Lesson 5 — Transseries

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IMS summer school

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Exp-log fields

Definition

Consider an ordered field K *with a partial function* $\exp: K \rightarrow K$ *such that*

E1. $\exp 0 = 1$.

E2. $\exp y = \exp(y - x) \exp x$ *for all* $x, y \in \operatorname{dom} \exp$.

E3. $\exp x \geqslant 1 + x + \cdots + \frac{1}{(n-1)!} x^{n-1}$ *for all* $x \in \operatorname{dom} x$ *and* $n \in \mathbb{N}$.

We call $\exp$ *an* ***exponential function****. Such a function is necessarily injective and its partial inverse is called a* ***logarithmic function****.*

If $\operatorname{dom} \exp = K$ *and* $\operatorname{im} \exp = K^{>0}$*, then* K *is called an* ***exp-log*** *field.*

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Proposition

$\mathbb{R}$ *is an exp-log field.*

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Partial logarithmic function. Let $f \in \mathbb{L}^{>0}$. Then

$$\begin{aligned} f \; &= \; \mathfrak{d}_f c_f (1 + \delta), \\ & \quad \mathfrak{d}_f = x^{\alpha_0} \cdots (\log_r x)^{\alpha_r}, \; c_f \in \mathbb{R}^{>}, \; \delta \in \mathbb{L}^{\prec 1} \\ \log f \; &:= \; \alpha_0 \log x + \cdots + \alpha_r \log_{r+1} x + \log c_f + \log (1 + z) \circ \delta. \end{aligned}$$

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Definition

Consider a logarithmic function $\log: \mathbb{T}^{>0} \to \mathbb{T}$ *extending the one on* $\mathbb{R}^{>0}$, *such that*

T1. $\operatorname{dom} \log = \mathbb{T}^{>0}$.

T2. $\log \mathfrak{m} \in \mathbb{T}_{>} := \{f \in \mathbb{T} : \operatorname{supp} f \succ 1\}$ *for all* $\mathfrak{m} \in \mathfrak{T}$.

T3. $\log(1+\varepsilon) = \log(1+z) \circ \varepsilon$ *for all* $\varepsilon \in \mathbb{T}^{\prec 1}$.

Then we say that $\mathbb{T} = \mathbb{R}[[\mathfrak{T}]]_{\mathscr{S}}$ *is a* ***field of*** $\mathscr{S}$***-based transseries***.

Exponential extensions

Given a field of transseries $\mathbb{T} = \mathbb{R}[[\mathfrak{T}]]_{\mathscr{S}}$, consider:

$$\begin{aligned} \mathfrak{T}_{\exp} &:= \exp \mathbb{T}_{\succ} \\ e^{\varphi} \preccurlyeq e^{\psi} &\Leftrightarrow \varphi \leqslant \psi \\ \mathbb{T}_{\exp} &:= \mathbb{R}[[\mathfrak{T}_{\exp}]]_{\mathscr{S}} \end{aligned}$$

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$$\begin{aligned} \log \colon \mathbb{T}_{\exp}^{>0} &\rightarrow \mathbb{T}_{\exp} \\ \underbrace{e^{\varphi}}_{\mathfrak{T}_{\exp}} \underbrace{c}_{\mathbb{R}^{>0}} (1 + \underbrace{\delta}_{\mathbb{T}_{\exp}^{<1}}) &\mapsto \underbrace{\varphi}_{\mathbb{T}_{\succ}} + \underbrace{\log c}_{\mathbb{R}} + \underbrace{\log(1+z) \circ \delta}_{\mathbb{T}_{\exp}^{<1}}. \end{aligned}$$

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Proposition

The exponential extension $\mathbb{T}_{\exp}$ of $\mathbb{T}$ is again a field of transseries.

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Then $\mathfrak{S} \subseteq \mathfrak{m}\{\mathfrak{e}_1, \ldots, \mathfrak{e}_k\}^*$ for $\mathfrak{m} \in \mathfrak{T}$, $\mathfrak{e}_1, \ldots, \mathfrak{e}_k \in \mathfrak{T}^{\prec 1}$.

