Fuchsian holonomic sequences*

Joris van der Hoeven

CNRS, École polytechnique, Institut Polytechnique de Paris
Laboratoire d'informatique de l'École polytechnique (LIX, UMR 7161)
1, rue Honoré d'Estienne d'Orves
Bâtiment Alan Turing, CS35003
91120 Palaiseau, France

Email: vdhoeven@lix.polytechnique.fr

Many sequences that arise in combinatorics and the analysis of algorithms turn out to be holonomic (note that some authors prefer the terminology D-finite). In this paper, we study various basic algorithmic problems for such sequences $(f_n)_{n\in\mathbb{N}}$: how to compute their asymptotics for large n? How to evaluate f_n efficiently for large n and/or large precisions p? How to decide whether $f_n>0$ for all n?

We restrict our study to the case when the generating function $f=\sum_{n\in\mathbb{N}} f_n z^n$ satisfies a Fuchsian differential equation (often it suffices that the dominant singularities of f be Fuchsian). Even in this special case, some of the above questions are related to long-standing problems in number theory. We will present algorithms that work in many cases and we carefully analyze what kind of oracles or conjectures are needed to tackle the more difficult cases.

1. Introduction

1.1. Statement of the problems

Let $\mathbb{K}$ be a subfield of $\mathbb{C}$. A sequence $(f_n)_{n\in\mathbb{N}}\in\mathbb{C}^{\mathbb{N}}$ is said to be *holonomic* over $\mathbb{K}$ if it satisfies a difference equation

$$\Sigma_s(n)\, f_{n+s}+\cdots+\Sigma_0(n)\, f_n=0, \qquad (1.1)$$

where $\Sigma=\Sigma_s\sigma^s+\cdots+\Sigma_0\in\mathbb{K}[n][\sigma]$ is a linear difference operator in $\sigma: n\mapsto n+1$ with $\Sigma_s\neq 0$. (Note that some authors prefer the terminology D-finite or P-finite instead of holonomic.) Many interesting sequences from combinatorics, the analysis of algorithms, and number theory are holonomic [46, 13]. We say that (f_n) is $\mathbb{K}$-*holonomic* if $f_n\in\mathbb{K}$ for all $n\in\mathbb{N}$. We say that the equation (1.1) is *non-degenerate* if $\Sigma_s(n)\neq 0$ for all $n\in\mathbb{N}$. In that case, the sequence is entirely determined by its first s coefficients and (f_n) is $\mathbb{K}$-holonomic if and only if $(f_0,\ldots,f_{s-1})\in\mathbb{K}^s$.

Throughout this paper, we assume that $\mathbb{K}=\mathbb{Q}^{\text{alg}}$ is the field of algebraic numbers. Given a non-degenerate $\mathbb{K}$-holonomic sequence $(f_n)\in\mathbb{K}^{\mathbb{N}}$, one may raise several natural questions:

Q1. How to compute the asymptotic expansion of f_n when $n\to\infty$?

Q2. What kind of constants coefficients can occur in the asymptotic expansion of f_n?

Q3. How to compute terms f_n of the sequence efficiently as a function of n?

Q4. How to decide whether $f_n>0$ or $f_n\geqslant 0$ for large, all, or infinitely many $n\in\mathbb{N}$? (For this question, we assume that $f_n\in\mathbb{Q}^{\text{alg}}\cap\mathbb{R}$ for all n.)

*. This article has been written using GNU TeXmacs [25].

These questions are related and can be further refined. For instance, it is natural to compute terms f_n as elements of $\mathbb{K}$. However, if n becomes large, then the bit-size of f_n is generally at least proportional to n. When that happens, it may be preferable to switch to a floating point representation. If we have an asymptotic expansion of f_n with suitable error bounds, then we may exploit that to quickly compute floating point approximations of the f_n for large n. Similarly, if the dominant term of the expansion of f_n is positive and provably dominates the other terms, then we may deduce that $f_n > 0$ for large n.

Assuming that the generating function

$$f(z) := \sum_{n \in \mathbb{N}} f_n z^n$$

is analytic at the origin, it is well known that the asymptotic behavior of the sequence (f_n) is closely related to the behavior of f near its dominant singularities (i.e. the singularities of smallest absolute value). As will be recalled in section 2.1, the generating function f is again holonomic in the sense that it satisfies a non-trivial linear differential equation with polynomial coefficients. Holonomic functions can be evaluated extremely efficiently and their singularities are well understood.

In this paper we will restrict our attention to the special case when at least the dominant singularities of f are Fuchsian (see section 2.2 for a definition). In that case, the behavior of f near its dominant singularities becomes much simpler and the evaluation of f near these singularities even more efficient.

In their full generality, the questions **Q1**, **Q3**, and **Q4** turn out to be very difficult, even in the Fuchsian case. Indeed, if f is actually a rational function, then the last question **Q4** is related to Skolem's problem, which asks whether $f_n = 0$ for some $n \in \mathbb{N}$. Now if f is a rational function, then so is $g(z) := -\sum_{n \in \mathbb{N}} f_n^2 z^n$ and $f_n = 0 \Longleftrightarrow g_n \geqslant 0$ for all $n \in \mathbb{N}$. The hard cases for Skolem's problem occur when f has several dominant singularities. One archetype example is

$$f_n \;=\; \lambda \cos(\alpha n) + \mu \cos(\beta n) + \nu \cos(\gamma n), \tag{1.2}$$

with $\mathrm{e}^{\mathrm{i}\alpha}, \mathrm{e}^{\mathrm{i}\beta}, \mathrm{e}^{\mathrm{i}\gamma}, \lambda, \mu, \nu \in \mathbb{K} \cap \mathbb{R}$. A variant of this problem is also relevant for the question **Q3**: if certain terms of this sequence (1.2) can become “absurdly small”, then the computation of floating point approximations for these terms may take much longer than expected, since we need to compute with a precision that exceeds the order of cancellation. We refer to [32] for more information on Skolem's problem and to [40] for some recent related progress in the context of hypergeometric sequences.

On the positive side, examples like (1.2) are fairly pathological, so it remains reasonable to hope answering our questions for most practical examples from combinatorics or the analysis of algorithms. One interesting concrete example was studied in [35]. The authors exploit the fact that the then open problem about the uniqueness of the Canham model for biomembranes reduces to proving the positivity of a certain holonomic sequence. They solve the latter problem using singularity analysis [13] and techniques for reliable numerical computations with holonomic functions [9, 20, 21, 37, 36]. In the present paper, among other things, we will extend this approach to more general holonomic sequences. Independently of this work, the implementations from [35] were further extended in [10]. Note that other approaches to automatically prove the positivity of sequences were proposed in [17, 31].

In fact, the main purpose of this paper is to provide the best possible answers to questions **Q1**–**Q4** as long as we do not run into number theoretic trouble. We will also identify the precise nature of possible trouble, thereby clarifying which problems need to be overcome if we want to give even better answers. Most of our results rely on well known techniques. Our main contribution is therefore a detailed analysis of how to answer the questions **Q1**–**Q4** as well as possible using these techniques.

1.2. Overview of our contributions

Let us briefly outline the structure of this paper. Section 2 contains reminders about holonomic functions, Fuchsian singularities, and holonomic constants.

In section 3, we start with questions **Q1** and **Q2**. In order to obtain asymptotic expansions, we use Cauchy's classical contour integral formula for f_n and deform the contour into a finite number of loop integrals around the smallest singularities of f (section 3.1). Each of these loop integrals is a truncated Mellin-style integral, whose asymptotic expansion can be computed using classical formulas (section 3.2). (In the context of difference equations, note that some authors use the terminology "Pincherle transform" [42] instead of "Mellin transform".) Adding up the contributions from each of the singularities, we obtain an asymptotic expansion for f_n (Theorem 3.2). The coefficients of this asymptotic expansion can be computed explicitly and expressed in terms of (non-singular) holonomic constants and values of higher derivatives of $\gamma(z) = \Gamma(z)^{-1}$ at points in $\mathbb{K}$. Using reliable numeric techniques from [21, 35], one may also compute explicit bounds for the error of the asymptotic expansion (section 3.3).

Unfortunately, Theorem 3.2 is imperfect in the sense that some of the terms in the "asymptotic expansion" for f_n may cancel out (in which case the bound for the error becomes larger than the expansion itself). This may actually happen in three different ways that will be analyzed in detail in section 4.

First of all, consider a holonomic function g that converges on the closed unit disk $\bar{\mathcal{D}}_1$. Its value $g(1)$ at $z=1$ is a holonomic constant (and any non-singular holonomic constant can be obtained in this way). Now $f = g/(1-z)$ is also a holonomic function and the first term of the asymptotic expansion of f_n is given by $f_n \sim g(1)$ if and only if $g(1) \neq 0$. This shows that we need a zero-test for holonomic constants in order to detect cancellations of terms in the asymptotic expansion of f_n.

A second kind of cancellation may occur between distinct terms in the asymptotic expansion and only for certain values of n. Consider for example

$$\begin{aligned} f(z) &= \frac{1}{1-z} + \frac{1}{1+z} + \frac{1}{1-z/2} \\ f_n &= 1^n + (-1)^n + 2^{-n}. \end{aligned}$$

Then the dominant terms 1^n and $(-1)^n$ cancel out for odd values of n. We call this phenomenon "resonance" and the good news is that it can be entirely eliminated by considering a finite number of cases in which f_n is replaced by a subsequence $f_{\Pi n + \rho}$ with $\Pi \in \mathbb{N}^>$ and $\rho \in \{0, \ldots, \Pi - 1\}$ (see section 4.2).

A non-periodic variant of resonance is "quasi-resonance". Consider for instance the sequence $(f_n)_{n \in \mathbb{N}}$ from (1.2). We say that this sequence is quasi-resonant if, for every $C > 0$ and $\kappa \in \mathbb{R}$, there exist an infinite number of $n > 0$ with $|f_n| < C n^{-\kappa}$. In fact, we conjecture that quasi-resonance implies resonance (Conjecture 4.5), but a proof seems currently out of reach.

Assuming a zero-test for holonomic constants and the absence of quasi-resonance, it is possible to automatically compute the asymptotic expansion of f_n in a strong sense, without suffering from uncontrolled cancellations (Theorem 4.6). Under these hypotheses, we also show in section 5 that p-bit floating point approximations for f_n can be computed in smoothly linear time $O(p\,(\log p)^2 \log\,(n p) \log\log n)$: see Theorem 5.2. This bound is uniform as long as $\log n = O(p)$.

In section 6 we turn to question **Q4**. Whenever we can compute an asymptotic expansion f_n^{as} of f_n for which the error $|f_n - f_n^{\mathrm{as}}|$ is strictly smaller than $|f_n|$ for $n \geqslant n_0$, the positivity of f_n can be deduced from the positivity of f_n^{as} for $n \geqslant n_0$ and the positivity of $f_0, \ldots, f_{n_0-1}$. However, for a sequence $(f_n)_{n \in \mathbb{N}}$ like (1.2) and $\kappa > 2$, it can be hard decide whether $f_n + |\lambda| + |\mu| + |\nu| > n^{-\kappa}$ for all $n \geqslant n_0$: what we need here is an even more precise version of Conjecture 4.5. Nevertheless, this example is fairly pathological. For any $\varepsilon > 0$ and $\kappa > 0$, we always have $f_n + |\lambda| + |\mu| + |\nu| + \varepsilon > n^{-\kappa}$ for all sufficiently large n. Conversely, if α, β, γ, and $2\,\pi$ are $\mathbb{Q}$-linearly independent, then $f_n + |\lambda| + |\mu| + |\nu| - \varepsilon < n^{-\kappa}$ for infinitely many n. In section 6, we will show that something similar holds in general, by relying on sequence counterparts of results from [19].

For our partial answers to questions **Q1–Q4**, we only need the dominant singularities of f to be Fuchsian, except in the case of cancellations that also require the examination of subdominant singularities. In the last section 7, we finally mention a few interesting results that hold if f is globally Fuchsian. From bases of local solutions of the differential equation for f, it is then classical that we may construct a basis of solutions to the difference equation (1.1) using Mellin transforms based at the corresponding singularities [42, 38, 15]. We show that this theory can be made fully effective and also develop a difference counterpart for the concept of transition matrices from [21]. The Mellin transforms can still be applied in the case when n is replaced with a complex variable u such that $\operatorname{Re} u$ is sufficiently large. This can be used for the construction and efficient evaluation of meromorphic solutions to the difference equation $\Sigma_s(u)\,\varphi(u+s) + \cdots + \Sigma_0(u)\,\varphi(s) = 0$.

One obvious limitation of the present paper is that we only consider the case when the dominant singularities of f are Fuchsian. The irregular case has been studied extensively from a theoretical point of view [39, 16, 11, 2, 28, 29, 30]. In a forthcoming work, we intend to investigate this case from a similar perspective as in this paper.

A minor restriction of this paper is that we assumed $\mathbb{K}$ to be the field of algebraic numbers. This enables us to prove a softly optimal uniform complexity bound for the computation of a p-bit floating point approximation of f_n. For more general subfields of $\mathbb{Q} \subseteq \mathbb{C}$, the results in this paper still go through, using ideas from [9], but the uniform complexity bounds in section 5.3 have to be replaced by $\tilde{O}(p^{3/2} \log n)$.

2. PRELIMINARIES

2.1. Holonomic functions

A function $f(z)$ is said to be *holonomic* over $\mathbb{K}$ if it satisfies a differential equation

$$L_r(z)\, f^{(r)}(z) + \cdots + L_0(z)\, f(z) = 0, \tag{2.1}$$

where $L = L_r \partial^r + \cdots + L_0 \in \mathbb{K}[z][\partial]$ is a linear differential operator in $\partial = \partial / \partial z$ with $L_r \neq 0$. Without loss of generality, we may assume that we normalized L so that $\gcd(L_0, \ldots, L_r) = 1$. We say that f is $\mathbb{K}$-*holonomic* if $F(z) := (f(z), \ldots, f^{(r-1)}(z)) \in \mathbb{K}^r$ at a certain non-singular point $z \in \mathbb{K}$.

