Surreal numbers as hyperseries

VINCENT BAGAYOKO
Département de Mathématique
Université de Mons, Le Pentagone
20, Place du Parc
B-7000 Mons, Belgique
Email: vincent.bagayoko@umons.ac.be

JORIS VAN DER HOEVEN
CNRS, LIX
Campus de l'École polytechnique
1, rue Honoré d'Estienne d'Orves
Bâtiment Alan Turing, CS35003
91120 Palaiseau, France
Email: vdhoeven@lix.polytechnique.fr

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Surreal numbers form the ultimate extension of the field of real numbers with infinitely large and small quantities and in particular with all ordinal numbers. Hyperseries can be regarded as the ultimate formal device for representing regular growth rates at infinity. In this paper, we show that any surreal number can naturally be regarded as the value of a hyperseries at the first infinite ordinal ω. This yields a remarkable correspondence between two types of infinities: numbers and growth rates.

1. INTRODUCTION

1.1. Toward a unification of infinities

At the end of the 19-th century, two theories emerged for computations with infinitely large quantities. The first one was due to du Bois-Reymond [19, 20, 21], who developed a "calculus of infinities" to deal with the growth rates of functions in one real variable at infinity. The second theory of "ordinal numbers" was proposed by Cantor [13] as a way to count beyond the natural numbers and to describe the sizes of sets in his recently introduced set theory.

Du Bois-Reymond's original theory was partly informal and not to the taste of Cantor, who misunderstood it [25]. The theory was firmly grounded and further developed by Hausdorff and Hardy. Hausdorff formalized du Bois-Reymond's "orders of infinity" in Cantor's set-theoretic universe [24]. Hardy focused on the computational aspects and introduced the differential field of *logarithmico-exponential functions* [28, 29]: such a function is constructed from the real numbers and an indeterminate x (that we think of as tending to infinity) using the field operations, exponentiation, and the logarithm. Subsequently, this led to the notion of a Hardy field [12].

As to Cantor's theory of ordinal numbers, Conway proposed a dramatic generalization in the 1970s. Originally motivated by game theory, he introduced the proper class **No** of *surreal numbers* [14], which simultaneously contains the set $\mathbb{R}$ of all real numbers and the class **On** of all ordinals. This class comes with a natural ordering and arithmetic operations that turn **No** into a non-Archimedean real closed field. In particular, $\omega+\pi$, ω^{-1}, $\sqrt{\omega}$, $\omega^{\omega}-3\,\omega^2$ are all surreal numbers, where ω stands for the first infinite ordinal.

Conway's original definition of surreal numbers is somewhat informal and draws inspiration from both Dedekind cuts and von Neumann's construction of the ordinals:

> "If L and R are any two sets of (surreal) numbers, and no member of L is $\geqslant$ any member of R, then there is a (surreal) number $\{L \mid R\}$. All (surreal) numbers are constructed in this way."

The notation $\{\mid\}$ is called *Conway's bracket*. Conway proposed to consider $\{L \mid R\}$ as the *simplest* number between L and R. Indeed, it turns out that one may define a partial ordering $\sqsubset$ on **No** with $\{L \mid R\} \sqsubseteq a$ for any number $a \in \mathbf{No}$ with $L < a < R$. This so-called *simplicity relation* has the additional property that any $a \in \mathbf{No}$ can be written canonically as

$$\begin{aligned} a &= \{a_L \mid a_R\} \\ a_L &:= \{b \in \mathbf{No} : b < a, b \sqsubset a\} \\ a_R &:= \{b \in \mathbf{No} : b > a, b \sqsubset a\}. \end{aligned}$$

One may regard $a_L \cup a_R$ as the set of surreal numbers that were defined before a when using Conway's recursive definition. Conway's bracket is uniquely determined by the simplicity relation $\sqsubset$ and *vice versa*.

The ring operations on **No** are defined in a recursive way that is both very concise and intuitive: given $x = \{x_L \mid x_R\}$ and $y = \{y_L \mid y_R\}$, we define

$$\begin{aligned} 0 &:= \{\mid\} \\ 1 &:= \{0 \mid\} \\ -x &:= \{-x_R \mid -x_L\} \\ x+y &:= \{x_L + y, x + y_L \mid x_R + y, x + y_R\} \\ xy &:= \{x'y + xy' - x'y', x''y + xy'' - x''y'' \mid x'y + xy'' - x'y'', x''y + xy' - x''y'\} \\ & \qquad (x' \in x_L, x'' \in x_R, y' \in y_L, y'' \in y_R). \end{aligned}$$

It is quite amazing that these definitions coincide with the traditional definitions when x and y are real, but that they also work for the ordinal numbers and beyond. Subsequently, Gonshor also showed how to extend the real exponential function to **No** [26] and this extension preserves all first order properties of exp [16]. Simpler accounts and definitions of exp can be found in [37, 9].

The theory of Hardy fields focuses on the study of growth properties of germs of actual real differentiable functions at infinity. An analogue formal theory arose after the introduction of *transseries* by Dahn and Göring [15] and, independently, by Écalle [22, 23]. Transseries are a natural generalization of the above definition of Hardy's logarithmico-exponential functions, by also allowing for infinite sums (modulo suitable precautions to ensure that such sums make sense). One example of a transseries is

$$f = \mathrm{e}^{\mathrm{e}^x + 2\frac{\mathrm{e}^x}{x} + 6\frac{\mathrm{e}^x}{x^2} + \cdots} + \mathrm{e}^{\mathrm{e}^{\pi \log\log x} - \sqrt{\log x}} - \sqrt{7} + \frac{1}{\log x} + \frac{1}{(\log x)^2} + \cdots.$$

In particular, any transseries can be written as a generalized series $f = \sum_{\mathfrak{m} \in \mathfrak{T}} f_{\mathfrak{m}} \mathfrak{m}$ with real coefficients $f_{\mathfrak{m}} \in \mathbb{R}$ and whose (trans)monomials $\mathfrak{m} \in \mathfrak{T}$ are exponentials of other (generally "simpler") transseries. The support $\operatorname{supp} f := \{\mathfrak{m} \in \mathfrak{T} : f_{\mathfrak{m}} \neq 0\}$ of such a series should be *well based* in the sense that it should be well ordered for the opposite ordering of the natural ordering $\preccurlyeq$ on the group of transmonomials $\mathfrak{T}$. The precise definition of a transseries depends on further technical requirements on the allowed supports. But for all reasonable choices, "the" resulting field $\mathbb{T}$ of transseries possesses a lot of closure properties: it is ordered and closed under derivation, composition, integration, and functional inversion [22, 30, 18]; it also satisfies an intermediate value property for differential polynomials [32, 3].

It turns out that surreal numbers and transseries are similar in many respects: both **No** and $\mathbb{T}$ are real closed fields that are closed under exponentiation and taking logarithms of positive elements. Surreal numbers too can be represented uniquely as Hahn series $\sum_{\mathfrak{m} \in \mathbf{Mo}} a_{\mathfrak{m}} \mathfrak{m}$ with real coefficients $a_{\mathfrak{m}} \in \mathbb{R}$ and monomials in a suitable multiplicative subgroup **Mo** of $\mathbf{No}^{>}$. Any transseries $f \in \mathbb{T}$ actually naturally induces a surreal number $f(\omega) \in \mathbf{No}$ by substituting ω for x and the map $f \longmapsto f(\omega)$ is injective [11].

But there are also differences. Most importantly, elements of $\mathbb{T}$ can be regarded as functions that can be derived and composed. Conversely, the surreal numbers **No** come equipped with the Conway bracket. In fact, it would be nice if any surreal number could naturally be regarded as the value $f(\omega)$ of a unique transseries f at ω. Indeed, this would allow us to transport the functional structure of $\mathbb{T}$ to the surreal numbers. Conversely, we might equip the transseries with a Conway bracket and other exotic operations on the surreal numbers. The second author conjectured the existence of such a correspondence between **No** and a suitably generalized field of the transseries [32, page 16]; see also [2] for a more recent account.

Now we already observed that at least *some* surreal numbers $a \in \mathbf{No}$ can be written uniquely as $a = f(\omega)$ for some transseries $f \in \mathbb{T}$. Which numbers and what kind of functions do we miss? Since a perfect correspondence would induce a Conway bracket on $\mathbb{T}$, it is instructive to consider subsets $L, R \subseteq \mathbb{T}$ with $L < R$ and examine which natural growth orders might fit between L and R.

One obvious problem with ordinary transseries is that there exists no transseries that grows faster than any iterated exponential $x, \mathrm{e}^x, \mathrm{e}^{\mathrm{e}^x}, \ldots$. Consequently, there exists no transseries $f \in \mathbb{T}$ with $f(\omega) = \{\omega, \mathrm{e}^{\omega}, \mathrm{e}^{\mathrm{e}^{\omega}}, \ldots \mid \}$. A natural candidate for a function that grows faster than any iterated exponential is the first *hyperexponential* E_ω, which satisfies the functional equation

$$E_\omega(x+1) \;=\; \exp E_\omega(x).$$

It was shown by Kneser [33] that this equation actually has a real analytic solution on $\mathbb{R}^{>}$. A natural hyperexponential E_ω on $\mathbf{No}^{>,\succ} := \{c \in \mathbf{No} : c > \mathbb{R}\}$ was constructed more recently in [8]. In particular, $E_\omega(\omega) = \{\omega, \mathrm{e}^{\omega}, \mathrm{e}^{\mathrm{e}^{\omega}}, \ldots \mid \}$.

More generally, one can formally introduce the transfinite sequence $(E_\alpha)_{\alpha \in \mathbf{On}}$ of *hyperexponentials* of arbitrary strengths α, together with the sequence $(L_\alpha)_{\alpha \in \mathbf{On}}$ of their functional inverses, called *hyperlogarithms*. Each E_{ω^n} with $n \in \mathbb{N}^{>}$ satisfies the equation

$$E_{\omega^n}(x+1) \;=\; E_{\omega^{n-1}}(E_{\omega^n}(x))$$

and there again exist real analytic solutions to this equation [38]. The function E_{ω^ω} does not satisfy any natural functional equation, but we have the following infinite product formula for the derivative of every hyperlogarithm L_α:

$$L_\alpha'(x) \;=\; \prod_{\beta < \alpha} \frac{1}{L_\beta(x)}.$$

We showed in [6] how to define $E_\alpha(a)$ and $L_\alpha(a)$ for any $\alpha \in \mathbf{On}$ and $a \in \mathbf{No}^{>,\succ}$.

The traditional field $\mathbb{T}$ of transseries is not closed under hyperexponentials and hyperlogarithms, but it is possible to define generalized fields of *hyperseries* that do enjoy this additional closure property. Hyperserial grow rates were studied from a formal point of view in [22, 23]. The first systematic construction of hyperserial fields of strength $\alpha < \omega^\omega$ is due to Schmeling [38]. In this paper, we will rely on the more recent constructions from [17, 7] that are fully general. In particular, the surreal numbers **No** form a hyperserial field in the sense of [7], when equipped with the hyperexponentials and hyperlogarithms from [6].

A less obvious problematic cut $L < R$ in the field of transseries $\mathbb{T}$ arises by taking

$$\begin{aligned} L &= \left\{ \sqrt{x}, \sqrt{x} + \mathrm{e}^{\sqrt{\log x}}, \sqrt{x} + \mathrm{e}^{\sqrt{\log x} + \mathrm{e}^{\sqrt{\log \log x}}}, \ldots \right\} \\ R &= \left\{ 2\sqrt{x}, \sqrt{x} + \mathrm{e}^{2\sqrt{\log x}}, \sqrt{x} + \mathrm{e}^{\sqrt{\log x} + \mathrm{e}^{2\sqrt{\log \log x}}}, \ldots \right\}. \end{aligned}$$

Here again, there exists no transseries $f \in \mathbb{T}$ with $L < f < R$. This cut has actually a natural origin, since any "tame" solution of the functional equation

$$f(x) = \sqrt{x} + \mathrm{e}^{f(\log x)} \tag{1.1}$$

lies in this cut. What is missing here is a suitable notion of "nested transseries" that encompasses expressions like

$$f = \sqrt{x} + \mathrm{e}^{\sqrt{\log x} + \mathrm{e}^{\sqrt{\log \log x} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}. \tag{1.2}$$

This type of cuts were first considered in [30, Section 2.7.1]. Subsequently, the second author and his former PhD student Schmeling developed an abstract notion of generalized fields of transseries [31, 38] that may contain nested transseries. However, it turns out that expressions like (1.2) are ambiguous: one may construct fields of transseries that contain arbitrarily large sets of pairwise distinct solutions to (1.1).

In order to investigate this ambiguity more closely, let us turn to the surreal numbers. The above cut $L < R$ induces a cut $L(\omega) < R(\omega)$ in **No**. Nested transseries solutions f to the functional equation (1.1) should then give rise to surreal numbers $f(\omega)$ with $L(\omega) < f(\omega) < R(\omega)$ and such that $f(\omega) - \sqrt{\omega}, \log(f(\omega) - \sqrt{\omega}) - \mathrm{e}^{\sqrt{\log \omega}}, \ldots$ are all monomials in **Mo**. In [5, Section 8], we showed that those numbers $f(\omega)$ actually form a class **Ne** that is naturally parameterized by a surreal number (**Ne** forms a so-called *surreal substructure*). Here we note that analogue results hold when replacing Gonshor's exponentiation by Conway's ω-map $a \in \mathbf{No} \longmapsto \omega^a$ (which generalizes Cantor's ω-map when $a \in \mathbf{On}$). This was already noted by Conway himself [14, pages 34–36] and further worked out by Lemire [34, 35, 36]. Section 6 of the present paper will be devoted to generalizing the result from [5, Section 8] to nested hyperseries.

Besides the two above types of superexponential and nested cuts, no other examples of "cuts that cannot be filled" come naturally to our mind. This led the second author to conjecture [32, page 16] that there exists a field $\mathbb{H}$ of suitably generalized hyperseries in x such that each surreal number can uniquely be represented as the value $f(\omega)$ of a hyperseries $f \in \mathbb{H}$ at $x = \omega$. In order to prove this conjecture, quite some machinery has been developed since: a systematic theory of surreal substructures [5], sufficiently general notions of hyperserial fields [17, 7], and definitions of $(E_\alpha)_{\alpha \in \mathbf{No}}$ on the surreals that give **No** the structure of a hyperserial field [8, 6].

Now one characteristic property of generalized hyperseries in $\mathbb{H}$ should be that they can uniquely be described using suitable expressions that involve x, real numbers, infinite summation, hyperlogarithms, hyperexponentials, and a way to disambiguate nested expansions. The main goal of this paper is to show that any surreal number can indeed be described uniquely by a hyperserial expression of this kind in ω. This essentially solves the conjecture from [32, page 16] by thinking of hyperseries in $\mathbb{H}$ as surreal numbers in which we replaced ω by x. Of course, it remains desirable to give a formal construction of $\mathbb{H}$ that does not involve surreal numbers and to specify the precise kind of properties that our "suitably generalized" hyperseries should possess. We intend to address this issue in a forthcoming paper.

Other work in progress concerns the definition of a derivation and a composition on $\mathbb{H}$. Now Berarducci and Mantova showed how to define a derivation on **No** that is compatible with infinite summation and exponentiation [10]. In [4, 1], it was shown that there actually exist many such derivations and that they all satisfy the same first order theory as the ordered differential field $\mathbb{T}$. However, as pointed out in [2], Berarducci and Mantova's derivation does not obey the chain rule with respect to E_ω. The hyperserial derivation that we propose to construct should not have this deficiency and therefore be a better candidate for *the* derivation on **No** with respect to ω.

1.2. Outline of our results and contributions

In this paper, we will strongly rely on previous work from [5, 17, 7, 8, 6]. The main results from these previous papers will be recalled in Sections 2, 3, and 4. For the sake of this introduction, we start with a few brief reminders.

The field of *logarithmic hyperseries* $\mathbb{L}$ was defined and studied in [17]. It is a field of Hahn series $\mathbb{L}=\mathbb{R}[[\mathfrak{L}]]$ in the sense of [27] that is equipped with a logarithm $\log\colon \mathbb{L}^{>} \longrightarrow \mathbb{L}$, a derivation $\partial\colon \mathbb{L} \longrightarrow \mathbb{L}$, and a composition $\circ\colon \mathbb{L} \times \mathbb{L}^{>,\succ} \longrightarrow \mathbb{L}$. Moreover, for each ordinal $\alpha \in \mathbf{On}$, it contains an element ℓ_α such that

$$\begin{aligned}
\ell_1 \circ f &= \log f\\
\ell_{\omega^{\mu+1}} \circ \ell_{\omega^\mu} &= \ell_{\omega^{\mu+1}} - 1\\
\log \ell_\alpha' &= -\sum_{\beta<\alpha} \ell_{\beta+1}.
\end{aligned}$$

for all $f \in \mathbb{L}^{>,\succ}$ and all ordinals α, μ. Moreover, if the Cantor normal form of α is given by $\alpha = \sum_{i=1}^{p} \omega^{\mu_i} n_i$ with $\mu_1 < \cdots < \mu_p$, then we have

$$\ell_\alpha = \ell_{\omega^{\mu_1}}^{\circ n_1} \circ \cdots \circ \ell_{\omega^{\mu_p}}^{\circ n_p}.$$

The derivation and composition on $\mathbb{L}$ satisfy the usual rules of calculus and in particular a formal version of Taylor series expansions.

In [7], Kaplan and the authors defined the concept of a *hyperserial field* to be a field $\mathbb{T} = \mathbb{R}[[\mathfrak{T}]]$ of Hahn series with a logarithm $\log\colon \mathbb{T}^{>} \longrightarrow \mathbb{T}$ and a composition law $\circ\colon \mathbb{L} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$, such that various natural compatibility requirements are satisfied. For every ordinal α, we then define the *hyperlogarithm* L_α of strength α by $L_\alpha\colon \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}^{>,\succ}$; $f \longmapsto \ell_\alpha \circ f$. We showed in [6] how to define bijective hyperlogarithms $L_\alpha\colon \mathbf{No}^{>,\succ} \longrightarrow \mathbf{No}^{>,\succ}$ for which **No** has the structure of a hyperserial field. For every ordinal α, the functional inverse $E_\alpha\colon \mathbf{No}^{>,\succ} \longrightarrow \mathbf{No}^{>,\succ}$ of L_α is called the *hyperexponential* of strength α.

The main aim of this paper is to show that any surreal number $a \in \mathbf{No}$ is not just an abstract hyperseries in the sense of [6], but that we can regard it as a hyperseries in ω. We will do this by constructing a suitable unambiguous description of a in terms of ω, the real numbers, infinite summation, the hyperexponentials, and the hyperlogarithms.

If $a = f(\omega)$ for some ordinary transseries f, then the idea would be to expand a as a linear combination of monomials, then to rewrite every monomial as an exponential of a transseries, and finally to recursively expand these new transseries. This process stops whenever we hit an iterated logarithm of ω.

In fact, this transserial expansion process works for any surreal number $a \in \mathbf{No}$. However, besides the iterated logarithms (and exponentials) of ω, there exist other monomials $\mathfrak{a} \in \mathbf{Mo}^{\succ} := \{\mathfrak{m} \in \mathbf{Mo} : \mathfrak{m} \succ 1\}$ such that $L_n(\mathfrak{a})$ is a monomial for all $n \in \mathbb{N}$. Such monomials are said to be *log-atomic*. More generally, given $\mu \in \mathbf{On}$, we say that $\mathfrak{a}$ is $L_{<\omega^\mu}$*-atomic* if $L_\alpha(\mathfrak{a}) \in \mathbf{Mo}$ for all $\alpha < \omega^\mu$. We write $\mathbf{Mo}_{\omega^\mu}$ for the set of such numbers. If we wish to further expand an $L_{<\omega^\mu}$-atomic monomial $\mathfrak{a}$ as a hyperseries, then it is natural to pick μ such that $\mathfrak{a}$ is not $L_{<\omega^{\mu+1}}$-atomic, to recursively expand $b := L_{\omega^\mu}$, and then to write $\mathfrak{a} = E_{\omega^\mu}(b)$.

Unfortunately, the above idea is slightly too simple to be useful. In order to expand monomials as hyperseries, we need something more technical. In Section 5, we show that every non-trivial monomial $\mathfrak{m} \in \mathbf{Mo} \setminus \{1\}$ has a unique expansion of exactly one of the two following forms:

$$\mathfrak{m} \;=\; \mathrm{e}^{\psi}\,(L_\beta(\omega))^\iota, \tag{1.3}$$

where $\mathrm{e}^\psi \in \mathbf{Mo}$, $\iota \in \{-1, 1\}$, and $\beta \in \mathbf{On}$, with $\operatorname{supp} \psi \succ \log(L_\beta(\omega))$; or

$$\mathfrak{m} \;=\; \mathrm{e}^{\psi}\,(L_\beta(E_\alpha(u)))^\iota, \tag{1.4}$$

where $\mathrm{e}^\psi \in \mathbf{Mo}$, $\iota \in \{-1, 1\}$, $\beta \in \mathbf{On}$, $\alpha \in \omega^{\mathbf{On}}$ with $\beta\,\omega < \alpha$, $\operatorname{supp} \psi \succ \log(L_\beta(E_\alpha(u)))$, and where $E_\alpha\, u$ lies in $\mathbf{Mo}_\alpha \setminus L_{<\alpha}\,\mathbf{Mo}_{\alpha\omega}$. Moreover, if $\alpha = 1$ then it is imposed that $\psi = 0$, $\iota = 1$, and that u cannot be written as $u = \varphi + \varepsilon\,\mathfrak{b}$ where $\varphi \in \mathbf{No}$, $\varepsilon \in \{-1, 1\}$, $\mathfrak{b} \in \mathbf{Mo}_\omega$, and $\mathfrak{b} \prec \operatorname{supp} \varphi$.

After expanding $\mathfrak{m}$ in the above way, we may pursue with the recursive expansions of ψ and u as hyperseries. Our next objective is to investigate the shape of the recursive expansions that arise by doing so. Indeed, already in the case of ordinary transseries, such recursive expansions may give rise to nested expansions like

$$\sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} + \mathrm{e}^{\sqrt{\log\log \omega} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}} \tag{1.5}$$

One may wonder whether it is also possible to obtain expansions like

$$\sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} + \mathrm{e}^{\sqrt{\log\log \omega} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}} \cdot^{\cdot^{\cdot}} + \log\log\log \omega} + \log\log \omega} + \log \omega. \tag{1.6}$$

Expansions of the forms (1.5) and (1.6) are said to be *well-nested* and *ill-nested*, respectively. The axiom **T4** for fields of transseries in [38] prohibits the existence of ill-nested expansions. It was shown in [10] that **No** satisfies this axiom **T4**.

The definition of hyperserial fields in [6] does not contain a counterpart for the axiom **T4**. The main goal of section 4 is to generalize this property to hyperserial fields and prove the following theorem:

THEOREM 1.1. *Every surreal number is well-nested.*

Now there exist surreal numbers for which the above recursive expansion process leads to a nested expansion of the form (1.5). In [5, Section 8], we proved that the class **Ne** of such numbers actually forms a *surreal substructure*. This means that $(\mathbf{No}, \leqslant, \sqsubseteq)$ is isomorphic to $(\mathbf{Ne}, \leqslant, \sqsubseteq_{\mathbf{Ne}})$ for the restriction $\sqsubseteq_{\mathbf{Ne}}$ of $\sqsubseteq$ to **Ne**. In particular, although the nested expansion (1.5) is inherently ambiguous, elements in **Ne** are naturally parameterized by surreal numbers in **No**.

The main goal of Section 6 is to prove a hyperserial analogue of the result from [5, Section 8]. Now the expansion (1.5) can be described in terms of the sequence $\sqrt{\omega}, \sqrt{\log \omega}, \sqrt{\log \log \omega}, \ldots$. More generally, in Section 6 we the define the notion of a *nested sequence* in order to describe arbitrary nested hyperserial expansions. Our main result is the following:

THEOREM 1.2. *Any nested sequence* Σ *induces a surreal substructure* **Ne** *of nested hyperseries.*

In Section 7, we reach the main goal of this paper, which is to uniquely describe any surreal number as a generalized hyperseries in ω. This goal can be split up into two tasks. First of all, we need to specify the hyperserial expansion process that we informally described above and show that it indeed leads to a hyperserial expansion in ω, for any surreal number. This will be done in Section 7.2, where we will use labeled trees in order to represent hyperserial expansions. Secondly, these trees may contain infinite branches (also called paths) that correspond to nested numbers in the sense of Section 6. By Theorem 1.2, any such nested number can uniquely be identified using a surreal parameter. By associating a surreal number to each infinite branch, this allows us to construct a unique *hyperserial description* in ω for any surreal number and prove our main result:

THEOREM 1.3. *Every surreal number has a unique hyperserial description. Two numbers with the same hyperserial description are equal.*

2. ORDERED FIELDS OF WELL-BASED SERIES

2.1. Well-based series

Let $(\mathfrak{M}, \times, 1, \prec)$ be a totally ordered (and possibly class-sized) abelian group. We say that $\mathfrak{S} \subseteq \mathfrak{M}$ is *well-based* if it contains no infinite ascending chain (equivalently, this means that $\mathfrak{S}$ is well-ordered for the opposite ordering). We denote by $\mathbb{R}[[\mathfrak{M}]]$ the class of functions $f: \mathfrak{M} \longrightarrow \mathbb{R}$ whose support

$$\operatorname{supp} f \;:=\; \{\mathfrak{m} \in \mathfrak{M} : f(\mathfrak{m}) \neq 0\}$$

is a *well-based*. The elements of $\mathfrak{M}$ are called monomials and the elements in $\mathbb{R}^{\neq} \mathfrak{M}$ are called *terms*. We also define

$$\operatorname{term} f \;:=\; \{f_{\mathfrak{m}} \mathfrak{m} : \mathfrak{m} \in \operatorname{supp} f\},$$

and elements $\tau \in \operatorname{term} f$ are called terms in f.

We see elements f of $\mathbb{S}$ as formal *well-based series* $f = \sum_{\mathfrak{m}} f_{\mathfrak{m}} \mathfrak{m}$ where $f_{\mathfrak{m}} := f(\mathfrak{m}) \in \mathbb{R}$ for all $\mathfrak{m} \in \mathfrak{M}$. If $\operatorname{supp} f \neq \emptyset$, then $\mathfrak{d}_f := \max \operatorname{supp} f \in \mathfrak{M}$ is called the *dominant monomial* of f. For $\mathfrak{m} \in \mathfrak{M}$, we define $f_{\succ \mathfrak{m}} := \sum_{\mathfrak{n} \succ \mathfrak{m}} f_{\mathfrak{n}} \mathfrak{n}$ and $f_{\succ} := f_{\succ 1}$. For $f, g \in \mathbb{S}$, we sometimes write $f + g = f \mathbin{+\!\!\!\!+} g$ if $\operatorname{supp} g \prec f$. We say that a series $g \in \mathbb{S}$ is a *truncation* of f and we write $g \trianglelefteq f$ if $\operatorname{supp}(f - g) \succ g$. The relation $\trianglelefteq$ is a well-founded partial order on $\mathbb{S}$ with minimum 0.

By [27], the class $\mathbb{S}$ is field for the pointwise sum

$$(f+g) := \sum_{\mathfrak{m}} (f_{\mathfrak{m}}+g_{\mathfrak{m}})\,\mathfrak{m},$$

and the Cauchy product

$$fg := \sum_{\mathfrak{m}} \left(\sum_{\mathfrak{u}\mathfrak{v}=\mathfrak{m}} f_{\mathfrak{u}} g_{\mathfrak{v}}\right) \mathfrak{m},$$

where each sum $\sum_{\mathfrak{u}\mathfrak{v}=\mathfrak{m}} f_{\mathfrak{u}} g_{\mathfrak{v}}$ is finite. The class $\mathbb{S}$ is actually an ordered field, whose positive cone $\mathbb{S}^{>}=\{f\in\mathbb{S}:f>0\}$ is defined by

$$\mathbb{S}^{>} := \{f\in\mathbb{S}:f\neq 0\wedge f_{\mathfrak{d}_f}>0\}.$$

The ordered group $(\mathfrak{M},\times,\prec)$ is naturally embedded into $(\mathbb{S}^{>},\times,<)$.

The relations $\prec$ and $\preccurlyeq$ on $\mathfrak{M}$ extend to $\mathbb{S}$ by

$$\begin{aligned} f \prec g &\iff \mathbb{R}^{>}|f| < |g| \\ f \preccurlyeq g &\iff \exists r\in\mathbb{R}^{>},\ |f| \leqslant r|g|. \end{aligned}$$

We also write $f\asymp g$ whenever $f\preccurlyeq g$ and $g\preccurlyeq f$. If f,g are non-zero, then $f\prec g$ (resp. $f\preccurlyeq g$, resp. $f\asymp g$) if and only if $\mathfrak{d}_f\prec\mathfrak{d}_g$ (resp. $\mathfrak{d}_f\preccurlyeq\mathfrak{d}_g$, resp. $\mathfrak{d}_f=\mathfrak{d}_g$).

We finally define

$$\begin{aligned} \mathbb{S}_{\succ} &:= \{f\in\mathbb{S}:\operatorname{supp} f\subseteq\mathfrak{M}^{\succ}\} \\ \mathbb{S}^{\prec} &:= \{f\in\mathbb{S}:\operatorname{supp} f\subseteq\mathfrak{M}^{\prec}\} = \{f\in\mathbb{S}:f\prec 1\}, \quad \text{and} \\ \mathbb{S}^{>,\succ} &:= \{f\in\mathbb{S}:f>\mathbb{R}\} = \{f\in\mathbb{S}:f\geqslant 0\wedge f\succ 1\}. \end{aligned}$$

Series in $\mathbb{S}_{\succ}$, $\mathbb{S}^{\prec}$ and $\mathbb{S}^{>,\succ}$ are respectively called *purely large*, *infinitesimal*, and *positive infinite*.