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For $n \in \mathbb{N}$ with $\mathfrak{m}, \mathfrak{e}_1, \ldots, \mathfrak{e}_k \in \mathfrak{L}_{\exp,\overset{n\times}{\ldots},\exp}$, we have $f \in \mathbb{L}_{\exp,\overset{n\times}{\ldots},\exp}$. □

Well-based transseries

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$$\begin{aligned} \mathfrak{T}_0 &:= \mathfrak{L} \\ \mathfrak{T}_{\alpha+1} &:= \mathfrak{T}_{\alpha,\exp} \\ \mathfrak{T}_\lambda &:= \bigcup_{\alpha<\lambda} \mathfrak{T}_\alpha \end{aligned}$$

$$\begin{aligned} \mathbb{T}_0 &:= \mathbb{R}[[\mathfrak{T}_0]] & & \\ \mathbb{T}_{\alpha+1} &:= \mathbb{R}[[\mathfrak{T}_{\alpha+1}]] &=& \ \mathbb{T}_{\alpha,\exp} \\ \mathbb{T}_\lambda &:= \mathbb{R}[[\mathfrak{T}_\lambda]] &\supsetneq& \ \bigcup_{\alpha<\lambda} \mathbb{T}_\alpha \end{aligned}$$

Well-based transseries

$$\begin{array}{lll} \mathfrak{T}_0 & := & \mathfrak{L} \\ \mathfrak{T}_{\alpha+1} & := & \mathfrak{T}_{\alpha,\exp} \\ \mathfrak{T}_\lambda & := & \bigcup_{\alpha<\lambda} \mathfrak{T}_\alpha \end{array} \qquad \begin{array}{lllll} \mathbb{T}_0 & := & \mathbb{R}[[\mathfrak{T}_0]] & & \\ \mathbb{T}_{\alpha+1} & := & \mathbb{R}[[\mathfrak{T}_{\alpha+1}]] & = & \mathbb{T}_{\alpha,\exp} \\ \mathbb{T}_\lambda & := & \mathbb{R}[[\mathfrak{T}_\lambda]] & \supsetneq & \bigcup_{\alpha<\lambda} \mathbb{T}_\alpha \end{array}$$

Proposition

For $\alpha < \beta$, we have $\mathbb{T}_\alpha \subsetneq \mathbb{T}_\beta$.

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Proposition

For $\alpha < \beta$, we have $\mathbb{T}_\alpha \subsetneq \mathbb{T}_\beta$.

Corollary

There is no non-trivial well-based field of transseries that is closed under exponentiation.

Upward and downward shifting

Logarithmic transseries

$$x^{\alpha_0} \cdots (\log_r x)^{\alpha_r} \xrightarrow{\cdot \circ \log} (\log x)^{\alpha_0} \cdots (\log_{r+1} x)^{\alpha_r} \in \mathfrak{T}_0$$

$$x^{\alpha_0} \cdots (\log_r x)^{\alpha_r} \xrightarrow{\cdot \circ \exp} \mathrm{e}^{\alpha_0 x} x^{\alpha_1} \cdots (\log_{r-1} x)^{\alpha_r} \in \mathfrak{T}_1$$

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Strong linearity: $\mathbb{T}_0 \xrightarrow{\cdot \circ \log} \mathbb{T}_0$ and $\mathbb{T}_0 \xrightarrow{\cdot \circ \exp} \mathbb{T}_1$

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Inductive step

For $\varphi \in \mathbb{T}_{\alpha,\succ}$, $\quad \mathrm{e}^{\varphi} \in \mathfrak{T}_{\alpha+1}$, $\quad \varphi \circ \log \in \mathbb{T}_{\alpha,\succ}$, $\quad \varphi \circ \exp \in \mathbb{T}_{\beta,\succ}$, $\quad \beta = \begin{cases} \alpha+1 & \text{if } \alpha < \omega \\ \alpha & \text{otherwise} \end{cases}$

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$$\mathrm{e}^{\varphi}\circ\log := \mathrm{e}^{\varphi\circ\log} \in \mathfrak{T}_{\alpha+1}$$

