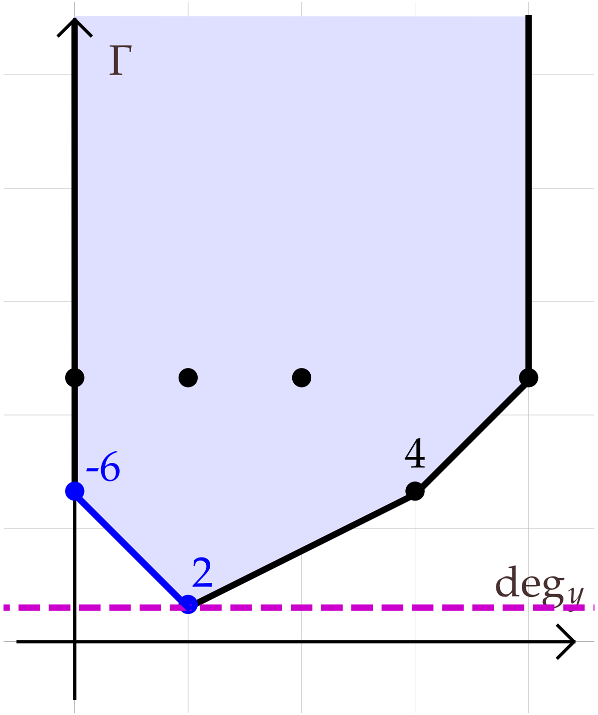
Let

$$\begin{aligned} y &:= z^{\gamma} u \\ Q &:= P_{\times z^{\gamma}} \\ R &:= \mathfrak{d}_Q^{-1} Q. \end{aligned}$$

Then $(\star)$ is equivalent to

$$R(u) = 0 \qquad (u \prec 1)$$

We have $\deg_{\prec 1} R = \operatorname{val} N_R = \deg N_R = 1$ and $\mathfrak{d}_R = 1$.



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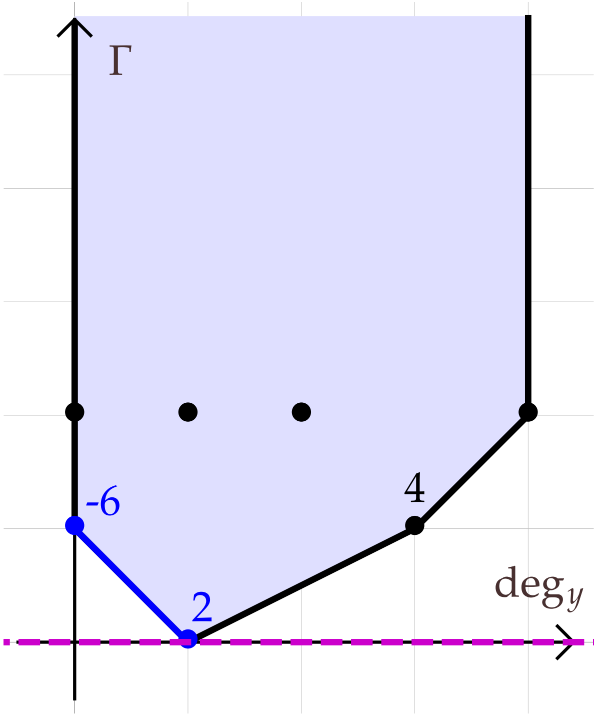
Then $(\star)$ is equivalent to

$$R(u) = 0 \qquad (u \prec 1)$$

We have $\deg_{\prec 1} R = \operatorname{val} N_R = \deg N_R = 1$ and $\mathfrak{d}_R = 1$.

The polynomial R is in **Hensel position**.