2.2. Well-based families

If $(f_i)_{i\in I}$ is a family in $\mathbb{S}$, then we say that $(f_i)_{i\in I}$ is *well-based* if

i. $\bigcup_{i\in I}\operatorname{supp} f_i$ is well-based, and

ii. $\{i\in I:\mathfrak{m}\in\operatorname{supp} f_i\}$ is finite for all $\mathfrak{m}\in\mathfrak{M}$.

Then we may define the sum $\sum_{i\in I} f_i$ of $(f_i)_{i\in I}$ as the series

$$\sum_{i\in I} f_i := \sum_{\mathfrak{m}} \left(\sum_{i\in I} (f_i)_{\mathfrak{m}}\right) \mathfrak{m}.$$

If $\mathbb{U}=\mathbb{R}[[\mathfrak{N}]]$ is another field of well-based series and $\Psi:\mathbb{S}\longrightarrow\mathbb{U}$ is $\mathbb{R}$-linear, then we say that Ψ is *strongly linear* if for every well-based family $(f_i)_{i\in I}$ in $\mathbb{S}$, the family $(\Psi(f_i))_{i\in I}$ in $\mathbb{U}$ is well-based, with

$$\Psi\left(\sum_{i\in I} f_i\right) = \sum_{i\in I} \Psi(f_i).$$

2.3. Logarithmic hyperseries

The field $\mathbb{L}$ of *logarithmic hyperseries* plays an important role in the theory of hyperseries. Let us briefly recall its definition and its most prominent properties from [17].

Let α be an ordinal. For each $\gamma<\alpha$, we introduce the formal hyperlogarithm $\ell_\gamma:=L_\gamma x$ and define $\mathfrak{L}_{<\alpha}$ to be the group of formal power products $\mathfrak{l}=\prod_{\gamma<\alpha}\ell_\gamma^{\mathfrak{l}_\gamma}$ with $\mathfrak{l}_\gamma\in\mathbb{R}$. This group comes with a monomial ordering $\succ$ that is defined by

$$\mathfrak{l}\succ 1 \iff \mathfrak{l}_{\min\{\gamma<\alpha:\mathfrak{l}_\gamma\neq 0\}}>0.$$

By what precedes, $\mathbb{L}_{<\alpha}:=\mathbb{R}[[\mathfrak{L}_{<\alpha}]]$ is an ordered field of well-based series. If α,β are ordinals with $\beta<\alpha$, then we define $\mathfrak{L}_{[\beta,\alpha)}$ to be the subgroup of $\mathfrak{L}_{<\alpha}$ of monomials $\mathfrak{l}$ with $\mathfrak{l}_\gamma=0$ whenever $\gamma<\beta$. As in [17], we define

$$\begin{aligned}\mathbb{L}_{[\beta,\alpha)} &:= \mathbb{R}[[\mathfrak{L}_{[\beta,\alpha)}]]\\ \mathfrak{L} &:= \bigcup_{\alpha\in\mathbf{On}}\mathfrak{L}_{<\alpha}\\ \mathbb{L} &:= \mathbb{R}[[\mathfrak{L}]].\end{aligned}$$

We have natural inclusions $\mathfrak{L}_{[\beta,\alpha)}\subseteq\mathfrak{L}_{<\alpha}\subset\mathfrak{L}$, hence natural inclusions $\mathbb{L}_{[\beta,\alpha)}\subseteq\mathbb{L}_{<\alpha}\subset\mathbb{L}$.

The field $\mathbb{L}_{<\alpha}$ is equipped with a derivation $\partial:\mathbb{L}_{<\alpha}\longrightarrow\mathbb{L}_{<\alpha}$ which satisfies the Leibniz rule and which is strongly linear: for each $\gamma<\alpha$, we first define the logarithmic derivative of ℓ_γ by $\ell_\gamma^\dagger:=\prod_{\iota\leqslant\gamma}\ell_\iota^{-1}\in\mathfrak{L}_{<\alpha}$. The derivative of a logarithmic hypermonomial $\mathfrak{l}\in\mathfrak{L}_{<\alpha}$ is next defined by

$$\partial\mathfrak{l} := \left(\sum_{\gamma<\alpha}\mathfrak{l}_\gamma\ell_\gamma^\dagger\right)\mathfrak{l}.$$

Finally, this definition extends to $\mathbb{L}_{<\alpha}$ by strong linearity. Note that $\partial\ell_\gamma=\frac{1}{\prod_{\iota<\gamma}\ell_\iota}$ for all $\gamma<\alpha$. For $f\in\mathbb{L}_{<\alpha}$ and $k\in\mathbb{N}$, we will sometimes write $f^{(k)}:=\partial^k f$.

Assume that $\alpha=\omega^\nu$ for a certain ordinal ν. Then the field $\mathbb{L}_{<\alpha}$ is also equipped with a composition $\circ:\mathbb{L}_{<\alpha}\times\mathbb{L}_{<\alpha}^{>,\succ}\longrightarrow\mathbb{L}_{<\alpha}$ that satisfies:

- For $g\in\mathbb{L}_{<\alpha}^{>,\succ}$, the map $\mathbb{L}_{<\alpha}\longrightarrow\mathbb{L}_{<\alpha}; f\longmapsto f\circ g$ is a strongly linear embedding [17, Lemma 6.6].
- For $f\in\mathbb{L}_{<\alpha}$ and $g,h\in\mathbb{L}_{<\alpha}^{>,\succ}$, we have $g\circ h\in\mathbb{L}_{<\alpha}^{>,\succ}$ and $f\circ(g\circ h)=(f\circ g)\circ h$ [17, Proposition 7.14].
- For $g\in\mathbb{L}_{<\alpha}^{>,\succ}$ and successor ordinals $\mu<\nu$, we have $\ell_{\omega^\mu}\circ\ell_{\omega^{\mu_-}}=\ell_{\omega^\mu}-1$ [17, Lemma 5.6].

The same properties hold for the composition $\circ:\mathbb{L}\times\mathbb{L}^{>,\succ}\longrightarrow\mathbb{L}$, when α is replaced by $\mathbf{On}$. For $\gamma<\alpha$, the map $\mathbb{L}_{<\alpha}\longrightarrow\mathbb{L}_{<\alpha}; f\longmapsto f\circ\ell_\gamma$ is injective, with range $\mathbb{L}_{[\gamma,\alpha)}$ [17, Lemma 5.11]. For $g\in\mathbb{L}_{[\gamma,\alpha)}$, we define $g^{\uparrow\gamma}$ to be the unique series in $\mathbb{L}_{<\alpha}$ with $g^{\uparrow\gamma}\circ\ell_\gamma=g$.

3. SURREAL NUMBERS AS A HYPERSERIAL FIELD

3.1. Surreal numbers

Following [26], we define **No** as the class of sequences

$$a:\alpha\longmapsto\{-1,1\}$$

of "signs" $-1,+1$ indexed by arbitrary ordinals $\alpha\in\mathbf{On}$. We will write $\operatorname{dom}a\in\mathbf{On}$ for the domain of such a sequence and $a[\beta]\in\{-1,1\}$ for its value at $\beta\in\operatorname{dom}a$. Given sign sequences a and b, we define

$$a\sqsubseteq b \iff \operatorname{dom}a\subseteq\operatorname{dom}b\wedge(\forall\beta\in\operatorname{dom}a,\ a[\beta]=b[\beta])$$

Conway showed how to define an ordering, an addition, and a multiplication on **No** that give **No** the structure of a real closed field [14]. See [5, Section 2] for more details about the interaction between $\sqsubseteq$ and the ordered field structure of **No**. By [14, Theorem 21], there is a natural isomorphism between **No** and the ordered field of well-based series $\mathbb{R}[[\mathbf{Mo}]]$, where **Mo** is a certain subgroup of $(\mathbf{No}^{>}, \times, <)$. We will identify those two fields and thus regard **No** as a field of well-based series with monomials in **Mo**.

The partial order $(\mathbf{No}, \sqsubseteq)$ contains an isomorphic copy of $(\mathbf{On}, \in)$ obtained by identifying each ordinal α with the constant sequence $(1)_{\beta<\alpha}$ of length α. We will write $\boldsymbol{\nu} \leqslant \mathbf{On}$ to specify that $\boldsymbol{\nu}$ is either an ordinal or the class of ordinals. The ordinal ω, seen as a surreal number, is the simplest element, or $\sqsubseteq$-minimum, of the class $\mathbf{No}^{>,\succ}$.

For $\alpha, \beta \in \mathbf{On}$, we write $\alpha \dotplus \beta$ and $\alpha \dot{\times} \beta$ for the non-commutative ordinal sum and product of α and β, as defined by Cantor. The surreal sum and product $\alpha + \beta$ and $\alpha\beta$ coincide with the commutative Hessenberg sum and product of ordinals. In general, we therefore have $\alpha + \beta \neq \alpha \dotplus \beta$ and $\alpha\beta \neq \alpha \dot{\times} \beta$.

For $\gamma \in \mathbf{On}$, we write ω^{γ} for the ordinal exponentiation of base ω at γ. Gonshor also defined an exponential function on **No** with range $\mathbf{No}^{>}$. One should not confuse ω^{γ} with $\exp(\gamma \log \omega)$, which yields a different number, in general. We define

$$\omega^{\mathbf{On}} := \{\omega^{\gamma} : \gamma \in \mathbf{On}\},$$

Recall that every ordinal γ has a unique Cantor normal form

$$\gamma = \omega^{\eta_1} n_1 + \cdots + \omega^{\eta_r} n_r,$$

where $r \in \mathbb{N}$, $n_1, \ldots, n_r \in \mathbb{N}^{>0}$ and $\eta_1, \ldots, \eta_r \in \mathbf{On}$ with $\eta_1 > \cdots > \eta_r$. The ordinals η_i are called the *exponents* of γ and the integers n_i its *coefficients*. We write $\rho \ll \sigma$ (resp. $\rho \leqq \sigma$) if each exponent η_i of the Cantor normal form of σ satisfies $\rho \prec \omega^{\eta_i}$ (resp. $\rho \preccurlyeq \omega^{\eta_i}$).

If γ, β are ordinals, then we write $\gamma \prec \beta$ if $\gamma \mathbb{N} < \beta$, we write $\gamma \preccurlyeq \beta$ if there exists an $n \in \mathbb{N}$ with $\gamma \leqslant \beta n$, and we write $\gamma \asymp \beta$ if both $\gamma \preccurlyeq \beta$ and $\gamma \preccurlyeq \beta$ hold. The relation $\preccurlyeq$ is a quasi-order on **On**. For $\eta, \beta, \gamma \in \mathbf{On}$ with $\beta \geqq \omega^{\eta}$ and $\gamma \preccurlyeq \omega^{\eta}$, we have $\beta + \gamma = \beta \dotplus \gamma$. In particular, we have $\gamma + 1 = \gamma \dotplus 1$ for all $\gamma \in \mathbf{On}$.

If $\mu \in \mathbf{On}$ is a successor, then we define μ_{-} to be the unique ordinal with $\mu = \mu_{-} + 1$. We also define $\mu_{-} := \mu$ if μ is a limit. Similarly, if $\alpha = \omega^{\mu}$, then we write $\alpha_{/\omega} := \omega^{\mu_{-}}$.

3.2. Hyperserial structure on No

We already noted that Gonshor constructed an exponential and a logarithm on **No** and $\mathbf{No}^{>}$, respectively. We defined hyperexponential and hyperlogarithmic functions of all strengths on $\mathbf{No}^{>,\succ}$ in [6]. In fact, we showed [6, Theorem 1.1] how to construct a composition law $\circ : \mathbb{L} \times \mathbf{No}^{>,\succ} \longrightarrow \mathbf{No}$ with the following properties:

C1. For $f \in \mathbb{L}$, $g \in \mathbb{L}^{>,\succ}$ and $a \in \mathbf{No}^{>,\succ}$, we have $g \circ a \in \mathbf{No}^{>,\succ}$ and

$$f \circ (g \circ a) = (f \circ g) \circ a.$$

C2. For $a \in \mathbf{No}^{>,\succ}$, the function $\mathbb{L} \longrightarrow \mathbf{No}; f \longmapsto f \circ a$ is a strongly linear field morphism.

C3. For $f \in \mathbb{L}$, $a \in \mathbf{No}^{>,\succ}$ and $\delta \in \mathbf{No}$ with $\delta \prec a$, we have

$$f \circ (a + \delta) = \sum_{k \in \mathbb{N}} \frac{f^{(k)} \circ a}{k!} \delta^k.$$

C4. For $\gamma \in \mathbf{On}$ and $a, b \in \mathbf{No}^{>,\succ}$ with $a < b$, we have $\ell_{\gamma} \circ a < \ell_{\gamma} \circ b$.

C5. For $\gamma \in \mathbf{On}$ and $a \in \mathbf{No}^{>,\succ}$, there is $b \in \mathbf{No}^{>,\succ}$ with $a = \ell_{\gamma} \circ b$.

Note that the composition law on $\mathbb{L}$ also satisfies **C1** to **C4** (but not **C5**), with each occurrence of **No** being replaced by $\mathbb{L}$.

3.3. Hyperlogarithms

For $\gamma \in \mathbf{On}$, we write L_γ for the function $\mathbf{No}^{>,\succ} \longrightarrow \mathbf{No}^{>,\succ}; a \longmapsto \ell_\gamma \circ a$, called the *hyperlogarithm of strength* γ. By **C4** and **C5**, this is a strictly increasing bijection. We sometimes write $L_\gamma a := L_\gamma(a)$ for $a \in \mathbf{No}^{>,\succ}$. We write E_γ for the functional inverse of L_γ, called the *hyperexponential of strength* γ.

For γ, ρ with $\rho \lessdot \gamma$, the relation $\ell_{\gamma+\rho} = \ell_\rho \circ \ell_\gamma$ in $\mathbb{L}$, combined with **C3**, yields

$$\forall a \in \mathbf{No}^{>,\succ}, \ L_{\gamma+\rho} a \ = \ L_\gamma L_\rho a, \tag{3.1}$$

For $\eta \in \mathbf{On}$, the relation $\ell_{\omega^{\eta+1}} \circ \ell_{\omega^\eta} = \ell_{\omega^{\eta+1}} - 1$ in $\mathbb{L}$, combined with **C3**, yields

$$\forall a \in \mathbf{No}^{>,\succ}, \ L_{\omega^{\eta+1}}(L_{\omega^\eta}(a)) \ = \ L_{\omega^{\eta+1}}(a) - 1, \tag{3.2}$$

and we call this relation the *functional equation* for $L_{\omega^{\eta+1}}$.

Let $a \in \mathbf{No}^>$ and write $r_a := a_{\mathfrak{d}_a}$ for the coefficient in $\mathfrak{d}_a$ in the Hahn series representation of a. There is a unique infinitesimal number ε_a with $a = r_a \mathfrak{d}_a (1 + \varepsilon_a)$. We write $\log_{\mathbb{R}}$ for the natural logarithm on $\mathbb{R}^> \subset \mathbf{No}$. The function defined by

$$\log a \ := \ L_1(\mathfrak{d}_a) + \log_{\mathbb{R}} r_a + \sum_{k \in \mathbb{N}} \frac{(-1)^k}{k+1} \varepsilon_a^{k+1}, \tag{3.3}$$

is called the *logarithm* on $\mathbf{No}^>$. This is a strictly increasing morphism $(\mathbf{No}^>, +) \longrightarrow (\mathbf{No}, +)$ which extends L_1. It also coincides with the logarithm on $\mathbf{No}^>$ that was defined by Gonshor.

3.4. Atomicity

Given $\mu \leqslant \mathbf{On}$, we write $\mathbf{Mo}_{\omega^\mu}$ for the class of numbers $a \in \mathbf{No}^{>,\succ}$ with $L_\gamma a \in \mathbf{Mo}^\succ$ for all $\gamma < \omega^\mu$. Those numbers are said to be $L_{<\omega^\mu}$*-atomic* and they play an important role in this paper. Note that $\mathbf{Mo}_1 = \mathbf{Mo}^\succ$ and

$$L_{\omega^\eta} \mathbf{Mo}_{\omega^{\eta+1}} \ = \ \mathbf{Mo}_{\omega^{\eta+1}}$$

for all $\eta \in \mathbf{On}$, in view of (3.1). There is a unique $L_{<\mathbf{On}}$-atomic number [6, Proposition 6.20], which is the simplest positive infinite number ω.

Each hyperlogarithmic function L_{ω^η} with $\eta > 0$ is essentially determined by its restriction to $\mathbf{Mo}_{\omega^\eta}$, through a generalization of (3.3). More precisely, for $a \in \mathbf{No}^{>,\succ}$, there exist $\gamma < \omega^\eta$ and $\mathfrak{a} \in \mathbf{Mo}_{\omega^\eta}$ with $\delta := L_\gamma(a) - L_\gamma(\mathfrak{a}) \prec L_\gamma(a)$. Moreover, the family $(((\ell_{\omega^\eta}^{\uparrow\gamma})^{(k)} \circ L_\gamma(\mathfrak{a}))\, \delta^k)_{k \in \mathbb{N}^>}$ is well-based, and the hyperlogarithm $L_{\omega^\eta}(a)$ is given by

$$L_{\omega^\eta}(a) \ = \ L_{\omega^\eta}(\mathfrak{a}) + \sum_{k \in \mathbb{N}^>} \frac{(\ell_{\omega^\eta}^{\uparrow\gamma})^{(k)} \circ L_\gamma(\mathfrak{a})}{k!} \delta^k. \tag{3.4}$$

3.5. Hyperexponentiation

DEFINITION 3.1. [7, Definition 6.10] *We say that* $\varphi \in \mathbf{No}^{>,\succ}$ *is* **1-truncated** *if* $\operatorname{supp} \varphi \succ 1$, *i.e. if* φ *is positive and purely large. For* $0 < \eta \in \mathbf{On}$, *we say that* $\varphi \in \mathbf{No}^{>,\succ}$ *is* $\boldsymbol{\omega^\eta}$**-truncated** *if*

$$\forall \mathfrak{m} \in \operatorname{supp} \varphi_{\prec}, \ \forall \gamma < \omega^\eta, \quad \varphi \ < \ \ell_{\omega^\eta}^{\uparrow\gamma} \circ \mathfrak{m}^{-1}.$$

If $E_{\omega^\eta}(\varphi)$ is defined, then φ is ω^η-truncated if and only if $\operatorname{supp}\varphi \succ 1/L_\gamma(E_{\omega^\eta}(\varphi))$*, for all $\gamma<\omega^\eta$.*

Given $\beta=\omega^\eta$ with $\eta\in\mathbf{On}$, we write $\mathbf{No}_{\succ,\beta}$ for the class of β-truncated numbers. Note that $\mathbf{No}_{\succ,1}=\mathbf{No}_\succ\cap\mathbf{No}^{>,\succ}$. We will sometimes write $E_\beta(\varphi)=:E_\beta^\varphi$ when $\varphi\in\mathbf{No}_{\succ,\beta}$. For $a\in\mathbf{No}^{>,\succ}$, there is a unique $\trianglelefteq$-maximal truncation $\sharp_\beta(a)$ of a which is β-truncated. By [7, Proposition 7.17], the classes

$$\{b\in a+\mathbf{No}^{\prec}: b=a\vee(\exists\gamma<\beta, b<\ell_\beta^{\uparrow\gamma}\circ|a-b|^{-1})\} \tag{3.5}$$

with $a\in\mathbf{No}^{>,\succ}$ form a partition of $\mathbf{No}^{>,\succ}$ into convex subclasses. Moreover, the series $\sharp_\beta(a)$ is both the unique β-truncated element and the $\trianglelefteq$-minimum of the convex class containing a. We have

$$\mathbf{No}_{\succ,\beta} \;=\; L_\beta\,\mathbf{Mo}_\beta$$

by [6, Proposition 7.6]. This allows us to define a map $\mathfrak{d}_\beta:\mathbf{No}^{>,\succ}\longrightarrow\mathbf{Mo}_\beta$ by $\mathfrak{d}_\beta(b):=E_\beta^{\sharp_\beta(L_\beta b)}$. In other words,

$$E_\beta^{\sharp_\beta(a)} \;=\; \mathfrak{d}_\beta(E_\beta(a)),$$

for all $a\in\mathbf{No}^{>,\succ}$ (see also [7, Corollary 7.23]).

The formulas (3.3) and (3.4) admit hyperexponential analogues. For all $a\in\mathbf{No}^{>,\succ}$, there is a $\gamma<\beta$ with $\varepsilon:=a-\sharp_\beta(a)\prec\frac{\ell_\beta'}{\ell_\gamma'}\circ E_\beta^{\sharp_\beta(a)}$. For any such γ, there is a family $(t_{\gamma,k})_{k\in\mathbb{N}}\in\mathbb{L}_{<\beta}^{\mathbb{N}}$ with $t_0=\ell_\gamma$ such that $\big((t_{\gamma,k}\circ E_\beta^{\sharp_\beta(a)})\,\varepsilon^k\big)_{k\in\mathbb{N}}$ is well-based and

$$E_\beta\,a \;=\; E_\gamma\left(\sum_{k\in\mathbb{N}}\frac{t_{\gamma,k}\circ E_\beta^{\sharp_\beta(a)}}{k!}\,\varepsilon^k\right). \tag{3.6}$$

See [7, Section 7.1] for more details on $(t_{\gamma,k})_{k\in\mathbb{N}}$. The number $L_\gamma E_\beta^{\sharp_\beta(a)}$ is a monomial with $L_\gamma E_\beta^{\sharp_\beta(a)}\succ(t_{\gamma,1}\circ E_\beta^{\sharp_\beta(a)})\,\varepsilon\succ(t_{\gamma,2}\circ E_\beta^{\sharp_\beta(a)})\,\varepsilon^2\succ\cdots$, so

$$L_\gamma\,E_\beta^{\sharp_\beta(a)} \;\trianglelefteq\; L_\gamma\,E_\beta\,a. \tag{3.7}$$

4. SURREAL SUBSTRUCTURES

In [5], we introduced the notion of *surreal substructure*. A surreal substructure is a subclass $\mathbf{S}$ of $\mathbf{No}$ such that $(\mathbf{No},\leqslant,\sqsubseteq)$ and $(\mathbf{S},\leqslant,\sqsubseteq)$ are isomorphic. The isomorphism $\mathbf{No}\longrightarrow\mathbf{S}$ is unique and denoted by $\Xi_\mathbf{S}$. For the study of $\mathbf{No}$ as a hyperserial field, many important subclasses of $\mathbf{No}$ turn out to be surreal substructures. In particular, given $\alpha=\omega^\nu\in\mathbf{On}$, it is known that the following classes are surreal substructures:

- The classes $\mathbf{No}^>$, $\mathbf{No}^{>,\succ}$ and $\mathbf{No}^{\prec}$ of positive, positive infinite and infinitesimal numbers.
- The classes $\mathbf{Mo}$ and $\mathbf{Mo}^{\succ}$ of monomials and infinite monomials.
- The classes $\mathbf{No}_\succ$ and $\mathbf{No}_\succ^>$ of purely infinite and positive purely infinite numbers.
- The class $\mathbf{Mo}_\alpha$ of $L_{<\alpha}$-atomic numbers.
- The class $\mathbf{No}_{\succ,\alpha}$ of α-truncated numbers.

We will prove in Section 6 that certain classes of nested numbers also form surreal substructures.

4.1. Cuts

Given a subclass $\mathbf{X}$ of $\mathbf{No}$ and $a \in \mathbf{X}$, we define

$$\begin{array}{rclcrcl} a_L^{\mathbf{X}} & := & \{b \in \mathbf{X} : b < a \wedge b \sqsubseteq a\} & \qquad & a_L & := & a_L^{\mathbf{No}} \\ a_R^{\mathbf{X}} & := & \{b \in \mathbf{X} : b > a \wedge b \sqsubseteq a\} & & a_R & := & a_R^{\mathbf{No}} \\ a_{\sqsubset}^{\mathbf{X}} & := & a_L^{\mathbf{X}} \cup a_R^{\mathbf{X}} & & a_{\sqsubset} & := & a_{\sqsubset}^{\mathbf{No}} \end{array}$$

If $\mathbf{X}$ is a subclass of $\mathbf{No}$ and L, R are subsets of $\mathbf{X}$ with $L < S$, then the class

$$(L \mid R)_{\mathbf{X}} := \{a \in \mathbf{X} : (\forall l \in L, l < a) \wedge (\forall r \in R, a < r)\}$$

is called a *cut* in $\mathbf{X}$. If $(L \mid R)_{\mathbf{X}}$ contains a unique simplest element, then we denote this element by $\{L \mid R\}_{\mathbf{X}}$ and say that (L, R) is a *cut representation* (of $\{L \mid R\}_{\mathbf{X}}$) in $\mathbf{X}$. These notations naturally extend to the case when $\mathbf{L}$ and $\mathbf{R}$ are subclasses of $\mathbf{X}$ with $\mathbf{L} < \mathbf{R}$.

A surreal substructure $\mathbf{S}$ may be characterized as a subclass of $\mathbf{No}$ such that for all cut representations (L, R) in $\mathbf{S}$, the cut $(L \mid R)_{\mathbf{S}}$ has a unique simplest element [5, Proposition 4.7].

Let $\mathbf{S}$ be a surreal substructure. Note that we have $a = \{a_L^{\mathbf{S}} \mid a_R^{\mathbf{S}}\}$ for all $a \in \mathbf{S}$. Let $a \in \mathbf{S}$ and let (L, R) be a cut representation of a in $\mathbf{S}$. Then (L, R) is *cofinal with respect to* $(a_L^{\mathbf{S}}, a_R^{\mathbf{S}})$ in the sense that L has no strict upper bound in $a_L^{\mathbf{S}}$ and R has no strict lower bound in $a_R^{\mathbf{S}}$ [5, Proposition 4.11(b)].

4.2. Cut equations

Let $\mathbf{X} \subseteq \mathbf{No}$ be a subclass, let $\mathbf{T}$ be a surreal substructure and $F: \mathbf{X} \longrightarrow \mathbf{T}$ be a function. Let λ, ρ be functions defined for cut representations in $\mathbf{X}$ such that $\lambda(L, R), \rho(L, R)$ are subsets of $\mathbf{T}$ whenever (L, R) is a cut representation in $\mathbf{X}$. We say that (λ, ρ) is a *cut equation* for F if for all $a \in \mathbf{X}$, we have

$$\lambda(a_L^{\mathbf{X}}, a_R^{\mathbf{X}}) < \rho(a_L^{\mathbf{X}}, a_R^{\mathbf{X}}), \qquad F(a) = \{\lambda(a_L^{\mathbf{X}}, a_R^{\mathbf{X}}) \mid \rho(a_L^{\mathbf{X}}, a_R^{\mathbf{X}})\}_{\mathbf{T}}.$$

Elements in $\lambda(a_L^{\mathbf{X}}, a_R^{\mathbf{X}})$ (resp. $\rho(a_L^{\mathbf{X}}, a_R^{\mathbf{X}})$) are called *left* (resp. *right*) *options* of this cut equation at a. We say that the cut equation is *uniform* if

$$\lambda(L, R) < \rho(L, R), \qquad F(\{L \mid R\}_{\mathbf{X}}) = \{\lambda(L, R) \mid \rho(L, R)\}_{\mathbf{T}}$$

for all cut representations (L, R) in $\mathbf{X}$. For instance, given $r \in \mathbb{R}$, consider the translation $T_r: \mathbf{No} \longrightarrow \mathbf{No}; a \longmapsto a + r$ on $\mathbf{No}$. By [26, Theorem 3.2], we have the following uniform cut equation for T_r on $\mathbf{No}$:

$$\forall a \in \mathbf{No}, \quad a + r = \{a_L + r, a + r_L \mid a + r_R, a_R + r\}. \tag{4.1}$$

Let $\nu \in \mathbf{On}$ with $\nu > 0$ and set $\alpha := \omega^{\nu}$. We have the following uniform cut equations for L_α on $\mathbf{Mo}_\alpha$ and E_α on $\mathbf{No}_{>,\alpha}$ [6, Section 8.1]:

$$\begin{aligned} \forall \mathfrak{a} \in \mathbf{Mo}_\alpha, \quad L_\alpha \mathfrak{a} &= \{L_\alpha \mathfrak{a}_L^{\mathbf{Mo}_\alpha} \mid L_\alpha \mathfrak{a}_R^{\mathbf{Mo}_\alpha}, L_{<\alpha} \mathfrak{a}\}_{\mathbf{No}_{>,\alpha}} && (4.2)\\ &= \{\mathcal{L}_\alpha L_\alpha \mathfrak{a}_L^{\mathbf{Mo}_\alpha} \mid \mathcal{L}_\alpha L_\alpha \mathfrak{a}_R^{\mathbf{Mo}_\alpha}, L_{<\alpha} \mathfrak{a}\}. && (4.3)\\ \forall \varphi \in \mathbf{No}_{>,\alpha}, \quad E_\alpha^\varphi &= \left\{E_{X_\alpha} \mathfrak{d}_\alpha(\varphi), E_\alpha^{\varphi_L^{\mathbf{No}_{>,\alpha}}} \,\middle|\, E_\alpha^{\varphi_R^{\mathbf{No}_{>,\alpha}}}\right\}_{\mathbf{Mo}_\alpha} && (4.4)\\ &= \left\{E_{<\alpha} \varphi, \mathcal{E}_\alpha E_\alpha^{\varphi_L^{\mathbf{No}_{>,\alpha}}} \,\middle|\, \mathcal{E}_\alpha E_\alpha^{\varphi_R^{\mathbf{No}_{>,\alpha}}}\right\}. && (4.5) \end{aligned}$$

where

$$\mathrm{X}_\alpha = \begin{cases} \{0\} & \text{if } \nu \text{ is a limit} \\ \{\omega^\mu n : n \in \mathbb{N}\} & \text{if } \nu = \mu + 1 \text{ is a successor.} \end{cases}$$

4.3. Function groups

A *function group* $\mathcal{G}$ on a surreal substructure $\mathbf{S}$ is a set-sized group of strictly increasing bijections $\mathbf{S} \longrightarrow \mathbf{S}$ under functional composition. We see elements f, g of $\mathcal{G}$ as actions on $\mathbf{S}$ and sometimes write fg and fa for $a \in \mathbf{S}$ rather than $f \circ g$ and $f(a)$. We also write f^{inv} for the functional inverse of $f \in \mathcal{G}$.

