Lesson 8 — Valued fields

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Valued fields

Definition

Let K be a field and Γ a totally ordered abelian group.

*A **valuation** is a map $v: K \to \Gamma \cup \{\infty\}$ such that*

- $v(a) = \infty$ *if and only if* $a = 0$;
- $v(ab) = v(a) + v(b)$;
- $v(a+b) \geqslant \min(v(a), v(b))$ *with equality if* $v(b) \neq v(a)$.

In that case, we define

$$\begin{aligned} \mathcal{O}_K &:= \{a \in K : v(a) \geqslant 0\} && \text{the \textbf{valuation ring}} \\ \mathcal{o}_K &:= \{a \in K : v(a) > 0\} && \text{its \textbf{maximal ideal}} \\ \boldsymbol{k}_K &:= \mathcal{O}_K / \mathcal{o}_K && \text{its \textbf{residue field}} \end{aligned}$$

Convention. We will usually assume that $\Gamma = v(K^{\neq 0})$.

Examples

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Well-based series. $K:=R[[z^\Gamma]]$, R field, Γ totally ordered group.

$$z^\alpha \succcurlyeq z^\beta \iff \alpha \leqslant \beta$$
$$v(f) := \alpha, \qquad \text{for } f\in K^{\neq 0} \text{ with } \mathfrak{d}_f = x^\alpha.$$

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***p*-adic numbers.** $K = \mathbb{Q}_p$, $\Gamma := \mathbb{Z}$, p-adic valuation.

Asymptotic relations

Let K be a valued field. For $x,y \in K$, we define

$$\begin{aligned} x \prec y &\iff v(x) > v(y) \iff x \in \mathcal{O} y \wedge y \neq 0 \\ x \preccurlyeq y &\iff v(x) \geqslant v(y) \iff x \in \mathcal{O} y \\ x \asymp y &\iff v(x) = v(y) \iff x \preccurlyeq y \preccurlyeq x \\ x \sim y &\iff x - y \prec x. \end{aligned}$$

Note. The axioms of valued fields can be reformulated in terms of $\preccurlyeq$.

Both points of views are essentially equivalent.

Always remind the reversal of the ordering.

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Proof. Embed increasingly large subgroups Δ of Γ into G.
Given $\Gamma \supsetneq \Delta \hookrightarrow G$ and $\gamma \in \Gamma \setminus \Delta$, let $k \in \mathbb{N}$ with $k\mathbb{Z} = \{n \in \mathbb{Z} : n\gamma \in \Delta\}$.
Take $\mathfrak{z}^{\gamma} \in G$ with $v(\mathfrak{z}^{\gamma}) = \gamma$ such that $(\mathfrak{z}^{\gamma})^k = \mathfrak{z}^{k\gamma}$ whenever $k > 0$. Apply Zorn. □

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Examples:

- $G = K^{\neq 0}$ for an algebraically closed valued field K.
- $G = K^{>0}$ for a real closed field K.

Newton polynomials

Let K be a valued field and $P \in K[Y]^{\neq 0}$. We extend the valuation v to $K[Y]$ by

$$v(P_d Y^d + \cdots + P_0) \;:=\; \min(v(P_d), \ldots, v(P_0)).$$

We also define the relation $\propto$ on $\boldsymbol{k}[Y]$ by

$$A \;\propto\; B \;\Longleftrightarrow\; (\exists \lambda \in \boldsymbol{k}^{\neq 0})\; B = \lambda A.$$

The **projective Newton polynomial** $N_{\propto}(P) \in \boldsymbol{k}[Y]/{\propto}$ is defined by

$$N_{\propto}(P) \;:=\; \overline{a P}/{\propto}, \qquad \text{where } a \in K \text{ is such that } a P \asymp 1.$$

The **monic Newton polynomial** $N_{\mathrm{mon}}(P) \in \boldsymbol{k}[Y]$ is the monic polynomial with

$$N_{\mathrm{mon}}(P)/{\propto} \;=\; N_{\propto}(P)$$

If K has a monomial group, then we define the **Newton polynomial** $N(P) \in \boldsymbol{k}[Y]$ by

$$N(P) \;:=\; \overline{\mathfrak{z}^{-v(P)} P}.$$

Newton degree

Given $P \in K[Y]$ and $\gamma \in \Gamma$, one may consider the asymptotic equation

$$P(y) = 0, \qquad v(y) > \gamma.$$

The **Newton degrees** of this equation is defined by

$$\deg_{>\gamma} P := \operatorname{val} N_{\propto}(P_{\times a})$$

where $a \in K^{\neq 0}$ is such that $v(a) = \gamma$.