Consider a quasi-linear equation

$$P(y) = 0 \qquad (y \prec z^{\gamma}) \qquad (\star)$$

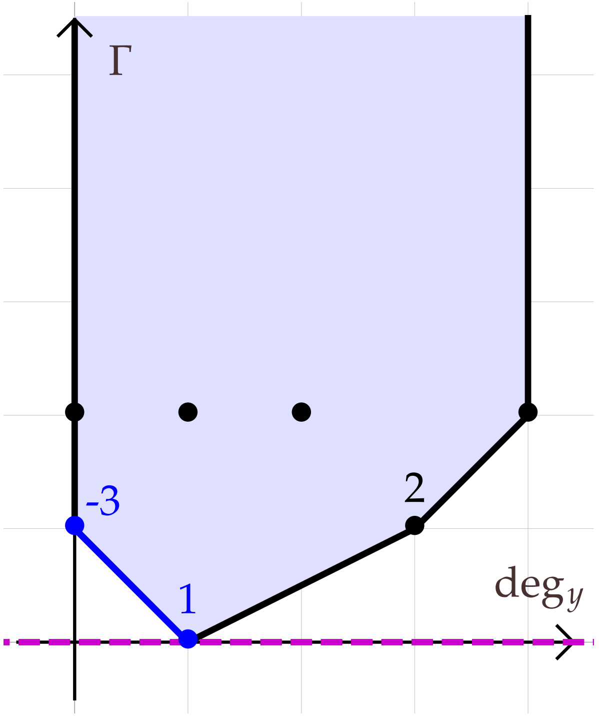
Let

$$\begin{aligned} y &:= z^{\gamma} u \\ Q &:= P_{\times z^{\gamma}} \\ R &:= \mathfrak{d}_Q^{-1} Q \\ S &:= N_{R,1}^{-1} R. \end{aligned}$$

Then $(\star)$ is equivalent to

$$S(u) = 0 \qquad (u \prec 1)$$

We have $\operatorname{val} N_S = \deg N_S = 1$, $\mathfrak{d}_S = 1$, and $N_{S,1} = 1$.



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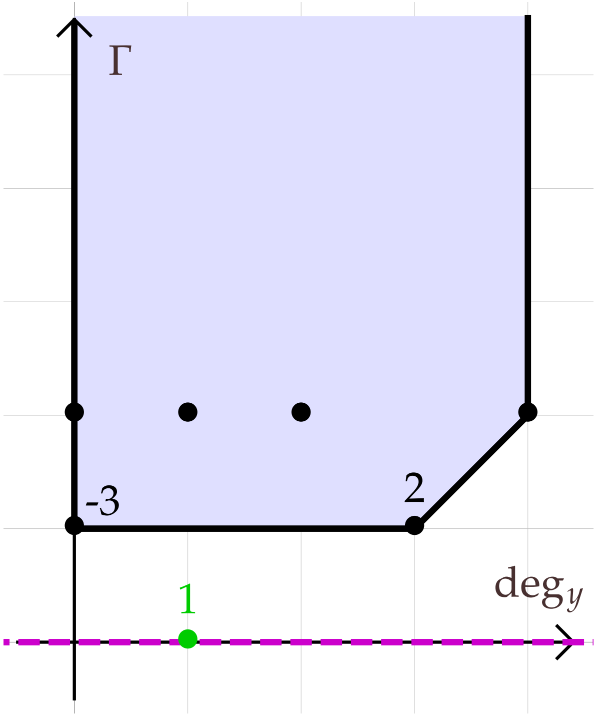
$$\begin{aligned} y &:= z^{\gamma} u \\ Q &:= P_{\times z^{\gamma}} \\ R &:= \mathfrak{d}_Q^{-1} Q \\ S &:= N_{R,1}^{-1} R \\ T &:= Y - S. \end{aligned}$$

Then $(\star)$ is equivalent to

$$u \;=\; T(u) \qquad (u \prec 1)$$

We have $T \prec 1$. $u = -3z + 2zu^3 + O(z^2)$



Fixed point theorem

Theorem

Let $P \in K[[z^{\Gamma}]][Y]$ *be such that* $P \prec 1$. *Then*

$$y = P(y) \qquad (y \prec 1)$$

has a unique solution $y \in K[[z^{\Gamma}]]$.

Fixed point theorem

Theorem

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Let $\mathfrak{M}$ *be a monomial monoid and let* $P \in K[[\mathfrak{M}]][Y]$ *be such that* $P \prec 1$ *(i.e.* $\operatorname{supp} P \prec 1$*).*
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We admit Kruskal's theorem: the set $\mathfrak{S}^{\top}$ is well-based for some natural $\preccurlyeq$.