Given such a function group $\mathcal{G}$, the collection of classes

$$\mathcal{G}[a] := \{b \in \mathbf{S} : \exists f, g \in \mathcal{G}, fa \leqslant b \leqslant ga\}$$

with $a \in \mathbf{S}$ forms a partition of $\mathbf{S}$ into convex subclasses. For subclasses $\mathbf{X} \subseteq \mathbf{S}$, we write $\mathcal{G}[\mathbf{X}] := \bigcup_{a \in \mathbf{X}} \mathcal{G}[a]$. An element $a \in \mathbf{S}$ is said to be *$\mathcal{G}$-simple* if it is the simplest element inside $\mathcal{G}[a]$. We write $\mathbf{Smp}_{\mathcal{G}}$ for the class of $\mathcal{G}$-simple elements. Given $a \in \mathbf{S}$, we also define $\pi_{\mathcal{G}}(a)$ to be the unique $\mathcal{G}$-simple element of $\mathcal{G}[a]$. The function $\pi_{\mathcal{G}}$ is a non-decreasing projection of $(\mathbf{S}, \leqslant)$ onto $(\mathbf{Smp}_{\mathcal{G}}, \leqslant)$. The main purpose of function groups is to define surreal substructures:

PROPOSITION 4.1. [5, Theorem 6.7 and Proposition 6.8] *The class $\mathbf{Smp}_{\mathcal{G}}$ is a surreal substructure. We have the uniform cut equation*

$$\forall z \in \mathbf{No}, \quad \Xi_{\mathbf{Smp}_{\mathcal{G}}} z = \{\mathcal{G}\Xi_{\mathbf{Smp}_{\mathcal{G}}} z_L \mid \mathcal{G}\Xi_{\mathbf{Smp}_{\mathcal{G}}} z_R\}_{\mathbf{S}}. \tag{4.6}$$

Note that for $a, b \in \mathbf{Smp}_{\mathcal{G}}$, we have $a < b$ if and only if $\mathcal{G}a < \mathcal{G}b$. We have the following criterion to identify the $\mathcal{G}$-simple elements inside $\mathbf{S}$.

PROPOSITION 4.2. [5, Lemma 6.5] *An element a of $\mathbf{S}$ is $\mathcal{G}$-simple if and only if there is a cut representation (L, R) of a in $\mathbf{S}$ with $\mathcal{G}L < a < \mathcal{G}R$. Equivalently, the number $a \in \mathbf{S}$ is $\mathcal{G}$-simple if and only if $\mathcal{G}a_L^{\mathbf{S}} < a < \mathcal{G}a_R^{\mathbf{S}}$.*

Given X, Y be sets of strictly increasing bijections $\mathbf{S} \longrightarrow \mathbf{S}$, we define

$$\begin{aligned} X \trianglelefteq Y &\overset{\text{def}}{\Longleftrightarrow} \forall a \in \mathbf{S}, \forall f \in X, \exists g \in Y, \ fa \leqslant ga \\ X \ntrianglelefteq Y &\overset{\text{def}}{\Longleftrightarrow} X \trianglelefteq Y \text{ and } Y \trianglelefteq X \\ X \leqslant Y &\overset{\text{def}}{\Longleftrightarrow} \forall a \in \mathbf{S}, \forall f \in X, \forall g \in Y, \ fa \leqslant ga \\ X < Y &\overset{\text{def}}{\Longleftrightarrow} \forall a \in \mathbf{S}, \forall f \in X, \forall g \in Y, \ fa < ga. \end{aligned}$$

If $X \trianglelefteq Y$, then we say that Y is *pointwise cofinal* with respect to X. For $f, g \in \mathbf{S}$, we also write $f < Y$ or $X < g$ instead of $\{f\} < Y$ and $X < \{g\}$.

Given a function group $\mathcal{G}$ on $\mathbf{S}$, we define a partial order $<$ on $\mathcal{G}$ by $f < g \Longleftrightarrow \{f\} < \{g\}$. We will frequently rely on the elementary fact that this ordering is compatible with the group structure in the sense that

$$\forall f, g, h \in \mathcal{G}, \quad g > \mathrm{id}_{\mathbf{S}} \Longleftrightarrow fgh > fh.$$

Given a set X of strictly increasing bijections $\mathbf{S} \longrightarrow \mathbf{S}$, we define $\langle X \rangle$ to be the smallest function group on $\mathbf{S}$ that is generated by X, i.e. $\langle X \rangle := \{f_1 \circ \cdots \circ f_n : n \in \mathbb{N}, f_1, \ldots, f_n \in X \cup X^{\mathrm{inv}}\}$.

4.4. Remarkable function groups

The examples of surreal substructures from the beginning of this section can all be obtained as classes $\mathbf{Smp}_{\mathcal{G}}$ of $\mathcal{G}$-simplest elements for suitable function groups $\mathcal{G}$ that act on $\mathbf{No}$, $\mathbf{No}^{>}$, or $\mathbf{No}^{>,\succ}$, as we will describe now. Given $c \in \mathbb{R}$ and $r \in \mathbb{R}^{>}$, we first define

$$\begin{array}{rcll} T_r & := & a \longmapsto a + c & \text{acting on } \mathbf{No} \text{ or } \mathbf{No}^{>,\succ} \\ H_c & := & a \longmapsto r\,a & \text{acting on } \mathbf{No}^{>} \text{ or } \mathbf{No}^{>,\succ} \\ P_c & := & a \longmapsto a^r & \text{acting on } \mathbf{No}^{>} \text{ or } \mathbf{No}^{>,\succ}. \end{array}$$

For $\alpha = \omega^{\nu} \in \mathbf{On}$, we then have the following function groups

$$\begin{array}{rcl} \mathcal{T} & := & \{T_c : c \in \mathbb{R}\} \\ \mathcal{H} & := & \{H_r : r \in \mathbb{R}^{>}\} \\ \mathcal{P} & := & \{P_r : r \in \mathbb{R}^{>}\} \\ \mathcal{E}_{\alpha} & := & \langle E_{\gamma} H_r L_{\gamma} : \gamma < \alpha, r \in \mathbb{R}^{>} \rangle \\ \mathcal{L}_{\alpha} & := & L_{\alpha}\, \mathcal{E}_{\alpha}\, E_{\alpha}. \end{array}$$

Now the action of $\mathcal{T}$ on $\mathbf{No}$ yields the surreal substructure $\mathbf{No}_{\succ} := \mathbf{Smp}_{\mathcal{T}}$ as class of $\mathcal{T}$-simplest elements. All examples from the beginning of this section can be obtained in a similar way:

- The action of $\mathcal{T}$ on $\mathbf{No}$ (resp. $\mathbf{No}^{>,\succ}$) yields $\mathbf{No}_{\succ}$ (resp. $\mathbf{No}_{\succ}^{>}$).
- The action of $\mathcal{H}$ on $\mathbf{No}^{>}$ (resp. $\mathbf{No}^{>,\succ}$) yields $\mathbf{Mo}$ (resp. $\mathbf{Mo}^{\succ}$).
- The action of $\mathcal{P}$ on $\mathbf{No}^{>,\succ}$ yields $\mathbf{Mo} \prec \mathbf{Mo} = E_1^{\mathbf{Mo}^{\succ}}$.
- The action of $\mathcal{E}_{\alpha}$ on $\mathbf{No}^{>,\succ}$ yields $\mathbf{Mo}_{\alpha}$.
- The action of $\mathfrak{L}_{\alpha}$ on $\mathbf{No}^{>,\succ}$ yields $\mathbf{No}_{\succ,\alpha}$.

We have

$$\begin{array}{rcl} \pi_{\mathbf{Smp}_{\mathcal{E}_{\alpha}}} & = & \mathfrak{d}_{\alpha} \\ \pi_{\mathbf{Smp}_{\mathcal{L}_{\alpha}}} & = & \sharp_{\alpha}. \end{array}$$

Let $E_{<\alpha} = \{E_{\gamma} : \gamma < \alpha\}$ and $L_{<\alpha} = \{L_{\gamma} : \gamma < \alpha\}$. We will need a few inequalities from [6]. The first one is immediate by definition and the fact that $\mathcal{H} < E_{\alpha}$. The others are [6, Lemma 6.9, Lemma 6.11, and Proposition 6.17], in that order:

$$\mathcal{E}_{\alpha\omega} < E_{\alpha} \tag{4.7}$$

$$E_{<\alpha} < E_{\alpha} H_2 L_{\alpha} \tag{4.8}$$

$$\langle E_{\gamma} : \gamma < \alpha \rangle \lneqq \mathcal{E}_{\alpha} \quad \text{if } \nu \text{ is a limit} \tag{4.9}$$

$$\forall \gamma < \rho < \alpha, \forall r, s > 1, \quad E_{\gamma} H_r L_{\gamma} < E_{\rho} H_s L_{\rho}. \tag{4.10}$$

From (4.10), we also deduce that

$$\{E_{\gamma} H_r L_{\gamma} : \gamma < \alpha, r \in \mathbb{R}\} \lneqq \mathcal{E}_{\alpha}. \tag{4.11}$$

5. WELL-NESTEDNESS

In this section, we prove Theorem 1.1, i.e. that each number is well-nested. In Section 5.1 we start with the definition and study of hyperserial expansions. We pursue with the study of paths and well-nestedness in Section 5.2.

The general idea behind our proof of Theorem 1.1 is as follows. Assume for contradiction that there exists a number a that is not well-nested and choose a simplest (i.e. $\sqsubseteq$-minimal) such number. By definition, a contains a so-called "bad path". For the ill-nested number a from (1.6), that would be the sequence

$$\mathrm{e}^{\sqrt{\log \omega}+\mathrm{e}^{\sqrt{\log\log \omega}+\mathrm{e}^{\cdot^{\cdot^{\cdot}}}+\log\log\log\omega}+\log\log\omega}, \mathrm{e}^{\sqrt{\log\log \omega}+\mathrm{e}^{\cdot^{\cdot^{\cdot}}}+\log\log\log\omega}, \mathrm{e}^{\cdot^{\cdot^{\cdot}}}, \ldots$$

From this sequence, we next construct a "simpler" number like

$$a' \;:=\; \sqrt{\omega}+\mathrm{e}^{\sqrt{\log \omega}+\mathrm{e}^{\sqrt{\log\log \omega}+\mathrm{e}^{\cdot^{\cdot^{\cdot}}}+\log\log\log\omega}}$$

that still contains a bad path

$$\mathrm{e}^{\sqrt{\log \omega}+\mathrm{e}^{\sqrt{\log\log \omega}+\mathrm{e}^{\cdot^{\cdot^{\cdot}}}+\log\log\log\omega}}, \mathrm{e}^{\sqrt{\log\log \omega}+\mathrm{e}^{\cdot^{\cdot^{\cdot}}}+\log\log\log\omega}, \mathrm{e}^{\cdot^{\cdot^{\cdot}}}, \ldots,$$

thereby contradicting the minimality assumption on a. In order to make this idea work, we first need a series of "deconstruction lemmas" that allow us to affirm that a' is indeed simpler than a; these lemmas will be listed in Section 5.3. We will also need a generalization $\lesssim$ of the relation $\blacktriangleleft$ that was used by Berarducci and Mantova to prove the well-nestedness of **No** as a field of transseries; this will be the subject of Section 5.4. We prove Theorem 1.1 in Section 5.5. Unfortunately, the relation $\lesssim$ does not have all the nice properties of $\blacktriangleleft$. For this reason, Sections 5.4 and 5.5 are quite technical.

5.1. Hyperserial expansions

Recall that any number can be written as a well-based series. In order to represent numbers as hyperseries, it therefore suffices to devise a means to represent the infinitely large monomials $\mathfrak{m}$ in $\mathbf{Mo}^{\succ}$. We do this by taking a hyperlogarithm $L_\alpha\,\mathfrak{m}$ of the monomial and then recursively applying the same procedure for the monomials in this new series. This procedure stops when we encounter a monomial in $L_{\mathbf{On}}\,\omega$.

Technically speaking, instead of directly applying a hyperlogarithm L_α to the monomial, it turns out to be necessary to first decompose $\mathfrak{m}$ as a product $\mathfrak{m}=\mathrm{e}^{\psi}\,\mathfrak{n}$ and write $\mathfrak{n}$ as a hyperexponential (or more generally as the hyperlogarithm of a hyperexponential). This naturally leads to the introduction of *hyperserial expansions* of monomials $\mathfrak{m}\in\mathbf{Mo}^{\neq 1}$, as we will detail now.

DEFINITION 5.1. *We say that a purely infinite number* $\varphi\in\mathbf{No}_{>}$ *is* **tail-atomic** *if* $\varphi=\psi +\!\!\!+\, \iota\,\mathfrak{a}$, *for certain* $\psi\in\mathbf{No}_{>}$, $\iota\in\{-1,1\}$, *and* $\mathfrak{a}\in\mathbf{Mo}_{\omega}$.

DEFINITION 5.2. *Let* $\mathfrak{m}\in\mathbf{Mo}^{\neq 1}$. *Assume that there are* $\psi\in\mathbf{No}_{>}$, $\iota\in\{-1,1\}$, $\alpha\in\{0\}\cup\omega^{\mathbf{On}}$, $\beta\in\mathbf{On}$ *and* $u\in\mathbf{No}$ *such that*

$$\mathfrak{m} \;=\; \mathrm{e}^{\psi}\,(L_\beta\,E_\alpha^u)^{\iota}, \tag{5.1}$$

with $\operatorname{supp}\psi \succ L_{\beta+1}\,E_\alpha^u$. *Then we say that* (5.1) *is a* **hyperserial expansion of type I** *if*

- $\beta\,\omega<\alpha$;
- $E_\alpha^u\in\mathbf{Mo}_\alpha\setminus L_{<\alpha}\,\mathbf{Mo}_{\alpha\omega}$;
- $\alpha=1 \Longrightarrow (\psi=0$ *and* u *is not tail-atomic*).

We say that (5.1) *is a* **hyperserial expansion of type II** *if* $\alpha=0$ *and* $u=\omega$*, so that* $E_\alpha^u=\omega$ *and*

$$\mathfrak{m} = \mathrm{e}^{\psi}(L_\beta\,\omega)^\iota. \tag{5.2}$$

Formally speaking, hyperserial expansions can be represented by tuples $(\psi,\iota,\alpha,\beta,u)$. By convention, we also consider

$$1 = \mathrm{e}^0(L_0\,E_0\,0)^0,$$

to be a hyperserial expansion of the monomial $\mathfrak{m}=1$; this expansion is represented by the tuple $(0,0,0,0,0)$.

Example 5.3. We will give a hyperserial expansion for the monomial

$$\mathfrak{m} = \exp(2\,E_\omega\,\omega-\sqrt{\omega}+L_{\omega+1}\,\omega),$$

and show how it can be expressed as a hyperseries. Note that

$$u := \log\mathfrak{m} = 2\,E_\omega\,\omega-\sqrt{\omega}+L_{\omega+1}\,\omega$$

is tail-atomic since $L_\omega\,\omega$ is log-atomic. Now $L_\omega\,\omega=L_\omega\,\omega$ is a hyperserial expansion of type II and we have $L_{\omega+1}\,\omega\prec E_\omega\,\omega$, $\sqrt{\omega}$. Hence $\mathfrak{m}=\mathrm{e}^{2E_\omega\omega-\sqrt{\omega}}(L_\omega\,\omega)$ is a hyperserial expansion.

Let $\psi:=2\,E_\omega\,\omega-\sqrt{\omega}$, so $\mathfrak{m}=\mathrm{e}^{\psi}(L_\omega\,\omega)$. We may further expand each monomial in $\operatorname{supp}\psi$. We clearly have $E_\omega\,\omega\in\mathbf{Mo}_{\omega^2}$. We claim that $E_\omega\,\omega\in\mathbf{Mo}_{\omega^2}\setminus L_{<\omega^2}\,\mathbf{Mo}_{\omega^3}$. Indeed, if we could write $E_\omega\,\omega=L_n\,L_{\omega m}\,\mathfrak{a}$ for some $\mathfrak{a}\in\mathbf{Mo}_{\omega^3}$ and $n,m\in\mathbb{N}^>$, then $\omega=L_\omega(L_n\,L_{\omega m}\,\mathfrak{a})=L_{\omega(m+1)}\,\mathfrak{a}-n$ and $L_{\omega(m+1)}\,\mathfrak{a}$ would both be monomials, which cannot be. Note that $E_\omega\,\omega=E_{\omega^2}(L_{\omega^2}\,E_\omega\,\omega)=E_{\omega^2}^{L_{\omega^2}\omega+1}$, so $E_\omega\,\omega=E_{\omega^2}^{L_{\omega^2}\omega+1}$ is a hyperserial expansion of type I. We also have $\sqrt{\omega}=\exp\left(\frac{1}{2}\log\omega\right)$ where $\frac{1}{2}\log\omega$ is tail expanded. Thus $\sqrt{\omega}=E_1^{\frac{1}{2}\log\omega}$ is a hyperserial expansion. Note finally that $\log\omega=L_1\,\omega$ is a hyperserial expansion. We thus have the following “recursive” expansion of $\mathfrak{m}$:

$$\mathfrak{m} = \mathrm{e}^{2E_{\omega^2}^{L_{\omega^2}\omega+1}-E_1^{\frac{1}{2}L_1\omega}}(L_\omega\,\omega). \tag{5.3}$$

LEMMA 5.4. *Any* $\mathfrak{m}\in\mathbf{Mo}$ *has a hyperserial expansion.*

Proof. We first prove the result for $\mathfrak{m}\in\mathbf{Mo}_\omega$, by induction with respect to the simplicity relation $\sqsubseteq$. The $\sqsubseteq$-minimal element of $\mathbf{Mo}_\omega$ is ω, which satisfies (5.2) for $\psi=\beta=0$ and $\iota=1$. Consider $\mathfrak{m}\in\mathbf{Mo}_\omega\setminus\{\omega\}$ such that the result holds on $\mathfrak{m}_\sqsubset^{\mathbf{Mo}_\omega}$. By [6, Proposition 6.20], the monomial $\mathfrak{m}$ is not $L_{<\mathbf{On}}$-atomic. So there is a maximal $\lambda\in\omega^{\mathbf{On}}$ with $\mathfrak{m}\in\mathbf{Mo}_\lambda$, and we have $\lambda\geqslant\omega$ by our hypothesis.

If there is no ordinal $\gamma<\lambda$ such that $E_\gamma^{\mathfrak{m}}\in\mathbf{Mo}_{\lambda\omega}$, then we have $\mathfrak{m}\in\mathbf{Mo}_\lambda\setminus L_{<\lambda}\,\mathbf{Mo}_{\lambda\omega}$. So setting $\alpha=\lambda$, $\beta=0$ and $u=L_\lambda\,\mathfrak{m}$, we are done. Otherwise, let $\gamma<\lambda$ be such that $\mathfrak{a}:=E_\gamma^{\mathfrak{m}}\in\mathbf{Mo}_{\lambda\omega}$. We cannot have $\gamma=0$ by definition of λ. So there is a unique ordinal η and a unique natural number $n\in\mathbb{N}^>$ such that $\gamma=\gamma'+\omega^\eta n$ and $\gamma'\gg\omega^\eta$. Note that $\lambda\geqslant\omega^{\eta+1}$. We must have $\lambda=\omega^{\eta+1}$: otherwise, $L_{\omega^{\eta+1}}\,\mathfrak{m}=L_{\gamma'+\omega^{\eta+1}}(\mathfrak{a})+n$ where $L_{\omega^{\eta+1}}\,\mathfrak{m}$ and $L_{\gamma'+\omega^{\eta+1}}\,\mathfrak{a}$ are monomials. We deduce that $\gamma'=0$ and $\gamma=\omega^\eta n$. Note that $L_\lambda\,\mathfrak{a}\asymp L_\lambda\,\mathfrak{m}$, $\lambda<\lambda\,\omega$, and $\mathfrak{a}\in\mathbf{Mo}_{\lambda\omega}$, so $\mathfrak{a}=\mathfrak{d}_{\lambda\omega}(\mathfrak{m})$. We deduce that $\mathfrak{a}\sqsubset\mathfrak{m}$. The induction hypothesis yields a hyperserial expansion $\mathfrak{a}=\mathrm{e}^{\psi}(L_\beta\,E_\alpha^u)^\iota$. Since $\mathfrak{a}$ is log-atomic, we must have $\psi=0$ and $\iota=1$. If $\mathfrak{a}=L_\beta\,\omega$, then $\beta\gtrapprox\lambda_{/\omega}=\omega^\eta$, since $\mathfrak{a}\in\mathbf{Mo}_{\lambda\omega}$. Thus $\mathfrak{m}=L_\gamma\,\mathfrak{a}=L_{\beta+\gamma}\,\omega$ is a hyperexponential expansion of type II. If $\mathfrak{a}=L_\beta\,E_\alpha^u$, then likewise $\beta\gtrapprox\omega^\eta$ and thus $\mathfrak{m}=L_{\beta+\gamma}\,E_\alpha^u$ is a hyperexponential expansion of type I. This completes the inductive proof.

Now let $\mathfrak{m}\in\mathbf{Mo}^{\neq}\setminus\mathbf{Mo}_{\omega}$ and set $\varphi:=\log\mathfrak{m}$. If φ is tail-atomic, then there are $\psi\in\mathbf{No}_{>}$, $\iota\in\{-1,1\}$ and $\mathfrak{a}\in\mathbf{Mo}_{\omega}$ with $\varphi=\psi+\!\!\!+\,\iota\,\mathfrak{a}$. Applying the previous arguments to $\mathfrak{a}$, we obtain elements $\alpha\geqslant\omega,\beta,u$ with $\mathfrak{a}=L_{\beta}E_{\alpha}^{u}$ and $\beta\,\omega<\alpha$, or an ordinal β with $\mathfrak{a}=L_{\beta}\,\omega$. Then $\mathfrak{m}=\mathrm{e}^{\psi}(L_{\beta}E_{\alpha}^{u})^{\iota}$ or $\mathfrak{m}=\mathrm{e}^{\psi}(L_{\beta}\,\omega)^{\iota}$ is a hyperserial expansion. If φ is not tail-atomic, then we have $\mathfrak{m}=E_{1}^{\varphi}$ is a hyperserial expansion of type I. □

LEMMA 5.5. *Consider a hyperserial expansion* $\mathfrak{a}=L_{\beta}E_{\alpha}^{u}$*. Let* $\mu>0$ *and define* $\mu_{-}:=\mu-1$ *if* μ *is a successor ordinal and* $\mu_{-}:=\mu$ *if* μ *is a limit ordinal. Let*

$$\begin{aligned}\beta &=: \beta'+\beta'' \quad \textit{where}\\ \beta' &:= \beta_{\succcurlyeq\omega^{\mu_{-}}} \geqq \omega^{\mu_{-}} \quad \textit{and}\\ \beta'' &:= \beta_{<\omega^{\mu_{-}}} < \omega^{\mu_{-}}.\end{aligned}$$

a) *Then* $\mathfrak{a}$ *is* $L_{<\omega^{\mu}}$*-atomic if and only if* $\beta''=0$ *and either* $\alpha\geqslant\omega^{\mu}$ *or* $\alpha=0$*.*

b) *If* $\alpha\geqslant\omega^{\mu}$*, then* $\mathfrak{d}_{\omega^{\mu}}(\mathfrak{a})=L_{\beta'}E_{\alpha}^{u}$*.*

Proof. We first prove *a*). Assume that $\mathfrak{a}$ is $L_{<\omega^{\mu}}$-atomic. Assume for contradiction that $\beta''\neq0$ and let $\omega^{\eta}m$ denote the least non-zero term in the Cantor normal form of β''. Since $\beta''<\omega^{\mu_{-}}$, we have $\omega^{\eta+1}<\omega^{\mu}$ so $L_{\omega^{\eta+1}}\mathfrak{a}$ is a monomial. But $L_{\omega^{\eta+1}}\mathfrak{a}=L_{\beta''_{>\omega^{\eta}}}E_{\alpha}^{u}-m$ where $L_{\beta''_{>\omega^{\eta}}}E_{\alpha}^{u}$ is a monomial: a contradiction. So $\beta''=0$. If $\alpha=0$ then we are done. Otherwise $E_{\alpha}^{u}\notin\mathbf{Mo}_{\alpha\omega}$, so we must have $\alpha\,\omega>\omega^{\mu}$, whence $\alpha\geqslant\omega^{\mu}$. Conversely, assume that $\alpha\geqslant\omega^{\mu}$ or $\alpha=0$, and that $\beta''=0$. If $\alpha\neq0$, then then for all $\gamma<\omega^{\mu}$, we have $L_{\gamma}\,\mathfrak{a}=L_{\beta'+\gamma}E_{\alpha}^{u}$ where $\beta'+\gamma<\alpha$, so $L_{\gamma}\,\mathfrak{a}$ is a monomial, whence $\mathfrak{a}\in\mathbf{Mo}_{\omega^{\mu}}$. If $\alpha=0$, then for all $\gamma<\omega^{\mu}$, we have $L_{\gamma}\,\mathfrak{a}=L_{\beta'+\gamma}\,\omega\in\mathbf{Mo}$, whence $\mathfrak{a}\in\mathbf{Mo}_{\omega^{\mu}}$. This proves *a*).