Equations of Newton degree one are said to be **quasi-linear**.

Definition

We say that K is ***henselian*** *if any quasi-linear equation has a solution in K.*

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Let $P \in K[Y]^{\neq 0}$ and $\gamma \in \Gamma$. If $\boldsymbol{k}$ is algebraically closed, then

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has exactly $\deg_{>\gamma} P$ solutions in K, when counting with multiplicities.

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Proof. Straightforward adaptation of proof from Lesson 4. □

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Corollary

a) *If* $\boldsymbol{k}$ *is algebraically closed, then so is* K.

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Proof of (b). Since $k[\mathrm{i}]$ is algebraically closed, so is $K[\mathrm{i}]$, by (a).

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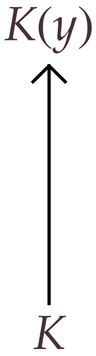
The complex roots of P in $K[\mathrm{i}]$ come in conjugate pairs.
If $\deg P$ is odd, this means that P has at least one root in K. □

Adjoining single elements to valued fields

Given a valued field K, describe the valued field extensions $L := K(y) \supseteq K$.



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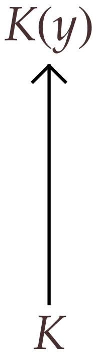

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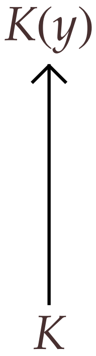

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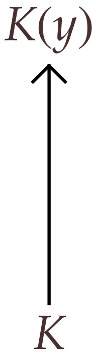

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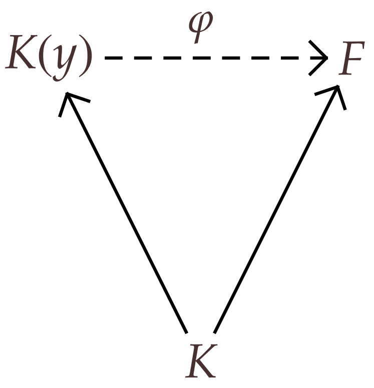

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How unique is the extension $K \subseteq K(y)$?

Given a valued field extension $F \supseteq K$ and $a \in F$ of "same type over K" as y, does there exist a unique embedding of valued fields $\varphi: K(y) \to F$ with $\varphi(y) = a$?

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- (a_ρ) is of **algebraic type** if there exists a $P \in K[Y]$ with $P(a_\rho) \rightsquigarrow 0$

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If (a_ρ) **pseudo-diverges**, then

- (a_ρ) is of **algebraic type** if there exists a $P \in K[Y]$ with $P(a_\rho) \rightsquigarrow 0$
- Otherwise, (a_ρ) is of **transcendental type**.

Transcendental immediate extensions

Lemma TR-IMM

Let (a_ρ) be pseudo-divergent of transcendental type. Then v extends to $K(Y)$ via

$$v(P) \;:=\; \text{eventual value of } v(P(a_\rho)), \qquad \text{for any } P \in K[Y].$$

The extension $L := K(Y) \supseteq K$ is immediate and $a_\rho \rightsquigarrow Y$ in L. Moreover, if $a_\rho \rightsquigarrow a$ in another immediate extension $F \supseteq K$, then there is a unique embedding $\varphi: L \to F$ over K with $\varphi(Y) = a$.