It is not hard to see that a sequence $(f_n)_{n\in\mathbb{N}}\in\mathbb{C}^{\mathbb{N}}$ is holonomic over $\mathbb{K}$ if and only if its generating function f is holonomic over $\mathbb{K}$. Indeed, using the dictionary

$$\begin{aligned} n &\longleftrightarrow z\partial \\ \sigma &\longleftrightarrow z^{-1} \end{aligned}$$

and modulo normalization, any non-zero operator $\Sigma\in\mathbb{K}[n][\sigma]$ can be rewritten as a non-zero operator $L\in\mathbb{K}[z][\partial]$ and *vice versa*. Then (1.1) holds if and only if (2.1) holds.

Example 2.1. The harmonic numbers $H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$ satisfy the difference equation

$$(n+2)\,H_{n+2}-(2n+3)\,H_{n+1}+(n+1)\,H_n=0$$

for all $n\in\mathbb{N}$ and the equation

$$(n+1)\,(n+2)\,H_{n+2}-(n+1)\,(2n+3)\,H_{n+1}+(n+1)^2 H_n=0$$

for all $n\in\mathbb{Z}$. According to the above dictionary, this yields the equation

$$(z\partial+1)\,((z\partial+2)(z^{-2}H)-(2z\partial+3)(z^{-1}H)+(z\partial+1)\,H) \;=\; 0$$

which can be rewritten as

$$z^{-1}\partial\,((z-1)^2\partial H+(z-1)\,H) \;=\; 0.$$

Taking $\Sigma=(n+2)\,\sigma^2-\sigma+n+1$, we thus get $L=(z-1)^2\partial^2+3\,(z-1)\,\partial+1$.

Remark 2.2. In the above example, we multiplied the equation by $n+1$ to make it hold for all $n\in\mathbb{Z}$ instead of all $n\in\mathbb{N}$. In general, given a difference equation (1.1) that holds for all $n\in\mathbb{N}$, we can transform it into a difference equation that holds for all $n\in\mathbb{Z}$ through multiplication by $(n+1)\cdots(n+s)$.

From an analytic perspective, if f is holomorphic at the origin, then we may retrieve the coefficients f_n using the Cauchy integral

$$f_n=\frac{1}{2\pi\mathrm{i}}\oint\frac{f(z)}{z^{n+1}}\,\mathrm{d}z, \tag{2.2}$$

where we integrate over a circle with center $z=0$ and a sufficiently small radius.

2.2. Fuchsian singularities

Consider a holonomic function f that satisfies (2.1). The only possible singularities of f are located at the roots of L_r or at infinity. Modulo a change of variables $z\leftrightarrow z+\alpha$ with $\alpha\in\mathbb{K}$ or $z\leftrightarrow z^{-1}$ (for a singularity at infinity), the study of f near such singularities can be reduced to the case of a singularity at the origin. If $L_r(0)=0$ and if (modulo multiplication by a power of z) we can rewrite the equation (2.1) as an equation

$$(A_r(z)\,(z\partial)^r+\cdots+A_0(z))(f)=0$$

with $A_0,\ldots,A_r\in\mathbb{K}[z]$ and $A_r(0)\neq 0$, then we say that L is *regular-singular* or *Fuchsian* at $z=0$. If this is the case at all singularities (modulo the above changes of variables), then we say that L is *Fuchsian*. Sometimes, we will also apply this terminology to solutions f of the equation $Lf=0$ or to the sequence $(f_n)_{n\in\mathbb{N}}$ instead of L. If f is holomorphic at the origin and the non-zero singularities of smallest absolute value of f are all Fuchsian[2.1], then we say that f (as well as the sequence $(f_n)_{n\in\mathbb{N}}$) is *dominant-Fuchsian*.

2.1. If f has no singularities in $\mathbb{C}$, then the assumption that f is Fuchsian at infinity implies that f is actually a polynomial. In what follows, we will discard this trivial case and assume that f has at least one singularity in $\mathbb{C}$.

If L is Fuchsian at the origin, then it is well known [14] that (2.1) has a fundamental system of local solutions $(h_{i,j})_{1\leqslant i\leqslant p,0\leqslant j<\nu_i}$ of the form

$$h_{i,j}\in z^{\kappa_i}((\log z)^j+z\,\mathbb{K}\{\{z\}\}[\log z]),$$

where $\kappa_i\in\mathbb{K}$, $z^{\mathbb{K}}=\{z^\lambda:\lambda\in\mathbb{K}\}$, $\nu_1+\cdots+\nu_p=r$, and $\mathbb{K}\{\{z\}\}$ denotes the ring of convergent power series in z. Moreover, there exists a unique such system of solutions (up to a permutation of indices) with the property that the coefficient of $z^{\kappa_i}(\log z)^j$ in $h_{i',j'}$ vanishes whenever $(i',j')\neq(i,j)$. We call it the *canonical system* of solutions at $z=0$. We call $z^{\kappa_i}(\log z)^j$ with $j<\nu_i$ a *fundamental monomial* for L.

If L is Fuchsian at a point $\alpha\in\mathbb{K}$, we thus have a corresponding canonical system of solutions $h_1^\alpha,\ldots,h_r^\alpha$ with

$$h_i^\alpha\in(z-\alpha)^{\mathbb{K}}\,\mathbb{K}\{\{z-\alpha\}\}[\log(z-\alpha)].$$

If L is Fuchsian at infinity, then we also have a corresponding canonical system of solutions $h_1^\infty,\ldots,h_r^\infty$ with

$$h_i^\infty\in z^{\mathbb{K}}\,\mathbb{K}\{\{z^{-1}\}\}[\log z].$$

Let H^α be the row vector with entries $h_1^\alpha,\ldots,h_r^\infty$. Given a local solution f to $Lf=0$ at $z=\alpha\in\mathbb{K}\cup\{\infty\}$, there exists a unique column vector $F(\alpha)\in\mathbb{C}^r$ such that $f=H^\alpha F(\alpha)$. We call $F(\alpha)$ the initial condition or *generalized value* of f at $z=\alpha$. This definition still makes sense at non-singular points, in which case $F(\alpha)$ is simply the column vector with entries $f(\alpha),f'(\alpha),\ldots,f^{(r-1)}(\alpha)/(r-1)!$.

2.3. Holonomic constants

For a precise answer to question **Q2**, it is important to first introduce various relevant classes of "holonomic constants" that can be obtained as values of holonomic functions. We will follow [26, section 4.4 and appendix B], while restricting us to non-singular and regular-singular holonomic constants.

Let $\mathcal{L}^{\mathrm{hol}}$ and $\mathcal{L}^{\mathrm{shol}}$ denote the sets of monic $L\in\mathbb{K}(z)[\partial]$ whose coefficients are respectively defined on $\bar{\mathcal{D}}_1:=\{z\in\mathbb{C}:|z|\leqslant1\}$ and $\mathcal{D}_1:=\{z\in\mathbb{C}:|z|<1\}$. Let $\mathcal{L}^{\mathrm{rhol}}$ be the set of $L\in\mathcal{L}^{\mathrm{shol}}$ such that L is at worst regular-singular at $z=1$. We also define $\mathcal{L}^{\mathrm{hola}}$ to be the set of monic operators $L\in\mathbb{K}(z)[\partial]$ whose coefficients are defined on $\bar{\mathcal{D}}_{0,1}\setminus\{0\}$ and such that L is at worst regular-singular at $z=0$.

We define $\mathcal{F}^{\mathrm{hol}}$, $\mathcal{F}^{\mathrm{rhol}}$, and $\mathcal{F}^{\mathrm{hola}}$ to be the sets of solutions $f\in\mathbb{K}\{\{z\}\}$ to an equation $Lf=0$, where $L\in\mathcal{L}^{\mathrm{hol}}$, $L\in\mathcal{L}^{\mathrm{rhol}}$, or $L\in\mathcal{L}^{\mathrm{hola}}$, respectively, and such that $\lim_{z\to1}f(z)$ exists. Then we define

$$\begin{aligned}\mathbb{K}^{\mathrm{hol}}&:=\{f(1):f\in\mathcal{F}^{\mathrm{hol}}\}\\ \mathbb{K}^{\mathrm{rhol}}&:=\{\lim_{z\to1}f(z):f\in\mathcal{F}^{\mathrm{rhol}}\}\\ \mathbb{K}^{\mathrm{hola}}&:=\{f(1):f\in\mathcal{F}^{\mathrm{hola}}\}\end{aligned}$$

We call $\mathbb{K}^{\mathrm{hol}}$ the set of *holonomic constants* and $\mathbb{K}^{\mathrm{hola}}$ the set of *Fuchsian holonomic constants*. Each of these three sets actually forms a ring [26, Proposition B.1]. Moreover, these three rings turn out to be closely related (see also [12]):

THEOREM 2.3. [26, Theorem B.5] *We have*

$$\mathbb{K}^{\mathrm{hol}}\subseteq\mathbb{K}^{\mathrm{rhol}}\subseteq\mathbb{K}^{\mathrm{hola}}\subseteq\mathcal{X}^{-1}\,\mathbb{K}^{\mathrm{hol}},$$

where

$$\mathcal{X}:=\{(1-\mathrm{e}^{-2\pi\mathrm{i}\alpha_1})\cdots(1-\mathrm{e}^{-2\pi\mathrm{i}\alpha_k}):\alpha_1,\ldots,\alpha_k\in(\mathbb{K}\cap\mathbb{R})\setminus\mathbb{Q}\}.$$

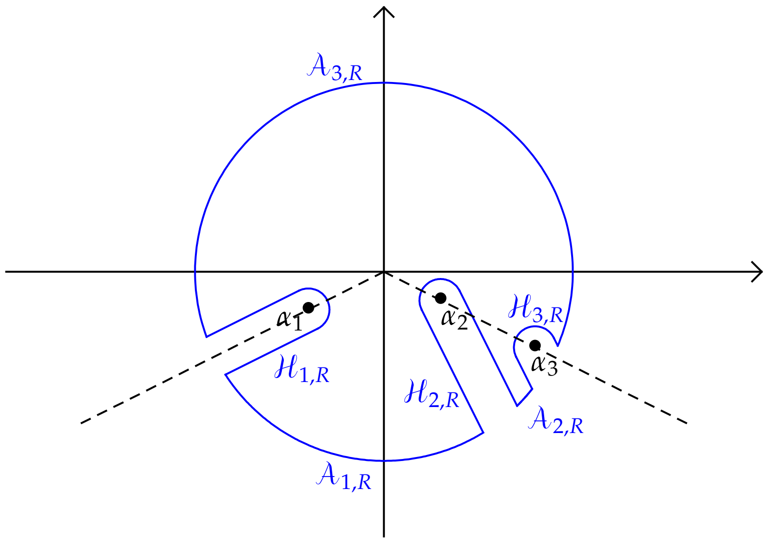

Figure 3.1. Deformation of a Cauchy contour into a contour of radius R that avoids a finite number of singularities. Since α_2 and α_3 are aligned with the origin, we slightly modified the directions of the corresponding truncated Hankel contours $\mathcal{H}_{2,R}$ and $\mathcal{H}_{3,R}$ to avoid collisions.

Now consider a holonomic sequence $(f_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ whose generating function f belongs to $\mathcal{F}^{\text{hola}}$. So $Lf = 0$ for some $L \in \mathcal{L}^{\text{hola}}$. Assume that L has a Fuchsian singularity at $\alpha \in \mathbb{K}^{\neq}$ and let $h_1^\alpha, \ldots, h_r^\alpha$ be the canonical system of local solutions at α. In particular, there exist unique $\lambda_1,\ldots,\lambda_r \in \mathbb{C}$ with $f = \lambda_1 h_1^\alpha + \cdots + \lambda_r h_r^\alpha$. In fact, we have $\lambda_1,\ldots,\lambda_r \in \mathbb{K}^{\text{hola}}$ (see [26, Proposition B.3]) and we can compute $\lambda_1,\ldots,\lambda_r$ using the algorithms from [21]. Here "computing" is understood in the following sense: for any $\varepsilon \in \mathbb{Q}^>$ and $i = 1,\ldots,r$, we can compute an approximation $\tilde{\lambda}_i \in \mathbb{Q}[\mathrm{i}]$ of λ_i with $|\tilde{\lambda}_i - \lambda_i| \leqslant \varepsilon$. Note that this does not imply the existence of a zero-test for the constants $\lambda_1,\ldots,\lambda_r$. The zero-test problem for holonomic constants will be discussed in section 4.4 below.

3. ASYMPTOTIC EXPANSIONS

3.1. Decomposing Cauchy contour integrals into Mellin integrals

The traditional method to determine the asymptotics of a sequence f_n whose generating function f is holomorphic at the origin is based on the Cauchy contour integral (2.2):

$$f_n = \frac{1}{2\pi \mathrm{i}} \oint \frac{f(z)}{z^{n+1}} \mathrm{d}z. \tag{3.1}$$

If f is holonomic, then f has only a finite number of singularities $\alpha_1,\ldots,\alpha_\ell$, which are all in $\mathbb{K}^{\neq}$. For some $R>0$ and $m \in \{1,\ldots,\ell\}$, assume that $|\alpha_i| < R$ for $i = 1,\ldots,m$ and $|\alpha_i| > R$ for $i = m+1,\ldots,\ell$. Then we may deform the contour from (3.1) into a contour that consists of m truncated Hankel contours $\mathcal{H}_{1,R},\ldots,\mathcal{H}_{m,R}$ and m residual arcs $\mathcal{A}_{1,R},\ldots,\mathcal{A}_{m,R}$ on the circle with center 0 and radius R: see Figure 3.1. Then (3.1) becomes:

$$f_n = \frac{1}{2\pi \mathrm{i}} \left(\sum_{k=1}^{m} \int_{\mathcal{H}_{k,R}} \frac{f(z)}{z^{n+1}} \mathrm{d}z + \sum_{k=1}^{m} \int_{\mathcal{A}_{k,R}} \frac{f(z)}{z^{n+1}} \mathrm{d}z \right). \tag{3.2}$$

We may chose the truncated Hankel contours $\mathcal{H}_i$ to depart radially from the origin toward infinity. In the degenerate case when the arguments of certain singularities $\alpha_i \neq \alpha_j$ coincide, we turn the contours clockwise by a fixed sufficiently small angle: see Figure 3.1.