Now assume that $\alpha\geqslant\omega^{\mu}$. So $L_{\beta'}E_{\alpha}^{u}$ is $L_{<\omega^{\mu}}$-atomic by *a*). If $\beta''=0$ then we conclude that $\mathfrak{a}=L_{\beta'}E_{\alpha}^{u}=\mathfrak{d}_{\omega^{\mu}}(\mathfrak{a})$. If $\beta''\neq0$, then let $\omega^{\eta}m$ denote the least non-zero term in its Cantor normal form. We have $\omega^{\eta+1}<\omega^{\mu}$ and $L_{\omega^{\eta+1}}\mathfrak{a}=L_{\omega^{\eta+1}}L_{\beta'}E_{\alpha}^{u}-m\asymp L_{\omega^{\eta+1}}L_{\beta'}E_{\alpha}^{u}$, so $L_{\beta'}E_{\alpha}^{u}=\mathfrak{d}_{\omega^{\mu}}(\mathfrak{a})$. □

COROLLARY 5.6. *Let* $\mu\in\mathbf{On}^{>}$*,* $\alpha:=\omega^{\mu}$*,* $\gamma<\alpha$*, and* $\mathfrak{b}\in\mathbf{Mo}_{\alpha\omega}$*. If* $L_{\gamma}\,\mathfrak{b}\in\mathbf{Mo}_{\alpha}\setminus\mathbf{Mo}_{\alpha\omega}$*, then* μ *is a successor ordinal and* $\gamma=\alpha_{/\omega}n$ *for some* $n\in\mathbb{N}^{>}$*.*

Proof. Since $L_{\gamma}\,\mathfrak{b}\in\mathbf{Mo}_{\alpha}\setminus\mathbf{Mo}_{\alpha\omega}$, we must have $\gamma\neq0$. By Lemma 5.4, we have a hyperserial expansion $\mathfrak{b}=\mathrm{e}^{\psi}(L_{\beta}E_{\rho}^{u})^{\iota}$. Since $\mathfrak{b}$ is log-atomic, we have $\log\mathfrak{b}=\psi+\!\!\!+\,\iota L_{\beta+1}E_{\alpha}^{u}\in\mathbf{Mo}$, whence $\psi=0$ and $\iota=1$. So $\mathfrak{b}=L_{\beta}E_{\rho}^{u}$. We have $\mathfrak{b}\in\mathbf{Mo}_{\alpha\omega}$ so by Lemma 5.5(a), we have $\beta\geqq\alpha$. It follows that $L_{\gamma}\,\mathfrak{b}=L_{\beta+\gamma}E_{\rho}^{u}$ is a hyperserial expansion. But then $L_{\beta+\gamma}E_{\rho}^{u}\in\mathbf{Mo}_{\alpha}$ and Lemma 5.5(*a*) imply that $\gamma\geqq\omega^{\mu-}$. The condition that $\gamma<\alpha$ now gives $\mu_{-}<\mu$, whence μ is a successor and $\gamma=\omega^{\mu-}n$ for a certain $n\in\mathbb{N}^{>}$, as claimed. □

LEMMA 5.7. *Any* $\mathfrak{m}\in\mathbf{Mo}$ *has a unique hyperserial expansion* (*that we will call* the *hyperserial expansion, henceforth*)*.*

Proof. Consider a monomial $\mathfrak{m}\neq1$ with

$$\mathfrak{m} \;=\; \mathrm{e}^{\psi}(L_{\beta}\,\mathfrak{a})^{\iota},$$

where $\psi\in\mathbf{No}_{>}$, $\iota\in\{-1,1\}$, $\beta,\alpha\in\omega^{\mathbf{On}}$, $\mathfrak{a}\in\mathbf{Mo}_{\alpha}$, $\beta\,\omega<\alpha$, and $\operatorname{supp}\psi>L_{\beta+1}\,\mathfrak{a}$. Assume for contradiction that we can write $\mathfrak{m}=\mathrm{e}^{\psi'}(L_{\beta'}E_{\alpha'}u')^{\iota'}$ as a hyperserial expansion of type I with $\alpha'<\alpha$. Note in particular that $\alpha>1$, so $L_{\beta+1}\,\mathfrak{a}$ is log-atomic. We have

$$\log\mathfrak{m} \;=\; \psi+\!\!\!+\,\iota L_{\beta+1}\,\mathfrak{a} \;=\; \psi'+\!\!\!+\,\iota' L_{\beta'+1}E_{\alpha'}^{u'}.$$

If $\alpha'=1$, then $\beta'=0$, $\psi'=0$, $\iota'=1$, and u' is not tail-atomic. But $\psi + \iota L_{\beta+1}\,\mathfrak{a} = u'$, where $L_{\beta+1}\,\mathfrak{a}\in\mathbf{Mo}_{\omega}$, so u' is tail-atomic: a contradiction. Hence $\alpha'>1$. Note that $\iota L_{\beta+1}\,\mathfrak{a}$ and $\iota' L_{\beta'+1}\,E_{\alpha'}^{u'}$ are both the least term of $\log \mathfrak{m}$. It follows that $\psi=\psi'$, $\iota=\iota'$, and

$$L_{\beta}\,\mathfrak{a} \;=\; L_{\beta'}\,E_{\alpha'}^{u'}. \tag{5.4}$$

Since $\beta'\omega<\alpha'$, we have

$$E_{\alpha'}^{u'} \;=\; \mathfrak{d}_{\alpha'}\big(L_{\beta'}\,E_{\alpha'}^{u'}\big) \;=\; \mathfrak{d}_{\alpha'}(L_{\beta}\,\mathfrak{a}).$$

Now $E_{\alpha'}^{u'}\notin\mathbf{Mo}_{\alpha'\omega}$, so $\mathfrak{d}_{\alpha'}(L_{\beta}\,\mathfrak{a})\neq\mathfrak{a}$ and thus $\beta\omega\geqslant\alpha'$. In particular $\beta>\beta'$. Taking hyperexponentials on both sides of (5.4), we may assume without loss of generality that $\beta'=0$ or that the least exponents η and η' in the Cantor normal forms of β resp. β' differ. If $\beta'=0$, then we decompose $b=\gamma\dotplus\omega^{\eta}n$ where $n\in\mathbb{N}^{>}$ and $\gamma\gg\omega^{\eta}$. Since $L_{\beta}\,\mathfrak{a}=E_{\alpha'}^{u'}\in\mathbf{Mo}_{\alpha'}\setminus\mathbf{Mo}_{\alpha'\omega}$, applying Lemma 5.5($a$) twice (for $\omega^{\mu}=\alpha'$ and $\omega^{\mu}=\alpha'\omega$) gives $\omega^{\eta+1}\geqslant\alpha'$ and $\omega^{\eta+1}\ngeqslant\alpha'\omega$, whence $\alpha'=\omega^{\eta+1}$. But then $E_{\alpha'}^{u'}=L_{\omega^{\eta}n}\,\mathfrak{b}$, where $\mathfrak{b}:=L_{\gamma}\,\mathfrak{a}\in\mathbf{Mo}_{\alpha'}\,\omega$ by Lemma 5.5(a). So $E_{\alpha'}^{u'}\in L_{<\alpha'}\,\mathbf{Mo}_{\alpha'\omega}$: a contradiction. Assume now that $\beta'\neq0$. Lemma 5.5(a) yields both $L_{\beta}\,\mathfrak{a}\in\mathbf{Mo}_{\omega^{\eta+1}}\setminus\mathbf{Mo}_{\omega^{\eta+2}}$ and $L_{\beta'}\,E_{\alpha'}^{u'}\in\mathbf{Mo}_{\omega^{\eta'+1}}\setminus\mathbf{Mo}_{\omega^{\eta'+2}}$, which contradicts (5.4).

Taking $\mathfrak{a}=\omega\in\mathbf{No}$ and $\alpha:=\max(\alpha'\omega,\beta\omega^2)$, this proves that no two hyperserial expansions of distinct types I and II can be equal. Taking $\mathfrak{a}=E_{\alpha}^{u}$ with $\alpha>\alpha'$, this proves that no two hyperserial expansions $\mathrm{e}^{\psi}(L_{\beta}\,E_{\alpha}^{u})^{\iota}, \mathrm{e}^{\psi'}\big(L_{\beta'}\,E_{\alpha'}^{u'}\big)^{\iota'}$ of type I with $\alpha\neq\alpha'$ can be equal.

The two remaining cases are hyperserial expansions of type II and hyperserial expansions $\mathrm{e}^{\psi}(L_{\beta}\,E_{\alpha}^{u})^{\iota}$ and $\mathrm{e}^{\psi'}\big(L_{\beta'}\,E_{\alpha'}^{u'}\big)^{\iota'}$ of type I with $\alpha=\alpha'$. Consider a monomial $\mathfrak{m}\in\mathbf{Mo}^{\neq}$ with the hyperserial expansions $\mathfrak{m}=\mathrm{e}^{\psi}(L_{\gamma}\,\omega)^{\iota}=\mathrm{e}^{\psi'}(L_{\gamma'}\,\omega)^{\iota'}$ of type II. As above we have $\psi=\psi'$, $\iota=\iota'$, and $L_{\gamma}\,\omega=L_{\gamma'}\,\omega$. We deduce that $\gamma=\gamma'$, so the expansions coincide.

Finally, consider a monomial $\mathfrak{m}\neq1$ with two hyperserial expansions of type I

$$\mathfrak{m} \;=\; \mathrm{e}^{\psi}(L_{\beta}\,E_{\alpha}^{u})^{\iota} \;=\; \mathrm{e}^{\psi'}(L_{\beta'}\,E_{\alpha}^{u'})^{\iota'}. \tag{5.5}$$

If $\alpha=1$, then we have $\psi=\psi'=0$ and $\beta=\beta'=0$ and $\iota=\iota'=1$, whence $u=u'$, so we are done.

Assume now that $\alpha>1$. Taking logarithms in (5.5), we see that $\psi=\psi'$, $\iota=\iota'$, and

$$L_{\beta}\,E_{\alpha}^{u} \;=\; L_{\beta'}\,E_{\alpha}^{u'}. \tag{5.6}$$

We may assume without loss of generality that $\beta\geqslant\beta'$. Assume for contradiction that $\beta>\beta'$. Taking hyperexponentials on both sides of (5.6), we may assume without loss of generality that $\beta'=0$ or that the least exponents η and η' in the Cantor normal forms of β resp. β' differ. On the one hand, Lemma 5.5(a) yields $L_{\beta}\,E_{\alpha}^{u}\in\mathbf{Mo}_{\omega^{\eta+1}}\setminus\mathbf{Mo}_{\omega^{\eta+2}}$. Note in particular that $L_{\beta}\,E_{\alpha}^{u}\notin\mathbf{Mo}_{\alpha}$, since $\beta\omega<\alpha$. On the other hand, if $\beta\neq0$, then Lemma 5.5(a) yields $L_{\beta'}\,E_{\alpha}^{u'}\in\mathbf{Mo}_{\omega^{\eta'}}\setminus\mathbf{Mo}_{\omega^{\eta'+1}}$; if $\beta'=0$, then $L_{\beta'}\,E_{\alpha}^{u'}\in\mathbf{Mo}_{\alpha}$. Thus (5.6) is absurd: a contradiction. We conclude that $\beta=\beta'$. Finally $E_{\alpha}^{u}=E_{\alpha}^{u'}$ yields $u=u'$, so the expansions are identical. □

LEMMA 5.8. *If $\mathfrak{m}=\mathrm{e}^{\psi}(L_{\beta}\,E_{\alpha}^{u})^{\iota}$ is a hyperserial expansion of type I, then we have*

$$\operatorname{supp}\psi\cap\operatorname{supp}u \;=\; \emptyset.$$

Proof. Assume for contradiction that $\mathfrak{n}\in\operatorname{supp}\psi\cap\operatorname{supp}u$. In particular $\mathfrak{n}>L_{\beta+1}\,E_{\alpha}^{u}$. Since $u>0$, there is $r\in\mathbb{R}^{>}$ with $u\geqslant r\,\mathfrak{n}$, so $L_{\beta+1}\,E_{\alpha}^{u}\succ\mathfrak{n}$: a contradiction. □

5.2. Paths and subpaths

Let λ be an ordinal with $0<\lambda\leqslant\omega$ and note that $i<1\dotplus\lambda\Longleftrightarrow(i\leqslant\lambda<\omega\vee i<\omega=\lambda)$ for all $i\in\mathbb{N}$. Consider a sequence

$$P = (P(i))_{i<\lambda} = (\tau_{P,i})_{i<\lambda} = (r_{P,i}\mathfrak{m}_{P,i})_{i<\lambda} \quad\text{in}\quad (\mathbb{R}^{\neq}\mathbf{Mo})^{\lambda}.$$

We say that P is a *path* if there exist sequences $(u_{P,i})_{i<1\dotplus\lambda}$, $(\psi_{P,i})_{i<1\dotplus\lambda}$, $(\iota_{P,i})_{i<\lambda}$, $(\alpha_{P,i})_{i<\lambda}$, and $(\beta_{P,i})_{i<1\dotplus\lambda}$ with

- $u_{P,0}=\tau_{P,0}$ and $\psi_{P,0}=0$;
- $\tau_{P,i}\in\operatorname{term}\psi_{P,i}$ or $\tau_{P,i}\in\operatorname{term}u_{P,i}$ for all $i<\lambda$;
- $\tau_{P,i}\in\mathbb{R}^{\neq}\cup\{\omega\}\Longrightarrow\lambda=i+1$ for all $i<\lambda$;
- For $i<\lambda$, the hyperserial expansion of $\mathfrak{m}_{P,i}$ is

$$\mathfrak{m}_{P,i} = \mathrm{e}^{\psi_{P,i+1}}(L_{\beta_{P,i}}E_{\alpha_{P,i}}^{u_{P,i+1}})^{\iota_{P,i}}.$$

We call λ the *length* of P and we write $|P|:=\lambda$. We say that P is *infinite* if $|P|=\omega$ and *finite* otherwise. We set $a_{P,0}:=a$. For $0<i<|P|$, we define

$$(s_{P,i},a_{P,i}) := \begin{cases}(-1,\psi_{P,i}) & \text{if } \mathfrak{m}_{P,i}\in\operatorname{supp}\psi_{P,i}\\ (1,u_{P,i}) & \text{if } \mathfrak{m}_{P,i}\in\operatorname{supp}u_{P,i}.\end{cases}$$

By Lemma 5.8, those cases are mutually exclusive so $(s_{P,i},a_{P,i})$ is well-defined. For $a\in\mathbf{No}$, we say that P is a path in a if $P(0)\in\operatorname{term}a$.

For $k\leqslant|P|$, we let $P_{\nearrow k}$ denote the path of length $|P|-k$ in $a_{P,k}$ with

$$\forall i < |P|-k, \tau_{P_{\nearrow k},i} := \tau_{P,k+i}.$$

Example 5.9. Let us find all the paths in the monomial $\mathfrak{m}$ of Example 5.3. We have a representation (5.3) of $\mathfrak{m}$ as a hyperseries

$$\mathfrak{m} = \mathrm{e}^{2E_{\omega^2}^{L_{\omega^2}\omega+1}-E_1^{\frac{1}{2}L_1\omega}}(L_\omega\,\omega)$$

which by Lemma 5.7 is unique. There are nine paths in $\mathfrak{m}$, namely

- one path $(\mathfrak{m})$ of length 1;
- three paths $(\mathfrak{m},2E_{\omega^2}^{L_{\omega^2}\omega+1})$, $\left(\mathfrak{m},-E_1^{\frac{1}{2}L_1\omega}\right)$, and $(\mathfrak{m},\omega)$ of length 2;
- three paths $(\mathfrak{m},2E_{\omega^2}^{L_{\omega^2}\omega+1},L_{\omega^2}\,\omega)$, $(\mathfrak{m},2E_{\omega^2}^{L_{\omega^2}\omega+1},1)$ and $\left(\mathfrak{m},-E_1^{\frac{1}{2}L_1\omega},\frac{1}{2}L_1\,\omega\right)$ of length 3;
- two paths $(\mathfrak{m},2E_{\omega^2}^{L_{\omega^2}\omega+1},L_{\omega^2}\,\omega,\omega)$ and $\left(\mathfrak{m},-E_1^{\frac{1}{2}L_1\omega},\frac{1}{2}L_1\,\omega,\omega\right)$ of length 4.

Note that the paths which cannot be extended into strictly longer paths are those whose last value is a real number or ω.

Infinite paths occur in so-called nested numbers that will be studied in more detail in Section 6.

DEFINITION 5.10. *Let $a\in\mathbf{No}$ and let P be a path in a. We say that an index $i<|P|$ is* **bad** *for (P,a) if one of the following conditions is satisfied*

1. *$\mathfrak{m}_{P,i}$ is not the $\preccurlyeq$-minimum of $\operatorname{supp}u_{P,i}$;*
2. *$\mathfrak{m}_{P,i}=\min\operatorname{supp}u_{P,i}$ and $\beta_{P,i}\neq0$;*

3. $\mathfrak{m}_{P,i} = \min \operatorname{supp} u_{P,i}$ *and* $\beta_{P,i} = 0$ *and* $r_{P,i} \notin \{-1, 1\}$;
4. $\mathfrak{m}_{P,i} = \min \operatorname{supp} u_{P,i}$ *and* $\beta_{P,i} = 0$ *and* $r_{P,i} \in \{-1, 1\}$ *and* $\mathfrak{m}_{P,i} \in \operatorname{supp} \psi_{P,i}$.

The index i is **good** *for (P, a) if it is not bad for (P, a).*

If P is infinite, then we say that it is **good** *if $(P, \tau_{P,0})$ is good for all but a finite number of indices. In the opposite case, we say that P is a* **bad** *path. An element $a \in \mathbf{No}$ is said to be* **well-nested** *every path in a is good.*

Remark 5.11. The above definition extends the former definitions of paths in [30, 38, 10]. More precisely, a path P with with $\alpha_{P,i} = 1$ (whence $\psi_{P,i} = 0$) for all $i < |P|$, corresponds to a path for these former definitions. The validity of the axiom **T4** for **No** means that those paths are good. With Theorem 1.1, we will extend this result to all paths.

LEMMA 5.12. *For $\mathfrak{m} \in (L_{\mathbf{On}}\, \omega)^{\pm 1}$ and for any path P in $\mathfrak{m}$, we have $|P| \leqslant 2$. For $a \in \mathbb{L} \circ \omega$ and for any path P in a, we have $|P| \leqslant 3$.*

Proof. Let $\mathfrak{l} \in \mathfrak{L} \setminus \{-1\}$ and let P be a path in $\mathfrak{l} \circ \omega$. If there is an ordinal γ with $\mathfrak{l} = \ell_\gamma$, then the hyperserial expansion of $\mathfrak{l} \circ \omega$ is $L_\gamma\, \omega$, so $|P| = 1$ if $\gamma = 0$ and $|P| = 2$ otherwise. If there is an ordinal γ with $\mathfrak{l} = \ell_\gamma^{-1}$, then the hyperserial expansion of $\mathfrak{l} \circ \omega$ is $(L_\gamma\, \omega)^{-1}$ and $|P| = 2$.

Assume now that $\mathfrak{l} \notin \ell_{\mathbf{On}}^{\pm 1}$. If $\log \mathfrak{l} \circ \omega$ is not tail-atomic, then hyperserial expansion of $\mathfrak{l} \circ \omega$ is $\mathfrak{l} \circ \omega = \mathrm{e}^{\log \mathfrak{l} \circ \omega}$. If $\log \mathfrak{l} \circ \omega$ is tail-atomic, then the hyperserial expansion of $\mathfrak{l} \circ \omega$ is $\mathfrak{l} \circ \omega = \mathrm{e}^{\psi \circ \omega} (\mathfrak{a} \circ \omega)^\iota$ for a certain log-atomic $\mathfrak{a} \in \mathbb{L}$. In both cases, $P_{\nearrow 1}$ is a path in some monomial in $L_{\mathbf{On}}\, \omega$, whence $|P_{\nearrow 1}| \leqslant 2$ and $|P| \leqslant 3$, by the previous argument. □

DEFINITION 5.13. *Let P, Q be paths. We say that Q is a* **subpath of** *P, or equivalently that P* **extends** *Q, if there exists a $k < |P|$ with $Q = P_{\nearrow k}$. For $a \in \mathbf{No}$, we say that Q is a* **subpath in** *a if there is a path P in a such that Q is a subpath of P. We say that P* **shares a subpath with** *a if there is a subpath of P which is a subpath in a.*

Let P be a finite path and let Q be a path with $Q(0) \in \operatorname{supp} u_{P,|P|} \cup \operatorname{supp} \psi_{P,|P|}$. Then we define $P * Q$ to be the path $(P(0), \ldots, P(|P|), Q(0), \ldots)$ of length $|P| + |Q|$.

LEMMA 5.14. *Let $\lambda \in \omega^{\mathbf{On}}$ and $\mathfrak{m} \in \mathbf{Mo}_\lambda$. Let P be a path in $\mathfrak{m}$ with $|P| > 1$. Then $P_{\nearrow 1}$ is a subpath in $L_\lambda\, \mathfrak{m}$.*

Proof. By Lemma 5.12, we have $\mathfrak{m} \notin (L_{\mathbf{On}}\, \omega)^{\pm 1}$. If $\mathfrak{m}$ has a hyperserial expansion of the form $\mathfrak{m} = \mathrm{e}^\psi (L_\gamma\, \omega)^\iota$, then $P_{\nearrow 1}$ must be a path in ψ. So ψ is non-zero and thus $\lambda = 1$. It follows that $P_{\nearrow 1}$ is a path in $\log \mathfrak{m} = \psi + \iota\, (L_\gamma\, \omega)^\iota$. Otherwise, let $\mathfrak{m} = \mathrm{e}^\psi (L_\beta\, E_\alpha^u)^\iota$ be the hyperserial expansion of $\mathfrak{m}$. If $P_{\nearrow 1}$ is a path in ψ, then it is a path in $\log \mathfrak{m}$ as above. Otherwise, it is a path in u. Assume that $\lambda = 1$. If $\alpha = 1$, then we have $\psi = 0$ and $\log \mathfrak{m} = \iota\, u$ so $P_{\nearrow 1}$ is a path in $\log \mathfrak{m}$. If $\alpha > 1$, then $\log \mathfrak{m} = \psi + \iota\, L_{\beta+1}\, E_\alpha^u$ where $L_{\beta+1}\, E_\alpha^u$ is a hyperserial expansion, so $P_{\nearrow 1}$ is a path in $\log \mathfrak{m}$. Assume now that $\lambda > 1$, so $\psi = 0$, $\iota = 1$, and $\alpha \geqslant \omega$. We must have $\beta \geqq \lambda_{/\omega}$ so there are $\beta' \in \mathbf{On}$ and $n \in \mathbb{N}$ with $\beta' \gg \lambda_{/\omega}$ and $\beta = \beta' + \lambda_{/\omega} n$. We have $L_\lambda\, \mathfrak{m} = L_{\beta' + \lambda}\, E_\alpha^u - n$ where $L_{\beta' + \lambda}\, E_\alpha^u$ is a hyperserial expansion, so $P_{\nearrow 1}$ is a path in $L_\lambda\, \mathfrak{m}$. □

LEMMA 5.15. *Let $a \in \mathbf{No}^{>,\succ}$, $\alpha \in \omega^{\mathbf{On}}$ and $k \in \mathbb{N}^{>}$. If P is a path in $\sharp_\alpha(a)$ with $|P| > 2$, then $P_{\nearrow 1}$ is a subpath in $\mathfrak{d}_{E_{\alpha k} a}$.*

Proof. We prove this by induction on $\alpha\, k$, for any number $a \in \mathbf{No}^{>,\succ}$. We consider $a \in \mathbf{No}^{>,\succ}$, and a fixed path P in $\sharp_\alpha(a)$ with $|P| > 1$.

Assume that $\alpha=k=1$. We have $\sharp_1(a)=a_{\succ}$ and $\mathfrak{d}_{\exp a}=\mathrm{e}^{a_{\succ}}$. Assume that $a_{\succ}=\psi+\!\!\!+\,\iota\,\mathfrak{a}$ for certain $\psi\in\mathbf{No}_{\succ}$, $\iota\in\{-1,1\}$, and $\mathfrak{a}\in\mathbf{Mo}_{\omega}$. Let $\mathfrak{a}=L_{\gamma}E_{\lambda}^{u}$ be the hyperserial expansion of $\mathfrak{a}$. If $\lambda=\omega$, then $\gamma=0$ and the hyperserial expansion of $\mathrm{e}^{\mathfrak{a}}$ is $\mathrm{e}^{\mathfrak{a}}=E_{\omega}^{u+1}$. Therefore $P_{\nearrow 1}$ is a subpath in $\mathfrak{d}_{\exp a}=\mathrm{e}^{\psi}(E_{\omega}^{u+1})^{\iota}$. If $\lambda>\omega$, then the hyperserial expansion of $\mathrm{e}^{\mathfrak{a}}$ is $\mathrm{e}^{\mathfrak{a}}=L_{\gamma+1}E_{\lambda}^{u}$. Therefore $P_{\nearrow 1}$ is a subpath in $\mathfrak{d}_{\exp a}=\mathrm{e}^{\psi}(L_{\gamma+1}E_{\lambda}^{u})^{\iota}$. Finally, if $\mathrm{e}^{a_{\succ}}$ is not tail-atomic, then $P_{\nearrow 1}$ is a subpath in $\mathfrak{d}_{\exp a}=(E_1^{\epsilon a_{\succ}})^{\epsilon}$, where $\epsilon\in\{-1,1\}$ is the sign of $a_{\succ}$.

Now assume that $\alpha=1$, $k>1$, and that the result holds strictly below k. We have $E_k\,a=E_{k-1}(\exp a)$ where $P_{\nearrow 1}$ is a subpath in $\mathfrak{d}_{\exp a}$ by the previous argument. We have $r\,\mathfrak{d}_{\exp a}\trianglelefteq\sharp_1(\exp a)$ for a certain $r\in\mathbb{R}^{\neq}$, so $Q:=(r\,\mathfrak{d}_{\exp a})*P_{\nearrow 1}$ is a path in $\sharp_1(\exp a)$. The induction hypothesis on $k-1$ implies that $Q_{\nearrow 1}=P_{\nearrow 1}$ is a subpath in $\mathfrak{d}_{E_k a}$.

Assume now that $\alpha\geqslant\omega$ and that the result holds strictly below α. Write $v:=\sharp_{\alpha}(a)$. By (3.7), there exist $\eta\in\mathbf{On}$, $n<\omega$, and $\delta\in\mathbf{No}$ with $\beta:=\omega^{\eta}<\alpha$ and

$$E_{\alpha}\,a \;=\; E_{\beta n}(L_{\beta n}\,E_{\alpha}^{v}+\!\!\!+\,\delta).$$

Assume for contradiction that there is a $\gamma\in\mathbf{On}$ with $E_{\alpha}^{v}=L_{\gamma}\,\omega$. We must have $\gamma\gtrsim\alpha_{/\omega}$, so there are a number $n\in\mathbb{N}$ and an ordinal $\gamma'\gtrsim\alpha$ with $\gamma=\gamma'+\alpha_{/\omega}\,n$. We have $v=L_{\gamma'+\alpha}\,\omega-n$. By Lemma 5.12, this contradicts the fact that $|P|>2$. So by Lemma 5.4, there exist $\beta\in\omega^{\mathbf{On}}$ and $\gamma\in\mathbf{On}$ with $\beta\geqslant\alpha$, $\gamma\,\omega<\beta$, $E_{\alpha}^{v}=L_{\gamma}\,E_{\beta}^{u}$, and $E_{\beta}^{u}\in\mathbf{Mo}_{\beta}\setminus L_{<\beta}\,\mathbf{Mo}_{\beta\omega}$. Since $E_{\alpha}^{v}\in\mathbf{Mo}_{\alpha}$, we must have $\gamma\gtrsim\alpha_{/\omega}$ so there are a number $n\in\mathbb{N}$ and an ordinal $\gamma'\gtrsim\alpha$ with $\gamma=\gamma'+\alpha_{/\omega}n$ (note that $n=0$ whenever $\alpha_{/\omega}=\alpha$). Thus $v+n=L_{\alpha}\,L_{\gamma'+\alpha_{/\omega}n}\,E_{\beta}^{u}+n=L_{\gamma'+\alpha}\,E_{\beta}^{u}$ is a monomial with hyperserial expansion $v+n=L_{\gamma'+\alpha}\,E_{\beta}^{u}$. There is no path in n of length >1, so P must be a path in $L_{\gamma'+\alpha}\,E_{\beta}^{u}$. We deduce that $P_{\nearrow 1}$ is a path in u. Consequently, $Q=(L_{\gamma}\,E_{\beta}^{u})*P_{\nearrow 1}$ is a path in E_{α}^{v} with $|Q|=|P|>1$. Applying n times Lemma 5.14, we deduce that $Q_{\nearrow 1}=P_{\nearrow 1}$ is a subpath in $L_{\beta n}\,E_{\alpha}^{v}$, hence in $\sharp_{\beta}(L_{\beta n}\,E_{\alpha}\,a)$. Consider a path R in $\sharp_{\beta}(L_{\beta n}\,E_{\alpha}\,a)$ with $P_{\nearrow 1}=R_{\nearrow i}$ for a certain $i>0$. Applying the induction hypothesis for $L_{\beta n}\,E_{\alpha}\,a$ and $\beta\,n$ in the roles of a and $\alpha\,k$, the path $R_{\nearrow 1}$ is a subpath in $\mathfrak{d}_{E_{\beta n}(L_{\beta n}E_{\alpha}a)}=\mathfrak{d}_{E_{\alpha}a}$. Therefore $P_{\nearrow 1}$ is a subpath in $\mathfrak{d}_{E_{\alpha}a}$. We deduce as in the case $\alpha=1$ that $P_{\nearrow 1}$ is a subpath in $\mathfrak{d}_{E_{\alpha k}a}$. □

LEMMA 5.16. *Let $\psi\in\mathbf{No}_{\succ}$, and $\mathfrak{m}\in\mathbf{Mo}^{\neq}$ with $\operatorname{supp}\psi\succ\log\mathfrak{m}$. Let P be a path in $\mathfrak{m}$ with $|P|>1$. Then $P_{\nearrow 1}$ is a subpath in $\mathrm{e}^{\psi}\,\mathfrak{m}$.*

Proof. Let $\mathfrak{m}=\mathrm{e}^{\varphi}(L_{\beta}\,E_{\alpha}^{u})^{\iota}$ be a hyperserial expansion. The condition $\operatorname{supp}\psi\succ\log\mathfrak{m}$ implies $\varphi+\psi=\varphi+\!\!\!+\,\psi$, whence $\mathrm{e}^{\psi}\,\mathfrak{m}=\mathrm{e}^{\psi+\!\!\!+\varphi}(L_{\beta}\,E_{\alpha}^{u})^{\iota}$ is also a hyperserial expansion. In particular $P_{\nearrow 1}$ is a subpath in $\mathrm{e}^{\psi}\,\mathfrak{m}$. □

COROLLARY 5.17. *Let $\alpha=\omega^{\nu}\in\mathbf{On}$, $\beta\in\mathbf{On}$ with $\beta<\alpha$, and $\varphi\in\mathbf{No}_{\succ,\alpha}$. If P is an infinite path, then P shares a subpath with φ if and only if it shares a subpath with $L_{\beta}\,E_{\alpha}^{\varphi}$.*

Proof. Write $\beta=\omega^{\eta_1}m_1+\cdots+\omega^{\eta_k}m_k$ in Cantor normal form, with $\eta_1>\cdots>\eta_k$ and $m_1,\ldots,m_k\in\mathbb{N}^{>}$ and let

$$\mathfrak{a}_i \;:=\; L_{\omega^{\eta_1}m_1+\cdots+\omega^{\eta_{i-1}}m_{i-1}}\,E_{\alpha}^{\varphi}$$

for all $i=1,\ldots,k$.

Assume that P shares a subpath with φ. In other words, there is a path R in φ which has a common subpath with P. The path R must be infinite, so by Lemma 5.15, it shares a subpath with $E_{\alpha}^{\varphi}=\mathfrak{a}_1$. Let us prove by induction on $i=1,\ldots,k$ that R shares a subpath with $E_{\alpha}^{\varphi}=\mathfrak{a}_i$. Assuming that this holds for $i<k$, we note that $\mathfrak{a}_i$ is $L_{<\omega^{\eta_{i-1}}\omega}$-atomic, hence $L_{<\omega^{\eta_i}}$-atomic. So P shares a subpath with $\mathfrak{a}_{i+1}$ by Lemma 5.14 and the induction hypothesis. We conclude by induction that P shares a subpath with $\mathfrak{a}_k=L_{\beta}\,E_{\alpha}^{\varphi}$.

Suppose conversely that P shares a subpath with $L_\beta E_\alpha^\varphi = \mathfrak{a}_k$. By induction on $i = k-1, \ldots, 1$, it follows from Lemma 5.15 that P shares a subpath with $\mathfrak{a}_i$. Applying Lemma 5.14 to $\mathfrak{a}_1 = E_\alpha^\varphi$, we conclude that P shares a subpath with φ. □

5.3. Deconstruction lemmas

In this subsection, we list several results on the interaction between the simplicity relation $\sqsubseteq$ and various operations in $(\mathbf{No}, +, \times, (L_\alpha)_{\alpha \in \mathbf{On}})$.