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Hence, $v(P(a)) = v(P(a_\rho)) = v(P)$, eventually, so $P(a) \neq 0$ and a is transcendental.
We conclude that $\exists!$ ring morphism $\varphi: L \to F$ with $\varphi(Y) = a$ and φ preserves v. $\square$

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yields an extension of v *to* L*. This extension* $L := K(Y) \supseteq K$ *is immediate and* $a_\rho \rightsquigarrow y$ *in* L*. Moreover, if* $a_\rho \rightsquigarrow a$ *and* $\mu(a) = 0$ *for* a *in another immediate extension* $F \supseteq K$*, there is a unique embedding* $\varphi : L \to F$ *over* K *with* $\varphi(y) = a$*.*

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But $Q(a_\rho)\,\mu(\alpha_\rho)$ is eventually increasing or eventually infinite.
Eventually, this yields $v(Q(a_\rho)\,\mu(a_\rho)) > v(R(a_\rho))$ and $v(s) + v(t) = v(R(a_\rho)) = v(s\,t)$. $\square$

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Proof. Combine Lemmas TR-IMM and ALG-IMM, and apply Zorn. □

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Proof. Any quasi-linear $y = P(y)$, $y \prec 1$ with no solution in K gives rise to a divergent pc-sequence (a_ρ) with $P(a_\rho) \rightsquigarrow 0$: $a_0 = 0$, $a_{\alpha+1} = P(a_\alpha)$, $a_\lambda := \ell$, whenever $(a_\alpha)_{\alpha<\lambda} \rightsquigarrow \ell$. □

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Proof. By what precedes and Newton polygon method. □

Adjoining transcendental residues

Lemma TR-RES

Define $v\colon K(Y)^{\neq 0} \to \Gamma$ *with* $v(P/Q) = v(P) - v(Q)$ *for* $P, Q \in K[Y]^{\neq 0}$ *and*

$$v(P) \;=\; \min\big(v(P_0), \ldots, v(P_d)\big), \qquad \text{for any } P = P_d Y^d + \cdots + P_0 \in K[Y].$$

Then $L := K(Y) \supseteq K$ *is a valued field extension with* $\boldsymbol{k}_L = \boldsymbol{k}(\bar{Y})$ *and* $\Gamma_L = \Gamma_K$.

For any valued field extension $F \supseteq K$ *with* $\Gamma_F = \Gamma_K$ *and* $a \in \mathcal{O}_L$ *such that* $\bar{a}$ *is transcendental over* $\boldsymbol{k}_K$, *there exists a unique valued field embedding* $\varphi\colon L \to F$ *over* K *with* $\varphi(Y) = a$.

Proof. $L \supseteq K$ is easily seen to be a valued field extension. Clearly, $\Gamma_L = \Gamma_K$.

Consider $A \in L$ with $v(A) = 0$. We claim that $\bar{A} \in \boldsymbol{k}(\bar{Y})$, which proves $\boldsymbol{k}_L = \boldsymbol{k}(\bar{Y})$.

Indeed, $A = P/Q$ with $P, Q \in K[Y]$ such that $v(P) = v(Q) = 0$.

Then $\bar{P}, \bar{Q} \in \boldsymbol{k}[\bar{Y}]^{\neq 0}$, so $\bar{A} = \bar{P}/\bar{Q} \in \boldsymbol{k}(\bar{Y})$.

Y, a transcendental over $K \Longrightarrow \exists!$ field embedding $L \to F$ over K with $\varphi(Y) = a$.

$v(a) = 0 \Longrightarrow v(P(a)) = \min\big(v(P_0), \ldots, v(P_d)\big)$ for any $P = P_d Y^d + \cdots + P_0 \in K[Y]$. □

Adjoining algebraic residues

Lemma ALG-RES

Let $\mu \in K[Y]$ with $v(\mu) = 0$ and $\bar{\mu} \in \boldsymbol{k}[\bar{Y}]$ irreducible of degree $d = \deg \mu$. Then $y := Y + (\mu)$ in $L := K[Y]/(\mu)$. Then $L \supseteq K$ is a valued field extension with $\boldsymbol{k}_L = \boldsymbol{k}[\bar{y}]/(\bar{\mu})$ and $\Gamma_L = \Gamma_K$ for

$$v(P(y)) \;=\; \min(v(P_0), \ldots, v(P_{d-1})), \qquad \text{for any } P \in K[Y]_d.$$

For any valued field extension $F \supseteq K$ with $\Gamma_F = \Gamma_K$ and $a \in \mathcal{O}_L$ such that $\boldsymbol{k}(\bar{a}) \cong \boldsymbol{k}_L$, there exists a unique valued field embedding $\varphi: L \to F$ over K with $\varphi(y) = a$.