Integrals of the form

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{H}_{k,R}}\frac{f(z)}{z^{n+1}}\mathrm{d}z \qquad (3.3)$$

are called *truncated Mellin integrals*[3.1]. As to the residual integral on $\mathcal{A}:=\mathcal{A}_{1,R}\cup\cdots\cup\mathcal{A}_{m,R}$, we have

$$\left|\frac{1}{2\pi\mathrm{i}}\sum_{k=1}^{m}\int_{\mathcal{A}_{k,R}}\frac{f(z)}{z^{n+1}}\mathrm{d}z\right| \leqslant \left|\frac{2\pi R}{2\pi\mathrm{i}}\cdot\frac{\|f\|_{\mathcal{A}}}{R^{n+1}}\right| = \frac{\|f\|_{\mathcal{A}}}{R^n},$$

where $\|f\|_{\mathcal{A}}:=\max_{z\in\mathcal{A}}|f(z)|$. Note that an upper bound for $\|f\|_{\mathcal{A}}$ can be computed efficiently using the algorithms from [20].

Most of the remainder of this section is dedicated to determining the asymptotics of integrals of the form (3.3) in the case when f is Fuchsian at α_k. The integrals for which $|\alpha_k|$ is smallest typically dominate the other ones, but cancellations may sometimes occur, in which case we will also need to examine the subdominant singularities. Let us consider one simple example of this phenomenon.

Example 3.1. The rational function

$$f = \frac{1}{1-z}+\frac{1}{1+z}+\frac{1}{1-z/2}$$

from the introduction is certainly holonomic, with singularities at $\alpha_1=-1$, $\alpha_2=1$, and $\alpha_3=2$. We have

$$f_n = (-1)^n+1^n+2^{-n},$$

where we note that $(-1)^n+1^n=0$ for odd values of n. In other words, the asymptotics of f_n depends on the parity of n:

$$f_n \approx 2+O(2^{-n}) \qquad (n\in 2\mathbb{N})$$
$$f_n \approx 2^{-n} \qquad (n\in 2\mathbb{N}+1)$$

3.2. Elementary Mellin integrals

Let us first study the very special case when

$$f = (\alpha-z)^{-\kappa}(\log(\alpha-z))^m$$

with $\alpha\in\mathbb{K}^{\neq}$, $\kappa\in\mathbb{K}$, and $m\in\mathbb{N}$. Explicit formulas for the asymptotics of the Taylor coefficients f_n are well known in this case, but it is convenient to recall the details of this computation. Modulo a change of variables $z=\alpha z'$, we may assume assume without loss of generality that $\alpha=1$.

In this special case, for $n>-\operatorname{Re}\kappa$, the contour integral (3.1) can rewritten into the full Mellin integral

$$f_n = \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{H}_1^{+\infty}}\frac{(1-z)^{-\kappa}(\log(1-z))^m}{z^{n+1}}\mathrm{d}z,$$

where $\mathcal{H}_1^{+\infty}$ is a Hankel contour that starts at $z=+\infty$, then turns clockwise around $z=1$, and finally returns to $z=+\infty$. Setting $z=\mathrm{e}^{-\zeta}$, this formula becomes

$$f_n = \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{H}_0^{-\infty}}\frac{(1-\mathrm{e}^{-\zeta})^{-\kappa}(\log(1-\mathrm{e}^{-\zeta}))^m}{\mathrm{e}^{-\zeta n}}\mathrm{d}\zeta,$$

3.1. The classical Mellin transform uses a straight integration path from $z=0$ to $+\infty$ instead of a Hankel contours around α_k. We will say that our Mellin integral is "based at α_k". The use of a Hankel contours extends the definition to the case when f is not integrable at α_k. In the context of difference equations, certain authors prefer the terminology "Laplace transform", "Pincherle transform", or "Nörlund transform" instead of "Mellin transform".

where $\mathcal{H}_0^{-\infty}$ is a Hankel contour that starts at $\zeta = -\infty$, then turns clockwise around $\zeta = 0$, and finally returns to $\zeta = -\infty$. We regard $(1 - \mathrm{e}^{-\zeta})^{-\kappa} (\log (1 - \mathrm{e}^{-\zeta}))^m$ as an element of $\zeta^{-\kappa} ((\log \zeta)^m + \zeta\, \mathbb{Q}\{\{\zeta\}\}[\log \zeta]_{<m})$, where $\mathbb{Q}\{\{\zeta\}\}[\log \zeta]_{<m}$ denotes the set of polynomials in $\mathbb{Q}\{\{\zeta\}\}[\log \zeta]$ of degree $<m$ in $\log \zeta$:

$$(1 - \mathrm{e}^{-\zeta})^{-\kappa} (\log (1 - \mathrm{e}^{-\zeta}))^m \;=\; \zeta^{-\kappa} \sum_{j \leqslant m} \sum_{i \in \mathbb{N}} \psi_{i,j} \zeta^i (\log \zeta)^j,$$

with $\psi_{0,j} = \delta_{j,m}$ (Kronecker delta) and $\psi_{i,m} = 0$ for all $i > 0$. Then

$$f_n \;=\; \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{H}_0^{-\infty}} \sum_{j \leqslant m} \sum_{i \in \mathbb{N}} \psi_{i,j} \zeta^{i-\kappa} (\log \zeta)^j \mathrm{e}^{\zeta n} \mathrm{d}\zeta. \tag{3.4}$$

Let

$$\gamma(\lambda) \;:=\; \frac{1}{\Gamma(\lambda)}$$

From the classical formula [45, Section 12.22]

$$\gamma(\lambda) \;=\; \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{H}_0^{-\infty}} \zeta^{-\lambda} \mathrm{e}^{\zeta} \mathrm{d}\zeta,$$

we deduce

$$\gamma^{(i)}(\lambda) \;=\; \frac{(-1)^i}{2\pi \mathrm{i}} \int_{\mathcal{H}_0^{-\infty}} (\log \zeta)^i \zeta^{-\lambda} \mathrm{e}^{\zeta} \mathrm{d}\zeta,$$

whence

$$\begin{aligned} \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{H}_0^{-\infty}} (\log \zeta)^j \zeta^{-\lambda} \mathrm{e}^{\zeta n} \mathrm{d}\zeta \;&=\; \sum_{i=0}^{j} \frac{(-1)^{j-i}}{2\pi \mathrm{i}} \binom{j}{i} (\log n)^{j-i} n^{\lambda-1} \int_{\mathcal{H}_0^{-\infty}} (\log \zeta)^i \zeta^{-\lambda} \mathrm{e}^{\zeta} \mathrm{d}\zeta \\ &=\; \sum_{i=0}^{j} (-1)^j \binom{j}{i} (\log n)^{j-i} n^{\lambda-1} \gamma^{(i)}(\lambda). \end{aligned} \tag{3.5}$$

When plugging this identity into (3.4), we obtain an asymptotic expansion

$$f_n \;\approx\; \frac{(-1)^m}{\Gamma(\kappa)} n^{\kappa-1} (\log n)^m + \sum_{j<m} \sum_{i \in \mathbb{N}} c_{i,j} n^{\kappa-1-i} (\log n)^j,$$

where $c_{i,j} \in \mathbb{K}[\gamma(\kappa), \gamma'(\kappa), \ldots, \gamma^{(m)}(\kappa)]$ can be computed explicitly. Note that the series that underlies this expansion usually diverges. It will be useful to introduce

$$\begin{aligned} \mathbb{K}[\gamma^{(\mathbb{N})}(\kappa)] \;&:=\; \mathbb{K}[\gamma(\kappa), \gamma'(\kappa), \gamma''(\kappa), \ldots] \\ \mathbb{K}[\gamma^{(\mathbb{N})}(\mathbb{K})] \;&:=\; \bigcup_{\kappa \in \mathbb{K}} \mathbb{K}[\gamma^{(\mathbb{N})}(\kappa)]. \end{aligned}$$

3.3. Effective local error bounds

Assume now that L has a Fuchsian singularity at $z = \alpha \in \mathbb{K}^{\neq}$ and consider a local solution f of the form

$$f \;\in\; (\alpha - z)^{-\kappa} \mathbb{K}\{\{z - \alpha\}\}[\log (z - \alpha)],$$

where $\alpha \in \mathbb{K}^{\neq}$, $\kappa \in \mathbb{K}$. We recall that general local solutions to $Lf = 0$ are linear combinations of local solutions of this special form. For some $\varepsilon > 0$ sufficiently small, consider

$$f_n^{\alpha,\varepsilon} \;:=\; \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{H}_\alpha^{\alpha(1+\varepsilon)}} \frac{f(z)}{z^{n+1}} \mathrm{d}z,$$

where $\mathcal{H}_\alpha^{\alpha(1+\varepsilon)}$ is a Hankel contour from $z=\alpha(1+\varepsilon)$ around $z=1$ and then back to $z=\alpha(1+\varepsilon)$. Our aim is to determine both the asymptotics of $f_n^{\alpha,\varepsilon}$ and an effective bound for the remainder.

As in section 3.2, we first reduce to the case when $\alpha=1$ and then perform the change of variables $z=\mathrm{e}^{-\zeta}$. Let $\psi(\zeta)=f(\mathrm{e}^{-\zeta})\in\zeta^{-\kappa}\,\mathbb{K}\{\{\zeta\}\}[\log\zeta]$ and $\varrho:=\log(1+\varepsilon)$, so that

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{H}_1^{1+\varepsilon}}\frac{f(z)}{z^{n+1}}\mathrm{d}z\;=\;\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{H}_0^{-\varrho}}\psi(\zeta)\,\mathrm{e}^{\zeta n}\mathrm{d}\zeta$$

Let $T\in\mathbb{N}$ be a truncation order and let

$$\tilde{\psi}\;=\;\zeta^{-\kappa}(\psi_0(\zeta)+\cdots+\psi_{T-1}(\zeta)\,\zeta^{T-1})$$

be the truncation of ψ modulo $o(\zeta^{T-\kappa})$, with $\psi_0,\ldots,\psi_{T-1}\in\mathbb{K}[\log\zeta]$. Using (3.5), we can explicitly compute

$$\tilde{f}_n^{1,\infty}\;:=\;\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{H}_0^{-\infty}}\tilde{\psi}(\zeta)\,\mathrm{e}^{\zeta n}\mathrm{d}\zeta$$

as an element of $n^{\kappa-1}\,\mathbb{K}[\gamma^{(\mathbb{N})}(\kappa)][n^{-1}][\log n]$.

Given $i,j\in\mathbb{N}$, let us now study the difference

$$\Delta\;:=\;\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{H}_0^{-\infty}}\zeta^{i-\kappa}(\log\zeta)^j\mathrm{e}^{\zeta n}\mathrm{d}\zeta-\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{H}_0^{-\varrho}}\zeta^{i-\kappa}(\log\zeta)^j\mathrm{e}^{\zeta n}\mathrm{d}\zeta.$$

We have

$$\Delta\;\leqslant\;\frac{\mathrm{e}^{\pi|\mathrm{Im}\,\kappa|}}{\pi}\int_\varrho^\infty\zeta^{i-\mathrm{Re}\,\kappa}(\pi+|\log\zeta|)^j\,\mathrm{e}^{-\zeta n}\mathrm{d}\zeta.$$

Moreover, assuming that $\varepsilon\leqslant1$, we have $|\zeta^{-1}|\leqslant2/\varepsilon$, $|\zeta\,\mathrm{e}^{-\zeta}|\leqslant\mathrm{e}^{-1}$, and $|(\pi+|\log\zeta|)\,\mathrm{e}^{-\zeta}|\leqslant 5+|\log\varepsilon|$ for $\zeta\in[\varrho,\infty)$, so

$$\Delta\;\leqslant\;\frac{1}{\pi n}\mathrm{e}^{\pi|\mathrm{Im}\,\kappa|}\left(\tfrac{2}{\varepsilon}\right)^{\mathrm{Re}\,\kappa-i}(5+|\log\varepsilon|)^j\,(1+\varepsilon)^{-n+j}\qquad(i\leqslant\mathrm{Re}\,\kappa)$$

$$\Delta\;\leqslant\;\frac{1}{\pi n}\mathrm{e}^{\pi|\mathrm{Im}\,\kappa|+\mathrm{Re}\,\kappa-i}(5+|\log\varepsilon|)^j\,(1+\varepsilon)^{-n-\mathrm{Re}\,\kappa+i+j}\qquad(i\geqslant\mathrm{Re}\,\kappa)$$

Since $\tilde{\psi}$ is a linear combination of functions $\zeta^{i-\kappa}(\log\zeta)^j$ with $i,j\in\mathbb{N}$, this allows us to compute an explicit bound

$$\left|\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{H}_0^{-\infty}}\psi(\zeta)\,\mathrm{e}^{\zeta n}\mathrm{d}\zeta-\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{H}_0^{-\varrho}}\tilde{\psi}(\zeta)\,\mathrm{e}^{\zeta n}\mathrm{d}\zeta\right|\;\leqslant\;C\,(1+\varepsilon)^{-n},$$

for a suitable constant $C>0$.

Since f is Fuchsian at $z=1$, by taking ε sufficiently small, we can use the algorithms from [21] to compute a bound

$$|\psi(\zeta)-\tilde{\psi}(\zeta)|\;\leqslant\;\tilde{B}\,|\zeta|^{T-\mathrm{Re}\,\kappa}|\log\zeta|^r$$

for all ζ with $|\zeta|\in(0,\rho]$ and $|\arg\zeta|\leqslant\pi$. Given $\nu>T-\mathrm{Re}\,\kappa+1$, we may further transform this into a bound

$$|\psi(\zeta)-\tilde{\psi}(\zeta)|\;\leqslant\;B\,|\zeta|^{\nu-1}.$$

If $\nu>0$, then

$$\begin{aligned}\left|\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{H}_0^{-\varrho}}(\psi(\zeta)-\tilde{\psi}(\zeta))\,\mathrm{e}^{\zeta n}\mathrm{d}\zeta\right|\;&\leqslant\;\frac{B}{\pi}\int_0^\varrho\zeta^{\nu-1}\mathrm{e}^{-\zeta n}\mathrm{d}\zeta\\&\leqslant\;\frac{B}{\pi}\Gamma(\nu)\,n^\nu.\end{aligned}\qquad(3.6)$$

Altogether, we may compute constants $B', C > 0$ and polynomials $c_0, \ldots, c_{T-1} \in \mathbb{K}[\gamma^{(\mathbb{N})}(\kappa)][\log n]$ such that

$$|f_n^{1,\varepsilon} - (c_0(\log n) + \cdots + c_{T-1}(\log n)\, n^{-(T-1)})\, n^{\kappa-1}| \leqslant B' n^{-\nu} + C\,(1+\varepsilon)^{-n},$$

for all $n > 0$. Such a bound can also be proved without the assumption $\nu > 0$, by doing $\lfloor 1-\nu \rfloor$ partial integrations before estimating the left-hand side of (3.6). We omit the details, since the case when $\nu > 0$ suffices for the proof of our main Theorem 3.2 below.