LEMMA 5.18. [26, Theorem 3.3] *For $a, b \in \mathbf{No}$, we have*

$$a \sqsubseteq b \iff -a \sqsubseteq -b.$$

LEMMA 5.19. [26, Theorem 5.12(a)] *For $\mathfrak{m} \in \mathbf{Mo}$ and $r \in \mathbb{R}^{\neq}$, we have*

$$\operatorname{sign}(r)\, \mathfrak{m} \sqsubseteq r \mathfrak{m}.$$

LEMMA 5.20. [5, Proposition 4.20] *Let $\varphi \in \mathbf{No}$. For δ, ε with $\delta, \varepsilon \prec \operatorname{supp} \varphi$, we have*

$$\varphi \mathbin{+\!\!\!+} \delta \sqsubseteq \varphi \mathbin{+\!\!\!+} \varepsilon \iff \delta \sqsubseteq \varepsilon.$$

LEMMA 5.21. [10, Corollary 4.21] *For $\mathfrak{m}, \mathfrak{n} \in \mathbf{Mo}$, we have*

$$\mathfrak{m} \sqsubseteq \mathfrak{n} \iff \mathfrak{m}^{-1} \sqsubseteq \mathfrak{n}^{-1}.$$

LEMMA 5.22. [10, Proposition 4.23] *Given φ, a, b in $\mathbf{No}_{\succ}$ with $a, b \prec \operatorname{supp} \varphi$, we have*

$$\mathrm{e}^a \sqsubseteq \mathrm{e}^b \Longrightarrow \mathrm{e}^{\varphi \mathbin{+\!\!\!+} a} \sqsubseteq \mathrm{e}^{\varphi \mathbin{+\!\!\!+} b}.$$

LEMMA 5.23. [10, Proposition 4.24] *Given $\mathfrak{m}, \mathfrak{n} \in \mathbf{Mo}^{\succ}$ with $\log \mathfrak{m} \prec \mathfrak{n}$, we have*

$$\mathfrak{m} \sqsubseteq \mathfrak{n} \Longrightarrow \mathrm{e}^{\mathfrak{m}} \sqsubseteq \mathrm{e}^{\mathfrak{n}}.$$

LEMMA 5.24. *Let $\varphi \in \mathbf{No}_{\succ}$ and $r \in \mathbb{R}^{\neq}$, let $\mathfrak{m}, \mathfrak{n} \in \mathbf{Mo}^{\succ} \cap \mathbf{No}^{\prec \operatorname{supp} \varphi}$ with $\mathfrak{m} \in \mathcal{E}_\omega[\mathfrak{n}]$, and let $\delta \in \mathbf{No}_{\succ}$ with $\delta \prec \operatorname{supp} \mathfrak{n}$. Then*

$$\mathfrak{m} \sqsubseteq \mathfrak{n} \Longrightarrow \mathrm{e}^{\varphi \mathbin{+\!\!\!+} \operatorname{sign}(r) \mathfrak{m}} \sqsubseteq \mathrm{e}^{\varphi \mathbin{+\!\!\!+} r \mathfrak{n} \mathbin{+\!\!\!+} \delta}.$$

Proof. The condition $\mathfrak{m} \in \mathcal{E}_\omega[\mathfrak{n}]$ yields $\log \mathfrak{m} \prec \mathfrak{n}$. We have $\mathrm{e}^{\mathfrak{m}} \sqsubseteq \mathrm{e}^{\mathfrak{n}}$ by Lemma 5.23. The identity $\mathrm{e}^{\mathbf{Mo}^{\succ}} = \mathbf{Smp}_{\mathcal{D}}$ implies that $\mathrm{e}^{\mathfrak{m}} \sqsubseteq \mathrm{e}^{|r| \mathfrak{n}}$, whence $\mathrm{e}^{\operatorname{sign}(r) \mathfrak{m}} \sqsubseteq \mathrm{e}^{r \mathfrak{n}}$ by Lemma 5.21. Consequently, $\mathrm{e}^{\varphi \mathbin{+\!\!\!+} \operatorname{sign}(r) \mathfrak{m}} \sqsubseteq \mathrm{e}^{\varphi \mathbin{+\!\!\!+} r \mathfrak{n}}$, by Lemma 5.22. Since $\mathrm{e}^0 = 1 \sqsubseteq \mathrm{e}^\delta \in \mathbf{No}^{>}$, we may apply Lemma 5.22 to $\varphi \mathbin{+\!\!\!+} r \mathfrak{n}$ and $\varphi \mathbin{+\!\!\!+} r \mathfrak{n} \mathbin{+\!\!\!+} \delta$ to obtain $\mathrm{e}^{\varphi \mathbin{+\!\!\!+} r \mathfrak{n}} \sqsubseteq \mathrm{e}^{\varphi \mathbin{+\!\!\!+} r \mathfrak{n} \mathbin{+\!\!\!+} \delta}$. We conclude using the transitivity of $\sqsubseteq$. □

LEMMA 5.25. *Let $\alpha \in \omega^{\mathbf{On}}$ with $\alpha > 1$. For $\varphi, \psi \in \mathbf{No}_{\succ, \alpha}$ with $L_\alpha E_{<\alpha} \varphi < \psi$, we have*

$$\varphi \sqsubseteq \psi \Longrightarrow E_\alpha^\varphi \sqsubseteq E_\alpha^\psi.$$

Proof. By (4.5), we have

$$E_\alpha \varphi = \left\{ E_{<\alpha} \varphi, \mathcal{E}_\alpha E_\alpha^{\varphi_L^{\mathbf{No}_{\succ,\alpha}}} \,\middle|\, \mathcal{E}_\alpha E_\alpha^{\varphi_R^{\mathbf{No}_{\succ,\alpha}}} \right\}.$$

Since $\varphi \sqsubseteq \psi$, we have $\varphi_L^{\mathbf{No}_{>,\alpha}} \subseteq \psi_L^{\mathbf{No}_{>,\alpha}}$ and $\varphi_R^{\mathbf{No}_{>,\alpha}} \subseteq \psi_R^{\mathbf{No}_{>,\alpha}}$, whence

$$\mathcal{E}_\alpha E_\alpha^{\varphi_L^{\mathbf{No}_{>,\alpha}}} < E_\alpha \psi < \mathcal{E}_\alpha E_\alpha^{\varphi_R^{\mathbf{No}_{>,\alpha}}}.$$

Furthermore, we have $L_\alpha E_{<\alpha} \varphi < \psi$, so $E_{<\alpha} \varphi < E_\alpha^\psi$. We conclude that $E_\alpha^\varphi \sqsubseteq E_\alpha^\psi$. □

5.4. Nested truncation

In [10, Section 8], the authors prove the well-nestedness axiom **T4** for **No** by relying on a well-founded partial order $\blacktriangleleft_{\mathrm{BM}}$ that is defined by induction. This relation has the additional property that

$$\forall a, b \in \mathbf{No}^{\neq}, \quad a \blacktriangleleft_{\mathrm{BM}} b \implies a \sqsubseteq b.$$

In this subsection, we define a similar relation $\lesssim$ on **No** that will be instrumental in deriving results on the structure of $(\mathbf{No}, (L_\gamma)_{\gamma \in \omega^{\mathbf{On}}})$. However, this relation does *not* satisfy $a \lesssim b \Longrightarrow a \sqsubseteq b$ for all $a, b \in \mathbf{No}$.

Given $a, b \in \mathbf{No}$, we define

$$a \lesssim b \overset{\text{def}}{\Longleftrightarrow} \exists n \in \mathbb{N}, \, a \lesssim_n b,$$

where $(\lesssim_n)_{n \in \mathbb{N}}$ is a sequence of relations that are defined by induction on n, as follows. For $n = 0$, we set $a \lesssim_0 b$, if $a \trianglelefteq b$ or if there exist decompositions

$$\begin{aligned} a &= \varphi \mathbin{+\!\!\!+} \operatorname{sign}(r)\, \mathfrak{m} \\ b &= \varphi \mathbin{+\!\!\!+} r\, \mathfrak{m} \mathbin{+\!\!\!+} \delta, \end{aligned}$$

with $r \in \mathbb{R}^{\neq}$ and $\mathfrak{m} \in \mathbf{Mo}$. Assuming that $\lesssim_n$ has been defined, we set $a \lesssim_{n+1} b$ if we are in one of the two following configurations:

Configuration I. We may decompose a and b as

$$\begin{aligned} a &= \varphi \mathbin{+\!\!\!+} \operatorname{sign}(r)\, \mathrm{e}^{\psi} (E_\alpha^u)^\iota && (5.7) \\ b &= \varphi \mathbin{+\!\!\!+} r\, \mathrm{e}^{\psi} (L_\beta E_\alpha^v)^\iota \mathbin{+\!\!\!+} \delta, && (5.8) \end{aligned}$$

where $r \in \mathbb{R}^{\neq}$, $\psi \in \mathbf{No}_{>}$, $\alpha \in \omega^{\mathbf{On}}$, $\beta \omega < \alpha$, $\iota \in \{-1, 1\}$, $u, v \in \mathbf{No}_{>,\alpha}$,

$$\operatorname{supp} \psi \succ \log E_\alpha^u, \, L_{\beta+1} E_\alpha^v,$$

and $u \lesssim_n v$. If $\alpha = 1$, then we also require that $\psi = 0$.

Configuration II. We may decompose a and b as

$$\begin{aligned} a &= \varphi \mathbin{+\!\!\!+} \operatorname{sign}(r)\, \mathrm{e}^{\psi} && (5.9) \\ b &= \varphi \mathbin{+\!\!\!+} r\, \mathrm{e}^{\psi'} \mathfrak{a}^\iota \mathbin{+\!\!\!+} \delta, && (5.10) \end{aligned}$$

where $r \in \mathbb{R}^{\neq}$, $\psi, \psi' \in \mathbf{No}_{>}$, $\iota \in \{-1, 1\}$, $\mathfrak{a} \in \mathbf{Mo}_\omega$, $\delta \in \mathbf{No}$, $\operatorname{supp} \psi' \succ \log \mathfrak{a}$, and $\psi \lesssim_n \psi'$.

Warning 5.26. Taking $\alpha = 1$ in the first configuration, we see that $\lesssim$ extends $\blacktriangleleft_{\mathrm{BM}}$. However, the relation $\lesssim$ is neither transitive nor anti-symmetric. Furthermore, as we already noted above, we do *not* have $\forall a, b \in \mathbf{No}, \; a \lesssim b \Longrightarrow a \sqsubseteq b$.

LEMMA 5.27. *Let $\alpha \in \omega^{\mathbf{On}}$. Let $a, b \in \mathbf{No}^{>,\succ}$ be numbers of the form*

$$\begin{aligned} a &= \varphi \mathbin{+\!\!\!+} r\, \mathfrak{m} \\ b &= \varphi \mathbin{+\!\!\!+} s\, \mathfrak{n} \mathbin{+\!\!\!+} \delta \end{aligned}$$

where $\varphi,\delta\in\mathbf{No}$, $r,s\in\mathbb{R}^{\neq}$ *with* $\operatorname{sign}(r)=\operatorname{sign}(s)$, *and* $\mathfrak{m},\mathfrak{n}\in\mathbf{Mo}^{\prec}$. *If* $\mathfrak{m}^{-1}<E_{\rho}\,\mathfrak{n}^{-1}$ *for sufficiently large* $\rho<\alpha$, *then*

$$b\;\in\;\mathbf{No}_{\succ,\alpha}\;\Longrightarrow\;a\;\in\;\mathbf{No}_{\succ,\alpha}.$$

Proof. Let $\nu\in\mathbf{On}$ and $\alpha:=\omega^{\nu}$. Assume for contradiction that $b\in\mathbf{No}_{\succ,\alpha}$ and $a\notin\mathbf{No}_{\succ,\alpha}$. Assume first that $a\trianglelefteq b$, so $b=a+\!\!\!+\,\delta$. Then $\operatorname{supp} b\succ\frac{1}{L_{<\alpha}E_{\alpha}b}$. Let $k\in\mathbb{N}^{>}$ be such that $a+\!\!\!+\,k\,\mathfrak{d}_{\delta}\geqslant b$. Since $\operatorname{supp}\,(a+\!\!\!+\,k\,\mathfrak{d}_{\delta})\subseteq\operatorname{supp} b$, we deduce that $\operatorname{supp}\,(a+k\,\mathfrak{d}_{\delta})\succ\frac{1}{L_{<\alpha}E_{\alpha}(a+k\,\mathfrak{d}_{\delta})}$, whence $a+k\,\mathfrak{d}_{\delta}\in\mathbf{No}_{\succ,\alpha}$. Modulo replacing b by $a+k\,\mathfrak{d}_{\delta}$, it follow that we may assume without loss of generality that $\delta=k\,\mathfrak{p}$ for some $k\in\mathbb{N}^{>}$ and some monomial $\mathfrak{p}$.

On the one hand, a is not α-truncated, so there are $\mathfrak{q}\in(\operatorname{supp}\varphi)_{\prec}$ and γ with $0<\gamma<\alpha$ and $a<L_{\alpha}^{\uparrow\gamma}(\mathfrak{q}^{-1})$. We may choose $\gamma=\omega^{\eta}n$ for certain $\eta<\nu$ and $n\in\mathbb{N}^{>}$, so $a<L_{\alpha}^{\uparrow\omega^{\eta}n}(\mathfrak{p}^{-1})$. On the other hand, $a+k\,\mathfrak{p}$ is α-truncated, so we have

$$a+k\,\mathfrak{p}\;>\;L_{\alpha}^{\uparrow\omega^{\eta}(n+\mathbb{N}^{>})}(\mathfrak{p}^{-1})\;>\;L_{\alpha}^{\uparrow\omega^{\eta}n}(\mathfrak{p}^{-1})\;>\;a.$$

We deduce that $k\,\mathfrak{p}>L_{\alpha}^{\uparrow\omega^{\eta}(n+\mathbb{N}^{>})}(\mathfrak{p}^{-1})-L_{\alpha}^{\uparrow\omega^{\eta}n}(\mathfrak{p}^{-1})$. If ν is a successor, then choosing $\eta=\nu_{-}$, we obtain $k\,\mathfrak{p}>L_{\alpha}^{\uparrow\omega^{\eta}n}(\mathfrak{p}^{-1})+\mathbb{N}^{>}-L_{\alpha}^{\uparrow\omega^{\eta}n}(\mathfrak{p}^{-1})$, so $k\,\mathfrak{p}\succ 1$: a contradiction. Otherwise, $k\,\mathfrak{p}>\ell_{[\omega^{\eta+1},\alpha)}^{-1}\circ\mathfrak{p}^{-1}$ by [7, (2.4)], where $\ell_{[\omega^{\eta+1},\alpha)}:=\prod_{\omega^{\eta+1}\leqslant\gamma<\alpha}\ell_{\gamma}$. Thus $k^{-1}\,\mathfrak{p}^{-1}<\ell_{[\omega^{\eta+1},\alpha)}\circ\mathfrak{p}^{-1}$, whence $k^{-1}\,\ell_{0}<\ell_{[\omega^{\eta+1},\alpha)}$: a contradiction.

We now treat the general case. By a similar argument as above, we may assume without loss of generality that $b=\varphi+\!\!\!+\,s\,\mathfrak{n}$. Assume that $b\leqslant a$. Since a is not α-truncated, there exists a $\gamma<\alpha$ with $\mathfrak{m}\prec(L_{\gamma}E_{\alpha}a)^{-1}\leqslant(L_{\gamma}E_{\alpha}b)^{-1}$, whence $\mathfrak{m}^{-1}>L_{\gamma}E_{\alpha}b$. But b is α-truncated, so $\mathfrak{n}^{-1}\prec L_{<\alpha}E_{\alpha}b$. In particular $\mathfrak{n}^{-1}\prec L_{\gamma}E_{\alpha}b$, so our hypothesis $\mathfrak{m}^{-1}<L_{\rho}\,\mathfrak{n}^{-1}$ implies that $\mathfrak{m}^{-1}\prec L_{\rho}L_{\gamma}E_{\alpha}b\preccurlyeq L_{\gamma}E_{\alpha}b$: a contradiction.

Assume now that $b>a$. As in the first part of the proof, there are $\eta<\nu$ and $n<n'<\omega$ with $\varphi+\!\!\!+\,s\,\mathfrak{n}>L_{\alpha}^{\uparrow\omega^{\eta}n'}(\mathfrak{n}^{-1})$ and $L_{\alpha}^{\uparrow\omega^{\eta}n}(\mathfrak{m}^{-1})>\varphi+\!\!\!+\,r\,\mathfrak{m}$. Recall that $\mathfrak{m}^{-1}<E_{\rho}\,\mathfrak{n}^{-1}$ for sufficiently large $\rho<\alpha$. Take $\eta<\nu$ and $n'<\omega$ such that

$$\begin{aligned}L_{\alpha}^{\uparrow\omega^{\eta}n'}(\mathfrak{n}^{-1})\;&>\;L_{\alpha}^{\uparrow\omega^{\eta}(n+1)}(\mathfrak{m}^{-1})\\ L_{\omega^{\eta}}\,\mathfrak{m}^{-1}\;&\prec\;\mathfrak{n}^{-1}\quad\text{if }\nu\text{ is a limit.}\end{aligned}\tag{5.11}$$

Then $b-a>L_{\alpha}^{\uparrow\omega^{\eta}(n+1)}(\mathfrak{m}^{-1})-L_{\alpha}^{\uparrow\omega^{\eta}n}(\mathfrak{m}^{-1})$. If ν is a successor, then choosing $\eta=\nu_{-}$ yields $b-a>1$, which contradicts the fact that $\mathfrak{m}$ and $\mathfrak{n}$ are infinitesimal. So ν is a limit. Writing $\mathfrak{q}:=\max\,(\mathfrak{m},\mathfrak{n})$, we have $b-a\asymp\mathfrak{q}$. As in the first part of the proof, we obtain $\mathfrak{q}\succcurlyeq\ell_{[\omega^{\eta+1},\alpha)}^{-1}\circ\mathfrak{m}^{-1}$, so $\mathfrak{q}^{-1}\preccurlyeq\ell_{[\omega^{\eta+1},\alpha)}\circ\mathfrak{m}^{-1}\prec\mathfrak{m}^{-1}$. In view of (5.11), we also obtain $\mathfrak{q}^{-1}\prec\mathfrak{n}^{-1}$, so $\mathfrak{q}^{-1}\prec\max\,(\mathfrak{m},\mathfrak{n})^{-1}$: a contradiction. □

LEMMA 5.28. *Let* $\alpha,\alpha'\in\omega^{\mathbf{On}}$ *with* $\alpha'\geqslant\alpha$. *For* $u,v\in\mathbf{No}^{>,\succ}$, *we have*

$$L_{\alpha}\,u\;<\;\mathcal{E}_{\alpha}\,v\;\Longrightarrow\;L_{\alpha'}\,E_{\alpha}\,u\;<\;\mathcal{E}_{\alpha'}\,E_{\alpha}\,v.$$

Proof. Assume that $L_{\alpha}\,u<\mathcal{E}_{\alpha}\,v$. Let $h\in\mathcal{E}_{\alpha'}$ and let h^{inv} be its functional inverse in $\mathcal{E}_{\alpha'}$. We have $h^{\mathrm{inv}}<E_{\alpha'}\,H_{2}\,L_{\alpha'}$ by (4.10, 4.11), whence $h>E_{\alpha'}\,H_{1/2}\,L_{\alpha'}$. Furthermore, $u<E_{\alpha}\,\mathcal{E}_{\alpha}\,v$, so

$$E_{\alpha}\,u\;<\;E_{\alpha}\,E_{\alpha}\,\mathcal{E}_{\alpha}\,v.\tag{5.12}$$

We want to prove that $E_{\alpha}\,u<(E_{\alpha'}\,h\,E_{\alpha})\,v$. By (5.12), it is enough to prove that there is a $g\in\mathcal{E}_{\alpha}$ such that the inequality $E_{\alpha}\,E_{\alpha}\,g\leqslant E_{\alpha'}\,h\,E_{\alpha}$ holds on $\mathbf{No}^{>,\succ}$.

Assume that $\alpha=\alpha'$. Setting $g:=H_{1/2}\in\mathcal{E}_{\alpha}$, we have $L_{\alpha}\,h\,E_{\alpha}>g$, whence $E_{\alpha}\,g\leqslant h\,E_{\alpha}$, and $E_{\alpha}\,E_{\alpha}\,g\leqslant E_{\alpha}\,h\,E_{\alpha}$.

Assume that $\alpha' > \alpha$. We have $E_{\alpha'} H_{1/2} > H_2$ so $E_{\alpha' 2} H_{1/2} > E_{\alpha'} H_2 > E_\alpha E_{\alpha'}$ by (4.8). Thus $E_{\alpha'} h > E_{\alpha' 2} H_{1/2} L_{\alpha'} > E_\alpha$. Consequently, $E_{\alpha'} h E_\alpha > E_\alpha E_\alpha$, as claimed. □

If a, b are numbers, then we write $[a \leftrightarrow b]$ for the interval $[\min(a,b), \max(a,b)]$.

PROPOSITION 5.29. *For $a, b, c \in \mathbf{No}$ with $a \lesssim c$ and $b \in [a \leftrightarrow c]$, any infinite path in a shares a subpath with b.*

Proof. We prove this by induction on n with $a \lesssim_n c$. Let P be an infinite path in a. Assume that $a \lesssim_0 c$. If $a \trianglelefteq c$, then we have $a \trianglelefteq b$ so P is a path in b. Otherwise, there are $\varphi, \delta \in \mathbf{No}$, $r \in \mathbb{R}^{\neq}$ and $\mathfrak{m} \in \mathbf{Mo}$ with $a = \varphi + \operatorname{sign}(r)\, \mathfrak{m}$ and $c = \varphi + r\, \mathfrak{m} + \delta$. Then $b = \varphi + s\, \mathfrak{n} + t$ for certain $t \in \mathbf{No}$, $s \in \mathbb{R}^{\neq}$ and $\mathfrak{n} \in \mathbf{Mo}$ with $s\, \mathfrak{n} \in [\operatorname{sign}(r)\, \mathfrak{m} \leftrightarrow r\, \mathfrak{m}]$. We must have $\mathfrak{n} = \mathfrak{m}$. If P is a path in φ, then it is a path in b. Otherwise, it is a path in $\operatorname{sign}(r)\, \mathfrak{m}$, so $P_{\nearrow 1}$ is a subpath in $s\, \mathfrak{m}$, hence in b.

We now assume that $a \lesssim_n c$ where $n > 0$ and that the result holds for all $a', b', c' \in \mathbf{No}$ and $k < n$ with $a' \lesssim_k c'$ and $b' \in [a' \leftrightarrow c']$. Assume first that (a, c) is in Configuration I, and write

$$\begin{array}{rcl} a & = & \varphi + \operatorname{sign}(r)\, \mathrm{e}^{\psi} (E_\alpha^u)^\iota \\ c & = & \varphi + r\, \mathrm{e}^{\psi} (L_\beta E_\alpha^v)^\iota + \delta \end{array} \quad \text{with} \quad u \lesssim_{n-1} v.$$

Then we can write $b = \varphi + s\, \mathfrak{m} + t$ like in the case when $n = 0$. If P is a path in φ, then it is a path in b. So we may assume that P is a path in $\operatorname{sign}(r)\, \mathrm{e}^{\psi} (E_\alpha^u)^\iota$. Note that we have $\mathfrak{m} \in [\mathrm{e}^{\psi} (E_\alpha^u)^\iota \leftrightarrow \mathrm{e}^{\psi} (L_\beta E_\alpha^v)^\iota]$. Setting $\mathfrak{n} := (\mathfrak{m}\, \mathrm{e}^{-\psi})^\iota \in [E_\alpha^u \leftrightarrow L_\beta E_\alpha^v]$, we observe that $\operatorname{supp} \log \mathfrak{n} \prec \operatorname{supp} \psi$, whence $\mathrm{e}^{\psi} \mathfrak{n}^\iota$ is the hyperserial expansion of $\mathfrak{m}$. If $P_{\nearrow 1}$ is a path in ψ, then it is a path in $\mathfrak{m}$.

Suppose that $P_{\nearrow 1}$ is not a path in ψ. Assume first that $\alpha = 1$, so $\psi = 0$, $\beta = 0$, and P is a path in $(E_1^u)^\iota$. Then Lemma 5.14 implies that $P_{\nearrow 1}$ is a subpath in ιu, so $P_{\nearrow 2}$ is a subpath in u. Otherwise, consider the hyperserial expansion $E_\alpha^u = L_{\beta'} E_{\alpha'}^w$, $E_{\alpha'}^w \in \mathbf{Mo}_{\alpha'} \setminus L_{<\alpha'} \mathbf{Mo}_{\alpha' \omega}$ of E_α^u. Since $P_{\nearrow 1}$ is not a path in ψ, it must be a path in w. The number $L_{\beta'} E_{\alpha'}^w$ is $L_{<\alpha}$-atomic, so we must have $\alpha' \geqslant \alpha$ and $\beta' \gtrsim \alpha_{/\omega}$. There are $n \in \mathbb{N}$ and $\beta'' \gg \alpha_{/\omega}$ such that $\beta' = \beta'' + \alpha_{/\omega} n$. Therefore $u = L_{\beta'' + \alpha} E_{\alpha'}^w - n$. It follows by Corollary 5.17 that $P_{\nearrow 1}$ shares a subpath with u, whence so does P.

Let $z := \sharp_\alpha (L_\alpha\, \mathfrak{n})$. Recall that $\mathfrak{n} \in [E_\alpha^u \leftrightarrow L_\beta E_\alpha^v]$, so $L_\alpha\, \mathfrak{n} \in [u \leftrightarrow L_\alpha L_\beta E_\alpha^v]$. Now (4.8) implies that $L_\beta E_\alpha^v \in \mathcal{E}_\alpha [E_\alpha^v]$, so $L_\alpha L_\beta E_\alpha^v \in L_\alpha \mathcal{E}_\alpha [E_\alpha^v] = \mathcal{L}_\alpha [v]$. The function $\sharp_\alpha = \pi_{\mathbf{Smp}_{\mathcal{L}_\alpha}}$ is non-decreasing, so $z = \sharp_\alpha (L_\alpha\, \mathfrak{n}) \in [u \leftrightarrow \sharp_\alpha (L_\alpha L_\beta E_\alpha^v)] = [u \leftrightarrow v]$. But $u \lesssim_{n-1} v$, so the induction hypothesis yields that $P_{\nearrow 2}$, and thus P, shares a subpath with z. We deduce with Lemma 5.15 that P shares a subpath with $\mathfrak{n}$, hence with b.

Assume now that (a, c) is in Configuration II, and write

$$\begin{array}{rcl} a & = & \varphi + \operatorname{sign}(r)\, \mathrm{e}^{\psi} \\ c & = & \varphi + r\, \mathrm{e}^{\psi'} \mathfrak{a}^\iota + \delta \end{array} \quad \text{with} \quad \psi \lesssim_{n-1} \psi'.$$

Note that we also have $\psi \lesssim_{n-1} \psi' + \iota \log \mathfrak{a}$. We may again assume that $P_{\nearrow 1}$ is a path in ψ. Write $b = \varphi + s'\, \mathfrak{q} + t'$, where $s' \in \mathbb{R}^{\neq}$, $t' \in \mathbf{No}$, and $\mathfrak{q} \in [\mathrm{e}^{\psi} \leftrightarrow \mathrm{e}^{\psi'} \mathfrak{a}^\iota] \cap \mathbf{Mo}$. Then $\log \mathfrak{q} \in [\psi \leftrightarrow \psi' + \iota \log \mathfrak{a}]$ where $\psi \lesssim_{n-1} \psi' + \iota \log \mathfrak{a}$. We deduce by induction that P shares a subpath with $\log \mathfrak{q}$. By Lemma 5.15, it follows that P shares a subpath with $\mathfrak{q}$, hence with b. This concludes the proof. □

LEMMA 5.30. *Let $\lambda, \alpha \in \omega^{\mathbf{On}}$ and $\beta \in \mathbf{On}$ with $\beta \omega < \alpha$. Let $a \in \mathbf{No}_{>,\lambda}$ be of the form*

$$a = \varphi + r \mathrm{e}^{\psi} (L_{\beta} E_{\alpha}^{b})^{\iota} + \delta,$$

with $\varphi \in \mathbf{No}$, $r \in \mathbb{R}^{\neq}$, $\psi \in \mathbf{No}_{>}$, $b \in \mathbf{No}_{>,\alpha}$, $\iota \in \{-1, 1\}$, $\delta \in \mathbf{No}$ and $\log L_{\beta} E_{\alpha}^{b} \prec \operatorname{supp} \psi$. Consider an infinite path P in $c \in \mathbf{No}_{>,\alpha}$ with $c \lesssim b$.

i. If $\log E_{\alpha}^{c} \not\prec \operatorname{supp} \psi$, then P shares a subpath with ψ.

ii. If $\log E_{\alpha}^{c} \prec \operatorname{supp} \psi$ and $\mathrm{e}^{\psi} (E_{\alpha}^{c})^{\iota} \not\prec \operatorname{supp} \varphi$, then P shares a subpath with φ.

iii. If $\log E_{\alpha}^{c} \prec \operatorname{supp} \psi$ and $\mathrm{e}^{\psi} (E_{\alpha}^{c})^{\iota} \prec \operatorname{supp} \varphi$ and $a' := \varphi + \operatorname{sign}(r) \mathrm{e}^{\psi} (E_{\alpha}^{c})^{\iota} \notin \mathbf{No}_{>,\lambda}$, then P shares a subpath with φ.