Note. μ is irreducible in $K[Y]$ since $\bar{\mu}$ is irreducible in $\boldsymbol{k}[\bar{Y}]$, by Gauss' lemma.

Proof. Similar to previous lemma, except for $v(s\,t) = v(s) + v(t)$ in L.
Any $s \in L$ can be decomposed $s = u\,\tilde{s}$ with $u \in K$ and $\tilde{s} \in L$ such that $v(\tilde{s}) = 0$.
Without loss of generality, we may therefore assume that $v(s) = v(t) = 0$.
Then $\bar{s}, \bar{t} \in \boldsymbol{k}_L^{\neq 0}$, so $\overline{s\,t} = \bar{s}\,\bar{t} \in \boldsymbol{k}_L^{\neq 0}$, hence $v(s\,t) = 0$. □

Adjoining "transcendental" elements to Γ

Lemma TR-VAL

Let $\Delta \supseteq \Gamma$ be a totally ordered group and $\gamma \in \Delta$ be such that $\Delta = \Gamma \oplus \mathbb{Z}\gamma$. Then there is a unique valued field extension $L := K(Y) \supseteq K$ with $v(Y) = \gamma$. It is given by

$$v(P) := \min\,(v(P_0), \ldots, v(P_d) + d\,\gamma), \qquad \textit{for all } P = P_d Y^d + \cdots + P_0 \in K[Y]^{\neq 0}.$$

Moreover, if $F \supseteq K$ is a valued field extension and $a \in F$ transcendental such that $v(a)$ and γ lie in the same cut over Γ, then $\exists!$ valued field embedding $\varphi\colon L \to F$ over K with $\varphi(a) = Y$.

Exercise. We also have $\boldsymbol{k}_L = \boldsymbol{k}_K$.

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Moreover, if $F \supseteq K$ is a valued field extension and $a \in F$ transcendental such that $v(a)$ and γ lie in the same cut over Γ, then $\exists!$ valued field embedding $\varphi: L \to F$ over K with $\varphi(a) = Y$.

Proof. For $P = P_d Y^d + \cdots + P_0 \in K[Y]^{\neq 0}$, there exists exactly one i with $v(P) = v(P_i) + i\gamma$.
Given a second $Q \in K[Y]^{\neq 0}$, there is also exactly one j with $v(Q) = v(Q_j) + j\gamma$.
One verifies that $v(PQ) = v(P_i Q_j) + (i+j)\gamma = v(P) + v(Q)$, so v_L is a valuation on L.

Y, a transcendental over $K \Longrightarrow \exists!$ field embedding $\varphi: L \to F$ over K with $\gamma(a) = Y$.
$v(P(a)) = \min(v(P_0), \ldots, v(P_d) + d\,v(a))$ for all $P = P_d Y^d + \cdots + P_0 \in K[Y]^{\neq 0}$.
Hence φ preserves v, since $v(a)$ and γ lie in the same cut over Γ. □

Adjoining "algebraic" elements to Γ

Lemma ALG-VAL

Let $\gamma \in d^{-1}\Gamma$ *be such that* $\Delta := \Gamma + \gamma\mathbb{Z} = \Gamma \cup \Gamma + \gamma \cup \cdots \cup \Gamma + (d-1)\gamma \supsetneq \Gamma$ *for* $d > 1$. *Let* $\xi \in K$ *be such that* $v(\xi) = d\gamma$ *and* $\mu := Y^d - \xi \in K[Y]$. *Let* $L := K[Y]/(\mu)$ *and* $y = Y + (\mu)$. *Then* $L \supseteq K$ *is a valued field extension for the valuation defined by*

$$v(P(y)) \;:=\; \min\,(v(P_0), \ldots, v(P_{p-1}) + (d-1)\gamma), \qquad \text{for all } P \in K[Y]_d^{\neq 0}.$$

Moreover, if $F \supseteq K$ *is a valued field extension and* $a \in F$ *satisfies* $a^d = \xi$, *then there exists a unique valued field embedding* $\varphi: L \to F$ *over* K *with* $\varphi(a) = y$.