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Logarithmic transseries

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Upward and downward shifting

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$$\mathrm{e}^{\varphi} \circ \log := \mathrm{e}^{\varphi \circ \log} \in \mathfrak{T}_{\alpha+1}$$

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Alternative notation: $\varphi\uparrow := \varphi \circ \exp$, $\varphi\downarrow := \varphi \circ \log$

Non-closure under exponentiation

$$f_\alpha := \sqrt{x} - \sum_{0<\beta<\alpha} \mathrm{e}^{f_\beta \circ \log}$$

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$$\beta < \alpha \Rightarrow f_\alpha < f_\beta$$

$$\operatorname{supp} f_\alpha \cong \alpha$$

Well-based transseries — continued

Proposition

Let $\mathscr{S}$ be the type of countable supports.

There exists a non-trivial field of $\mathscr{S}$-based transseries that is closed under exponentiation.

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Logarithmic depth $\operatorname{ld}(f)$ of $f \in \mathbb{T}$:= smallest $n \in \mathbb{N}$ such that $f \in \mathbb{E} \circ \log_n$ or infinity.

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$$\begin{aligned} \mathrm{ld}\,(\mathrm{e}^{\mathrm{e}^{3x}+2x} - x^3\,\mathrm{e}^x) &= 0 \\ \mathrm{ld}\,(x^x) &= 1 \\ \mathrm{ld}\,(x + \log x + \log\log x + \cdots) &= \infty. \end{aligned}$$

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\end{aligned}$$

Proposition

The field of well-based transseries of finite logarithmic depth is an exp-log field.

Alternative construction (Écalle, Dahn-Göring)

Exponential transseries.

$\mathbb{E}$:= smallest subset of $\mathbb{T}$ with $\mathbb{E} \supseteq x^{\mathbb{R}}$ that is closed under $\sum$ and $\exp$.

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$$\mathfrak{E}_0 = x^{\mathbb{R}} \qquad \mathbb{E}_0 = \mathbb{R}[[\mathfrak{E}_0]]$$

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$$\begin{aligned} \mathfrak{E}_0 &= x^{\mathbb{R}} & \mathbb{E}_0 &= \mathbb{R}[[\mathfrak{E}_0]] & \\ \mathfrak{E}_k &= x^{\mathbb{R}} \exp \mathbb{E}_{k-1,\succ} & \mathbb{E}_k &= \mathbb{R}[[\mathfrak{E}_k]] & k = 1, 2, \ldots \end{aligned}$$

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$$
\begin{aligned}
\mathfrak{E}_0 &= x^{\mathbb{R}} & \mathbb{E}_0 &= \mathbb{R}[[\mathfrak{E}_0]] \\
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\mathfrak{E} &= \mathfrak{E}_0 \cup \mathfrak{E}_1 \cup \cdots & \mathbb{E} &= \mathbb{R}[[\mathfrak{E}]] \\
& & &= \mathbb{E}_0 \cup \mathbb{E}_1 \cup \cdots
\end{aligned}
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Logarithmic closure.

$$\mathbb{T} = \mathbb{E} \cup \mathbb{E} \circ \log \cup \mathbb{E} \circ \log_2 \cup \cdots.$$

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Logarithmic closure.

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Level.

The **level** of $f \in \mathbb{T}$ is the largest $l \in \mathbb{Z}$ with $f \in \mathbb{E} \circ \exp_l$.
Here $\exp_l x = \log_{-l} x$ if $l < 0$.

Flatness relations. For $f, g \in \mathbb{T}^{\neq 0}$,

$$\begin{aligned} f &\lll g &\iff& \log|f| \prec \log|g| \\ f &\preccurlyeq\!\!\!\prec g &\iff& \log|f| \preccurlyeq \log|g| \\ f &\asymp\!\!\!\asymp g &\iff& \log|f| \asymp \log|g|. \end{aligned}$$

Flatness

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$$
\begin{aligned}
f \ll g &\iff \log|f| \prec \log|g| \\
f \lll g &\iff \log|f| \preccurlyeq \log|g| \\
f \asymp g &\iff \log|f| \asymp \log|g|.
\end{aligned}
$$