Proof. i. If $\log E_{\alpha}^{c} \not\prec \operatorname{supp} \psi$, then we have $\psi \neq 0$, so $\alpha > 1$. Let $\mathfrak{m} \in \operatorname{supp} \psi$ with $\log E_{\alpha}^{c} \succcurlyeq \mathfrak{m}$. Since $\log E_{\alpha}^{c}$ and $\mathfrak{m}$ are monomials, we have $\mathfrak{m} \leqslant \log E_{\alpha}^{c}$, whence $\mathrm{e}^{\mathfrak{m}} \leqslant E_{\alpha}^{c}$. Our assumption that $\mathfrak{m} \in \operatorname{supp} \psi \succ \log L_{\beta} E_{\alpha}^{b}$ also implies $\mathrm{e}^{\mathfrak{m}} \leqslant L_{\beta} E_{\alpha}^{b}$. Hence $\mathrm{e}^{\mathfrak{m}} \in [E_{\alpha}^{c} \leftrightarrow L_{\beta} E_{\alpha}^{b}]$. Now P shares a subpath with E_{α}^{c}, by Lemma 5.15. Since $E_{\alpha}^{c} \lesssim L_{\beta} E_{\alpha}^{b}$, Proposition 5.29 next implies that P shares a subpath with $\mathrm{e}^{\mathfrak{m}}$. Using Lemma 5.14, we conclude that P shares a subpath with $\mathfrak{m}$, and hence with ψ.

ii. Let $\mathfrak{m} \in \operatorname{supp} \varphi$ with $\mathfrak{m} \preccurlyeq \mathrm{e}^{\psi} (E_{\alpha}^{c})^{\iota}$. It is enough to prove that P shares a subpath with $\mathfrak{m}$. Since $\mathfrak{m}$, $\mathrm{e}^{\psi} (L_{\beta} E_{\alpha}^{b})^{\iota}$, and $\mathrm{e}^{\psi} (E_{\alpha}^{c})^{\iota}$ are monomials, we have $\mathrm{e}^{\psi} (L_{\beta} E_{\alpha}^{b})^{\iota} \leqslant \mathfrak{m} \leqslant \mathrm{e}^{\psi} (E_{\alpha}^{c})^{\iota}$. Let $\mathfrak{n} := (\mathrm{e}^{-\psi} \mathfrak{m})^{\iota}$, so that $\mathfrak{n} \in [L_{\beta} E_{\alpha}^{b} \leftrightarrow E_{\alpha}^{c}]$. In particular, we have $\operatorname{supp} \psi \succ \log \mathfrak{n} \succ 1$. Moreover $E_{\alpha}^{c} \lesssim L_{\beta} E_{\alpha}^{b}$, so using Lemma 5.15 and Proposition 5.29, we deduce in the same way as above that P shares a subpath with $\mathfrak{n}$. If $\mathfrak{n} \notin \mathbf{Mo}_{\omega}$, then $\mathfrak{m} = \mathrm{e}^{\psi + \iota \log \mathfrak{n}}$ is the hyperserial expansion of $\mathfrak{m}$, so P shares a subpath with $\mathfrak{m}$. If $\mathfrak{n} \in \mathbf{Mo}_{\omega}$, then the hyperserial expansion of $\mathfrak{n}$ must be of the form $\mathfrak{n} = E_{\beta'} E_{\alpha'}^{u}$, since otherwise $\log \mathfrak{n}$ would have at least two elements in its support. We deduce that P shares a subpath with u and that the hyperserial expansion of $\mathfrak{m}$ is $\mathrm{e}^{\psi} (E_{\beta'} E_{\alpha'}^{u})^{\iota}$. Therefore P shares a subpath with $\mathfrak{m}$.

iii. We assume that a' is not λ-truncated whereas $\log E_{\alpha}^{c} \prec \operatorname{supp} \psi$ and $\mathrm{e}^{\psi} (E_{\alpha}^{c})^{\iota} \prec \operatorname{supp} \varphi$. If $\lambda = 1$, then we must have $\mathrm{e}^{\psi} (E_{\alpha}^{c})^{\iota} \preccurlyeq 1$, which means that $\psi < 0$ or that $\psi = 0$ and $\iota = -1$. But then $\mathrm{e}^{\psi} (L_{\beta} E_{\alpha}^{b})^{\iota} \preccurlyeq 1$: a contradiction.

Assume that $\lambda > 1$. By Lemma 5.27, we may assume without loss of generality that $\delta = 0$. The assumption on a' and the fact that $a \in \mathbf{No}^{>,\succ}$ imply that φ is non-zero. Write

$$\begin{aligned} \mathfrak{p} &:= \mathrm{e}^{\psi} (E_{\alpha}^{c})^{\iota} \quad \text{and} \\ \mathfrak{q} &:= \mathrm{e}^{\psi} (L_{\beta} E_{\alpha}^{b})^{\iota}. \end{aligned}$$

So $a = \varphi + r \mathfrak{q}$ and $a' = \varphi + \operatorname{sign}(r) \mathfrak{p}$. Note that $\mathfrak{p}$ must be infinitesimal since a' is not λ-truncated. Thus $\mathfrak{q}$ is also infinitesimal. By Lemma 5.27, we deduce that $E_{<\lambda} \mathfrak{q}^{-1} < \mathfrak{p}^{-1}$. We have $\sharp_{\lambda}(a') \trianglelefteq a'$, so $\sharp_{\lambda}(a') = \varphi$, since a and $\varphi \trianglelefteq a$ are both λ-truncated. Since a' is not λ-truncated, there is an ordinal $\gamma < \lambda$ with $\mathfrak{p} \prec (L_{\gamma} E_{\lambda}^{\varphi})^{-1}$. If $\varphi \geqslant a$, then $\mathfrak{q} \succ (L_{<\lambda} E_{\lambda}^{a})^{-1}$, because a is λ-truncated. Thus $\mathfrak{q} \succ (L_{<\lambda} E_{\lambda}^{\varphi})^{-1}$. If $\varphi < a$, then $\varphi + (L_{<\lambda} E_{\lambda}^{\varphi})^{-1} \in \mathcal{L}_{\lambda}[\varphi] < \mathcal{L}_{\lambda}[a] \ni a = \varphi + r \mathfrak{q}$, because φ and a are λ-truncated. Now $r > 0$, since $\varphi < a$. We again deduce that $\mathfrak{q} \succ (L_{<\lambda} E_{\lambda}^{\varphi})^{-1}$.

In both cases, we have $L_{\gamma} E_{\lambda}^{\varphi} \in [\mathfrak{p}^{-1} \leftrightarrow \mathfrak{q}^{-1}]$ where $\mathfrak{p}^{-1} \lesssim \mathfrak{q}^{-1}$, so P shares a subpath with $L_{\gamma} E_{\lambda}^{\varphi}$, by Proposition 5.29. It follows by Corollary 5.17 that P shares a subpath with φ. $\square$

5.5. Well-nestedness

We now prove that every number is well-nested. Throughout this subsection, P will be an infinite path inside a number $a \in \mathbf{No}$. At the beginning of Section 5.2 we have shown how to attach sequences $(r_{P,i})_{i<\omega}$, $(\mathfrak{m}_{P,i})_{i<\omega}$, etc. to this path. In order to alleviate notations, we will abbreviate $r_i := r_{P,i}$, $\mathfrak{m}_i := \mathfrak{m}_{P,i}$, $u_i := u_{P,i}$, $\psi_i := \psi_{P,i}$, $\iota_i := \iota_{P,i}$, $\alpha_i := \alpha_{P,i}$, and $\beta_i := \beta_{P,i}$ for all $i \in \mathbb{N}$.

We start with a technical lemma that will be used to show that the existence of a bad path P in a implies the existence of a bad path in a strictly simpler number than a.

LEMMA 5.31. *Let $a \in \mathbf{No}$, let P be an infinite path in a and let $i \in \mathbb{N}$ such that every index $k \leqslant i$ is good for (P, a). For $k \leqslant i$, let $\varphi_k := (u_k)_{\succ \mathfrak{m}_k}$, $\varepsilon_k := r_k$, and $\rho_k := (u_k)_{\prec \mathfrak{m}_k}$, so that $\varepsilon_0, \ldots, \varepsilon_{i-1} \in \{-1, 1\}$ and*

$$\begin{aligned} u_k &= \varphi_k \mathbin{+\!\!\!+} \varepsilon_k \mathrm{e}^{\psi_{k+1}} (E_{\alpha_k}^{u_{k+1}})^{\iota_k} && (k<i) \\ u_i &= \varphi_i \mathbin{+\!\!\!+} r_i \mathrm{e}^{\psi_{i+1}} (L_{\beta_i} E_{\alpha_i}^{u_{i+1}})^{\iota_i} \mathbin{+\!\!\!+} \rho_i. \end{aligned}$$

Let $\chi \in \{0,1\}$ and let $c_i \in \mathbf{No}_{>,\alpha_{i-1}}$ be a number with $c_i \lesssim u_i$ and

$$c_i = \varphi_i \mathbin{+\!\!\!+} \chi \operatorname{sign}(r_i) \mathrm{e}^{\psi_{i+1}} \mathfrak{p}^{\iota_i}, \tag{5.13}$$

for a certain $\mathfrak{p} \in \mathbf{Mo}^{\succ}$ with $\log \mathfrak{p} \prec \operatorname{supp} \psi_{i+1}$, $\mathfrak{p} \sqsubseteq E_{\alpha_i}^{u_{i+1}}$ and $\mathfrak{p} \in \mathcal{E}_\omega[E_{\alpha_i}^{u_{i+1}}]$ whenever $\psi_{i+1} = 0$. For $k = i-1, \ldots, 0$, we define

$$c_k := \varphi_k + \varepsilon_k \mathrm{e}^{\psi_{k+1}} (E_{\alpha_k}^{c_{k+1}})^{\iota_k} \tag{5.14}$$

Assume that P shares a subpath with c_i. If P shares no subpath with any of the numbers $\varphi_0, \psi_1, \ldots, \varphi_{i-1}, \psi_i$, then we have $c_0 \sqsubseteq a$, and P shares a subpath with c_0.

Proof. Using backward induction on k, let us prove for $k = i-1, \ldots, 0$ that

$$\begin{aligned} L_{\alpha_k} c_{k+1} &< \mathcal{E}_{\alpha_k} u_{k+1} && (5.15)_k \\ \log E_{\alpha_k}^{c_{k+1}} &\prec \operatorname{supp} \psi_{k+1} && (5.16)_k \\ \mathrm{e}^{\psi_{k+1}} (E_{\alpha_k}^{c_{k+1}})^{\iota_k} &\prec \operatorname{supp} \varphi_k && (5.17)_k \\ c_k &\lesssim u_k && (5.18)_k \\ P \text{ shares a subpath with } c_{k+1} &&& (5.19)_k \\ c_{k+1} &\in \mathbf{No}_{>,\alpha_k} && (5.20)_k \\ c_{k+1} &\sqsubseteq u_{k+1} && (5.21)_k \end{aligned}$$

and that $(5.19)_k$ and $(5.21)_k$ also hold for $k = -1$.

We first treat the case when $k = i-1$. Note that $c_i \neq 0$ since it contains a subpath, so $\varphi_i \in \mathbf{No}^{>,\succ}$ or $\chi = 1$. From our assumption that $c_i = \varphi_i \mathbin{+\!\!\!+} \chi \operatorname{sign}(r_i) \mathrm{e}^{\psi_{i+1}} \mathfrak{p}^{\iota_i}$ and the fact that $\mathfrak{p} \in \mathcal{E}_\omega[E_{\alpha_i}^{u_{i+1}}]$ if $\psi_{i+1} = 0$, we deduce that $c_i \in \mathcal{E}_\omega[u_i]$. Hence $L_{\alpha_{i-1}} c_i < \mathcal{E}_{\alpha_{i-1}} u_i$ and $(5.15)_{i-1}$. Note that $(5.19)_{i-1}$ and $(5.20)_{i-1}$ follow immediately from the other assumptions on c_i. If $\chi = 0$ then $c_i = \varphi_i \trianglelefteq u_i$. If $\chi = 1$, then $\mathfrak{p} \sqsubseteq L_{\beta_i} E_{\alpha_i}^{u_{i+1}}$, since $L_{\beta_i} E_{\alpha_i}^{u_{i+1}} \in \mathcal{E}_{\alpha_i}[E_{\alpha_i}^{u_{i+1}}]$ and $\mathfrak{p} \sqsubseteq E_{\alpha_i}^{u_{i+1}} \sqsubseteq \mathcal{E}_{\alpha_i}[E_{\alpha_i}^{u_{i+1}}]$. Hence $\mathfrak{p}^{\iota_i} \sqsubseteq (L_{\beta_i} E_{\alpha_i}^{u_{i+1}})^{\iota_i}$ by Lemma 5.21 and $\operatorname{sign}(r_i) \mathrm{e}^{\psi_{i+1}} \mathfrak{p}^{\iota_i} \sqsubseteq r_i \mathrm{e}^{\psi_{i+1}} (L_{\beta_i} E_{\alpha_i}^{u_{i+1}})^{\iota_i}$ by Lemmas 5.19 and 5.22. Finally, $c_i \sqsubseteq u_i$ by Lemma 5.20, so $(5.21)_{i-1}$ holds in general. Recall that P is a subpath in c_i, but that it shares no subpath with ψ_i or φ_{i-1}. In view of $(5.20)_{i-1}$, we deduce $(5.16)_{i-1}$ from Lemma 5.30(i) and $(5.17)_{i-1}$ from Lemma 5.30(ii). Combining $(5.16)_{i-1}$, $(5.17)_{i-1}$ and $(5.20)_{i-1}$ with the relation $c_i \lesssim u_i$, we finally obtain $(5.18)_{i-1}$.

Let $k\in\{0,\ldots,i-1\}$ and assume that $(5.15–5.21)_\ell$ hold for all $\ell\geqslant k$. We shall prove $(5.15–5.21)_{k-1}$ if $k>0$, as well as $(5.19)_{-1}$ and $(5.21)_{-1}$. Recall that

$$c_k \;=\; \varphi_k+\varepsilon_k \mathrm{e}^{\psi_{k+1}}(E_{\alpha_k}^{c_{k+1}})^{\iota_k}.$$

$(5.15)_{k-1}$. Recall that $k>0$. If $\varphi_k\neq 0$ or $\psi_{k+1}\neq 0$, then $c_k\in\mathcal{P}[u_k]$ and $(5.16–5.17)_k$ imply $(5.15)_{k-1}$. Assume now that $\varphi_k=\psi_{k+1}=0$. It follows since $k>0$ that $\iota_k=1$, so $c_{k-1}=E_{\alpha_{k-1}}^{c_k}$ and $u_{k-1}=E_{\alpha_{k-1}}u_k$. Since $E_{\alpha_{k-1}}^{u_k}$ is a hyperserial expansion, we must have $u_k\notin\mathbf{Mo}_{\alpha_{k-1}\omega}$, so $\alpha_{k-1}\geqslant\alpha_k$. The result now follows from $(5.15)_k$ and Lemma 5.28.

$(5.19)_{k-1}$. We know by $(5.19)_k$ that P shares a subpath with c_{k+1}. Since $c_{k+1}\in\mathbf{No}_{\succ,\alpha_k}$, we deduce with Corollary 5.17 that P also shares a subpath with $E_{\alpha_k}^{c_{k+1}}$, hence with $(E_{\alpha_k}^{c_{k+1}})^{\iota_k}$. In view of $(5.16)_k$ and Lemma 5.16, we see that P shares a subpath with $\mathrm{e}^{\psi_{k+1}}(E_{\alpha_k}^{c_{k+1}})^{\iota_k}$. Hence $(5.17)_k$ gives that P shares a subpath with c_k.

$(5.16)_{k-1}$. By $(5.18)_k$, we have $c_k\lesssim u_k$. Now P shares a subpath with c_k by $(5.19)_k$, but it shares no subpath with ψ_k. Lemma 5.30(i) therefore yields the desired result $\log E_{\alpha_{k-1}}^{c_k}\prec\operatorname{supp}\psi_k$.

$(5.17)_{k-1}$. As above, P shares a subpath with c_k, but no subpath with φ_{k-1}. We also have $c_k\lesssim u_k$ and $\log E_{\alpha_{k-1}}^{c_k}\prec\operatorname{supp}\psi_k$, so $(5.17)_{k-1}$ follows from Lemma 5.30(ii).

$(5.18)_{k-1}$. We obtain $(5.18)_{k-1}$ by combining $(5.15–5.18)_k$ and $(5.20)_k$.

$(5.20)_{k-1}$. The path P shares a subpath with c_k, but no subpath with φ_k. By what precedes, we also have $\log E_{\alpha_k}^{c_{k+1}}\prec\operatorname{supp}\psi_k$ and $\mathrm{e}^{\psi_k}(E_{\alpha_k}^{c_{k+1}})^{\iota_k}\prec\operatorname{supp}\varphi_k$. Note finally that $u_k\in\mathbf{No}_{\succ,\alpha_{k-1}}$. Hence $c_k\in\mathbf{No}_{\succ,\alpha_{k-1}}$, by applying Lemma 5.30(iii) with α_k, α_{k-1}, u_k, u_{k+1}, and c_{k+1} in the roles of α, λ, a, b, and c.

$(5.21)_{k-1}$. It suffices to prove that $E_{\alpha_k}^{c_{k+1}}\sqsubseteq E_{\alpha_k}^{u_{k+1}}$, since

$$\begin{array}{rlll} & E_{\alpha_k}^{c_{k+1}} & \sqsubseteq\; E_{\alpha_k}^{u_{k+1}} & \\ \Longrightarrow & (E_{\alpha_k}^{c_{k+1}})^{\iota_k} & \sqsubseteq\; (E_{\alpha_k}^{u_{k+1}})^{\iota_k} & \text{(by Lemma 5.21)}\\ \Longrightarrow & \mathrm{e}^{\psi_{k+1}}(E_{\alpha_k}^{c_{k+1}})^{\iota_k} & \sqsubseteq\; \mathrm{e}^{\psi_{k+1}}(E_{\alpha_k}^{u_{k+1}})^{\iota_k} & \text{(by Lemma 5.22)}\\ \Longrightarrow & \varepsilon_k\mathrm{e}^{\psi_{k+1}}(E_{\alpha_k}^{c_{k+1}})^{\iota_k} & \sqsubseteq\; \varepsilon_k\mathrm{e}^{\psi_{k+1}}(E_{\alpha_k}^{u_{k+1}})^{\iota_k} & \\ \Longrightarrow & \varphi_k+\!\!\!+\varepsilon_k\mathrm{e}^{\psi_{k+1}}(E_{\alpha_k}^{c_{k+1}})^{\iota_k} & \sqsubseteq\; \varphi_k+\!\!\!+\varepsilon_k\mathrm{e}^{\psi_{k+1}}(E_{\alpha_k}^{u_{k+1}})^{\iota_k} & \text{(by Lemma 5.20)}\\ \Longrightarrow & c_k & \sqsubseteq\; u_k. & \end{array}$$

Assume that $\alpha_k>1$ and recall that

$$\begin{aligned} c_k &= \varphi_k+\!\!\!+\varepsilon_k\mathrm{e}^{\psi_{k+1}}(E_{\alpha_k}^{c_{k+1}})^{\iota_k}\\ c_{k+1} &= \varphi_{k+1}+\!\!\!+\varepsilon_{k+1}\mathrm{e}^{\psi_{k+2}}(E_{\alpha_{k+1}}^{c_{k+2}})^{\iota_{k+1}}. \end{aligned}$$

By Lemma 5.25, it suffices to prove that $c_{k+1}\sqsubseteq u_{k+1}$ and that $E_\gamma c_{k+1}<E_{\alpha_k}^{u_{k+1}}$ for all $\gamma<\alpha_k$. The first relation holds by $(5.21)_k$. By $(5.15)_k$, we have $L_{\alpha_k}c_{k+1}<\mathcal{E}_{\alpha_k}u_{k+1}$. Therefore $c_{k+1}<E_{\alpha_k}\frac{1}{2}u_{k+1}<L_{<\alpha_k}E_{\alpha_k}u_{k+1}$ by Lemma 5.25. This yields the result.

Assume now that $\alpha_k=1$. For $d=0,\ldots,i$, let

$$\begin{aligned} \mathfrak{c}_d &:= \mathfrak{d}_{c_d-\varphi_d}\\ \mathfrak{u}_d &:= \mathfrak{d}_{u_d-\varphi_d}. \end{aligned}$$

We will prove, by a second descending induction on $d=i,\ldots,k-1$, that the monomials $\mathfrak{c}_d$ and $\mathfrak{u}_d$ satisfy the premises of Lemma 5.24, i.e. $\mathfrak{c}_d,\mathfrak{u}_d\succ 1$, $\mathfrak{c}_d\in\mathcal{E}_\omega[\mathfrak{u}_d]$, and $\mathfrak{c}_d\sqsubseteq\mathfrak{u}_d$. It will then follow by Lemma 5.24 that $\mathrm{e}^{c_k}\sqsubseteq\mathrm{e}^{u_k}$, thus concluding the proof.

If $d = i$, then $\operatorname{supp} c_i, \operatorname{supp} u_i \succ 1$, because $\alpha_{i-1} = 1$. In particular $\mathfrak{c}_i, \mathfrak{u}_i \succ 1$. Moreover, $\mathfrak{c}_i \sqsubseteq \mathfrak{u}_i$ follows from our assumption that $\mathfrak{p} \sqsubseteq E_{\alpha_i}^{u_{i+1}}$, the fact that $E_{\alpha_i}^{u_{i+1}} \sqsubseteq \mathcal{E}_{\alpha_i}[E_{\alpha_i}^{u_{i+1}}] \ni L_{\beta_i} E_{\alpha_i}^{u_{i+1}}$, and Lemmas 5.22 and 5.21. If $\psi_{i+1} \neq 0$, then we have $\mathfrak{c}_i \in \mathcal{E}_\omega[\mathfrak{u}_i]$ because $\operatorname{supp} \psi_{i+1} \succ \log \mathfrak{p}, \log E_{\alpha_i}^{u_{i+1}}$. Otherwise, we have $\mathfrak{c}_i = \mathfrak{p} \in \mathcal{E}_\omega[E_{\alpha_i}^{u_{i+1}}] = \mathcal{E}_\omega[\mathfrak{u}_i]$.

Now assume that $d < i$, that the result holds for $d+1$, and that $\alpha_d = 1$. Again $\alpha_d = 1$ implies that $\mathfrak{c}_{d+1}, \mathfrak{u}_{d+1} \succ 1$. The relation $c_{d+1} \sqsubseteq u_{d+1}$ and Lemmas 5.18, 5.19, and 5.20 imply that $\mathfrak{c}_{d+1} \sqsubseteq \mathfrak{u}_{d+1}$. If $\psi_{d+2} \neq 0$, then $\mathfrak{c}_{d+1} \in \mathcal{E}_\omega[\mathfrak{u}_{d+1}]$ by $(5.16)_{d+1}$. Otherwise, we have $\iota_{d+1} = 1$, because $c_d \in \mathbf{No}_{\succ,1}$. Since $\alpha_d = 1$, the number $u_{d+1} = \varphi_{d+1} + \varepsilon_{d+1} E_{\alpha_{d+1}}^{u_{d+2}}$ is not tail-atomic, so we must have $\alpha_{d+1} = 1$. This entails that $\mathfrak{c}_{d+1} = \mathrm{e}^{c_{d+2}}$ and $\mathfrak{u}_{d+1} = \mathrm{e}^{u_{d+2}}$. By the induction hypothesis at $d+1$, we have $\mathfrak{c}_{d+2} \in \mathcal{E}_\omega[\mathfrak{u}_{d+2}]$. We deduce that $c_{d+2} \in \mathcal{E}_\omega[u_{d+2}]$, so

$$\mathfrak{c}_{d+1} \in \exp \mathcal{E}_\omega[u_{d+2}] = \mathcal{E}_\omega[\mathrm{e}^{u_{d+2}}] = \mathcal{E}_\omega[\mathfrak{u}_{d+1}].$$

It follows by induction that $(5.21)_{k-1}$ is valid.

This concludes our inductive proof. The lemma follows from $(5.21)_{-1}$ and $(5.19)_{-1}$. □

We are now in a position to prove our first main theorem.

Proof of Theorem 1.1. Assume for contradiction that the theorem is false. Let a be a $\sqsubseteq$-minimal ill-nested number and let P be a bad path in a. Let $i \in \mathbb{N}$ be the smallest bad index in (P, a). As in Lemma 5.31, we define $\varphi_k := (u_k)_{\prec \mathfrak{m}_k}$, $\rho_k := (u_k)_{\succ \mathfrak{m}_k}$, and $\varepsilon_k := r_k$ for all $k \leqslant i$. We may assume that $i > 0$, otherwise the number $c_0 := \varphi_0 + \operatorname{sign}(r_0)\, \mathrm{e}^{\psi_1} (E_{\alpha_0}^{u_1})^{\iota_0}$ is ill-nested and satisfies $c_0 \sqsubset a$: a contradiction.

Assume for contradiction that there is a $j < i$ such that φ_j or ψ_{j+1} is ill-nested. Set $\chi := 0$ if φ_j is ill-nested and $\chi := 1$ otherwise. If $\chi = 1$, then P cannot share a subpath with φ_j, so $\operatorname{supp} \varphi_j \succ \mathrm{e}^{\psi_{j+1}}$ by Lemma 5.30, and $\varphi_j + \varepsilon_j \mathrm{e}^{\psi_{j+1}}$ is ill-nested. In general, it follows that $c_j := \varphi_j + \chi \varepsilon_j \mathrm{e}^{\psi_{j+1}}$ is ill-nested. Let Q be a bad path in c_j and set $P' := (P(0), \ldots, P(j-1)) * Q$. Then we may apply Lemma 5.31 to j, c_j, and P' in the roles of i, c_i, and P. Since $c_j \neq u_j$, this yields an ill-nested number $c_0 \sqsubset a$: a contradiction.

Therefore the numbers $\varphi_0, \psi_1, \ldots, \varphi_{i-1}, \psi_i$ are well-nested. Since i is bad for (P, a), one of the four cases listed in Definition 5.10 must occur. We set

$$d_i := \begin{cases} \varphi_i + \operatorname{sign}(r_i)\, \mathrm{e}^{\psi_{i+1}} & \text{if Definition 5.10(4) occurs} \\ \varphi_i + \operatorname{sign}(r_i)\, \mathrm{e}^{\psi_{i+1}} (E_{\alpha_i}^{u_{i+1}})^{\iota_i} & \text{otherwise.} \end{cases}$$

By construction, we have $d_i \lesssim u_i$. Furthermore P shares a subpath with d_i, so there exists a bad path Q in d_i. We have $d_i \in \mathbf{No}_{\succ, \alpha_{j-1}}$ by Lemma 5.27. If Definition 5.10(4) occurs, then we must have $\psi_{i+1} \neq 0$ so d_i is written as in (5.13) with d_i in the role of c_i and $\mathfrak{p} = \chi = 1$. Otherwise, d_i is as in (5.13) for $\mathfrak{p} = E_{\alpha_i}^{u_{i+1}}$. Setting $P' := (P(0), \ldots, P(i-1)) * Q$, it follows that we may apply Lemma 5.31 to d_i and P' in the roles of c_i and P. We conclude that there exists an ill-nested number $d_0 \sqsubset a$: a contradiction. □

6. SURREAL SUBSTRUCTURES OF NESTED NUMBERS

In the previous section, we have examined the nature of infinite paths in surreal numbers and shown that they are ultimately "well-behaved". In this section, we work in the opposite direction and show how to construct surreal numbers that contain infinite paths of a specified kind. We follow the same method as in [5, Section 8].

Let us briefly outline the main ideas. Our aim is to construct "nested numbers" that correspond to nested expressions like

$$a = \sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} - \mathrm{e}^{\sqrt{\log\log \omega} + \mathrm{e}^{\sqrt{\log\log\log \omega} - \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}} \tag{6.1}$$

Nested expressions of this kind will be presented through so-called *coding sequences* Σ. Once we have fixed such a coding sequence Σ, numbers a of the form (6.1) need to satisfy a sequence of natural inequalities: for any $c \in \mathbb{R}$ with $c > 1$, we require that

$$\begin{array}{rcccl} c^{-1}\sqrt{\omega} & < & a & < & c\sqrt{\omega} \\ \sqrt{\omega} + \mathrm{e}^{c^{-1}\sqrt{\log \omega}} & < & a & < & \sqrt{\omega} + \mathrm{e}^{c\sqrt{\log \omega}} \\ \sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} - \mathrm{e}^{c\sqrt{\log\log \omega}}} & < & a & < & \sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} - \mathrm{e}^{c^{-1}\sqrt{\log\log \omega}}} \\ \sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} - \mathrm{e}^{\sqrt{\log\log \omega} + c^{-1}\mathrm{e}^{\sqrt{\log\log\log \omega}}}} & < & a & < & \sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} - \mathrm{e}^{\sqrt{\log\log \omega} + c\mathrm{e}^{\sqrt{\log\log\log \omega}}}} \\ & & \vdots & & \end{array}$$

Numbers that satisfy these constraints are said to be *admissible*. Under suitable conditions, the class **Ad** of admissible numbers forms a convex surreal substructure. This will be detailed in Sections 6.1 and 6.2, where we will also introduce suitable coordinates

$$\begin{array}{rcll} a_{;0} & = & \sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} - \mathrm{e}^{\sqrt{\log\log \omega} + \mathrm{e}^{\sqrt{\log\log\log \omega} - \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}} & = a \\ a_{;1} & = & \sqrt{\log \omega} - \mathrm{e}^{\sqrt{\log\log \omega} + \mathrm{e}^{\sqrt{\log\log\log \omega} - \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}} & = \log (a_{;0} - \sqrt{\omega}) \\ a_{;2} & = & \sqrt{\log\log \omega} + \mathrm{e}^{\sqrt{\log\log\log \omega} - \mathrm{e}^{\cdot^{\cdot^{\cdot}}}} & = \log \left(\sqrt{\log \omega} - a_{;1}\right) \\ & & \vdots & \end{array}$$

for working with numbers in **Ad**.