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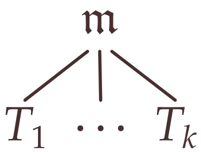

$$\Longrightarrow \quad \tau_T := P_{k,\mathfrak{m}} \mathfrak{m} \tau_{T_1} \cdots \tau_{T_k}.$$

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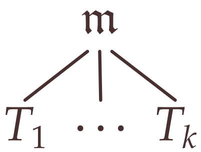

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Now $y = \sum_{T \in \mathfrak{S}^{\top}} \tau_T$ satisfies $y = P(y)$ and $y \prec 1$.

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The map $T \mapsto \tau_T$ is increasing, so $(\tau_T)_{T \in \mathfrak{S}^{\top}}$ is well-based.

Now $y = \sum_{T \in \mathfrak{S}^{\top}} \tau_T$ satisfies $y = P(y)$ and $y \prec 1$. Indeed:

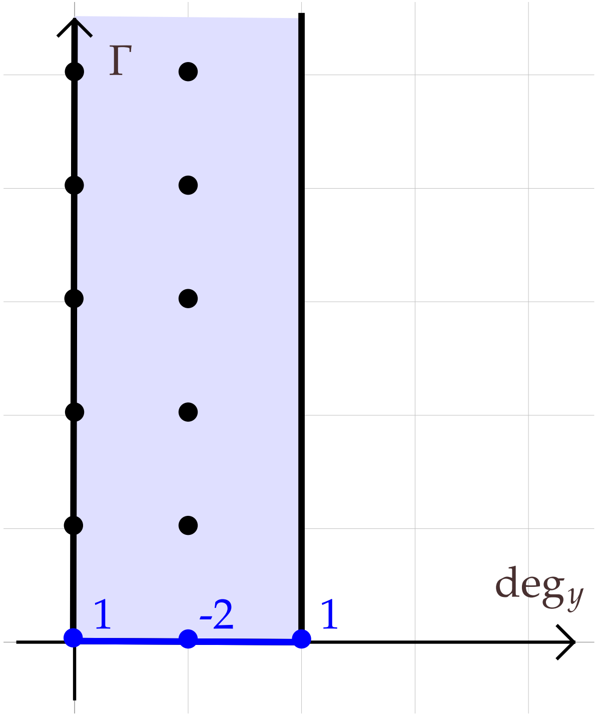
$$\begin{aligned} P(y) &= \sum_{k \in \mathbb{N}} \sum_{\mathfrak{m} \in \mathfrak{S}} P_{k,\mathfrak{m}} \mathfrak{m} \, y^k = \sum_{k \in \mathbb{N}} \sum_{\mathfrak{m} \in \mathfrak{S}} \sum_{T_1 \in \mathfrak{S}^{\top}} \cdots \sum_{T_k \in \mathfrak{S}^{\top}} P_{k,\mathfrak{m}} \mathfrak{m} \, \tau_{T_1} \cdots \tau_{T_k} \\ &= \sum_{T \in \mathfrak{S}^{\top}} \tau_T = y. \end{aligned}$$

□

Almost multiple solutions

Consider the equation

$$\left(y - \frac{1}{1-z}\right)^2 = z^{1000}.$$



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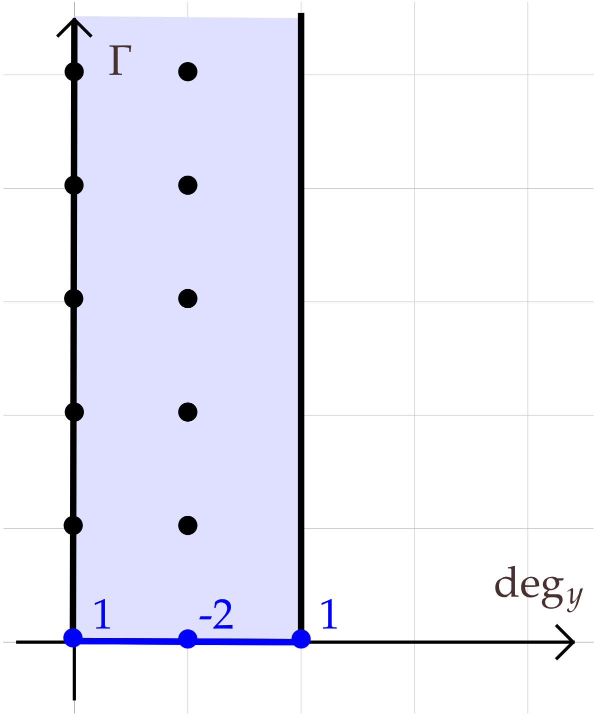
There is a unique starting term $y \sim 1$.