Recursive expansions. $x \ll \mathrm{e}^x$

$$
\frac{1}{1 - x^{-1} - \mathrm{e}^{-x}} = \frac{1}{1 - x^{-1}} + \left(\frac{1}{1 - x^{-1}}\right)^2 \mathrm{e}^{-x} + \left(\frac{1}{1 - x^{-1}}\right)^3 \mathrm{e}^{-2x} + \cdots
$$

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Recursive expansions. $x \lll \mathrm{e}^x$

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Flatness relations. For $f, g \in \mathbb{T}^{\neq 0}$,

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Recursive expansions. Let $b_1, \ldots, b_n \in \mathbb{T}^{\succ 1}$ with $b_1 \lll \cdots \lll b_n$. Then

$$\begin{aligned} \varphi: x_1^{\mathbb{R}} \dot{\times} \cdots \dot{\times} x_n^{\mathbb{R}} &\longrightarrow \mathbb{T} \\ x_1^{\alpha_1} \cdots x_n^{\alpha_n} &\longmapsto b_1^{\alpha_1} \cdots b_n^{\alpha_n} \end{aligned}$$

extends by strong linearity into an embedding

$$\hat{\varphi}: \mathbb{R}[[x_1^{\mathbb{R}} \dot{\times} \cdots \dot{\times} x_n^{\mathbb{R}}]]_{\mathscr{S}} \longrightarrow \mathbb{T}.$$

We define $\mathbb{R}[[b_1; \ldots; b_n]]_{\mathscr{S}} := \operatorname{im} \hat{\varphi}$.

Transbases

$\mathbb{T} = \mathbb{R}[[[x]]]$, the field of grid-based transseries.

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Definition

A ***transbasis*** *is a finite tuple* $\mathfrak{B} = (\mathfrak{b}_1, \ldots, \mathfrak{b}_n) \in \mathbb{T}^n$ *such that*

TB1. $\mathfrak{b}_1, \ldots, \mathfrak{b}_n \succ 1$ *and* $\mathfrak{b}_1 \lll \cdots \lll \mathfrak{b}_n$.

TB2. $\mathfrak{b}_1 = \exp_l x$ *for some* $l \in \mathbb{Z}$.

TB3. $\log \mathfrak{b}_i \in \mathbb{R}[[\mathfrak{b}_1; \ldots; \mathfrak{b}_{i-1}]]_{\succ}$, *for* $i = 2, \ldots, n$.

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TB3. $\log \mathfrak{b}_i \in \mathbb{R}[[\mathfrak{b}_1; \ldots; \mathfrak{b}_{i-1}]]_{>}$, *for* $i = 2, \ldots, n$.

- l is called the **level** of $\mathfrak{B}$.

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TB3. $\log \mathfrak{b}_i \in \mathbb{R}[[\mathfrak{b}_1; \ldots; \mathfrak{b}_{i-1}]]_{>}$, *for* $i = 2, \ldots, n$.

- l is called the **level** of $\mathfrak{B}$.
- $\mathfrak{B}$ is a transbasis **for** each element of $\mathbb{R}[[\mathfrak{B}^{\mathbb{R}}]] := \mathbb{R}[[\mathfrak{b}_1; \ldots; \mathfrak{b}_n]]$.

Transbases

$\mathbb{T} = \mathbb{R}[[[x]]]$, the field of grid-based transseries.

Definition

*A **transbasis** is a finite tuple $\mathfrak{B} = (\mathfrak{b}_1, \ldots, \mathfrak{b}_n) \in \mathbb{T}^n$ such that*

TB1. $\mathfrak{b}_1, \ldots, \mathfrak{b}_n > 1$ *and* $\mathfrak{b}_1 \ll \cdots \ll \mathfrak{b}_n$.