Exercise. We also have $\boldsymbol{k}_L = \boldsymbol{k}_K$.

Proof. Similar to the previous proof (exercise). □

Algebraic closure of valued fields

Theorem

If $\operatorname{char} K = \operatorname{char} \boldsymbol{k} = 0$*, then the valuation on* K *can be extended to the algebraic closure* K^{a} *of* K*. Any valued field embedding* $K \to F$ *into another algebraically closed field* F *extends to a valued field embedding* $K^{\mathrm{a}} \to F$*.*

Proof. Lemmas ALG-IMM, ALG-RES, ALG-VAL, and Zorn yield:

- An algebraic valued field extension $L \supseteq K$, such that
  - L is henselian (ALG-IMM).
  - $\boldsymbol{k}_L$ algebraically closed (ALG-RES).
  - Γ_L is divisible (ALG-VAL).
- Any valued field embedding $K \to F$ extends to a valued field embedding $K^{\mathrm{a}} \to F$. (See also below.)

Newton polygon methods $\Longrightarrow$ L is algebraically closed. $\square$

Basic model theoretic terminology

Languages

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$\mathcal{L}$-structures

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functions $f_j \colon M_{s_1} \times \cdots \times M_{s_n} \to M_t$ $(s_1, \ldots, s_n, t$ depend on i, $j)$. Morphisms, …

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Formed from $\mathscr{L}$, variables of the sorts S, and $\top$, $\bot$, $\neg$, $\vee$, $\wedge$, $=$, $\exists$, $\forall$.
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$\mathcal{L}$-theories

Let $\boldsymbol{M}$ be an $\mathscr{L}$-structure and Σ, Σ' sets of $\mathscr{L}$-formulas

$\boldsymbol{M} \models \Sigma$	$\boldsymbol{M}$ is a model for Σ	$\Sigma \models \Sigma'$	$\boldsymbol{M} \models \mathrm{T}$ whenever $\boldsymbol{M} \models \Sigma'$
$\mathrm{Th}(\boldsymbol{M})$	$\{\sigma : \boldsymbol{M} \models \sigma\}$	$\mathrm{Th}(\Sigma)$	$\{\sigma : \Sigma \models \sigma\}$

Basic concepts from model theory

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$\boldsymbol{M} \equiv \boldsymbol{N}$	$\mathrm{Th}(\boldsymbol{M}) = \mathrm{Th}(\boldsymbol{N})$	$\boldsymbol{M}$ and N are **elementary equivalent**
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Σ is **complete**	Σ has a model and $\Sigma \models \sigma$ or $\Sigma \models \neg\sigma$ for any formula σ
T is an $\mathscr{L}$-**theory**	$\mathrm{Th}(T) = T$
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T^* is a **model companion** of T	T^* model complete and Any model of T embeds into a model of T^*

Types and saturation

Let $\Phi = \Phi(x)$ be a set of formulas depending on free variables $x = (x_i)$ of sorts (s_i)
Let $\boldsymbol{M}$ be an $\mathscr{L}$-structure and $M_x := \prod_i M_{s_i}$.

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Proposition

Suppose that $\boldsymbol{M}$ is κ-saturated, κ is infinite, $A \subseteq M$ and x have size $<\kappa$.
Then every x-type over A in $\boldsymbol{M}$ is realized in $\boldsymbol{M}$.

Quantifier elimination and completeness

Theorem

Assume that Σ eliminates quantifiers and also has a model.

Then Σ is complete if and only if some $\mathscr{L}$-structure embeds into every model of Σ.