Almost multiple solutions

Consider the equation

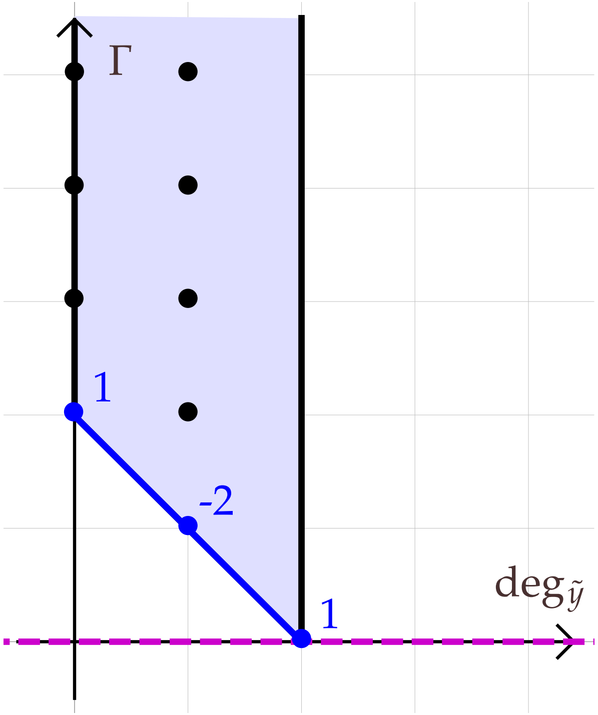
$$\left(y - \frac{1}{1-z}\right)^2 = z^{1000}.$$

There is a unique starting term $y \sim 1$. After

$$y = 1 + \tilde{y} \qquad (\tilde{y} \prec 1),$$

we obtain

$$\left(\tilde{y} - \frac{z}{1-z}\right)^2 = z^{1000} \qquad (\tilde{y} \prec 1).$$



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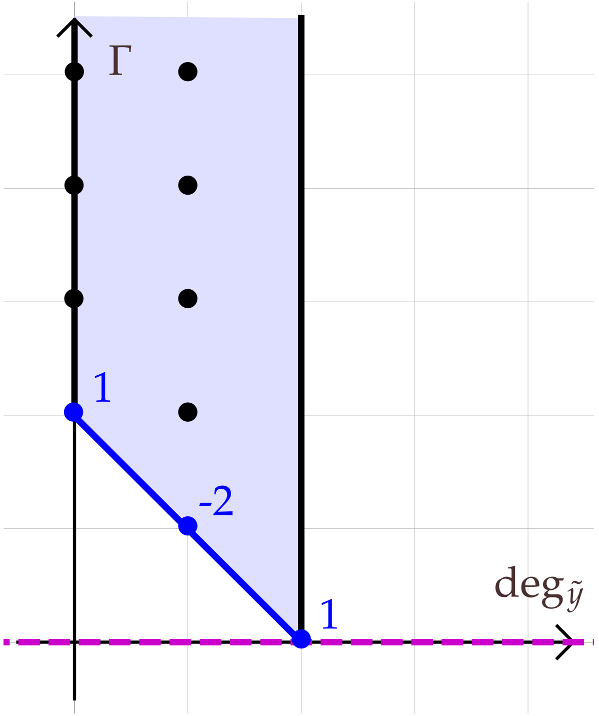