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$(x, e^{\sqrt{x}}, e^{x\sqrt{x}})$ is a transbasis for $e^{(x+1)^{3/2}} = e^{x^{3/2} + (3/2) x^{1/2} + c x^{-1/2} + \cdots}$

$(x, e^{(x + 3/2)\sqrt{x}})$ is a transbasis for $e^{(x+1)^{3/2}}$

$(\log x, x, e^x, x^x)$ is a transbasis of level -1 for $\Gamma(x)$

The incomplete transbasis theorem

Theorem

Let $\mathfrak{B} = (\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$ *be a transbasis of level* l *and* $f \in \mathbb{T}$ *a transseries of level* l'.

Then there exists a transbasis $\hat{\mathfrak{B}}$ *of level* $\min(l, l')$ *for* f *that extends* $\mathfrak{B}$.

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Proof in a special case. Assume that $f = \mathrm{e}^g$ and that $\mathfrak{B}$ is a transbasis for g.

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Let $k \in \{0, \ldots, n\}$ be minimal such that there exist $\lambda_{k+1}, \ldots, \lambda_n \in \mathbb{R}$ with

$$g = \lambda_n \log \mathfrak{b}_n + \cdots + \lambda_{k+1} \log \mathfrak{b}_{k+1} + \delta, \qquad \delta \prec \log \mathfrak{b}_{k+1}.$$

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If $\delta \preccurlyeq 1$, then

$$f = \mathrm{e}^\delta \mathfrak{b}_{k+1}^{\lambda_{k+1}} \cdots \mathfrak{b}_n^{\lambda_n} \in \mathbb{R}[[\mathfrak{B}^{\mathbb{R}}]].$$

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$$f = \mathrm{e}^{\delta} \mathfrak{b}_{k+1}^{\lambda_{k+1}} \cdots \mathfrak{b}_n^{\lambda_n} \in \mathbb{R}[[\mathfrak{B}^{\mathbb{R}}]].$$

Otherwise, let $i \leqslant k$ be such that $\log \mathfrak{b}_i \prec \delta \prec \log \mathfrak{b}_{i+1}$.

$$f = \mathrm{e}^{\delta_{\preccurlyeq}} \mathrm{e}^{\delta_{\succ}} \mathfrak{b}_{k+1}^{\lambda_{k+1}} \cdots \mathfrak{b}_n^{\lambda_n} \in \mathbb{R}[[\hat{\mathfrak{B}}^{\mathbb{R}}]]$$

$$\hat{\mathfrak{B}} := (\mathfrak{b}_1, \ldots, \mathfrak{b}_i, \mathrm{e}^{|\delta_{\succ}|}, \mathfrak{b}_{i+1}, \ldots, \mathfrak{b}_n).$$

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Otherwise, $\operatorname{supp} f \subseteq (\exp_l x)^{\mathbb{R}} \mathrm{e}^{g_0 + g_1 \mathbb{N} + \cdots + g_k \mathbb{N}}$ with $g_0, \ldots, g_k \in \mathbb{E}_{h-1} \circ \exp_l x$.

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Induction hypothesis + special case $\rightsquigarrow$ $\mathfrak{B}$ transbasis for $\mathrm{e}^{g_0}, \ldots, \mathrm{e}^{g_k}$.

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Hence $\mathfrak{B}$ is a transbasis for each monomial in $\operatorname{supp} f$.
And thus for f itself. $\square$

Strong derivations

$\mathfrak{M} \rightarrow$ totally ordered monomial group.

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Definition

*A **strong derivation** on* $\mathbb{R}[[\mathfrak{M}]]$ *is a map* $\partial\colon \mathbb{R}[[\mathfrak{M}]] \to \mathbb{R}[[\mathfrak{M}]]$ *such that*

D1. $\partial c = 0$ *for all* $c \in \mathbb{R}$.

D2. $\partial(fg) = (\partial f)g + f\,\partial g$ *for all* $f, g \in \mathbb{R}[[\mathfrak{M}]]$.

D3. ∂ *is strongly* $\mathbb{R}$*-linear.*

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Let $\partial\colon \mathfrak{M} \to \mathbb{R}[[\mathfrak{M}]]$ *be a mapping such that* $\partial(\mathfrak{m}\,\mathfrak{n}) = (\partial\mathfrak{m})\,\mathfrak{n} + \mathfrak{m}\,\partial\mathfrak{n}$ *for all* $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$.
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Notation. $f' = \partial f$ and $f^\dagger = \dfrac{\partial f}{f}$ if ∂ is clear from the context.