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- So does the theory ACF(0) of algebraically closed fields of characteristic zero.
- $\mathbb{Z}$ embeds into any (algebraically closed) field of characteristic zero.
- Hence ACF(0) is complete.

Test for quantifier-elimination

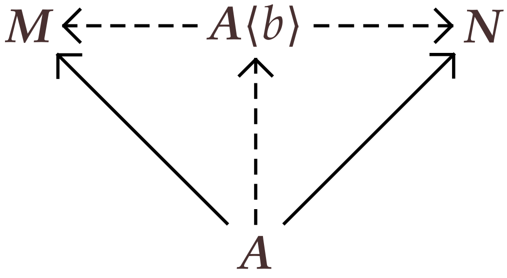
Theorem

Let Σ be given and suppose that

$\forall$
- $\boldsymbol{M} \models \Sigma$
- *proper substructure* $\boldsymbol{A} \subsetneq \boldsymbol{M}$
- $|A|^+$*-saturated model* $\boldsymbol{N}$ *of* Σ
- *embedding* $\iota \colon \boldsymbol{A} \hookrightarrow \boldsymbol{N}$

$\exists$
- $b \in M_s \setminus A_s$ *for some* $s \in S$
- *an extension* $\hat{\iota} \colon \boldsymbol{A}\langle b \rangle \hookrightarrow \boldsymbol{N}$ *of* ι

Then Σ *admits quantifier elimination.*



Test for model completeness

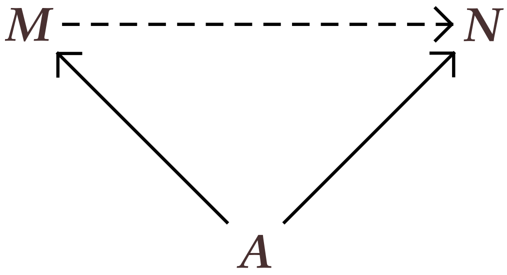
Theorem

Let Σ be given and suppose that

$\forall$
- $M \models \Sigma$
- $A \models \Sigma$ *with* $A \subseteq M$
- $|A|^+$*-saturated* $N \succcurlyeq A$
- *inclusion* $\iota\colon A \hookrightarrow N$

$\exists$ *an embedding* $\hat{\iota}\colon M \hookrightarrow N$ *that extends* ι

Then Σ is model complete.



Application to ACF

Theorem

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Proof. Let

- E be an algebraically closed field.
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Take $x \in A^{\neq 0}$ such that $a := x^{-1} \in E \setminus A$.

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Then ι uniquely extends into an embedding $\hat{\iota}: A[a] = A\,x^{-\mathbb{N}} \to F$ with $\hat{\iota}(a) = \iota(x)^{-1}$.

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Case 1. $K := A$ **is a field that is not algebraically closed**

Take $a \in K^{\mathrm{a}} \setminus K \subseteq E \setminus K$ with $P(a) = 0$ for some irreductible $\mu = \mu_d Y^d + \cdots + \mu_0 \in K[Y]$.

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Since F is algebraically closed, there exists a $b \in F$ with $\iota(\mu_d) b^d + \cdots + \iota(\mu_0) = 0$.

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Since $K[Y]/(\mu) \cong \iota(K)(b)$, we may extend ι into an embedding $\hat{\imath}: K(a) \to F$ with $\hat{\imath}(a) = b$.

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Let $a \in E \setminus K$. Then a is transcendental over K.

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Then $K[a] \cong \iota(K)[b]$, so we may extend ι into an embedding $\hat{\imath}\colon K[a] \to F$. $\square$

Application to ACVF

Theorem

The theory ACVF of algebraically closed valued fields eliminates quantifiers.

Note. ACVF can be modeled in the language $(K, \Gamma, +, -, \cdot, v, \leqslant_\Gamma, +_\Gamma, -_\Gamma)$.
Sometimes: extra sort for $\boldsymbol{k}$ (and extra component $\iota_{\boldsymbol{k}}\colon \boldsymbol{k}_A \to \boldsymbol{k}_F$).
Alternatively: one-sorted language $(K, +, -, \cdot, \preccurlyeq)$.