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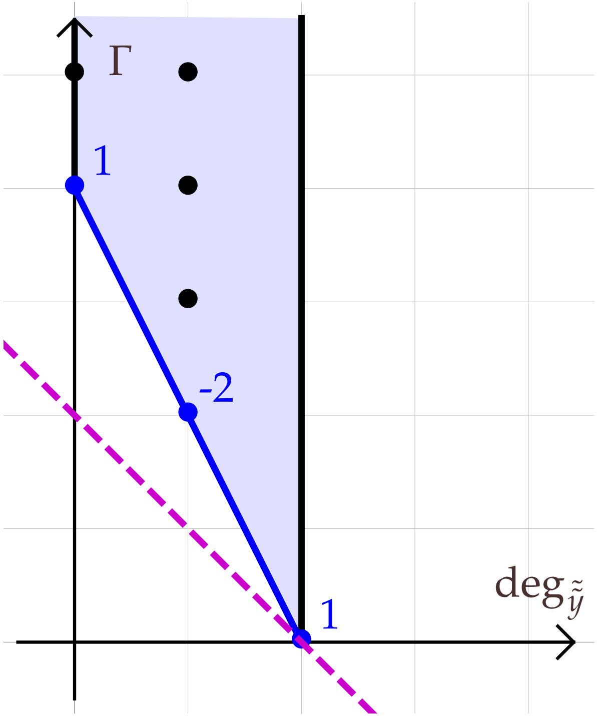
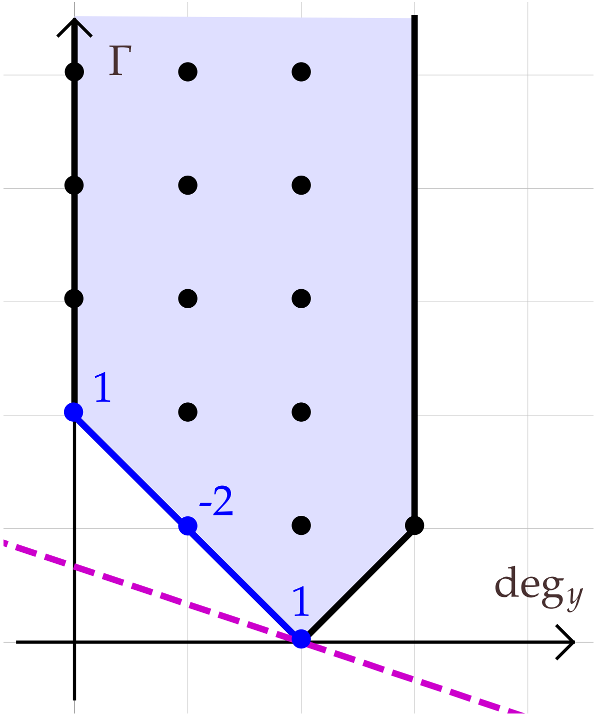


$$P(y) = 0 \qquad (y \prec z^{\gamma})$$

with unique d-fold starting term $y \sim c\,z^{\nu}$.

Then $N_{P_{\times z^{\nu}}} = \alpha\,(Y - c)^d$, where $d = \deg_{\prec z^{\gamma}} P$.



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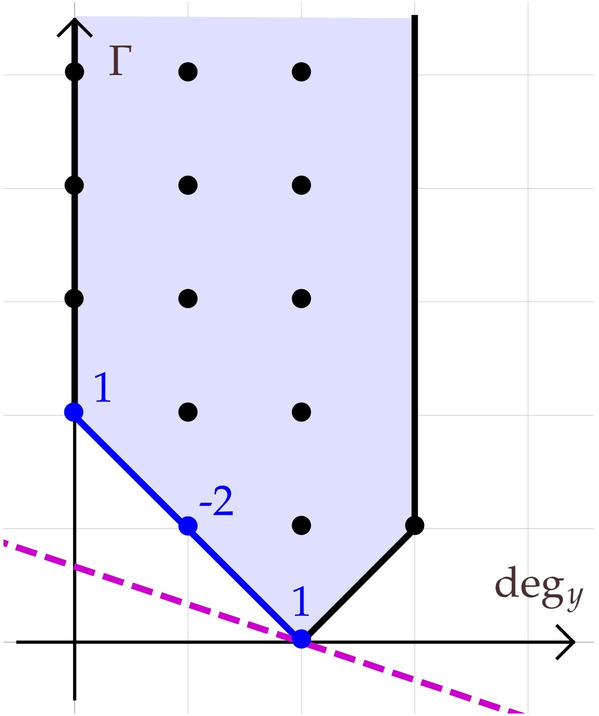
Note: char $K = 0 \Longrightarrow N_{P_{\times z^{\nu}}, d-1} = -d\,c \neq 0 \Longrightarrow P_{d-1} \neq 0$.