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Proof. Let $\mathfrak{G} \subseteq \mathfrak{M}$ be grid-based and let $\mathfrak{e}_1, \ldots, \mathfrak{e}_k \prec 1$ and $\mathfrak{f}$ be in $\mathfrak{M}$ with

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Strong derivations — continued

Proposition

Let $\partial : \mathfrak{M} \to \mathbb{R}[[\mathfrak{M}]]$ be a mapping such that $\partial(\mathfrak{m}\,\mathfrak{n}) = (\partial\mathfrak{m})\,\mathfrak{n} + \mathfrak{m}\,\partial\mathfrak{n}$ for all $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$. Then ∂ is a grid-based mapping that extends uniquely into a strong derivation on $\mathbb{R}[[\mathfrak{M}]]$.

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Hence $(\mathfrak{m}')_{\mathfrak{m} \in \mathfrak{G}}$ is grid-based.

Indeed, $\operatorname{supp} \mathfrak{G}' \subseteq \mathfrak{G} \operatorname{supp} \mathfrak{G}^\dagger$ is grid-based.

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Proof. Let $\mathfrak{S} \subseteq \mathfrak{M}$ be grid-based and let $\mathfrak{e}_1, \ldots, \mathfrak{e}_k \prec 1$ and $\mathfrak{f}$ be in $\mathfrak{M}$ with

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Indeed, $\operatorname{supp} \mathfrak{S}' \subseteq \mathfrak{S} \operatorname{supp} \mathfrak{S}^\dagger$ is grid-based.

Given $\mathfrak{v} \in \mathfrak{S}$, the $(\mathfrak{m}, \mathfrak{n}) \in \mathfrak{S} \times \operatorname{supp} \mathfrak{S}^\dagger$ with $\mathfrak{v} = \mathfrak{m}\,\mathfrak{n}$ form a finite antichain.

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Let $\partial\colon \mathfrak{M} \rightarrow \mathbb{R}[[\mathfrak{M}]]$ *be a mapping such that* $\partial(\mathfrak{m}\,\mathfrak{n}) = (\partial\mathfrak{m})\,\mathfrak{n} + \mathfrak{m}\,\partial\mathfrak{n}$ *for all* $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$. *Then* ∂ *is a grid-based mapping that extends uniquely into a strong derivation on* $\mathbb{R}[[\mathfrak{M}]]$.

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Strong derivations — continued

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Hence $(\mathfrak{m}')_{\mathfrak{m} \in \mathfrak{G}}$ is grid-based.

Uniqueness: $(f, g) \mapsto (f g)'$ and $(f, g) \mapsto f' g + f g'$ strongly bilinear, same on $\mathfrak{M}^2$. $\square$

Exp-log derivations

Definition

Assume that $\exp$ *is a partial exponential function on* $\mathbb{R}[[\mathfrak{M}]]$.

An ***exp-log derivation*** *on* $\mathbb{R}[[\mathfrak{M}]]$ *is a derivation* ∂ *that satisfies*

ED. $\partial \exp f = (\partial f) \exp f$, *for all* $f \in \operatorname{dom} \exp$.

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Proposition

Let $\partial : \mathfrak{M} \rightarrow \mathbb{R}[[\mathfrak{M}]]$ *be a strong derivation with* $\partial \log \mathfrak{m} = \mathfrak{m}^{\dagger}$ *for all* $\mathfrak{m} \in \mathfrak{M}$.

Then ∂ *is a strong exp-log derivation on* $\mathbb{R}[[\mathfrak{M}]]$.

Exp-log derivations

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Assume that $\exp$ *is a partial exponential function on* $\mathbb{R}[[\mathfrak{M}]]$.

An ***exp-log derivation*** *on* $\mathbb{R}[[\mathfrak{M}]]$ *is a derivation* ∂ *that satisfies*

ED. $\partial \exp f = (\partial f) \exp f$, *for all* $f \in \operatorname{dom} \exp$.