The notation (6.1) also suggests that each of the numbers $a_{;0} - \sqrt{\omega}$, $\sqrt{\log \omega} - a_{;1}$, … should be a monomial. An admissible number $a \in \mathbf{Ad}$ is said to be *nested* if this is indeed the case. The main result of this section is Theorem 1.2, i.e. that the class **Ne** of nested numbers forms a surreal substructure. In other words, the notation (6.1) is ambiguous, but can be disambiguated using a single surreal parameter.

6.1. Coding sequences

DEFINITION 6.1. *Let* $\Sigma := (\varphi_i, \varepsilon_i, \psi_i, \iota_i, \alpha_i)_{i \in \mathbb{N}} \in (\mathbf{No} \times \{-1, 1\} \times \mathbf{No} \times \{-1, 1\} \times \omega^{\mathbf{On}})^{\mathbb{N}}$. *We say that* Σ *is a* **coding sequence** *if for all* $i \in \mathbb{N}$, *we have*

a) $\psi_i \in \mathbf{No}_{\succ}$;

b) $\varphi_{i+1} \in \mathbf{No}_{\succ, \alpha_i} \cup \{0\}$;

c) $(\alpha_i = 1) \Longrightarrow (\psi_i = 0 \wedge (\psi_{i+1} = 0 \Longrightarrow \alpha_{i+1} = 1))$;

d) $(\varphi_{i+1} = \psi_{i+1} = 0) \Longrightarrow (\alpha_i \geqslant \alpha_{i+1} \wedge \varepsilon_{i+1} = \iota_{i+1} = 1)$;

e) $\exists j > i, (\varphi_j \neq 0 \vee \psi_j \neq 0)$.

Taking $\alpha_i = 1$ for all $i \in \mathbb{N}$, we obtain a reformulation of the notion of coding sequences in [5, Section 8.1]. If $\Sigma = (\varphi_i, \varepsilon_i, \psi_i, \iota_i, \alpha_i)_{i \in \mathbb{N}}$ is a coding sequence and $k \in \mathbb{N}$, then we write

$$\Sigma_{\nearrow k} := (\varphi_{k+i}, \varepsilon_{k+i}, \psi_{k+i}, \iota_{k+i}, \alpha_{k+i})_{i \in \mathbb{N}},$$

which is also a coding sequence.

LEMMA 6.2. *Let P be an infinite path in a number $a\in\mathbf{No}$ without any bad index for a. Let $\varphi_0 := a_{\succ\mathfrak{m}_{P,0}}$ and $\varphi_i := (a_{P,i})_{\succ\mathfrak{m}_{P,i}}$ for all $i\in\mathbb{N}^>$. Then $\Sigma_P := (\varphi_i, r_{P,i}, \psi_{P,i+1}, \iota_{P,i}, \alpha_{P,i})_{i\in\mathbb{N}}$ is a coding sequence.*

Proof. Let $i\in\mathbb{N}$. We have $r_{P,i}\in\{-1,1\}$ because i is a good index for (P,a). We have $\psi_{P,i+1}\in\mathbf{No}_{\succ}$ and $a_{P,i+1}\in\mathbf{No}_{\succ,\alpha_i}$ by the definition of hyperserial expansions. If $i>0$ and $\varphi_i\neq 0$, then we have $\varphi_i\in\mathbf{No}^{>,\succ}$ because $a_{P,i}\in\mathbf{No}^{>,\succ}$ by the definition of paths. Lemma 5.27 also yields $\varphi_i\in\mathbf{No}_{\succ,\alpha_i}$. This proves the conditions a) and b) for coding sequences. Assume that $\alpha_i=1$. Then by the definition of hyperserial expansions, we have $\psi_{P,i+1}=0$ and $u_{P,i+1}=a_{P,i+1}$ is not tail-atomic. Assume that $\psi_{P,i+2}=0$. Then $\operatorname{supp} u_{P,i+1}\succ 1$ so $\iota_{P,i+2}=1$. We have $u_{P,i+1}=\varphi_{i+1} + r_{P,i+1}\mathfrak{a}$ where $\mathfrak{a} := E_{\alpha_{P,i+1}}^{u_{P,i+2}}$ and $u_{P,i+1}$ is not tail-atomic. This implies that $\mathfrak{a}$ is not log-atomic, so $\alpha_{P,i+1}=1$. Thus c) is valid.

Assume that $\varphi_{i+1}=\psi_{P,i+2}=0$. Recall that $a_{P,i+1}=r_{P,i+1}(E_{\alpha_{P,i+1}}^{u_{P,i+2}})^{\iota_{P,i+1}}=u_{P,i+1}\in\mathbf{No}^{>,\succ}$, so $r_{P,i+1}=\iota_{P,i+1}=1$. Since $E_{\alpha_{P,i}}^{u_{P,i+1}}\notin\mathbf{Mo}_{\alpha_{P,i}\omega}$, we have $u_{P,i+1}\notin\mathbf{Mo}_{\alpha_{P,i}\omega}$, whence $\alpha_{P,i+1}\leqslant\alpha_{P,i}$. This proves d).

Assume now for contradiction that there is an $i_0\in\mathbb{N}$ with $\varphi_{P,j}=\psi_{P,j+1}=0$ for all $j>i_0$. By d), we have $r_{P,j}=\iota_{P,j}=1$ for all $j>i_0$, and the sequence $(\alpha_{P,j})_{j>i_0}$ is non-increasing, hence eventually constant. Let $i_1>i_0$ with $\alpha_{P,i_1}=\alpha_{P,j}$ for all $j>i_1$. For $k\in\mathbb{N}$, we have $a_{P,i_1}=E_{\alpha_{P,i_1}k}\,a_{P,i_1+k}$ so $a_{P,i_1}\in\bigcap_{k\in\mathbb{N}}E_{\alpha_{P,i_1}k}\mathbf{Mo}_{\alpha_{P,i_1}}=\mathbf{Mo}_{\alpha_{P,i_1}\omega}$. Therefore $E_{\alpha_{P,i_1}}^{a_{P,i_1+1}}$ is $L_{<\alpha_{P,i_1+1}\omega}$-atomic: a contradiction. We deduce that e) holds as well. □

We next fix some notations. For all $i,j\in\mathbb{N}$ with $i\leqslant j$, we define partial functions Φ_i, $\Phi_{i;}$ and $\Phi_{j;i}$ on $\mathbf{No}$ by

$$\begin{aligned}\Phi_i(a) &:= \varphi_i+\varepsilon_i\,\mathrm{e}^{\psi_{i-1}}(E_{\alpha_{i-1}}a)^{\iota_{i-1}},\\ \Phi_{j;i}(a) &:= (\Phi_i\circ\cdots\circ\Phi_{j-1})(a),\\ \Phi_{i;} &:= \Phi_{i;0}.\end{aligned}$$

The domains of these functions are assumed to be largest for which these expressions make sense. We also write

$$\begin{aligned}\sigma_{i;} = \sigma_{;i} &:= \prod_{k<i}\varepsilon_k\iota_k\\ \sigma_{j;i} = \sigma_{i;j} &:= \prod_{i\leqslant k<j}\varepsilon_k\iota_k\end{aligned}$$

We note that on their respective domains, the functions Φ_i, $\Phi_{i;}$, and $\Phi_{j;i}$ are strictly increasing if $\varepsilon_i\iota_i=1$, $\sigma_{i;}=1$, and $\sigma_{j;i}=1$, respectively, and strictly decreasing in the contrary cases. We will write $\Phi_{;i}$ and $\Phi_{i;j}$ for the partial inverses of $\Phi_{i;}$ and $\Phi_{j;i}$. We will also use the abbreviations

$$\begin{aligned}a_{i;} &:= \Phi_{i;}(a) & a_{j;i} &:= \Phi_{j;i}(a)\\ a_{;i} &:= \Phi_{;i}(a) & a_{i;j} &:= \Phi_{i;j}(a)\end{aligned}$$

For all $i\in\mathbb{N}$, we set

$$\begin{aligned}L_i^{[1]} &:= (\varphi_i-\sigma_{;i}\mathbb{R}^>\operatorname{supp}\varphi_i)_{i;} & R_i^{[1]} &:= (\varphi_i+\sigma_{;i}\mathbb{R}^>\operatorname{supp}\varphi_i)_{i;}\\ L_i^{[2]} &:= \left(\varphi_i+\varepsilon_i\,\mathrm{e}^{\psi_i-\varepsilon_i\sigma_{;i}\mathbb{R}^>\operatorname{supp}\psi_i}\right)_{i;} & R_i^{[2]} &:= \left(\varphi_i+\varepsilon_i\,\mathrm{e}^{\psi_i+\varepsilon_i\sigma_{;i}\mathbb{R}^>\operatorname{supp}\psi_i}\right)_{i;}\\ L_i^{[3]} &:= \begin{cases}\varnothing & \text{if } \varphi_{i+1}=0\\ & \text{or } \sigma_{;i+1}\varepsilon_{i+1}=-1\\ (\mathcal{L}_{\alpha_i}\varphi_{i+1})_{i+1;} & \text{otherwise}\end{cases} & R_i^{[3]} &:= \begin{cases}\varnothing & \text{if } \varphi_{i+1}=0\\ & \text{or } \sigma_{;i+1}\varepsilon_{i+1}=1\\ (\mathcal{L}_{\alpha_i}\varphi_{i+1})_{i+1;} & \text{otherwise}\end{cases}\\ L_i &:= L_i^{[1]}\cup L_i^{[2]}\cup L_i^{[3]} & R_i &:= R_i^{[1]}\cup R_i^{[2]}\cup R_i^{[3]}\\ L &:= \bigcup_{i\in\mathbb{N}}L_i. & R &:= \bigcup_{i\in\mathbb{N}}R_i.\end{aligned}$$

Note that

$$\begin{aligned} \varphi_i &= 0 \iff L_i^{[1]} = R_i^{[1]} = \emptyset \quad \text{and} \\ \psi_i &= 0 \iff L_i^{[2]} = R_i^{[2]} = \emptyset. \end{aligned}$$

The following lemma generalizes [5, Lemma 8.1].

LEMMA 6.3. *If $a \in (L|R)$, then $a_{;i}$ is well defined for all $i \in \mathbb{N}$.*

Proof. Let us prove the lemma by induction on i. The result clearly holds for $i=0$. Assuming that $a_{;i}$ is well defined, let $j>i$ be minimal such that $\varphi_j \neq 0$ or $\psi_j \neq 0$. Note that we have $\alpha_i \geqslant \alpha_{i+1} \geqslant \cdots \geqslant \alpha_j$, so $E_{\alpha_i} \circ E_{\alpha_{i+1}} \circ \cdots \circ E_{\alpha_j} = E_\gamma$ where $\gamma = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$. Applying $\Phi_{;i}$ to the inequality

$$L_j < a < R_j,$$

we obtain

$$\sigma_{;i} (L_j)_{;i} < \sigma_{;i} a_{;i} < \sigma_{;i} (R_j)_{;i}.$$

Now if $\varphi_j \neq 0$, then

$$\begin{aligned} (L_j)_{;i} &\supseteq \varphi_i + \varepsilon_i \mathrm{e}^{\psi_i} (E_\gamma(\varphi_j - \sigma_{;j} \mathbb{R}^{>} \operatorname{supp} \varphi_j))^{\iota_i} \\ (R_j)_{;i} &\supseteq \varphi_i + \varepsilon_i \mathrm{e}^{\psi_i} (E_\gamma(\varphi_j + \sigma_{;j} \mathbb{R}^{>} \operatorname{supp} \varphi_j))^{\iota_i}, \end{aligned}$$

whence

$$\sigma_{;i} \mathrm{e}^{\psi_i} (E_\gamma(\varphi_j - \sigma_{;j} \mathbb{R}^{>} \operatorname{supp} \varphi_j))^{\iota_i} < \sigma_{;i} \frac{a_{;i} - \varphi_i}{\varepsilon_i} < \sigma_{;i} \mathrm{e}^{\psi_i} (E_\gamma(\varphi_j + \sigma_{;j} \mathbb{R}^{>} \operatorname{supp} \varphi_j))^{\iota_i}.$$

Both in the cases when $\sigma_{;i} = 1$ and when $\sigma_{;i} = -1$, it follows that $((a_{;i} - \varphi_i) / \varepsilon_i \mathrm{e}^{\psi_i})^{\iota_i}$ is bounded from below by the hyperexponential E_γ of a number. Thus $a_{;j} = L_\gamma(((a_{;i} - \varphi_i) / (\varepsilon_i \mathrm{e}^{\psi_i}))^{\iota_i})$ is well defined and so is each $a_{;k}$ for $i \leqslant k < j$. If $\varphi_j = 0$, then we have $\psi_j \neq 0$ and

$$\begin{aligned} (L_j)_{;i} &\supseteq \varphi_i + \varepsilon_i \mathrm{e}^{\psi_i} \big(E_\gamma\big(\mathrm{e}^{\psi_j - \varepsilon_j \sigma_{;j} \mathbb{R}^{>} \operatorname{supp} \psi_j}\big)\big)^{\iota_i}, \\ (R_j)_{;i} &\supseteq \varphi_i + \varepsilon_i \mathrm{e}^{\psi_i} \big(E_\gamma\big(\mathrm{e}^{\psi_j + \varepsilon_j \sigma_{;j} \mathbb{R}^{>} \operatorname{supp} \psi_j}\big)\big)^{\iota_i}. \end{aligned}$$

Hence

$$\varepsilon_i \mathrm{e}^{\psi_i} \big(E_\gamma\big(\mathrm{e}^{\psi_j - \varepsilon_j \sigma_{;j} \mathbb{R}^{>} \operatorname{supp} \psi_j}\big)\big)^{\iota_i} < a_{;i} - \varphi_i < \varepsilon_i \mathrm{e}^{\psi_i} \big(E_\gamma\big(\mathrm{e}^{\psi_j + \varepsilon_j \sigma_{;j} \mathbb{R}^{>} \operatorname{supp} \psi_j}\big)\big)^{\iota_i}$$

Both in the cases when $\varepsilon_i = 1$ and when $\varepsilon_i = -1$, it follows that $((a_{;i} - \varphi_i) / \varepsilon_i \mathrm{e}^{\psi_i})^{\iota_i}$ is bounded from below by the hyperexponential E_γ of a number, so $a_{;j}$ is well defined and so is each $a_{;k}$ for $i \leqslant k < j$. □

6.2. Admissible sequences

DEFINITION 6.4. *Let $\Sigma := (\varphi_i, \varepsilon_i, \psi_i, \iota_i, \alpha_i)_{i \in \mathbb{N}}$ be a coding sequence and let $a \in \mathbf{No}$. We say that a is Σ-**admissible** if $a_{;i}$ is well defined for all $i \in \mathbb{N}$ and*

$$\begin{aligned} a_{;i} &= \varphi_i \mathbin{+\!\!\!+} \varepsilon_i \mathrm{e}^{\psi_i} (E_{\alpha_i} a_{;i+1})^{\iota_i}, \\ \operatorname{supp} \psi_i &\succ \log E_{\alpha_i} a_{;i+1}, \text{ and} \\ \varphi_{i+1} &\lhd \sharp_{\alpha_i}(a_{;i+1}) \quad \text{if } \varphi_{i+1} \neq 0. \end{aligned}$$

We say that Σ is **admissible** *if there exists a Σ-admissible number.*

Note that we do not ask that $\mathrm{e}^{\psi_i}(E_{\alpha_i} a_{;i+1})^{\iota_i}$ be a hyperserial expansion, nor even that $E_{\alpha_i} a_{;i+1}$ be a monomial. For the rest of the section, we fix a coding sequence $\Sigma = (\varphi_i, \varepsilon_i, \psi_i, \iota_i, \alpha_i)_{i \in \mathbb{N}}$. We write $\mathbf{Ad}$ for the class of Σ-admissible numbers. If $a \in \mathbf{Ad}$, then the definition of $\mathbf{Ad}$ implicitly assumes that $a_{;i}$ is well defined for all $i \in \mathbb{N}$. Note that if Σ is admissible, then so is $\Sigma_{\nearrow k}$ for $k \in \mathbb{N}$. We denote by $\mathbf{Ad}_{\nearrow k}$ the corresponding class of $\Sigma_{\nearrow k}$-admissible numbers.

The main result of this subsection is the following generalization of [5, Proposition 8.2]:

PROPOSITION 6.5. *We have* $\mathbf{Ad} = (L \mid R)$.

Proof. Let $a \in \mathbf{Ad} \cup (L \mid R)$ and let $i \in \mathbb{N}$. We have $a_{;i} \in \mathbf{No}^{>,\succ}$. If $\sigma_{;i} = 1$, then $\Phi_{i;}$ is strictly increasing so we have

$$\begin{aligned} L_i^{[1]} < a < R_i^{[1]} &\Longleftrightarrow (L_i^{[1]})_{;i} < a_{;i} < (R_i^{[1]})_{;i} \\ &\Longleftrightarrow \varphi_i - \mathbb{R}^{>} \operatorname{supp} \varphi_i < a_{;i} < \varphi_i + \mathbb{R}^{>} \operatorname{supp} \varphi_i \\ &\Longleftrightarrow a_{;i} - \varphi_i \prec \operatorname{supp} \varphi_i \\ &\Longleftrightarrow \varphi_i \trianglelefteq a_{;i}. \end{aligned}$$

If $\sigma_{;i} = -1$, then $\Phi_{i;}$ is strictly decreasing and likewise we obtain $L_{i;} < a < R_{i;} \Longleftrightarrow \varphi_i \trianglelefteq a_{;i}$.

We have $\iota_i \log E_{\alpha_i} a_{;i+1} = \iota_i \left(\log \frac{a_{;i} - \varphi_i}{\varepsilon_i \mathrm{e}^{\psi_i}} \right)$. If $\sigma_{;i} = 1$, then $\Phi_{i;}$ is strictly increasing so we have

$$\begin{aligned} L_i^{[2]} < a < R_i^{[2]} &\Longleftrightarrow \varphi_i + \varepsilon_i \mathrm{e}^{\psi_i - \varepsilon_i \mathbb{R}^{>} \operatorname{supp} \psi_i} < a_{;i} < \varphi_i + \varepsilon_i \mathrm{e}^{\psi_i + \varepsilon_i \mathbb{R}^{>} \operatorname{supp} \psi_i} \\ &\Longleftrightarrow -\mathbb{R}^{>} \operatorname{supp} \psi_i < \log \frac{a_{;i} - \varphi_i}{\varepsilon_i \mathrm{e}^{\psi_i}} < \mathbb{R}^{>} \operatorname{supp} \psi_i \\ &\Longleftrightarrow \operatorname{supp} \psi_i \succ \log \frac{a_{;i} - \varphi_i}{\varepsilon_i \mathrm{e}^{\psi_i}} \\ &\Longleftrightarrow \log E_{\alpha_i} a_{;i+1} \prec \operatorname{supp} \psi_i. \end{aligned}$$

Likewise, we have $L_i^{[2]} < a < R_i^{[2]} \Longleftrightarrow \log E_{\alpha_i} a_{;i+1} \prec \operatorname{supp} \psi_i$ if $\sigma_{;i} = -1$.

Assume that $\varphi_{i+1} \neq 0$ and $\sigma_{;i+1} = 1$. If $\varepsilon_{i+1} = 1$, then we have $a_{;i+1} > \varphi_{i+1}$. Hence

$$\begin{aligned} L_i^{[3]} \cup L_{i+1}^{[1]} < a < R_i^{[3]} \cup R_{i+1}^{[1]} &\Longleftrightarrow \mathcal{L}_{\alpha_i} \varphi_{i+1} < a_{;i+1} \wedge \varphi_{i+1} \trianglelefteq a_{;i+1} \\ &\Longleftrightarrow \varphi_{i+1} < \sharp_{\alpha_i}(a_{;i+1}) \wedge \varphi_{i+1} \trianglelefteq a_{;i+1} \\ &\Longleftrightarrow \varphi_{i+1} \vartriangleleft \sharp_{\alpha_i}(a_{;i+1}). \end{aligned}$$

If $\varepsilon_{i+1} = -1$, then we have $a_{;i+1} > \varphi_{i+1}$, whence

$$\begin{aligned} L_i^{[3]} \cup L_{i+1}^{[1]} < a < R_i^{[3]} \cup R_{i+1}^{[1]} &\Longleftrightarrow a_{;i+1} < \mathcal{L}_{\alpha_i} \varphi_{i+1} \wedge \varphi_{i+1} \trianglelefteq a_{;i+1} \\ &\Longleftrightarrow \sharp_{\alpha_i}(a_{;i+1}) < \varphi_{i+1} \wedge \varphi_{i+1} \trianglelefteq a_{;i+1} \\ &\Longleftrightarrow \varphi_{i+1} \vartriangleleft \sharp_{\alpha_i}(a_{;i+1}). \end{aligned}$$

Symmetric arguments apply when $\varphi_{i+1} \neq 0$ and $\sigma_{;i+1} = -1$.

We deduce by definition of $\mathbf{Ad}$ that $\mathbf{Ad} = \bigcap_{i \in \mathbb{N}} (L_i \mid R_i) = (L \mid R)$. □

As a consequence of this last proposition and [5, Proposition 4.29(a)], the class $\mathbf{Ad}$ is a surreal substructure if and only if Σ is admissible.

Example 6.6. Consider the coding sequence $\Sigma_0 = (\varphi_i, \varepsilon_i, \iota_i, \psi_i, \alpha_i)_{i \in \mathbb{N}}$ where for all $i \in \mathbb{N}$, we have

$$\begin{aligned} \varphi_i &= L_{\omega^{2i}} \omega + L_{\omega^{2i}2} \omega + L_{\omega^{2i}3} \omega + \cdots, \\ \varepsilon_i &= 1, \\ \psi_i &= L_{\omega^{2i+1}} \omega + L_{\omega^{2i+1}2} \omega + L_{\omega^{2i+1}3} \omega + \cdots, \\ \iota_i &= -1 \quad \text{and} \\ \alpha_i &= \omega^{2i+1}. \end{aligned}$$

We use the notations from Section 6.1. We claim that Σ_0 is admissible. Indeed for $i \in \mathbb{N}$, set

$$a_i := \varphi_0 + \mathrm{e}^{\psi_0} \left(E_\omega^{\varphi_1 + \mathrm{e}^{\psi_1} \left(E_{\omega^3}^{\cdot^{\cdot^{\cdot \varphi_i}}} \right)^{-1}} \right)^{-1}.$$

Given $j \in \mathbb{N}$ and $i > j$, we have $L_j < a_i$ and $a_i < R_j$. We deduce that $L < R$, whence Σ_0 is admissible.

LEMMA 6.7. *Let $a \in \mathbf{Ad}$ and $b \in \mathbf{No}$ be such that $a - \varphi_0$ and $b - \varphi_0$ have the same sign and the same dominant monomial. Then $b \in \mathbf{Ad}$.*

Proof. For $x, y \in \mathbf{No}^{\neq}$, we write $x \equiv y$ if $x \asymp y$ and x and y have the same sign. Let us prove by induction on $i \in \mathbb{N}$ that $b_{;i}$ is defined and that $a_{;i} - \varphi_i \equiv b_{;i} - \varphi_i$. Since this implies that $\varphi_i \trianglelefteq b_{;i}$, that $\log \frac{b_{;i} - \varphi_i}{\varepsilon_i \mathrm{e}^{\psi_i}} \prec \operatorname{supp} \psi_i$, and that $\varphi_i \vartriangleleft \sharp_{\alpha_{i-1}}(b_{;i})$ if $i > 0$, this will yield $b \in \mathbf{Ad}$.

The result follows from our hypothesis if $i = 0$. Assume now that $a_{;i} - \varphi_i \equiv b_{;i} - \varphi_i$ and let us prove that $a_{;i+1} - \varphi_{i+1} \equiv b_{;i+1} - \varphi_{i+1}$. Let

$$c_i := \left(\frac{b_{;i} - \varphi_i}{\varepsilon_i \mathrm{e}^{\psi_i}} \right).$$

We have $c_i \equiv \left(\frac{a_{;i} - \varphi_i}{\varepsilon_i \mathrm{e}^{\psi_i}} \right)^{\iota_i} = E_{\alpha_i} a_{;i+1} \in \mathbf{No}^{>, \succ}$, so $b_{;i+1} = L_{\alpha_i}(c_i)$ is defined. Moreover $c_i \in \mathcal{E}_{\alpha_i}[E_{\alpha_i} a_{;i+1}]$ so $b_{;i+1} \in \mathcal{L}_{\alpha_i}[a_{;i+1}]$. Since $\varphi_{i+1} \vartriangleleft \sharp_{\alpha_i}(a_{;i+1}) = \sharp_{\alpha_i}(b_{;i+1})$, we deduce that $b_{;i+1} - \varphi_{i+1} \sim a_{;i+1} - \varphi_{i+1}$, whence in particular $b_{;i+1} - \varphi_{i+1} \equiv a_{;i+1} - \varphi_{i+1}$. This concludes the proof. □

COROLLARY 6.8. *We have $\mathbf{Ad}_{\nearrow 1} = \mathcal{L}_{\alpha_0}[\mathbf{Ad}_{\nearrow 1}]$.*

Proof. For $b \in \mathbf{Ad}_{\nearrow 1}$, and $c \in \mathcal{L}_{\alpha_0}[b]$, we have $\varphi_1 \vartriangleleft \sharp_{\alpha_0}(b) = \sharp_{\alpha_0}(c)$ so $c - \varphi_1 \sim b - \varphi_1$. We conclude with the previous lemma. □

LEMMA 6.9. *For $a, b \in \mathbf{Ad}$ and $i \in \mathbb{N}^>$, we have $L_{\alpha_{i-1}} a_{;i} < \mathcal{E}_{\alpha_{i-1}} b_{;i}$.*

Proof. Let $j > i$ be minimal with $\varphi_j \neq 0$ or $\psi_j \neq 0$. We thus have $a_{;j}, b_{;j} \in \mathcal{P}[\varphi_j + \varepsilon_j \mathrm{e}^{\psi_j}]$ so $\log a_{;j} \prec b_{;j}$. We have $a_{;i} = E_{\alpha_i + \cdots + \alpha_{j-1}} a_{;j}$ and $b_{;i} = E_{\alpha_i + \cdots + \alpha_{j-1}} b_{;j}$ where $\alpha_i \geqslant \cdots \geqslant \alpha_j \geqslant 1$. We deduce by induction using Lemma 5.28 that $L_{\alpha_{i-1}} a_{;i} < \mathcal{E}_{\alpha_{i-1}} b_{;i}$. □

6.3. Nested sequences

In this subsection, we assume that Σ is admissible. For $k \in \mathbb{N}$ we say that a $\Sigma_{\nearrow k}$-admissible number a is $\Sigma_{\nearrow k}$*-nested* if we have $E_{\alpha_{k+i}} a_{k;i+1} \in \mathbf{Mo}_{\alpha_{k+i}} \setminus L_{<\alpha_{k+i}} \mathbf{Mo}_{\alpha_{k+i}\omega}$ for all $i \in \mathbb{N}$. We write $\mathbf{Ne}_{\nearrow k}$ for the class of $\Sigma_{\nearrow k}$-nested numbers. For $k = 0$ we simply say that a is Σ-nested and we write $\mathbf{Ne} := \mathbf{Ne}_{\nearrow 0}$.

DEFINITION 6.10. *We say that Σ is* **nested** *if for all $k \in \mathbb{N}$, we have*

$$\mathbf{Ad}_{\nearrow k} = \varphi_k + \varepsilon_k \mathrm{e}^{\psi_k} (E_{\alpha_k} \mathbf{Ad}_{\nearrow k+1})^{\iota_k}.$$

Note that the inclusion $\mathbf{Ad}_{\nearrow k} \subseteq \varphi_k + \varepsilon_k \mathrm{e}^{\psi_k} (E_{\alpha_k} \mathbf{Ad}_{\nearrow k+1})^{\iota_k}$ always holds. In [5, Section 8.4], we gave examples of nested and admissible non-nested sequences in the case of transseries, i.e. with $\alpha_i = 1$ for all $i \in \mathbb{N}$. We next give an example in the hyperserial case.

Example 6.11. We claim that the sequence Σ_0 from Example 6.6 is nested. Indeed, let $k \in \mathbb{N}$ and $a \in \mathbf{Ad}_{\nearrow k+1}$. We have $a = \varphi_{k+1} + \mathrm{e}^{\psi_{k+1}} (E_{\omega^{2k+3}} b)^{-1}$ for a certain $b \in \mathbf{No}^{>,\succ}$ with $b \asymp L_{\omega^{2k+4}} \omega$. Let us check that the conditions of Definition 6.4 are satisfied for $c := \varphi_k + \mathrm{e}^{\psi_k} (E_{\omega^{2k+1}} a)^{-1}$.

First let $\mathfrak{m} \in \operatorname{supp} \psi_k$. We want to prove that $\mathfrak{m} \succ \log E_{\omega^{2k+1}} a$. We have $\mathfrak{m} = L_{\omega^{2k+1} n} \omega$ for a certain $n \in \mathbb{N}^>$. Now $a < 2 L_{\omega^{2k+2}} \omega$, so $\log E_{\omega^{2k+1}} a \prec E_{\omega^{2k+1} 2}^{L_{\omega^{2k+2}} \omega} = L_{\omega^{2k+2}} (\omega + 2) \prec \mathfrak{m}$.