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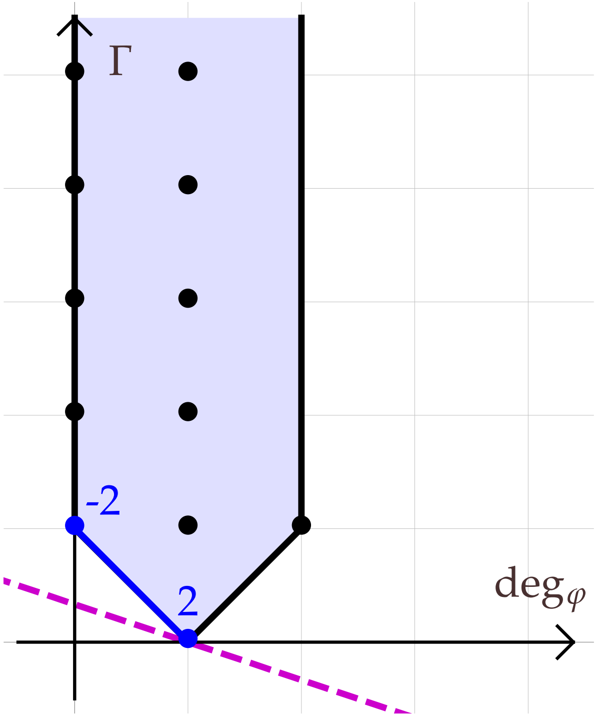
φ: the unique solution of the quasi-linear equation

$$Q(\varphi) := \frac{\partial^{d-1} P}{\partial Y^{d-1}}(\varphi) \qquad (\varphi \prec z^{\gamma})$$



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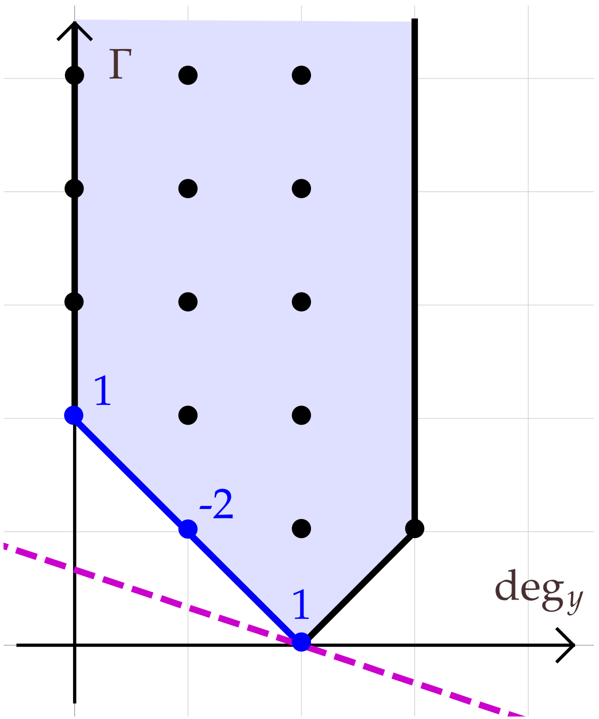
$$y = \varphi + \tilde{y} \qquad (\tilde{y} \prec z^{\nu})$$