Proposition

Let $\partial : \mathfrak{M} \to \mathbb{R}[[\mathfrak{M}]]$ *be a strong derivation with* $\partial \log \mathfrak{m} = \mathfrak{m}^{\dagger}$ *for all* $\mathfrak{m} \in \mathfrak{M}$.

Then ∂ *is a strong exp-log derivation on* $\mathbb{R}[[\mathfrak{M}]]$.

Proof. Let $f = c\,\mathfrak{m}\,(1 + \varepsilon)$, $c \in \mathbb{R}^{\neq 0}$, $\mathfrak{m} \in \mathfrak{M}$, $\varepsilon \in \mathbb{R}[[\mathfrak{M}]]^{\prec 1}$.

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$$(\log(1+\varepsilon))' \;=\; (\varepsilon - {}^1\!/_2\,\varepsilon^2 + {}^1\!/_3\,\varepsilon^3 + \cdots)' \;=\; \varepsilon'(1 - \varepsilon + \varepsilon^2 + \cdots) \;=\; (1+\varepsilon)^\dagger.$$

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$$(\log f)' = (\log c + \log \mathfrak{m} + \log(1+\varepsilon))' = \mathfrak{m}^\dagger + (1+\varepsilon)^\dagger = f^\dagger.$$

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On $\mathfrak{L}$, we must have

$$\partial(x^{\alpha_0} \cdots (\log_r x)^{\alpha_r}) = \left(\frac{\alpha_0}{x} + \frac{\alpha_1}{x \log x} + \cdots + \frac{\alpha_r}{x \log x \cdots \log_r x}\right) x^{\alpha_0} \cdots (\log_r x)^{\alpha_r}.$$

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Assume $\partial \colon \mathbb{T}_h \to \mathbb{T}_h$. On $\mathfrak{T}_{h+1} = \exp \mathbb{T}_{h,>}$, we must have $\partial \mathrm{e}^{\varphi} = (\partial \varphi) \mathrm{e}^{\varphi}$,

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Asymptotic properties of the derivation

Proposition

The derivation on $\mathbb{T}$ is ***asymptotic*** *and* ***positive*** *in the following sense:*

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Easy exercise if $n = 1$.

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$$\mathfrak{b}_1 \ll \mathfrak{b}_2 \ll \cdots \ll \mathfrak{b}_n$$

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Hence $\mathfrak{m}' \prec \mathfrak{n}'$ in all cases.

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Assume $f \prec g \not\asymp 1$.

For all $\mathfrak{n} \in \operatorname{supp}(g - \tau_g)$, we have $\mathfrak{n}' \prec \mathfrak{d}_g'$, whence $(g - \tau_g)' \prec \mathfrak{d}_g'$ and $g' \sim c_g \mathfrak{d}_g'$.

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We conclude by induction. □

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Corollary

Given $y \in \mathbb{T}$ and $r \in \mathbb{N}$, we have $y^{(r)} \preccurlyeq y^c$ for some $c \in \mathbb{Q}^{>0}$.

Integration

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There exists a unique strong map $\int : \mathbb{T} \to \mathbb{T}$ *with* $(\int f)' = f$ *and* $(\int f)_1 = 0$ *for all* $f \in \mathbb{T}$.
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Remainder shown at the end of Lesson 3. □

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We say that a difference operator $\sigma\colon \mathbb{R}[[\mathfrak{M}]] \rightarrow \mathbb{R}[[\mathfrak{N}]]$ *is **asymptotic** resp. **positive** if*

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Composition of transseries

Proposition

Given $g \in \mathbb{T}^{>\mathbb{R}} = \mathbb{R}^{\succ 1, >0}$*, there exists a unique strong exp-log difference operator* σ *on* $\mathbb{T}$ *with* $\sigma x = g$*. This operator is asymptotic and positive. For* $f \in \mathbb{T}$*, we define* $f \circ g := \sigma f$*.*

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Assume $\sigma : \mathbb{T}_h \rightarrow \mathbb{T}$. On $\mathfrak{T}_{h+1} = \exp \mathbb{T}_{h,\succ}$, we must have

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Properties of composition

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Proofs. See LNM 1888.