To conclude this section, let us finally consider a solution f to $Lf = 0$ with $f_n \in \mathbb{K}$ for all $n \in \mathbb{N}$. Let $h_1^\alpha, \ldots, h_r^\alpha$ be the fundamental system of local solutions to $Lh = 0$ with

$$h_i^\alpha \in (\alpha - z)^{-\kappa_i} \mathbb{K}\{\{z-\alpha\}\}[\log(z-\alpha)]$$

for $i = 1,\ldots,r$. Using the algorithms from [21], we may compute the unique constants $\lambda_1,\ldots, \lambda_r \in \mathbb{K}^{\text{hola}}$ such that $f = \lambda_1 h_1^\alpha + \cdots + \lambda_r h_r^\alpha$. Let $\nu > 0$ and let $T_1, \ldots, T_r$ be minimal integers with $\nu > T_i - \operatorname{Re} \kappa_i + 1$ for $i = 1, \ldots, r$. We have shown above how to compute polynomials $c_{i,0}, \ldots, c_{i,T_i-1} \in \mathbb{K}[\gamma^{(\mathbb{N})}(\kappa_i)][\log n]$ and constants B_i, C_i such that

$$|(h_i^\alpha)_n^{\alpha,\varepsilon} - (c_{i,0}(\log n) + \cdots + c_{i,T_i-1}(\log n)\, n^{-(T_i-1)})\, n^{\kappa_i-1} \alpha^{-n}| \leqslant B_i n^{-\nu} |\alpha|^{-n} + C_i |\alpha + \varepsilon\, \alpha|^{-n},$$

for all $n > 0$. Setting $B_* = |\lambda_1| B_1 + \cdots + |\lambda_r| B_r$ and $C_* = |\lambda_1| C_1 + \cdots + |\lambda_r| C_r$, it follows that

$$\left| f_n^{\alpha,\varepsilon} - \sum_{i,j<T_i} c_{i,j}(\log n)\, n^{\kappa_i-1-j} \alpha^{-n} \right| \leqslant B_* n^{-\nu} |\alpha|^{-n} + C_* |\alpha + \varepsilon\, \alpha|^{-n},$$

for all $n > 0$.

3.4. Asymptotic expansions with effective error bounds

Let us now return to the general setting from subsection 3.1: for some $R > 0$ and $m \in \{1,\ldots,\ell\}$, we assume that $|\alpha_i| < R$ for $i = 1, \ldots, m$ and $|\alpha_i| > R$ for $i = m+1, \ldots, \ell$. We also assume that each of the singularities α_i with $i \in \{1, \ldots, m\}$ is Fuchsian.

In the previous subsection, we have seen how to compute asymptotic expansions with effective error bounds for the truncated Hankel integrals

$$\frac{1}{2\pi \mathrm{i}} \int_{\mathcal{H}_{\alpha_i}^{\alpha_i(1+\varepsilon)}} \frac{f(z)}{z^{n+1}} \mathrm{d}z,$$

for $i = 1,\ldots,m$. Assuming that $|\alpha_i|\,(1+\varepsilon) \leqslant R$, the truncated Hankel contour $\mathcal{H}_{i,R}$ consists of $\mathcal{H}_{\alpha_i}^{\alpha_i(1+\varepsilon)}$ and two straight stretches between $\alpha_i(1+\varepsilon)$ and $\alpha_i(R/|\alpha_i|)$. Using the algorithms from [20], we may compute a bound $\|f\|_i$ for $|f|$ on these two stretches. Then we have

$$\begin{aligned}\left| \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{H}_{i,R}} \frac{f(z)}{z^{n+1}} \mathrm{d}z - \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{H}_{\alpha_i}^{\alpha_i(1+\varepsilon)}} \frac{f(z)}{z^{n+1}} \mathrm{d}z \right| &\leqslant \left| \frac{2}{2\pi} \int_{|\alpha_i|(1+\varepsilon)}^{R} \frac{\|f\|_i}{z^{n+1}} \mathrm{d}z \right| \\ &\leqslant \frac{\|f\|_i}{\pi\, |\alpha_i + \alpha_i\, \varepsilon|^n}.\end{aligned}$$

Combining this with the bounds for the residual integrals, this gives a first answer to questions **Q1** and **Q2**:

THEOREM 3.2. *Let f, m, ℓ, R, ε, and $\alpha_1, \ldots, \alpha_\ell$ be as in the text above. For any $\nu_1, \ldots, \nu_m \in \mathbb{R}$, we can compute an asymptotic expansion f_n^{as} for f_n in $\mathbb{K}[\gamma^{(\mathbb{N})}(\mathbb{K})][(\mathbb{K}^{\neq})^n, n^{\mathbb{K}}, \log n]$ together with bounds $B_i, C_i, E \in \mathbb{Q}^>$ and $n_0 \in \mathbb{N}$ such that*

$$|f_n - f_n^{\text{as}}| \leqslant \sum_{i=1}^m \frac{B_i}{n^{\nu_i} |\alpha_i|^n} + \sum_{i=1}^m \frac{C_i}{|\alpha_i + \alpha_i\, \varepsilon|^n} + \frac{E}{R^n}, \tag{3.7}$$

for all $n \geqslant n_0$.

Remark 3.3. In principle, the error at the right hand side of (3.7) can be replaced by a single term of the form $B\,n^{-\nu}\,|\alpha_i|^{-n}$, where $|\alpha_i|$ is smallest among $|\alpha_1|,\ldots,|\alpha_m|$. However, from a numerical point of view, some of the B_i and C_i may be very small (or even zero), in which case it may be preferable to use the error bound from the theorem. We will return to this issue in section 4.4 below.

4. PERIODIC OR QUASI-PERIODIC CANCELLATIONS

4.1. Classical results for rational functions

In Example 3.1, we have seen that the contributions of more than one dominant singularity may cancel out in a periodic fashion. Is it possible to predict when this phenomenon occurs? As demonstrated by Example 3.1, this is already an interesting question in the case when f is a rational function. In that case, exact cancellations can actually be predicted, as we will recall now. In section 4.2, we will deal more generally with approximate cancellations as in Example 3.1.

So consider a sequence $(f_n)_{n\in\mathbb{N}}\in\mathbb{K}^{\mathbb{N}}$ whose generating function is f is rational. Then the classical Skolem-Mahler-Lech theorem tells us that the zero-set $\{n\in\mathbb{N}:f_n=0\}$ is ultimately periodic. This was first proved by Skolem in the case when $\mathbb{K}=\mathbb{Q}$, next by Mahler for $\mathbb{K}=\mathbb{Q}^{\text{alg}}$, and finally by Lech for general fields $\mathbb{K}$ of characteristic zero:

THEOREM 4.1. [44, 34, 33] *Let $\mathbb{K}$ be a field of characteristic zero and let $(f_n)_{n\in\mathbb{N}}\in\mathbb{K}^{\mathbb{N}}$ be a sequence whose generating function f is rational. Then there exists a period $\Pi\in\mathbb{N}^{>}$ and finite sets $\Lambda\subseteq\{0,\ldots,\Pi-1\}$ and $\mathrm{X}\subseteq\mathbb{N}$ such that*

$$\{n\in\mathbb{N}:f_n=0\} \;=\; (\mathrm{P}+\Pi\,\mathbb{N})\cup\mathrm{X}. \tag{4.1}$$

The periodic part $\mathrm{P}+\Pi\,\mathbb{N}$ in the above decomposition is actually computable [4], whereas the computability of the exceptional part X is currently an open problem [41]. Let $\alpha_1,\ldots,\alpha_\ell$ be the singularities of f and let $\mathbb{U}:=\exp(2\,\pi\,\mathrm{i}\,\mathbb{Q})$. We say that f is *resonant* if $\alpha_i/\alpha_j\in\mathbb{U}$ for some $i\neq j$. Setting

$$\Pi \;:=\; \operatorname{lcm}\{q:\alpha_i/\alpha_j=\mathrm{e}^{2\pi\mathrm{i}p/q},\gcd(p,q)=1\}, \tag{4.2}$$

this is the case if and only if $\Pi>1$. Berstel and Mignotte proved the following:

THEOREM 4.2. [4] *Let $\mathbb{K}$ be a field of characteristic zero and let $(f_n)_{n\in\mathbb{N}}\in\mathbb{K}^{\mathbb{N}}$ be a sequence whose generating function f is rational. Assume that $f_n=0$ for infinitely many $n\in\mathbb{N}$ and let Π be defined as in (4.2). Then f is resonant and we may compute a finite set $\mathrm{P}\subseteq\{0,\ldots,\Pi-1\}$ such that (4.1) holds for some finite set $\mathrm{X}\subseteq\mathbb{N}$.*

It is instructive to detail the computability statements. Let $\mathbb{L}:=\mathbb{Q}(\alpha_1,\ldots,\alpha_\ell)$ and $d:=[\mathbb{L}:\mathbb{Q}]$. Consider $i\neq j$ for which $\alpha_i/\alpha_j=\mathrm{e}^{2\pi\mathrm{i}p/q}$ with $\gcd(p,q)=1$. Since $\mathbb{Q}(\alpha_i/\alpha_j)$ is a subfield of $\mathbb{L}$, we must have $[\mathbb{Q}(\alpha_i/\alpha_j)]=\varphi(q)\,|\,d$. Using that $\varphi(n)>n/(\mathrm{e}^{\Upsilon}\log\log n+3/\log\log n)$ for $q>2$ [43, Theorem 15], it follows that $q\leqslant\bar{q}:=6\,d\,\lceil\log\log\max(d,\mathrm{e}^{\mathrm{e}})\rceil$. This allows us to compute q as the smallest $k\in\{1,\ldots,\bar{q}\}$ with $(\alpha_i/\alpha_j)^k=1$. We conclude that Π is computable.

Now for every $\rho \in \{0, \ldots, \Pi - 1\}$, the generating function $g^{[\rho]}$ of $(f_{n\Pi+\rho})_{n\in\mathbb{N}}$ satisfies

$$g^{[\rho]}(z^{\Pi}) = \sum_{0\leqslant i<\Pi} f(\omega^i z)\,\omega^{-i\rho z}, \tag{4.3}$$

where $\omega = \mathrm{e}^{2\pi\mathrm{i}/\Pi}$. This allows us in particular to test whether $g^{[\rho]} = 0$ and to compute the set P. If needed, we may also deduce the minimal period Π' and corresponding P' for which (4.1) holds.

4.2. Removing resonance

Let us now consider an arbitrary holonomic sequence $(f_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ whose generating function f is convergent at the origin. Let $\alpha_1, \ldots, \alpha_\ell$ be the non-zero singularities of f. We define Π as in (4.2) and say that f is *resonant* if $\Pi > 1$. Note that the arguments at the end of the previous subsection still allow us to compute Π. As in the algebraic case, the coefficients of a resonant holonomic sequence can vanish in a periodic manner, or become disproportionally small on a regular basis. This problem can be removed through the consideration of subsequences and case separation, as in Example 3.1:

PROPOSITION 4.3. *With the above notations, the subsequence $(f_{n\Pi+\rho})_{n\in\mathbb{N}}$ is holonomic and non-resonant for any $\rho \in \{0, \ldots, \Pi - 1\}$.*

Proof. The generating function $g^{[\rho]}$ of $(f_{n\Pi+\rho})_{n\in\mathbb{N}}$ satisfies (4.3). Using standard closure properties, it follows that $(f_{n\Pi+\rho})_{n\in\mathbb{N}}$ is holonomic. The singularities of the right-hand side of (4.3) are contained in the set $\mathrm{A} := \{\alpha_i\,\omega^j : i \in \{1, \ldots, \ell\}, j \in \{0, \ldots, \Pi - 1\}\}$. Therefore the singularities of g are contained in the set $\mathrm{A}^{\Pi} = \{\alpha_1^{\Pi}, \ldots, \alpha_\ell^{\Pi}\}$. So it suffices to show that $\alpha_i^{\Pi}/\alpha_j^{\Pi} \notin \mathbb{U}$ whenever $\alpha_i^{\Pi} \neq \alpha_j^{\Pi}$. Assume for contradiction that $\alpha_i^{\Pi} \neq \alpha_j^{\Pi}$, but $\alpha_i^{k\Pi} = \alpha_j^{k\Pi}$ for some $k \geqslant 2$. Without loss of generality, we may assume that k is minimal with this property. But then we have $\alpha_i/\alpha_j = \mathrm{e}^{2\pi\mathrm{i}p/(k\Pi)}$ for some $p \in \mathbb{N}$ with $\gcd(p, k\Pi) = 1$, which implies $k\Pi | \Pi$ by (4.2): a contradiction. □

The above proposition has an operator counterpart. Let $\vartheta := z\,\partial$ and consider a monic operator $L(z, \vartheta) \in \mathbb{K}(z)[\vartheta]$ of order r with $Lf = 0$. Let $\alpha_1, \ldots, \alpha_\ell$ now be the singularities of L (i.e. the zeros of the dominators of its coefficients). We define Π as in (4.2) and say that f is *resonant* if $\Pi > 1$.