instead of

$$y = c z^{\nu} + \tilde{y} \qquad (\tilde{y} \prec z^{\nu}).$$



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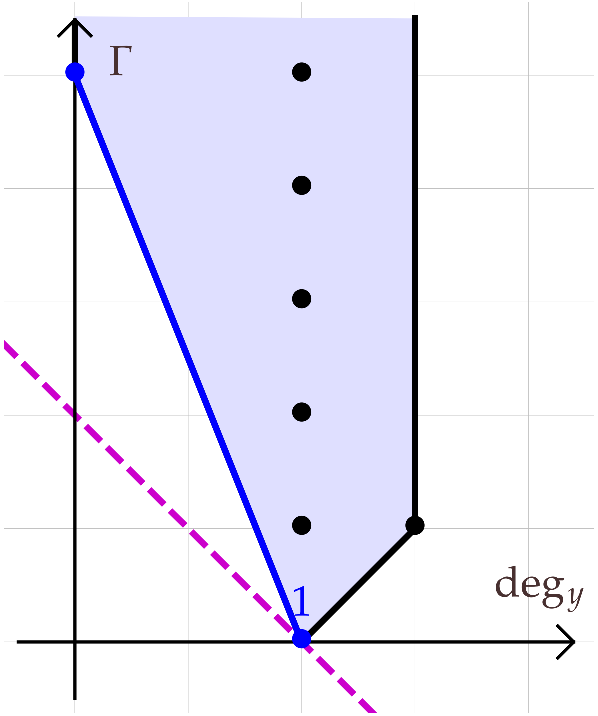

Unravelling

$$P(y) = 0 \qquad (y \prec z^{\gamma})$$

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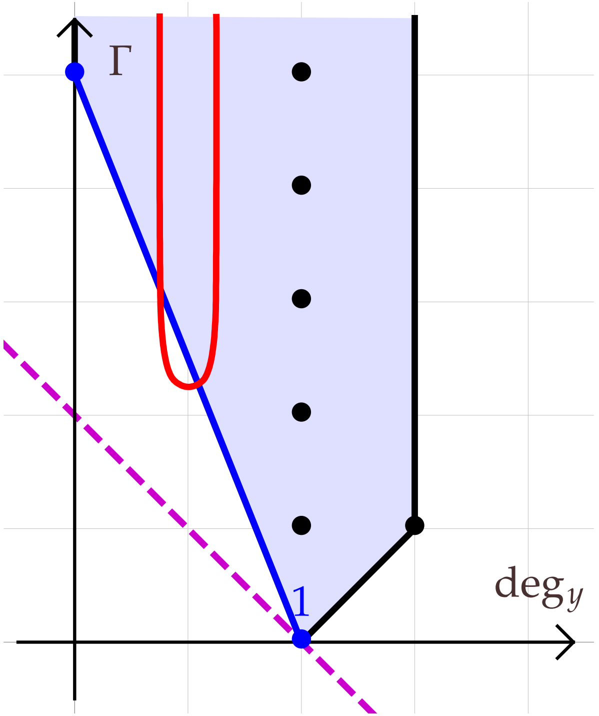
Then

$$P_{+\varphi}(\tilde{y}) = P(\varphi) + P'(\varphi)\tilde{y} + \cdots = 0 \qquad (\tilde{y} \prec z^{\nu}),$$

whence $P_{+\varphi,d-1} = 0$



Complete algorithm

Algorithm solve(P, z^γ)

INPUT: $P \in K[[z^\Gamma]][Y]$ and $z^\gamma \in z^\Gamma$ with $d := \deg_{\prec z^\gamma} P > 0$ and $\operatorname{char} K = 0$

OUTPUT: solutions $y_1, \ldots, y_d \in K[[z^\Gamma]]$ of $P(y) = 0, y \prec z^\gamma$, counted with multiplicities

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Final theorem

Let K be a field of characteristic zero.

Theorem

Let $P \in K[[z^\Gamma]][Y]^{\neq 0}$ and $z^\gamma \in z^\Gamma$. If K is algebraically closed and Γ divisible, then

$$P(y) = 0 \qquad (y \prec z^\gamma)$$

has exactly $\deg_{\prec z^\gamma} P$ solutions in $K[[z^\Gamma]]$, when counting with multiplicities.

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- Generalizations to $K[[z^\Gamma]]_{\mathscr{S}}$ and $K[\![z^\Gamma]\!]_{\mathscr{L}}$ instead of $K[[z^\Gamma]]$.