Proof. Let

- E be an algebraically closed valued field.
- $A \subseteq E$ a substructure, i.e. a "valued integral domain".
- F an algebraically closed valued field that is $|A|^+$-saturated.
- An embedding $\iota\colon A \to F$.

Problem: construct $y \in E \setminus A$ + embedding $\hat{\imath}\colon A[y] \to F$ that extends ι.

To easy notations, we may assume wlog that $A \subseteq F$ and that ι is the inclusion.

Proof that ACVF has QE — continued

Case 0. *A* is not a field

Let x be a non-invertible element of $A^{\neq 0}$ and take $y := x^{-1}$.
Let $\hat{\imath}\colon A[a] = A\,x^{-\mathbb{N}} \to F$ extend ι with $\hat{\imath}(a) = \iota(x)^{-1}$ (as for ACF).
Any element of $A[a]$ is of the form $c\,a^n = c\,x^{-n}$ for $c \in A$ and $n \in \mathbb{N}$.
Then $v(\hat{\imath}(c\,a^n)) = v(c\,x^{-n}) = v(c) - n\,v(x)$, both in $\Gamma_{A[a]} = \Gamma_A$ and in $\Gamma_F \supseteq \Gamma_A$.
Hence the embedding $\hat{\imath}$ preserves the valuation.

Case 1a. $K := A$ is a field, but $\boldsymbol{k}_K$ is not AC (algebraically closed).

Let $\mu \in K[Y]$ be monic with $\mu \preccurlyeq 1$ and $\bar{\mu}$ irreducible in $\boldsymbol{k}_K[Y]$. Let $y \in E$ be a root of μ.
Since F is AC, $\exists\, a \in F$ with $\mu(a) = 0$. Let $\hat{\imath}\colon K[y] \to F$ extend ι with $\hat{\imath}(y) = a$ (as for ACF).
Then $\boldsymbol{k}_K(\bar{a}) \cong \boldsymbol{k}_K(\bar{y})$ and $\hat{\imath}$ preserves the valuation by Lemma ALG-RES.

Case 1b. $K := A$ is a field, but Γ_K is not divisible.

Similar as above, with $\mu = Y^p - \xi$ for p prime and $\xi \in K$ such that $p^{-1} v(\xi) \notin \Gamma_K$.

Completeness

The valued field K has **characteristic** (m,n) if $\operatorname{char} K = m$ and $\operatorname{char} \boldsymbol{k}_K = n$.

Theorem

The theory $\mathrm{ACV}_{(m,n)}$ *of algebraically valued fields of characteristic* (m,n) *has QE and it is complete.*

QE. The characteristic of a valued field is conserved under the extensions. Hence the previous proof goes through for any fixed characteristic.

Completeness. Sufficient: a valued ring that embeds into any model of $\mathrm{ACV}_{(m,n)}$.

- If $m = n = 0$, then we may take $\mathbb{Z}$ with the trivial valuation.
- If $m = 0$ and $n = p$ is prime, then we may take $\mathbb{Z}$ with the p-adic valuation.
- If $m = n = p$ is prime, then we may take $\mathbb{F}_p$ with the trivial valuation. □

Valued ordered fields

Let $(K, \leqslant)$ be an ordered field (so $\mathbb{Q} \subseteq K$).

Given $X \subseteq K$, its **convex hull** is $\{a \in K : (\exists x, y \in X)\ x \leqslant a \leqslant y\}$.

Definition

Given a valuation v on K, we say that $(K, \leqslant, v)$ is an ***ordered valued field*** *if $\mathcal{O}_K$ is convex.*

Example. The "finest" valuation v with $\mathcal{O}_K = \mathrm{hull}(K)$ and $\mathcal{o}_K = \{a \in K : |a| < \mathbb{Q}^{>0}\}$.

Theorem

The theory RCVF of real closed valued fields eliminates quantifiers and is complete.

Proof. QE: similar as for ACVF. Completeness: $\mathbb{Z}$ embeds into any model. $\square$