Let us show how to compute annihilators for the generating functions $g^{[\rho]}$. First of all, for each $i \in \{0, \ldots, \Pi - 1\}$, we observe that $L^{\{i\}} f(\omega^i z) = 0$ for the monic operator $L^{\{i\}}(z, \vartheta) := L(\omega^i z, \vartheta)$. It follows that $\Lambda := \operatorname{lcm}(L, L^{\{1\}}, \ldots, L^{\{\Pi-1\}})$ is an annihilator for $g^{[0]}(z^{\Pi}), \ldots, g^{[\Pi-1]}(z^{\Pi})$. For each $i \in \{0, \ldots, \Pi - 1\}$, we next observe that $g^{[i]}(z^{\Pi}) = g^{[i]}((\omega^i z)^{\Pi})$, so $\Gamma := \gcd(\Lambda, \Lambda^{\{1\}}, \ldots, \Lambda^{\{\Pi-1\}})$ is still a monic annihilator of $g^{[i]}(z^{\Pi})$. Furthermore, since $\Gamma^{\{1\}} = \gcd(\Lambda^{\{1\}}, \ldots, \Lambda^{\{\Pi\}}) = \Gamma$, we must have $\Gamma \in \mathbb{K}(z^{\Pi})[\vartheta]$. Setting $u = z^{\Pi}$ and $\vartheta_u = u\,\partial/\partial u$, we finally note that $\vartheta = \Pi\,\vartheta_u$. This allows us to rewrite $\Gamma_u := \Pi^{-r}\Gamma$ as a monic operator in $\mathbb{K}(u)[\vartheta_u]$ with $\Gamma_u\, g^{[\rho]}(u) = 0$ for all $\rho \in \{0, \ldots, \Pi - 1\}$.

We note that the sets of non-zero singularities of Λ and Γ (i.e. the zeros of their denominators) both coincide with the set A from the proof of Proposition 4.3. Consequently, the set of non-zero singularities of Γ_u is A^{Π}. In other words, Γ_u is non-resonant.

Remark 4.4. We may regard the transformation that maps L to Γ_u as a differential counterpart of the Graeffe transform [27]. Indeed, if $f = P/Q$ is a rational function, then the denominator of $g^{[\rho]}$ must be the Π-fold Graeffe transform of Q for each $\rho \in \{0, \ldots, \Pi - 1\}$, up to constant multiples.

4.3. Quasi-resonance

Even for a non-resonant holonomic sequence, the contributions of the dominant singularities to its asymptotics may occasionally cancel out. For instance, theoretically speaking, the sequence f_n from (1.2) might occasionally become small, although this would necessarily happen in a non-periodic manner.

Let us make this unlikely phenomenon more precise in the case when $(f_n)_{n \in \mathbb{N}}$ is dominant-Fuchsian. Let $\alpha_1, \ldots, \alpha_m$ be the dominant singularities for this sequence (we assume that $m \geqslant 1$). We say that $(f_n)_{n \in \mathbb{N}}$ is *quasi-resonant* if for any constants $M > 0$ and $\kappa \in \mathbb{R}$, there exist infinitely many $n > 0$ with

$$|f_n| \; < \; \frac{M}{|\alpha_1|^n n^\kappa}. \tag{4.4}$$

In fact, we conjecture that this never happens:

CONJECTURE 4.5. (QUASI-RESONANCE CONJECTURE) *Assume that $(f_n)_{n \in \mathbb{N}}$ is a non-resonant dominant-Fuchsian holonomic sequence. Then $(f_n)_{n \in \mathbb{N}}$ is not quasi-resonant.*

Although Conjecture 4.5 is far beyond the current state of knowledge in number theory, let us provide meager evidence why we believe that it might hold. The conjecture is already interesting in the case when $f = \sum_{n \in \mathbb{N}} f_n z^n$ is a rational function. Even more specifically, assume that

$$f_n \; = \; \lambda \alpha^n - \mu \beta^n,$$

where $\alpha, \beta, \lambda, \mu \in \mathbb{K}^{\neq}$ are such that $|\alpha| = |\beta|$, but $\alpha / \beta \notin \mathbb{U}$. If $|\lambda| \neq |\mu|$, then we clearly have $|f_n| \geqslant ||\mu| - |\lambda|| \cdot |\alpha|^n$ for all n. Assume that $|\lambda| = |\mu|$ and that $\log(\lambda / \mu)$, $\log(\alpha / \beta)$, and $2 \pi \mathrm{i}$ are $\mathbb{Q}$-linearly independent. Then Baker's theorem [1] implies the existence of a (computable) constant $C > 0$ such that

$$\left| n \log \frac{\alpha}{\beta} + \log \frac{\lambda}{\mu} + 2 k \pi \mathrm{i} \right| \; > \; n^{-C}$$

for all but a finite number of $(n, k) \in \mathbb{N} \times \mathbb{Z}$. Taking exponentials, it follows that

$$\left| \frac{\lambda \alpha^n}{\mu \beta^n} - 1 \right| \; > \; \frac{n^{-C}}{2}$$

and

$$|f_n| \; > \; \frac{|\mu|}{2 n^C} |\alpha|^n$$

for all but a finite number of $n \in \mathbb{N}$. If $\log(\lambda / \mu) = \phi \log(\alpha / \beta)$ with $\phi \in \mathbb{Q}$, then Baker's theorem implies the existence of a constant $C > 0$ with

$$\left| (n + \phi) \log \frac{\alpha}{\beta} + 2 k \pi \mathrm{i} \right| \; > \; n^{-C}$$

for all but a finite number of $(n, k) \in \mathbb{N} \times \mathbb{Z}$. Using a similar reasoning as above, we deduce that our conjecture again holds in this particular case.

Of course, our general conjecture is far more ambitious and we have no convincing further evidence for it yet. It may be interesting to consider the slightly weaker conjecture for which (4.4) is replaced by

$$|f_n| < \frac{M}{|\alpha_1|^n \mathrm{e}^{\kappa(\log n)^\tau}} \tag{4.5}$$

for some fixed constant $\tau \geqslant 1$.

4.4. Asymptotic expansions

Using Proposition 4.3, we can generalize the technique from Example 3.1 and reduce the determination of the asymptotic expansion of a holonomic sequence $(f_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ to the non-resonant case, modulo a finite number of case separations.

In order to detect periodic cancellations we still need a way to decide whether a particular solution f to the equation $Lf=0$ in section 3.1 is actually analytic at a given regular singularity α of L. Now we may write $f=\lambda_1 h_1^\alpha+\cdots+\lambda_r h_r^\alpha$ as a $\mathbb{K}^{\text{hola}}$-linear combination of the canonical basis of local solutions $h_1^\alpha,\ldots,h_r^\alpha$ at α. Let $I\subseteq\{1,\ldots,r\}$ be the subset of indices i for which h_i^α is singular at α. Then f is analytic at α if and only if $\lambda_i=0$ for every $i\in I$.

In other words, if we have a way to decide whether a given Fuchsian holonomic constant $\mathbb{K}^{\text{hola}}$ is zero, then can determine the actual dominant singularity of f. By Theorem 2.3, it actually suffices to be able to decide whether a holonomic constant in $\mathbb{K}^{\text{hol}}$ is zero. We will denote by **Hol** an oracle to do this. Here we assume an exact representation for elements as $\mathbb{K}^{\text{hol}}$ as the value at 1 of a holonomic function in $\mathcal{F}^{\text{hol}}$ that is given *via* a vanishing operator in $\mathbb{K}(z)[\partial]$ and a finite number of initial conditions in $\mathbb{K}$.

It is interesting to point out that if we can determine the dominant singularity of a holonomic function that is analytic at the origin, then we also have an algorithm to test whether holonomic constants in $\mathbb{K}^{\text{hol}}$ are zero. Indeed, given $c\in\mathbb{K}^{\text{hol}}$, let $g\in\mathcal{F}^{\text{hol}}$ be such that $c=g(1)$. Then 1 is the dominant singularity of the holonomic function $g/(1-z)$ if and only if $c=0$.

The above considerations lead to the following variant of Theorem 3.2:

THEOREM 4.6. *Let f, m, ℓ, R, ε, and $\alpha_1,\ldots,\alpha_\ell$ be as in section 2.3. Assume that we have an oracle* **Hol***, that f is non-resonant, and that f is singular at α_k for some $k\in\{1,\ldots,m\}$. Modulo reordering indices, assume that $k=1$, $|\alpha_1|=\cdots=|\alpha_p|$, and $|\alpha_i|\neq|\alpha_1|$ for $i>p$. Then, for any $\nu\in\mathbb{R}$, we can compute an asymptotic expansion f_n^{as} for f_n in $\mathbb{K}[\gamma^{(\mathbb{N})}(\mathbb{K})][\alpha_1^{-n},\ldots,\alpha_p^{-n},n^{\mathbb{K}},\log n]$ together with $B\in\mathbb{Q}^>$ and $n_0\in\mathbb{N}$ such that*

$$|f_n-f_n^{\text{as}}| \leqslant \frac{B}{n^\nu|\alpha_1|^n}, \tag{4.6}$$

for all $n\geqslant n_0$. Moreover, if Conjecture 4.5 holds, then we may require in addition that

$$|f_n| > \frac{B}{2n^\nu|\alpha_1|^n}$$

for all sufficiently large n.

5. UNIFORMLY FAST EVALUATION

In this section, we investigate question **Q3**. We assume that $(f_n)_{n\in\mathbb{N}}\in\mathbb{K}^{\mathbb{N}}$ satisfies the difference equation (1.1). We also assume that its generating function f is convergent at the origin and that it satisfies a holonomic equation (2.1) that is Fuchsian at the origin. Our goal is to compute f_n for large n using a precision of p bits, with a good uniform complexity in both n and p.

5.1. Preliminaries

Before we proceed, let us briefly recall some notations and basic facts about fast arithmetic. We define $\mathsf{M}(p)$ to be the time that is needed to multiply two p-bit integers and we make that customary assumption that $\mathsf{M}(p)/p$ is non-decreasing. It was shown recently in [18] that we may take $\mathsf{M}(p) = O(p \log p)$.

Approximate computations with real and complex numbers can be done using either fixed point or floating point arithmetic. Let $\mathbb{D} := \mathbb{Z}\,2^{\mathbb{Z}}$ be the set of *dyadic* numbers. A p-bit *fixed point approximation* of a complex number $x \in \mathbb{R}$ is a number $\tilde{z} \in \mathbb{D}$ with $|\tilde{z} - z| \leqslant 2^{-p}$. A p-bit *floating point approximation* of $z \in \mathbb{C}$ is a number $\tilde{z} = m\,2^e$ with $m \in \mathbb{Z}[\mathrm{i}]\,2^{-p}$, $e \in \mathbb{Z}$, $|\tilde{z} - z| \leqslant 2^{e-p}$, $|m| \leqslant 1$, and either $|z| \geqslant 2^{e-1}$ or $|z| = 0$.

It is well known [8] that p-bit approximations of bounded exponentials and logarithms can be computed in time $O(\mathsf{M}(p) \log p)$, both for fixed point and floating point representations. Here a bounded exponential is a number e^z with $z \in \mathbb{D}[\mathrm{i}]$ and $|z| \leqslant B$ for some fixed constant $B > 0$. Similarly, a bounded logarithm is a number $\log z$ with $B^{-1} \leqslant |z| \leqslant B$. For non-zero $z \in \mathbb{C}$, this allows us to convert between p-bit floating point approximations of z and p-bit fixed point approximations of $\log z$.

We also observe that, given $\alpha \in \mathbb{Z}\,2^{-p}$ with $|\alpha| \leqslant 2^p$, we can compute a p-bit approximation of $\mathrm{e}^{\alpha \mathrm{i}}$ in time $O(\mathsf{M}(p) \log p)$. Indeed, it suffices to compute $k := \lfloor \alpha / (2\pi) \rceil$ and $\beta := \alpha - 2\pi k$ with $|\beta| \leqslant \pi$ and then use the fact that $\mathrm{e}^{\alpha \mathrm{i}} = \mathrm{e}^{\beta \mathrm{i}}$.

We will finally need the following result that was essentially proved in [21].

PROPOSITION 5.1. *Let $L = \partial^r + L_{r-1}\partial^{r-1} + \cdots + L_0 \in \mathbb{K}(z)[\partial]$ be Fuchsian at the origin. Let $(h_{i,j})_{1 \leqslant i \leqslant \mu, 0 \leqslant j < \nu_i}$ be the canonical solutions of $Lf = 0$ at the origin and let $z^{\kappa_i} (\log z)^j$ be the dominant monomial of $h_{i,j}$. Let $\alpha_1, \ldots, \alpha_\ell$ be the non-zero singularities of L, let $\rho < \min\{|\alpha_1|, \ldots, |\alpha_\ell|\}$, and let $M := \max_{i,j,t} \max_{|z| \leqslant \rho} |h_{i,j,k}(z)|$, where $h_{i,j} z^{-\kappa_i} = h_{i,j,0} + \cdots + h_{i,j,j} (\log z)^j$. Let σ be the total bit-size of the operator L and let $z \in \mathbb{Z}[\mathrm{i}]\,2^{-p}$ with $|z| \leqslant \rho$. Then we may compute a p-bit fixed point approximation of $h_{i,j}(z)$ using $O(\mathsf{M}((\sigma + \log p)\, p \log p))$ bit-operations, for all i, j. This complexity is uniform in p and σ, provided that $\alpha_1, \ldots, \alpha_\ell$, $\kappa_1, \ldots, \kappa_\mu$, $\nu_1, \ldots, \nu_\mu$ remain fixed and $\log(M+1) = O(p)$.*

Proof. We approximate $f(z)$ using the technique from [21, section 3]. The matrices A_k from (3.10) and (3.12) in that section have size $O(\sigma + \log k)$. Since $\log(M+1) = O(p)$, it suffices to evaluate $A_k \cdots A_1$ for $k = O(p)$ in order to obtain p-bit approximations for the values $h_{i,j,t}(z)$. Using binary splitting, this requires $O(\mathsf{M}((\sigma + \log p)\, p \log p))$ bit operations. Since p-bit approximations of $z^{\kappa_i} (\log z)^t$ can be computed using $O(\mathsf{M}(p) \log p)$ bit operations for all $t < \nu_i$, this allows us to compute a p-bit fixed point approximation of $h_{i,j}(z)$ in time $O(\mathsf{M}((\sigma + \log p)\, p \log p))$. □

5.2. Exact computation of the terms of a holonomic sequence

If $n = O(p)$, then the most efficient strategy is to compute f_n exactly as an element of $\mathbb{K}$ and convert the result into a fixed point or floating point approximation. In fact, as in [20, 21], it suffices to compute in the algebraic number field $\mathbb{L}$ generated by the coefficients of Σ and the initial conditions $f_0, \ldots, f_t$ with $t = \max(k \in \mathbb{N} : k \geqslant s \vee \Sigma_s(k) = 0)$. Let us recall from [9, 21] how to compute f_n using binary splitting.

For each $N \in \mathbb{N}$, let F_N be the column vector with entries $f_N, \ldots, f_{N+s-1}$. If $N \geqslant t$, then

$$F_{N+1} = \Delta_{N \to N+1} F_N,$$

where

$$\Delta_{N\to N+1} = \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ -\frac{\Sigma_0(N)}{\Sigma_s(N)} & -\frac{\Sigma_1(N)}{\Sigma_s(N)} & \cdots & -\frac{\Sigma_{s-1}(N)}{\Sigma_s(N)} \end{pmatrix}. \tag{5.1}$$

More generally, for $k\in\mathbb{N}$ and $\Delta_{N\to N+k}:=\Delta_{N+k-1\to N+k}\cdots\Delta_{N\to N+1}$, we have

$$F_{N+k} = \Delta_{N\to N+k}F_N$$

and $\Delta_{N\to N+k}$ can be computed efficiently using binary splitting:

$$\Delta_{N\to N+k} = \Delta_{N+\lfloor k/2\rfloor\to N+k}\Delta_{N\to N+\lfloor k/2\rfloor}.$$

Since f is Fuchsian at the origin, the bit-size of the entries of $\Delta_{N\to N+k}$ is bounded by $O(k\log(N+k))$. Using a classical complexity analysis [9, 21], it follows that $\Delta_{N\to N+k}$ can be computed in time $O(\mathsf{M}(k\log(N+k))\log k)$. In particular, we may compute f_n in time $O(\mathsf{M}(n\log^2 n))$. Converting the result into p-bit fixed point or floating point notation can be done in time $O(\mathsf{M}(p))$.

5.3. Fast computation of floating point approximations

Having dealt with the case when $n=O(p)$ in the previous subsection, let us now assume that $p=o(n)$ and $\log n=O(p)$. If f has a dominant singularity at α, then f_n typically grows like α^{-n}, in first approximation. Consequently, a p-bit fixed point approximation of f_n typically requires $\Theta(n)$ bits if $|\alpha|<1$. In particular, unless $n=\tilde{O}(p)$, then it is hopeless to compute such approximations in smoothly linear time in p. From now on, we will focus on the computation of a p-bit floating point approximation of f_n. Under the assumption that $(f_n)_{n\in\mathbb{N}}$ is not quasi-resonant, we will show that such an approximation can be computed in uniform smoothly linear time.

So assume that $(f_n)_{n\in\mathbb{N}}$ is not quasi-resonant and let $\alpha_1,\ldots,\alpha_\ell$ be the singularities of f. Assume that $\alpha_1,\ldots,\alpha_m$ are the dominant singularities of f with $m\geqslant 1$. We compute f_n using

$$f_n = \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{H}_1\cup\mathcal{A}_1\cup\cdots\cup\mathcal{H}_m\cup\mathcal{A}_m}\frac{f(z)}{z^{n+1}}\mathrm{d}z,$$

where $\mathcal{H}_i$ is an axial truncated Hankel contour around $z=\alpha_i$ until $z=(1+\delta)\,\alpha_i$ and $\mathcal{A}_i$ is a circular arc around 0 from $(1+\delta)\,\alpha_i$ to $(1+\delta)\,\alpha_{i+1}$ (or to $(1+\delta)\,\alpha_1$ if $i=m$), for $i=1,\ldots,m$. Let $q\geqslant p$ be a temporarily increased working precision with $\log n=O(q)$ and $q=O(p)$. We will specify q later. For some sufficiently small $\epsilon>0$ with $\epsilon<1/2$ and any $n\geqslant q/\epsilon$, we take

$$\delta := \frac{q}{n}.$$

Since f is Fuchsian at $\alpha_1,\ldots,\alpha_m$, we may compute positive constants ρ, C, and μ such that

$$|f(z)| \leqslant C\,\delta^{-\mu}$$

for all $z\in\mathcal{A}_1\cup\cdots\cup\mathcal{A}_m$ (uniformly, for all $\delta\leqslant\epsilon$). It follows that

$$\left|\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{A}_1\cup\cdots\cup\mathcal{A}_m}\frac{f(z)}{z^{n+1}}\mathrm{d}z\right| \leqslant C\frac{\delta^{-\mu}(1+\delta)^{-n}}{|\alpha_1|^n} \leqslant 2C\frac{\delta^{-\mu}\mathrm{e}^{-q}}{|\alpha_1|^n}.$$

For $i=1,\ldots,m$, we have

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{H}_i}\frac{f(z)}{z^{n+1}}\mathrm{d}z = (\psi_i(\delta)-\psi_i(\delta\,\mathrm{e}^{-2\pi\mathrm{i}}))\,\alpha_i^{-n} = -\psi_i(\delta\,\mathrm{e}^{-2\pi\mathrm{i}})\,\alpha_i^{-n},$$

where

$$\psi_i(w) \;:=\; \frac{1}{2\pi\mathrm{i}}\int_\delta^w \frac{f((1+w)\,\alpha_i)}{(1+w)^{n+1}}\,\mathrm{d}w.$$

We note that the integrand $f((1+w)\,\alpha_i)\,(1+w)^{-(n+1)}$ satisfies a holonomic equation whose total size is bounded by $O(\log n)$. Consequently, the same holds for ψ_i. Since $(1+w)^{-(n+1)}$ is analytic at $w=0$ and bounded by $O(\mathrm{e}^q)=\mathrm{e}^{O(p)}$ for $|w|\leqslant\delta$, the uniformity assumptions of Proposition 5.1 are satisfied. It follows that we may compute a q-bit fixed point approximation of $\psi_i(\delta\,\mathrm{e}^{-2\pi\mathrm{i}})$ using $O(\mathsf{M}(q\log n\log q))$ bit operations. We may also compute a q-bit fixed point approximation of $(\alpha_1/\alpha_i)^{-n}$ using $O(\mathsf{M}(q\log q))$ bit operations. Altogether, this allows us to compute a number $v_i\in\mathbb{Z}[\mathrm{i}]\,2^{\mathbb{Z}}$ with

$$|v_i-\psi_1(\delta\,\mathrm{e}^{-2\pi\mathrm{i}})\,(\alpha_1/\alpha_i)^{-n}| \;<\; 2^{-q}$$

using $O(\mathsf{M}(q\log n\log q))$ bit operations. By construction, it follows that

$$|f_n\,\alpha_1^n-(v_1+\cdots+v_m)| \;\leqslant\; m\,2^{-q}+2\,C\,\delta^{-\mu}\,\mathrm{e}^{-q},$$

provided that $n\geqslant q/\epsilon$. Since $(f_n)_{n\in\mathbb{N}}$ is not quasi-resonant, there also exist constants $M>0$, $\kappa\in\mathbb{R}$, and $n_0>0$ such that

$$|f_n\,\alpha_1^n| \;>\; M\,n^{-\kappa}$$

for all $n\geqslant n_0$. In order to obtain p-bit floating point approximations of $f_n\,\alpha_1^n$ and then f_n, it suffices to chose q in such a way that

$$m\,2^{-q}+2\,C\,\delta^{-\mu}\,\mathrm{e}^{-q} \;\leqslant\; \tfrac{1}{2}M\,n^{-\kappa}\,2^{-p}.$$

Using that $m\,2^{-q}+2\,C\,\delta^{-\mu}\,\mathrm{e}^{-q}\leqslant 2\,(m+C)\,n^{\mu}\,2^{-q}$, this is certainly the case if we take

$$q \;:=\; p+(\kappa+\mu)\log_2 n+\log_2\frac{m+C}{M}+2.$$

Note that we have indeed have $q=O(p)$ for this choice of q. In fact, we even have $q\sim p$ as soon as $\log n=o(p)$.

In combination with the results from section 5.2, we have proved the following:

THEOREM 5.2. *Let $(f_n)_{n\in\mathbb{N}}$ be a dominant-Fuchsian holonomic sequence that is not quasi-resonant. Then there exists an algorithm to compute a p-bit floating point approximation of f_n using $O(\mathsf{M}(p\log p\log(n\,p)))$ bit operations. This bound is uniform in p and n, provided that* $\log n=O(p)$.

COROLLARY 5.3. *Let $(f_n)_{n\in\mathbb{N}}$ be a holonomic sequence and assume Conjecture 4.5. Then there exists an algorithm to compute a p-bit floating point approximation of f_n using $O(\mathsf{M}(p\log p\log(n\,p)))$ bit operations. This bound is uniform in p and n, provided that* $\log n=O(p)$.

Remark 5.4. Note that the operator L with $Lf=0$ may have singularities β with $|\beta|<|\alpha_1|$, as long as the particular solution f remains analytic at β. Assuming that $(f_n)_{n\in\mathbb{N}}$ is Fuchsian and that we have an oracle **Hol**, we may verify whether this is the case by checking that the coefficients of the singular canonical solutions h_i^β in f all vanish. Note that our theorem and its corollary only claim the *existence* of an efficient algorithm to compute f_n; for this, we do not need the oracle **Hol**.

Remark 5.5. In practice, for the fast evaluation of a Fuchsian holonomic sequence $(f_n)_{n\in\mathbb{N}}$, we first make it non-resonant using the pre-treatment from section 4.2, then apply the algorithm from the previous subsection for the evaluation of f_n, while falling back on the slower algorithm from section 5.2 whenever we detect massive cancellation. In particular, this mixed strategy takes care of exceptional values of n for which f_n vanishes. Whenever this algorithm does not run in time $O(\mathsf{M}(p\log p\log(n p)))$ when $n\to\infty$, we note that $(f_n)_{n\in\mathbb{N}}$ would actually provide an explicit counterexample to Conjecture 4.5.

Remark 5.6. If we replace (4.4) by (4.5) for some fixed constant $\tau\geqslant 1$, then the theorem and its corollary still hold, but the complexity bound becomes $O(\mathsf{M}(p(\log p)^\tau\log(n p)))$.

6. POSITIVITY TESTING

In this section, we study question **Q4**. We assume that $(f_n)_{n\in\mathbb{N}}\in(\mathbb{K}\cap\mathbb{R})^{\mathbb{N}}$ is a holonomic sequence whose generating function f is convergent at the origin. Modulo the pre-treatment from section 4.2, we may assume without loss of generality that $(f_n)_{n\in\mathbb{N}}$ is non-resonant. We also assume that f is dominant-Fuchsian. Our aim is to decide whether $f_n\geqslant 0$ or $f_n>0$ for all $n\in\mathbb{N}$ or for all sufficiently large n.

6.1. A density theorem for sequences

Our positivity test will rely on a way to compute limsups and liminfs of certain oscillating sequences. For this, we will adapt results from [6, 19]. Recall that a *Hardy field* is a field of germs of differentiable real functions at infinity that is closed under differentiation [7]. In particular, $\mathbb{R}(x,\log x)$ and $\mathbb{R}(x^{\mathbb{R}\cap\mathbb{K}},\log x)$ are Hardy fields. The following is a direct consequence of [19, Theorem 3].

THEOREM 6.1. *Let $g_1,\ldots,g_k$ be real functions whose germs at infinity belong to a Hardy field and such that $g_1\prec\cdots\prec g_k$, i.e. $g_1=o(g_2),\ldots,g_{k-1}=o(g_k)$. For each $i\in\{1,\ldots,k\}$, let $\lambda_{i,1},\ldots,\lambda_{i,r_i}$ be $\mathbb{Q}$-linearly independent numbers in $\mathbb{R}$. Consider the function*

$$\psi(x) \ := \ \left(\mathrm{e}^{g_1(\lambda_{1,1}x)\mathrm{i}},\ldots,\mathrm{e}^{g_1(\lambda_{1,r_1}x)\mathrm{i}},\ldots,\mathrm{e}^{g_k(\lambda_{k,1}x)\mathrm{i}},\ldots,\mathrm{e}^{g_k(\lambda_{k,r_k}x)\mathrm{i}}\right)$$

from $\mathbb{R}$ into the torus $\mathbb{T}^D:=(\mathrm{e}^{\mathbb{R}\mathrm{i}})^D$ of dimension $D=r_1+\cdots+r_k$. Then $\psi([x_0,\infty))$ is dense in $\mathbb{T}^D$ for any $x_0\in\mathbb{R}$.

What we really need is a counterpart of this theorem for sequences:

COROLLARY 6.2. *With the notations of the theorem, assume that $g_k(x)=x$ and that $\lambda_{k,1},\ldots,\lambda_{k,r_k},2\pi$ are $\mathbb{Q}$-linearly independent. Then $\psi(\{n_0,n_0+1,\ldots\})$ is dense in $\mathbb{T}^D$ for all $n_0\in\mathbb{N}$.*

Proof. For $w=(w_1,\ldots,w_D),w'=(w_1',\ldots,w_D')\in\mathbb{T}^D$, let

$$|w'-w| \ := \ \max(|w_1'-w_1|,\ldots,|w_D'-w_D|).$$

We also define

$$\psi^\sharp(x) \ := \ \left(\mathrm{e}^{g_1(\lambda_{1,1}x)\mathrm{i}},\ldots,\mathrm{e}^{g_1(\lambda_{1,r_1}x)\mathrm{i}},\ldots,\mathrm{e}^{g_k(\lambda_{k,1}x)\mathrm{i}},\ldots,\mathrm{e}^{g_k(\lambda_{k,r_k}x)\mathrm{i}},\mathrm{e}^{2\pi x\mathrm{i}}\right).$$

Let $n_0 \in \mathbb{N}$ and let Λ be any constant with $\Lambda > \max(|\lambda_{k,1}|, \ldots, |\lambda_{k,r_k}|, 2\pi)$. If n_0 is sufficiently large, then $|\psi^\sharp(x') - \psi^\sharp(x)| \leqslant \Lambda |x' - x|$ for all $x \geqslant x' \geqslant n_0$. By the theorem, the image $\psi^\sharp([n_0, \infty))$ is dense in $\mathbb{T}^{D+1}$. Given $\varepsilon > 0$ and $w \in \mathbb{T}^D$, we may thus find an $x \geqslant n_0$ such that

$$|\psi^\sharp(x) - w^\sharp| < \frac{\varepsilon}{2\Lambda},$$

where $w^\sharp := (w, 1)$. Let $n \geqslant n_0$ be an integer with minimal distance to x. Since $|\psi^\sharp(x) - w^\sharp| < \varepsilon / (2\Lambda)$, we have in particular $|\psi^\sharp(x)_{D+1} - w^\sharp_{D+1}| = |\psi^\sharp(x)_{D+1} - 1| = |\mathrm{e}^{2\pi x \mathrm{i}} - 1| < \varepsilon / (2\Lambda)$, whence $|x - n| < \varepsilon / (2\Lambda)$. It follows that

$$|\psi^\sharp(x) - \psi^\sharp(n)| \leqslant \Lambda |x - n| < \frac{\varepsilon}{2},$$

whence

$$\begin{aligned} |\psi(n) - w| &\leqslant |\psi^\sharp(n) - w^\sharp| \\ &\leqslant |\psi^\sharp(n) - \psi^\sharp(x)| + |\psi^\sharp(x) - w^\sharp| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2\Lambda} < \varepsilon. \end{aligned}$$

This shows that $\psi(\{n_0, n_0 + 1, \ldots\})$ is indeed dense in $\mathbb{T}^D$. □

We will also need the following counterpart of [19, Theorem 5]:

THEOREM 6.3. *Let $g_1, \ldots, g_k \in (\mathbb{K} \cap \mathbb{R})(x^{\mathbb{K} \cap \mathbb{R}}, \log x)$ be real functions with $g_1 \prec \cdots \prec g_k$. For each $i \in \{1, \ldots, k\}$, let $\lambda_{i,1}, \ldots, \lambda_{i,r_i} \in \mathbb{K} \cap \mathbb{R}$. Consider the function*

$$\psi(x) := \left(\mathrm{e}^{g_1(\lambda_{1,1} x)\mathrm{i}}, \ldots, \mathrm{e}^{g_1(\lambda_{1,r_1} x)\mathrm{i}}, \ldots, \mathrm{e}^{g_k(\lambda_{k,1} x)\mathrm{i}}, \ldots, \mathrm{e}^{g_k(\lambda_{k,r_k} x)\mathrm{i}}\right)$$

from $\mathbb{R}^>$ into the torus $\mathbb{T}^D := (\mathrm{e}^{\mathbb{R}\mathrm{i}})^D$ of dimension $D = r_1 + \cdots + r_k$. Let

$$P \in \mathbb{K}[Z_{1,1}^{\mathbb{Z}}, \ldots, Z_{1,r_1}^{\mathbb{Z}}, \ldots, Z_{k,1}^{\mathbb{Z}}, \ldots, Z_{k,r_k}^{\mathbb{Z}}]$$

be a Laurent polynomial that takes only real values on $\mathbb{T}^D$. Then $\limsup_{n \to \infty} P(\psi(n))$ and $\limsup_{n \to \infty} P(\psi(n))$ are both computable numbers in $\mathbb{K} \cap \mathbb{R}$.

Proof. Using the rewriting techniques from the proof of [19, Theorem 5], we first reduce the general case to the case when $\lambda_{i,1}, \ldots, \lambda_{i,r_i}$ are $\mathbb{Q}$-linearly independent for $i = 1, \ldots, k$. (Note that there exists an algorithm to find $\mathbb{Q}$-linear dependencies between algebraic numbers, so we do not need the general oracle to find $\mathbb{Q}$-linear dependencies between exp-log constants.)

We next reduce to the case when $\lambda_{k,1}, \ldots, \lambda_{k,r_k}, 2\pi$ are $\mathbb{Q}$-linearly independent. Assume on the contrary that $c_1 \lambda_{k,1} + \cdots + c_{r_k} \lambda_{k,r_k} + 2\pi c_0 = 0$ with $c_0, \ldots, c_{r_k} \in \mathbb{Z}$ and $c_0 \neq 0$. Without loss of generality, we may assume that $c_{r_k} \neq 0$. Taking $\lambda'_{k,i} = \lambda_{k,i} / c_{r_k}$ for $i = 1, \ldots, r_k - 1$, we may rewrite $\mathrm{e}^{\lambda_{k,i} x \mathrm{i}} = (\mathrm{e}^{\lambda'_{k,i} x \mathrm{i}})^{c_{r_k}}$ and $\mathrm{e}^{\lambda_{k,r_k} x \mathrm{i}} = \mathrm{e}^{-c_0 2\pi \mathrm{i} / c_{r_k}} (\mathrm{e}^{\lambda'_{k,1} x \mathrm{i}})^{-c_1} \cdots (\mathrm{e}^{\lambda'_{k,r_k-1} x \mathrm{i}})^{-c_{r_k-1}}$ as Laurent polynomials in $\mathbb{K}\left[\mathrm{e}^{\lambda'_{k,1} x i}, \ldots, \mathrm{e}^{\lambda'_{k,r_k-1} x i}\right]$, while performing the corresponding corresponding substitutions in P. We repeat this procedure with $\lambda'_{k,1}, \ldots, \lambda'_{k,r_k-1}$ in the role of $\lambda_{k,1}, \ldots, \lambda_{k,r_k}$ until $\lambda_{k,1}, \ldots, \lambda_{k,r_k}, 2\pi$ are $\mathbb{Q}$-linearly independent.

After the above reductions, we are in a position to apply Corollary 6.2. This yields

$$\begin{aligned} \limsup_{n \to \infty} P(\psi(n)) &= \sup_{z \in \mathbb{T}^D} P(z) \\ \liminf_{n \to \infty} P(\psi(n)) &= \inf_{z \in \mathbb{T}^D} P(z). \end{aligned}$$

Now $\sup_{z \in \mathbb{T}^D} P(z) \in \mathbb{K}$ and $\inf_{z \in \mathbb{T}^D} P(z) \in \mathbb{K}$ can be computed using classical algorithms from effective real algebraic geometry [3]. □

6.2. Positivity testing

Let $\alpha_1, \ldots, \alpha_m \in \mathbb{K}$ be the dominant singularities of f. By Theorem 4.6, we may compute constants $t_1, \ldots, t_m \in \mathbb{N}$, $c_{i,j} \in \mathbb{K} \cap \mathbb{R}$, $\kappa_{i,j} \in \mathbb{K}$, $\nu \in \mathbb{N}$, $B > 0$, $n_0 \in \mathbb{N}$, and $\kappa \in \mathbb{K} \cap \mathbb{R}$ such that

$$\left| f_n - \sum_{1 \leqslant i \leqslant m} \sum_{1 \leqslant j \leqslant t_i} c_{i,j} n^{\kappa_{i,j}} \alpha_i^{-n} (\log n)^\nu \right| \leqslant B n^\kappa \alpha_1^{-n} (\log n)^{\nu - 1} \tag{6.1}$$

for all $n \geqslant n_0$ and $\operatorname{Re} \kappa_{i,j} = \kappa$ for all i, j. Setting

$$\phi_n = \sum_{1 \leqslant i \leqslant m} \sum_{1 \leqslant j \leqslant t_i} c_{i,j} n^{\kappa_{i,j} - \kappa} \left(\frac{\alpha_i}{|\alpha_i|} \right)^{-n},$$

we may rewrite (6.1) as

$$|f_n - \phi_n |\alpha_1|^{-n} n^\kappa (\log n)^\nu| \leqslant B n^\kappa \alpha_1^{-n} (\log n)^{\nu - 1}. \tag{6.2}$$

Now observe that ϕ_n can be interpreted as a polynomial

$$\phi_n \in \mathbb{K}[(\mathrm{e}^{\arg \kappa_{i,j} (\log n) \mathrm{i}})_{i,j}, (\mathrm{e}^{\arg \alpha_i n \mathrm{i}})_i].$$

Since $\phi_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, this allows us to apply Theorem 6.3 and compute $\phi^* := \limsup_{n \to \infty} \phi_n$ and $\phi_* := \liminf_{n \to \infty} \phi_n$. If $\phi_* > 0$, then (6.2) yields $f_n > 0$ for all $n > \mathrm{e}^{B / \phi_*}$. If $\phi_* < 0$, then $\phi_n < \phi_* / 2$ for infinitely many n. For any $n > \mathrm{e}^{-2B / \phi_*}$ with $\phi_n < \phi_* / 2$, the relation (6.2) then yields $f_n < 0$.

The only remaining case is when $\phi_* = 0$. In lucky cases, we may look at the next subdominant term of the asymptotic expansion of f_n and prove the positivity of $f_n - \phi_n |\alpha_1|^{-n} n^\kappa (\log n)^\nu$ in a similar way as above. However, in general, the positivity of f_n can be hard to decide. In fact, a general decision procedure would allow us to answer difficult questions about diophantine approximability. For instance, given a real algebraic number α, the positivity of the holonomic sequence

$$\log^2 n - \cos(n \log \alpha) \log^2 n - \frac{1}{n^2}$$

is related to a rate of diophantine approximability of α by $p / q \in \mathbb{Q}$ of the form

$$\forall p, q \in \mathbb{Z}^{\neq}, \quad \left| \alpha - \frac{p}{q} \right| > \frac{c}{q^2 \log q}.$$

We do not know of a general decision procedure for this kind of inequalities.

7. FUCHSIAN HOLONOMIC SEQUENCES

Let f be the generating function of a holonomic sequence $(f_n)_{n \in \mathbb{N}}$. So far, we have mainly been interested in the case when f is convergent at the origin and dominant-Fuchsian (possibly modulo a reduction to the non-resonant case as in section 4.2). This is indeed sufficient for obtaining information about $(f_n)_{n \in \mathbb{N}}$ through its asymptotic properties. In this section, we assume that f is globally Fuchsian and study nice additional properties that hold in this case.

For simplicity, we assume that $(f_n)_{n \in \mathbb{N}}$ satisfies a non-degenerate difference equation (1.1). We normalize Σ to make it divisible by $(n+1) \cdots (n+s)$ and we let $L = L_r \partial^r + \cdots + L_0 \in \mathbb{K}(z)[\partial]$ be the corresponding differential operator with $Lf = 0$, as constructed in section 2.1. We let $\alpha_1, \ldots, \alpha_\ell$ be the singularities of L.

7.1. Full Mellin integrals

First of all, for Fuchsian L, the truncated Mellin integrals from (3.3) tend to a limit

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{H}_k}\frac{f(z)}{z^{n+1}}\mathrm{d}z \tag{7.1}$$

when R tends to infinity and n is sufficiently large. This leads to the exact representation

$$f = \frac{1}{2\pi\mathrm{i}}\sum_{k=1}^{\ell}\int_{\mathcal{H}_k}\frac{f(z)}{z^{n+1}}\mathrm{d}z. \tag{7.2}$$

We call (7.1) a full *Mellin integral* that is based at α_k.

Remark 7.1. We may take $\mathcal{H}_k$ to be the contours from infinity to a point $\tilde{\alpha}_k$ close to α_k, which next performs a complete circular turn around α_k, and then goes back to infinity along the same direction where it came from. Then we have

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{H}_k}\frac{f(z)}{z^{n+1}}\mathrm{d}z = \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{C}_k}\frac{f(z)}{z^{n+1}}\mathrm{d}z + \frac{1}{2\pi\mathrm{i}}\int_{\tilde{\alpha}_k}^{\infty}\frac{(\Delta_{\alpha_k}f)(z)}{z^{n+1}}\mathrm{d}z, \tag{7.3}$$

where $\mathcal{C}_k$ denotes the circle around α_k and $\Delta_{\alpha_k}f$ denotes the difference between the analytic continuations of f that get around α_k on the left and right, respectively. Note that $\Delta_{\alpha_k}f$ is a solution of the same differential equation as f, i.e. $L\Delta_{\alpha_k}f=0$. The formula (7.3) is convenient for machine computations due to the fact that we only have a single stretch going to infinity.

More generally, for an analytic function φ on $\mathcal{H}_k$ and with $\varphi(z)=O(|z|^{O(1)})$ at infinity, we define

$$(\mathcal{M}_k\varphi)_n := \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{H}_k}\frac{\varphi(z)}{z^{n+1}}\mathrm{d}z.$$

Then we note that

$$\begin{aligned}(\mathcal{M}_k\varphi)_{n+1} &= (\mathcal{M}_k(z^{-1}\varphi))_n\\ n(\mathcal{M}_k\varphi)_n &= (\mathcal{M}_k(z\varphi'))_n,\end{aligned}$$

where the second relation is proved using integration by parts. Consequently, the sequence $((\mathcal{M}_kf)_n)_{n\in\mathbb{N}}$ satisfies the same recurrence relation (1.1) as $(f_n)_{n\in\mathbb{N}}$.

7.2. Canonical solutions via Mellin transforms

We observe that (7.1) only depends on the behavior of f at the singularity α_k. Expressing f in terms of the canonical basis $h_1^{\alpha_k},\ldots,h_r^{\alpha_k}$ of local solutions to $Lh=0$ at α_k,

$$f = c_1h_1^{\alpha_k}+\cdots+c_rh_r^{\alpha_k}, \qquad c_1,\ldots,c_r\in\mathbb{K}^{\mathrm{rhol}},$$

and setting

$$\phi_i^{\alpha_k} := \mathcal{M}_kh_i^{\alpha_k}, \qquad i=1,\ldots,r,$$

it follows that

$$\mathcal{M}_kf = c_1\phi_1^{\alpha_k}+\cdots+c_r\phi_r^{\alpha_k}. \tag{7.4}$$

Here we note that $\phi_i^{\alpha_k}=0$ whenever $h_i^{\alpha_k}$ is analytic at α_k. Moreover, we recall that

$$h_i^{\alpha_k}\in(z-\alpha_k)^{-\kappa}((\log(z-\alpha_k))^j+z\,\mathbb{K}\{\{z-\alpha_k\}\}[\log(z-\alpha_k)]),$$

for some $\kappa \in \mathbb{K}$ and $j \in \mathbb{N}$. If $\kappa \notin \mathbb{N}$ or $j \neq 0$, then the sequence $\phi_i^{\alpha_k}$ has a non-zero formal transseries in $\alpha_k^{-n} n^{\kappa-1} \mathbb{K}[\gamma^{(\mathbb{N})}(\kappa)][[n^{-1}]][\log n]$ as its asymptotic expansion, by the formulas from subsection 3.2. We will sometimes identify $\phi_i^{\alpha_k}$ with this transseries.

Let $\phi_1^*, \ldots, \phi_{s'}^*$ be the collection of all non-zero $\phi_i^{\alpha_k}$ with $k \in \{1, \ldots, \ell\}$ and $i \in \{1, \ldots, r\}$. Since the dominant monomials of $\phi_1^*, \ldots, \phi_{s'}^*$ are pairwise distinct when considering them as transseries, these sequences are $\mathbb{C}$-linearly independent. From (7.2) and (7.4), we also know that any sequence solution of (1.1) can be written as a $\mathbb{C}$-linear combination of $\phi_1^*, \ldots, \phi_{s'}^*$. Conversely, we noted at the end of subsection 7.1 that each ϕ_i^* is actually a solution of (1.1). This shows that $\phi_1^*, \ldots, \phi_{s'}^*$ forms basis of the solution space of (1.1) in $\mathbb{C}^{\mathbb{N}}$. Since we assumed Σ to be non-degenerate, this space has dimension s, so $s' = s$. We denote by Φ^* the row vector with entries $\phi_1^*, \ldots, \phi_s^*$.

7.3. Canonical formal solutions at infinity

We have shown that (1.1) has a basis of formal transseries solutions

$$\phi_1^*, \ldots, \phi_s^* \in \mathbb{K}^n n^{\mathbb{K}} \mathbb{K}[\gamma^{(\mathbb{N})}(\mathbb{K})][[n^{-1}]][\log n].$$

In fact, it is well known [5] that we may compute a canonical system of formal solutions in $\mathbb{K}^n n^{\mathbb{K}} \mathbb{K}[[n^{-1}]][\log n]$, similar to the ones that we saw in the differential case in subsection 2.2. Let us briefly describe how to do this.

We have seen that for any singularity α_k of L, there exists at least one formal solution of the form $(\phi_i^*)_n \in \alpha_k^{-n} n^{\mathbb{K}} \mathbb{K}[\gamma^{(\mathbb{N})}(\mathbb{K})][[n^{-1}]][\log n]$. Modulo a transformation $\sigma \leftrightarrow \alpha_k^{-1} \sigma$, we may assume without loss of generality that $\alpha_k = 1$. We next replace σ by $\exp(n^{-1} \vartheta_n)$ in Σ, where $\vartheta_n = n \, \partial / \partial n$. After multiplication by a suitable power of n, this yields an operator

$$n^{\mu} \Sigma \;=\; \sum_{i \in \mathbb{N}} n^{-i} \Lambda_i(\vartheta_n) \;\in\; \mathbb{K}[\vartheta_n][[n^{-1}]]$$

with $\Lambda_0(\vartheta_n) \neq 0$. Whenever we have a formal solution ϕ_i^* with $(\phi_i^*)_n \asymp n^{\kappa} (\log n)^j$, then this implies that Λ_0 is divisible by $(\vartheta_n - \kappa)^{j+1}$. Inversely, if $\kappa \in \mathbb{K}$ is a root of multiplicity ν of Λ_0 in ϑ_n, then for any $j < \nu$, there exists a unique formal solution $f \in n^{\kappa} (\log n)^j + n^{\kappa-1} \mathbb{K}[[n^{-1}]][\log n]$ to (1.1). Indeed, writing $f_n = n^{\kappa} \sum_{j \in \mathbb{N}} \varphi_j n^{-j}$ with $\varphi_j \in \mathbb{K}[\log n]$, we have

$$n^{\mu} (\Sigma f)_n \;=\; n^{\kappa} \sum_{i,j \in \mathbb{N}} n^{-i-j} \Lambda_i(\vartheta_n + \kappa - j)(\varphi_j) \;=\; 0,$$

which yields the recurrence relation

$$\Lambda_0(\vartheta_n + \kappa - m)(\varphi_m) \;=\; \sum_{1 \leqslant i \leqslant m} \Lambda_i(\vartheta_n + \kappa - m + i)(\varphi_{m-i}) \tag{7.5}$$

for the computation of the coefficients φ_m. The solution is unique when requiring that $\varphi_0 = (\log n)^j$ and that φ_m is divisible by $(\log n)^p$ whenever $k - m$ is a root of multiplicity p of Λ_0 in ϑ_n.

Let $\phi_1^{\infty}, \ldots, \phi_s^{\infty}$ be the collection of all formal solutions of (1.1) in $\mathbb{K}^n n^{\mathbb{K}} \mathbb{K}[[n^{-1}]][\log n]$ that we obtain in the way that we just described, with $\phi_i^{\infty} \asymp \phi_i^*$. Since the dominant monomials of these solutions are pairwise distinct, this again forms a fundamental system of solutions; we call it the *canonical system of solutions* of (1.1) at infinity and we denote by Φ^{∞} the row vector with entries $\phi_1^{\infty}, \ldots, \phi_s^{\infty}$. Since Φ^* and Φ^{∞} are both fundamental systems of solutions, there exists a matrix $\mathrm{M} \in \mathbb{C}^{s \times s}$ with

$$\Phi^* \;=\; \Phi^{\infty} \mathrm{M}. \tag{7.6}$$

Since Φ^* has coefficients in $\mathbb{K}[\gamma^{(\mathbb{N})}(\mathbb{K})]$ and Φ has coefficients in $\mathbb{K}$, the matrix M must actually be in $\mathbb{K}[\gamma^{(\mathbb{N})}(\mathbb{K})]^{r\times r}$.

For machine computations, the recurrence relation (7.5) is not very efficient if we want to compute a large number of terms. In that case it is better to rewrite the original equation (1.1) with respect to $t=n^{-1}$, which transforms the shift operator σ into $t\mapsto\frac{t}{1+t}$. Compositions of a power series with $\frac{t}{1+t}$ can be computed efficiently in a relaxed manner using the algorithm from [22, section 3.4.2]. The equation (1.1) is not necessarily "recursive", so it is not always possible to directly solve it using the techniques from [22]. Nevertheless, it can always be rewritten as a recursive equation using the algorithms from [24]. Altogether, this allows us to compute the first N coefficients of the canonical solutions $\phi_1^\infty,\ldots,\phi_s^\infty$ in time $O(\mathsf{M}(N^2\log^3 N))$, which is softly optimal in the bit-size of the result.

Remark 7.2. The equation (7.6) can be used to map formal transseries solutions of (1.1) to actual holonomic sequences. It is interesting to note that this association actually preserves all difference ring operations, in a similar way as accelero-summation in the differential setting [23].

7.4. Transition matrices

Let us briefly recall the concept of a transition matrix, which forms an important ingredient for the efficient evaluation of holonomic functions in [9, 20, 21]. We will use the notations H^α and $F(\alpha)$ from section 2.2 for canonical systems of local solutions and generalized values at α.

Given a non-singular path $\alpha\rightsquigarrow\beta$ between two non-singular points $\alpha,\beta\in\mathbb{K}\cup\{\infty\}$, the analytic continuation of the canonical solutions at α can be expressed as linear combinations of the canonical solutions at β. In other words, there exists a matrix $\Delta_{\alpha\to\beta}\in\mathbb{C}^{r\times r}$ with

$$H^\alpha = H^\beta\Delta_{\alpha\rightsquigarrow\beta}.$$

We call $\Delta_{\alpha\rightsquigarrow\beta}$ the *transition matrix* along the path $\alpha\rightsquigarrow\beta$. We naturally have the relation

$$\Delta_{\alpha\rightsquigarrow\beta\rightsquigarrow\gamma} = \Delta_{\beta\rightsquigarrow\gamma}\Delta_{\alpha\rightsquigarrow\beta}$$

for composed paths. In terms of generalized values of a solution f to (2.1), we also obtain

$$F(\beta) = \Delta_{\alpha\rightsquigarrow\beta}F(\alpha).$$

These notions extend to the case when the paths start and/or end at singular points, modulo the precaution that we specify the angles that are used to approach the singularities (in order to determine the branch of the logarithm).

7.5. Transition matrices for sequences

Let us now study the analogue of transition matrices for sequences. Given $N\in\mathbb{N}$ and $i\in\{0,\ldots,s-1\}$, assume that (1.1) has a unique solution $(f_n)_{n\in\mathbb{N}}$ with $f_{N+i}=1$ and $f_{N+j}=0$ for $j\in\{0,\ldots,s-1\}\setminus\{i\}$. Then we will denote this solution by $(\phi_{i,n}^N)_{n\in\mathbb{N}}$. We denote by Φ^N the row vector with entries $\phi_0^N,\ldots,\phi_{s-1}^N$ if these solutions are all defined and call it the canonical system of solutions at $n=N$. Given a general solution $(f_n)_{n\in\mathbb{N}}$ to (1.1), we call the column vector F_N with entries $f_N,\ldots,f_{N+s-1}$ the generalized value of $(f_n)_{n\in\mathbb{N}}$ at $n=N$, so that

$$f = \Phi^N F_N.$$

By definition, the F_N satisfy a recurrence relation

$$F_{N+1} = \Delta_{N\to N+1} F_N,$$

where $\Delta_{N\to N+1}$ was defined in (5.1). More generally, for $k\in\mathbb{N}$ and $\Delta_{N\to N+k} := \Delta_{N+k-1\to N+k}\cdots\Delta_{N\to N+1}$, we have

$$F_{N+k} = \Delta_{N\to N+k} F_N.$$

Setting $\Delta_{N\to N+k} := \Delta_{N+k\to N}^{-1}$, this relation extends to the case when $k\in\mathbb{Z}$. Dually, we also have

$$\Phi^N = \Phi^{N+k}\Delta_{N\to N+k}.$$

We call $\Delta_{N\to N+k}$ the *transition matrix* between $n=N$ and $n=N+k$. We have seen in section 5.2 how to compute $\Delta_{N\to N+k}$ in time $O(\mathsf{M}(k\log(N+k)\log k))$ using binary splitting.

7.6. Transition matrices at infinity

In section 7.2, we introduced the canonical system of solutions Φ^∞ of (1.1) at infinity. Given a solution $(f_n)_{n\in\mathbb{N}}$ of (1.1), this leads to the corresponding notion of generalized value $F_\infty\in\mathbb{C}^s$ at infinity with

$$f = \Phi^\infty F_\infty. \tag{7.7}$$

For any $N\in\mathbb{N}$, the matrix $\Delta_{N\to\infty}$ with

$$\begin{aligned} F_\infty &= \Delta_{N\to\infty} F_N,\\ \Phi^N &= \Phi^\infty \Delta_{N\to\infty}\end{aligned}$$

is called the *transition matrix* between N and $n=\infty$, whenever it exists.

Using a combination of the techniques so far, we may compute the transition matrix $\Delta_{N\to\infty}$ as follows. Consider one of the canonical solutions $(f_n)_{n\in\mathbb{N}} = (\phi_{i,n}^N)_{n\in\mathbb{N}}$ of (1.1) at $n=N$ and the corresponding power series solution $f\in\mathbb{K}\{\{z\}\}$ of (2.1) at the origin. Given one of the singularities α_k of L, we may use the algorithms from [21] to compute the transition matrix $\Delta_{0\to\alpha_k}$ for L and then re-express f as a $\mathbb{K}^{\text{hola}}$-linear combination of the canonical solutions $h_1^{\alpha_k},\ldots,h_r^{\alpha_k}$ at $z=\alpha_k$. The collection of these relations yields f as a $\mathbb{K}^{\text{hola}}$-linear combination of the canonical solutions $\phi_1^*,\ldots,\phi_s^*$. Using the methods from sections 7.2 and 7.3, we finally obtain f as a $\mathbb{K}^{\text{hola}}[\gamma^{(\mathbb{N})}(\mathbb{K})]$-linear combination of $\phi_1^\infty,\ldots,\phi_s^\infty$. Doing this for each ϕ_i^N with $i=1,\ldots,s$, this yields the transition matrix $\Delta_{N\to\infty}\in\mathbb{K}^{\text{hola}}[\gamma^{(\mathbb{N})}(\mathbb{K})]^{s\times s}$. For fixed N and large p, we may compute 2^{-p}-approximations of the entries of $\Delta_{N\to\infty}$ in time $O(\mathsf{M}(p\log^3 p))$, using the algorithms from [23].

7.7. Analytic solutions to the difference equation

Our main focus in this paper is on the study of sequence solutions to holonomic equations (1.1). Yet, it is interesting to note that the definition of Mellin transforms of the canonical solutions $\varphi := h_i^{\alpha_k}$ generalizes to complex numbers u for which z^{u+1} decreases sufficiently fast on $\mathcal{H}_k$:

$$(\mathcal{M}_k\varphi)(u) := \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{H}_k}\frac{\varphi(z)}{z^{u+1}}\mathrm{d}z. \tag{7.8}$$

Applying this to the theory from section 7.2, this yields a fundamental system of analytic solutions to the difference equation

$$\Sigma_s(u) f(u+s) + \cdots + \Sigma_0(u) f(u) = 0$$

that extends our fundamental system of sequence solutions. For $u \in \mathbb{K}$, we also note that the integrand of (7.8) is still holonomic over $\mathbb{K}$ and Fuchsian. Consequently, we may evaluate it with a precision of p bits in time $O(\mathsf{M}(p \log^2 p))$, using the algorithms from [21]. For general $u \in \mathbb{C}$, the evaluation can be done in time $\tilde{O}(p^{3/2})$ using the baby-step-giant-step technique from [9]. It is also possible to extend the uniform complexity analysis from section 5 to this setting, but additional care is needed for the treatment of non-real arguments u. We intend to carry out the detailed analysis in a forthcoming paper.

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