Secondly, let $\mathfrak{n} \in \operatorname{supp} \varphi_k$. We want to prove that $\mathfrak{n} \succ \mathrm{e}^{\psi_k} (E_{\omega^{2k+1}} a)^{-1}$. We have $\mathfrak{n} = L_{\omega^{2k} n} \omega$ for a certain $n \in \mathbb{N}^>$. Then $\mathrm{e}^{\psi_k} (E_{\omega^{2k+1}} a)^{-1} \prec \mathrm{e}^{2\psi_k}$ by the previous paragraph. Now $2 \psi_k + \mathbb{N} < 3 L_{\omega^{2k+1}} \omega$ so $\mathrm{e}^{2\psi_k} \prec \mathrm{e}^{3 L_{\omega^{2k+1}} \omega} \prec \mathfrak{n}$.

Finally, we claim that $\varphi_{k+1} \lhd \sharp_{\omega^{2k+1}}(a)$. This is immediate since the dominant term τ of $\mathrm{e}^{\psi_{k+1}} (E_{\omega^{2k+3}} b)^{-1}$ is positive infinite, so $\varphi_{k+1} \lhd \varphi_{k+1} + \tau \trianglelefteq \sharp_{\omega^{2k+1}}(a)$. Therefore Σ_0 is nested.

A crucial feature of nested sequences is that they are sufficient to describe nested expansions. This is the content of Theorem 6.15 below.

LEMMA 6.12. *Let $b \in \mathbf{Ad}_{\nearrow 1}$. If $\alpha_0 > 1$, or $\alpha_0 = 1$ and $b_\succ$ is not tail-atomic, then the hyperserial expansion of $E_{\alpha_0} \sharp_{\alpha_0}(b)$ is*

$$E_{\alpha_0} \sharp_{\alpha_0}(b) \;=\; E_{\alpha_0}^{\sharp_{\alpha_0}(b)}$$

If $\alpha_0 = 1$, $b_\succ = \psi + \iota \mathfrak{b}$ is tail-atomic, and $\mathrm{e}^{\mathfrak{b}} = L_\beta E_\alpha^u$ is a hyperserial expansion, then $\psi \in \mathbf{Ad}_{\nearrow 1}$ and the hyperserial expansion of $\exp b_\succ$ *is*

$$\exp b_\succ \;=\; \mathrm{e}^{\psi} (L_\beta E_\alpha^u)^\iota.$$

Proof. Recall that $\sharp_1(b) = b_\succ$. By Corollary 6.8, we have $\sharp_{\alpha_0}(b) \in \mathbf{Ad}_{\nearrow 1}$,. So we may assume without loss of generality that $b = \sharp_{\alpha_0}(b)$.

We claim that $E_{\alpha_0}^b \in \mathbf{Mo}_{\alpha_0} \setminus L_{<\alpha_0} \mathbf{Mo}_{\alpha_0 \omega}$. Assume for contradiction that $E_{\alpha_0}^b \in L_{<\alpha_0} \mathbf{Mo}_{\alpha_0 \omega}$ and write $E_{\alpha_0}^b = L_\gamma \mathfrak{a}$ accordingly. Then Corollary 5.6 implies that $\gamma = 0$, in which case we define $n := 0$, or $\alpha_0 = \omega^{\mu+1}$ for some ordinal μ and $\gamma = (\alpha_0)_{/\omega} n$ for some $n \in \mathbb{N}^>$. Therefore $E_{\alpha_0}^{b+n} \in \mathbf{Mo}_{\alpha_0 \omega}$, so $b + n \in \mathbf{Mo}_{\alpha_0 \omega}$. This implies that

$$b \;=\; (b + n) + (-n).$$

Recall that $\varphi_1 \lhd b$. Assume that $n = 0$, so $\varphi_1 = 0$. Since b is log-atomic, we also have $\psi_1 = 0$. Let $j > 1$ be minimal with $\varphi_j \neq 0$ or $\psi_j \neq 0$. We have $\alpha_1 \geqslant \cdots \geqslant \alpha_{j-1}$ and $b_{1;j} = L_{\alpha_1 + \cdots + \alpha_{j-1}} b \in \mathbf{Mo}_{\alpha_{j-1} \omega}$. In particular, the number $b_{1;j}$ is log-atomic. If $\varphi_j \neq 0$, this contradicts the fact that $\varphi_j \lhd b_{1;j}$. If $\psi_j \neq 0$, then $\operatorname{supp} \psi_j \succ \log ((b_{1;j} \mathrm{e}^{-\psi_j})^{\iota_j})$ implies

$$\log b_{1;j} \;=\; \psi_j + \log ((b_{1;j} \mathrm{e}^{-\psi_j})^{\iota_j}).$$

But then $\log b_{1;j}$ is not a monomial: a contradiction. Assume now that $n > 0$. So $\varphi_1 = b + n$ and $b = \varphi_1 + (-n)$. But then $b_{1;2}$ is not defined: a contradiction. We conclude that $E_{\alpha_0}^b \notin L_{<\alpha_0} \mathbf{Mo}_{\alpha_0 \omega}$.

If $\alpha_0 > 1$, or if $\alpha_0 = 1$ and b is not tail-atomic, then our claim yields the result. Assume now that $\alpha_0 = 1$ and that $b = \psi + \iota \mathfrak{b}$ is tail-atomic where $\iota \in \{-1, 1\}$, $\psi \in \mathbf{No}_\succ$, and $\mathrm{e}^{\mathfrak{b}} = L_\beta E_\alpha^u \in \mathbf{Mo}_\omega$ is a hyperserial expansion. Then the hyperserial expansion of $\exp b$ is $\exp b = \mathrm{e}^\psi (L_\beta E_\alpha^u)^\iota$.

We next show that $\psi \in \mathbf{Ad}_{\nearrow 1}$. If $\mathfrak{b} \notin \mathrm{e}^{\psi_1} (E_{\alpha_1} \mathbf{Ad}_{\nearrow 2})^{\iota_1}$, then $\varphi_1 \lhd \psi$, and we conclude with Lemma 6.7 that $\psi \in \mathbf{Ad}_{\nearrow 1}$. Assume for contradiction that $\mathfrak{b} \in \mathrm{e}^{\psi_1} (E_{\alpha_1} \mathbf{Ad}_{\nearrow 2})^{\iota_1}$. Since $\mathfrak{b}$ is log-atomic, we must have $\psi_1 = 0$. By the definition of coding sequences, this implies that $\iota_1 = 1$ and $\alpha_1 = 1$. So $b = \varphi_1 + \varepsilon_1 \exp(b_{1;2})$, whence $\psi = \varphi_1$, $\iota = \varepsilon_1$, and $\mathfrak{b} = \exp(b_{1;2})$. In particular the number $b_{1;2}$ is log-atomic, hence tail-atomic. Since $b_{1;2} \in \mathbf{Ad}_{\nearrow 2}$, the claim in the second paragraph of the proof, applied to $\Sigma_{\nearrow 1}$, gives $E_1^{b_{1;2}} \notin \mathbf{Mo}_{\omega}$. But then also $\mathfrak{b} \notin \mathbf{Mo}_{\omega}$: a contradiction. ◻

We pursue with two auxiliary results that will be used order to construct a infinite path required in the proof of Theorem 6.15 below.

LEMMA 6.13. *For $a \in \mathbf{Ad}$, there is a finite path P in a with $u_{P,|P|} \in \mathbf{Ad}_{\nearrow 1} - \mathbb{N}$ or $\psi_{P,|P|} \in \mathbf{Ad}_{\nearrow 1} - \mathbb{N}$.*

Proof. By Lemma 5.16, it is enough to find such a path in $E_{\alpha_0} a_{;1}$. Write $\alpha_0 =: \omega^{\mu}$. Assume first that $\mu = 0$, so $\alpha_0 = 1$ and $\psi_0 = 0$. If $(a_{;1})_{\succ}$ is not tail-atomic, then the hyperserial expansion of $\exp\,(a_{;1})_{\succ}$ is $\exp\,(a_{;1})_{\succ} = E_1^{(a_{;1})_{\succ}}$ and $r\, E_1^{(a_{;1})_{\succ}}$ is the dominant term of $\exp a_{;1}$ for some $r \in \mathbb{R}^{\neq}$. Then the path P with $|P| = 1$ and $\tau_{P,0} := r\, E_1^{(a_{;1})_{\succ}}$ satisfies $u_{P,|P|} = (a_{;1})_{\succ} \in \mathbf{Ad}_{\nearrow 1}$. If $(a_{;1})_{\succ}$ is tail-atomic, then there exist $\psi \in \mathbf{Ad}_{\nearrow 1}$, $\iota \in \{-1, 1\}$ and $\mathfrak{a} \in \mathbf{Mo}_{\omega}$ such that the hyperserial expansion of $\exp\,(a_{;1})_{\succ}$ is $\exp\,(a_{;1})_{\succ} = \mathrm{e}^{\psi}\, \mathfrak{a}^{\iota}$. Let $r\, \mathrm{e}^{\psi}\, \mathfrak{a}^{\iota}$ be a term in $\exp a_{;1}$ with $r \in \mathbb{R}^{\neq}$. Then the path P with $|P| = 1$ and $P(0) := r\, \mathrm{e}^{\psi}\, \mathfrak{a}^{\iota}$ satisfies $\psi_{P,|P|} = \psi \in \mathbf{Ad}_{\nearrow 1} - \mathbb{N}$.

Assume now that $\mu > 0$. In view of (3.6), we recall that there are an ordinal $\lambda < \alpha_0$ and a number δ with

$$E_{\alpha_0} a_{;1} \;=\; E_{\lambda}(L_{\lambda}\, E_{\alpha_0}^{\sharp_{\alpha_0}(a_{;1})} + \delta).$$

If μ is a limit ordinal, then by Lemma 6.12, we have a hyperserial expansion $\mathfrak{m} := L_{\lambda}\, E_{\alpha_0}^{\sharp_{\alpha_0}(a_{;1})}$. Let $\tau \in \operatorname{term} \sharp_{\alpha_0}(a_{;1})$ and set $Q(0) = \mathfrak{m}$ and $Q(1) := \tau$, so that Q is a path in $\mathfrak{m}$. By Lemma 5.15, there is a subpath in $E_{\alpha_0} a_{;1}$, hence also a path P in $E_{\alpha_0} a_{;1}$, with $\tau_{P,|P|-1} = \mathfrak{m}$. So $u_{P,|P|} = \sharp_{\alpha_0}(a_{;1}) \in \mathbf{Ad}_{\nearrow 1}$. If μ is a successor ordinal, then we may choose $\lambda = \omega^{\mu_-} n$ for a certain $n \in \mathbb{N}$. By Lemma 6.12, we have a hyperserial expansion $\mathfrak{m} := E_{\alpha_0}^{\sharp_{\alpha_0}(a_{;1}) - n}$. As in the previous case, there is a path P in $E_{\alpha_0} a_{;1}$ with $\tau_{P,|P|} = \mathfrak{m}$, whence $u_{P,|P|} = \sharp_{\alpha_0}(a_{;1}) - n \in \mathbf{Ad}_{\nearrow 1} - \mathbb{N}$. ◻

COROLLARY 6.14. *For $a \in \mathbf{Ad}$ and $k \in \mathbb{N}$, there is a finite path P in a with $|P| \geqslant k$ and $u_{P,|P|} \in \mathbf{Ad}_{\nearrow k} - \mathbb{N}$ or $\psi_{P,|P|} \in \mathbf{Ad}_{\nearrow k} - \mathbb{N}$.*

Proof. This is immediate if $k = 0$. Assume that the result holds at k and pick a corresponding path P with $u_{P,|P|} \in \mathbf{Ad}_{\nearrow k} - \mathbb{N}$ (resp. $\psi_{P,|P|} \in \mathbf{Ad}_{\nearrow k} - \mathbb{N}$). Note that the dominant term τ of $u_{P,|P|} - \varphi_k$ (resp. $\psi_{P,|P|} - \varphi_k$) lies in $\varepsilon_k \mathrm{e}^{\psi_k} (E_{\alpha_k} \mathbf{Ad}_{\nearrow k+1})^{\iota_k}$ by Lemma 6.7. Moreover τ is a term of $u_{P,|P|}$ (resp. $\psi_{P,|P|}$). By the previous lemma, there is a path Q in τ with $u_{Q,|Q|} \in \mathbf{Ad}_{\nearrow k+1} - \mathbb{N}$ or $\psi_{Q,|Q|} \in \mathbf{Ad}_{\nearrow k+1} - \mathbb{N}$, so $(P(0), \ldots, P(|P|-1), Q(0)) * Q$ satisfies the conditions. ◻

THEOREM 6.15. *There is a $k \in \mathbb{N}$ such that $\Sigma_{\nearrow k}$ is nested.*

Proof. Assume for contradiction that this is not the case. This means that the set Δ of indices $d \in \mathbb{N}$ such that we do not have $\mathbf{Ad}_{\nearrow d} = \varphi_d + \varepsilon_d\, \mathrm{e}^{\psi_d} (E_{\alpha_d} \mathbf{Ad}_{\nearrow d+1})^{\iota_d}$ is infinite. We write $\Delta = \{d_i : i \in \mathbb{N}\}$ where $d_0 < d_1 < \cdots$. Fix $a \in \mathbf{Ad}$ and let $d := d_i \in \Delta$. Let $u \in \mathbf{Ad}_{\nearrow d+1}$ such that

$$\varphi_d + \varepsilon_d\, \mathrm{e}^{\psi_d} (E_{\alpha_d} u)^{\iota_d} \;\notin\; \mathbf{Ad}_{\nearrow d}, \tag{6.2}$$

let $n \in \mathbb{N}$ and let P be any finite path with

$$u_{P,|P|} \;=\; \varphi_d + \varepsilon_d\, \mathrm{e}^{\psi_d} (E_{\alpha_d} u)^{\iota_d} - n.$$

We claim that we can extend P to a path Q with $|Q|>|P|$, $u_{Q,|Q|}\in\mathbf{Ad}_{\nearrow d_{i+3}}-\mathbb{N}$ and such that $|P|$ is a bad index in Q. Indeed, in view of Definition 6.4 for $\mathbf{Ad}_{\nearrow d}$, the relation (6.2) translates into the following three possibilities:

- There is an $\mathfrak{n}\in\operatorname{supp}\psi_d$ with $\mathfrak{n}\preccurlyeq\log E_{\alpha_d}u$. We then have $\log E_{\alpha_d}a_{;d+1}\prec\mathfrak{n}\preccurlyeq\log E_{\alpha_d}u$. By Lemma 6.7 and the convexity of $\mathbf{Ad}_{\nearrow d+1}$, we deduce that $\iota_d(\psi_d)_{\mathfrak{n}}\mathfrak{n}$ lies in the class $\iota_d\log E_{\alpha_d}\mathbf{Ad}_{\nearrow d+1}$, so $\mathrm{e}^{(\psi_d)_{\mathfrak{n}}\mathfrak{n}}\in(E_{\alpha_d}\mathbf{Ad}_{\nearrow d+1})^{\iota_d}$. By Corollary 6.14 for the admissible sequence starting with $(0,1,0,\iota_d,\alpha_d)$ and followed by $\Sigma_{\nearrow d+1}$, there is a finite path R_0 in $\mathrm{e}^{(\psi_d)_{\mathfrak{n}}\mathfrak{n}}$ with $|R_0|\geqslant d_{i+3}-d>2$ and $u_{R_0,|R_0|}\in\mathbf{Ad}_{\nearrow d_{i+3}}-\mathbb{N}$. Taking the logarithm and using Lemma 5.14, we obtain a finite path R_1 in $(\psi_d)_{\mathfrak{n}}\mathfrak{n}$, hence in ψ_d, with $|R_1|\geqslant 2$ and $u_{R_1,|R_1|}=u_{R_0,|R_0|}\in\mathbf{Ad}_{\nearrow d_{i+3}}-\mathbb{N}$. Write $(E_{\alpha_d}a_{;d+1})^{\iota_d}=r\,\mathfrak{m}+\rho$ where $r\in\mathbb{R}^{\neq}$ and $\mathfrak{m}\in\mathbf{Mo}^{\neq}$. Then $\log\mathfrak{m}\asymp E_{\alpha_d}a_{;d+1}\prec\operatorname{supp}\psi_d$, so the hyperserial expansion of $\mathrm{e}^{\psi_d}\mathfrak{m}$ has one of the following forms

$$\begin{aligned}\mathrm{e}^{\psi_d}\mathfrak{m} &= \mathrm{e}^{\psi_d+\delta}(L_\beta E_\alpha^u)^\iota \quad \text{or}\\ \mathrm{e}^{\psi_d}\mathfrak{m} &= \left(E_1^{\psi_d+\delta}\right)^\iota\end{aligned}$$

  where $(L_\beta E_\alpha^u)^\iota$ is a hyperserial expansion and δ is purely large. In both cases, the path $R=(\varepsilon_d r\,\mathrm{e}^{\psi_d}\mathfrak{m})*R_1$ is a finite path R in $\varepsilon_d\mathrm{e}^{\psi_d}(E_{\alpha_d}a_{;d+1})^{\iota_d}$ with $u_{R,|R|}=u_{R_1,|R_1|}\in\mathbf{Ad}_{\nearrow d_{i+3}}-\mathbb{N}$. Since $R(0)$ is a term in $u_{P,|P|}$, we may consider the path $Q:=P*R$. Moreover, since $\tau_{Q,|P|}$ is a term in $\psi_d=\psi_{Q,|P|}$, the index $|P|$ is bad for Q.

- We have $\log E_{\alpha_d}u\prec\operatorname{supp}\psi_d$, but there is an $\mathfrak{m}\in\operatorname{supp}\varphi_d$ with $\mathfrak{m}\preccurlyeq\mathrm{e}^{\psi_d}(E_{\alpha_d}u)^{\iota_d}$. We then have $\mathrm{e}^{\psi_d}(E_{\alpha_d}a_{;d+1})^{\iota_d}\prec\varphi_{\mathfrak{m}}\mathfrak{m}\preccurlyeq\mathrm{e}^{\psi_d}(E_{\alpha_d}u)^{\iota_d}$. By Lemma 6.7 and the convexity of $\mathbf{Ad}_{\nearrow d+1}$, we deduce that $(\varphi_d)_{\mathfrak{m}}\mathfrak{m}$ lies in $\mathrm{e}^{\psi_d}(E_{\alpha_d}\mathbf{Ad}_{\nearrow d+1})^{\iota_d}$. So $L_{\alpha_d}((\mathrm{e}^{-\psi_d}(\varphi_d)_{\mathfrak{m}}\mathfrak{m})^{\iota_d})$ lies in $\mathbf{Ad}_{\nearrow d+1}$. But then also $v:=\sharp_{\alpha_d}(L_{\alpha_d}((\mathrm{e}^{-\psi_d}(\varphi_d)_{\mathfrak{m}}\mathfrak{m})^{\iota_d}))$ lies in $\mathbf{Ad}_{\nearrow d+1}$ by Corollary 6.8. By Corollary 6.14, there is a finite path R_0 in v with $|R_0|>2$ and $u_{R_0,|R_0|}\in\mathbf{Ad}_{\nearrow d_{i+3}}-\mathbb{N}$. Applying Lemma 5.15 to this path R_0 in v, we obtain is a finite path R_1 in $(\mathrm{e}^{-\psi_d}(\varphi_d)_{\mathfrak{m}}\mathfrak{m})^{\iota_d}$ with $u_{R_1,|R_1|}\in\mathbf{Ad}_{\nearrow d_{i+3}}-\mathbb{N}$. Since $(\varphi_d)_{\mathfrak{m}}\mathfrak{m}\in\mathrm{e}^{\psi_d}(E_{\alpha_d}\mathbf{Ad}_{\nearrow d+1})^{\iota_d}$, we have $\operatorname{supp}\psi_d\succ\mathrm{e}^{-\psi_d}(\varphi_d)_{\mathfrak{m}}\mathfrak{m}$. So Lemma 5.16 implies that there is a finite path R in $(\varphi_d)_{\mathfrak{m}}\mathfrak{m}$, hence in φ_d, with $u_{R,|R|}\in\mathbf{Ad}_{\nearrow d_{i+3}}-\mathbb{N}$. We have $R(0)\in\operatorname{term}\varphi_d\setminus\mathbb{R}\subseteq\operatorname{term}u_{P,|P|}$, so $Q:=P*R$ is a path. Write τ for the dominant term of $\varepsilon_d\mathrm{e}^{\psi_d}(E_{\alpha_d}u')^{\iota_d}$. The index $|P|$ is a bad in Q because $\tau_{Q,|P|}$ and τ both lie in $\operatorname{term}a_{Q,|P|}$, and $\tau_{Q,|P|}\succ\tau$.

- We have $\log E_{\alpha_d}u\prec\operatorname{supp}\psi_d$ and $\operatorname{supp}\varphi_d\succ\mathrm{e}^{\psi_d}(E_{\alpha_d}u)^{\iota_d}$, but $\varphi_{d+1}=\sharp_{\alpha_d}(\varphi_{d+1}+\varepsilon_{d+1}\mathrm{e}^{\psi_{d+1}}(E_{\alpha_{d+1}}u)^{\iota_{d+1}})$. By the definition of α_d-truncated numbers, there is a $\beta<\alpha_d$ with

$$\mathrm{e}^{\psi_{d+1}}(E_{\alpha_{d+1}}u)^{\iota_{d+1}}\prec\frac{1}{L_\beta E_{\alpha_d}^{\varphi_{d+1}}}\prec\mathrm{e}^{\psi_{d+1}}(E_{\alpha_{d+1}}a_{;d+2})^{\iota_{d+1}}.$$

  Using the convexity of $\mathbf{Ad}_{\nearrow d+2}$, it follows that $L_\beta E_{\alpha_d}^{\varphi_{d+1}}\in\mathrm{e}^{-\psi_{d+1}}(E_{\alpha_{d+1}}\mathbf{Ad}_{\nearrow d+2})^{-\iota_{d+1}}$. By similar arguments as above (using Corollary 6.14 and Lemmas 5.15 and 5.14), we deduce that there is a finite path R in φ_{d+1} with $u_{R,|R|}\in\mathbf{Ad}_{\nearrow d_{i+2}}-\mathbb{N}$. As in the previous case $Q:=P*R$ is a path and $|P|$ is a bad index in Q.

Consider a $b\in\mathbf{Ad}_{\nearrow d_1-1}$ and the path $P_0:=(\tau_{a-\varphi_{d_0}})$ in b. So P is a finite path with $u_{P_0,|P_0|}\in\mathbf{Ad}_{\nearrow d_1}$. Thus there exists a path P_1 which extends P_0 with $u_{P_1,|P_1|}\in\mathbf{Ad}_{\nearrow d_3}$, where $|P_0|$ is a bad index in P. Repeating this process iteratively for $i=2,3,\ldots$, we construct a path P_i that extends P_{i-1} and such that $u_{P_i,|P_i|}\in\mathbf{Ad}_{\nearrow d_{2i+1}}$ and such that $|P_{i-1}|$ is a bad index in P_i. At the limit, this yields an infinite path Q in a that extends each of the paths P_i. This path Q has a cofinal set of bad indices, which contradicts Theorem 1.1. We conclude that there is a $k\in\mathbb{N}$ such that $\Sigma_{\nearrow k}$ is nested. □

LEMMA 6.16. *Assume that Σ is nested. Then we have* $\mathbf{Ad} = \varphi_0 + \varepsilon_0 \mathrm{e}^{\psi_0} (\mathcal{E}_{\alpha_0}[E_{\alpha_0} \mathbf{Ad}_{\nearrow 1}])^{\iota_0}$.

Proof. Note that $\mathcal{E}_{\alpha_0}[E_{\alpha_0} \mathbf{Ad}_{\nearrow 1}] = E_{\alpha_0} \mathcal{L}_{\alpha_0}[\mathbf{Ad}_{\nearrow 1}]$. The result thus follows from Corollary 6.8 and the assumption that Σ is nested. □

LEMMA 6.17. *Assume that Σ is nested. Let $k \in \mathbb{N}$, $a \in \mathbf{Ad}$ and $c_k \in \mathbf{No}$ with*

$$c_k \;=\; \varphi_k +\!\!\!\!+\, \varepsilon_k \mathrm{e}^{\psi_k} \mathfrak{p}^{\iota_k} \tag{6.3}$$

for a certain $\mathfrak{p} \in \mathbf{Mo}^{\succcurlyeq}$ with $\mathfrak{p} \sqsubseteq E_{\alpha_j} a_{;k+1}$ and $\mathfrak{p} \in \mathcal{E}_{\omega}[E_{\alpha_k} a_{;k+1}]$ whenever $\psi_k = 0$. If $c_k \in \mathbf{Ad}_{\nearrow k}$, then we have

$$(c_k)_{k;} \;\sqsubseteq\; a.$$

Proof. The proof is similar to the proof of Lemma 5.31. We have $a_{;k} = \varphi_k +\!\!\!\!+\, \varepsilon_k \mathrm{e}^{\psi_k} (E_{\alpha_k} a_{;k+1})^{\iota_k}$ and we must have $\operatorname{supp} \psi_k \succ \log \mathfrak{p}$ since $c_k = \varphi_k +\!\!\!\!+\, \varepsilon_k \mathrm{e}^{\psi_k} \mathfrak{p}^{\iota_k} \in \mathbf{Ad}_{\nearrow k}$. If follows from the deconstruction lemmas in Section 5.3 that $c_k \sqsubseteq a_{;k}$. This proves the result in the case when $k = 0$.

Now assume that $k > 0$. Setting $c_{k-p} := \Phi_{k-p;k}(c_k)$, let us prove by induction on $p \leqslant k$ that

$$\begin{aligned} c_{k-p} &\in \mathbf{Ad}_{\nearrow k-p} \\ c_{k-p} &\in \mathbf{No}_{\succ, \alpha_{k-p-1}} \\ c_{k-p} &\sqsubseteq a_{;k-p}. \end{aligned}$$

For $p = k$, the last relation yields the desired result.

If $p = 0$, then we have $c_k \in \mathbf{Ad}_{\nearrow k}$ by assumption and we have shown above that $c_k \sqsubseteq a_{;k}$. We have $\varphi_k \vartriangleleft \sharp_{\alpha_{k-1}}(c_k)$ and $\mathrm{e}^{\psi_k} \mathfrak{p}^{\iota_j}$ is a monomial, so (6.3) yields $c_k = \sharp_{\alpha_{k-1}}(c_k) \in \mathbf{No}_{\succ, \alpha_{k-1}}$. This deals with the case $p = 0$. In addition, we have $c_k > 0$ because $k > 0$ and $c_k \in \mathbf{Ad}_{\nearrow k}$. Let us show that

$$\log c_k \;\prec\; a_{;k}. \tag{6.4}$$

If $\varphi_k \neq 0$, then this follows from the facts that $\varphi_k \vartriangleleft a_{;k}$ and $\varphi_k \vartriangleleft c_k$. If $\varphi_k = 0$ and $\psi_k \neq 0$, then $\log (c_k / \varepsilon_k) \sim \psi_k \sim \log (a_{;k} / \varepsilon_k) \prec a_{;k}$. If $\varphi_k = \psi_k = 0$, then $a_{;k} = E_{\alpha_k} a_{;k+1}$ and $c_k = \mathfrak{p} \in \mathcal{E}_{\omega}[a_{;k}]$, so $\log c_k \prec a_{;k}$.

Assume now that $0 < p \leqslant k$ and that the induction hypothesis holds for all smaller p. We have

$$c_{k-p} \;=\; \Phi_{k-p}(c_{k-p+1}) \;=\; \varphi_k + \varepsilon_k \mathrm{e}^{\psi_k} (E_{\alpha_{k-p}}^{c_{k-p+1}})^{\iota_k} \tag{6.5}$$

Since Σ is nested, we immediately obtain $\varphi_{k-p} \vartriangleleft \sharp_{\alpha_{k-p-1}}(c_{k-p})$, whence $c_{k-p} \in \mathbf{No}_{\succ, \alpha_{k-p-1}}$ as above. Since $c_{k-p-1} \in \mathbf{Ad}_{\nearrow (k-p-1)}$ and Σ is nested, we have $c_{k-p} \in \mathbf{Ad}_{\nearrow (k-p)}$. Using (6.5), (6.4), and the decomposition lemmas, we observe that the relation $c_{k-p} \sqsubseteq a_{;k-p}$ is equivalent to

$$E_{\alpha_{k-p}}^{c_{k-p+1}} \;\sqsubseteq\; E_{\alpha_{k-p}} a_{;k-p+1}. \tag{6.6}$$

We have $c_{k-p+1} \sqsubseteq a_{;k-p+1}$, so $c_{k-p+1} \sqsubseteq \sharp_{\alpha_{k-p}}(a_{;k-p+1})$. Note that

$$E_{\alpha_{k-p}}^{\sharp_{\alpha_{k-p}}(a_{;k-p+1})} \;=\; \mathfrak{d}_{\alpha_{k-p}}(E_{\alpha_{k-p}} a_{;k-p+1}) \;\sqsubseteq\; E_{\alpha_{k-p}} a_{;k-p+1}.$$

So it is enough, in order to derive (6.6), to prove that $E_{\alpha_{k-p}}^{c_{k-p+1}} \sqsubseteq E_{\alpha_{k-p}}^{\sharp_{\alpha_{k-p}}(a_{;k-p+1})}$. Now

$$L_{\alpha_{k-p}} c_{k-p+1} \;<\; \mathcal{E}_{\alpha_{k-p}} \sharp_{\alpha_{k-p}}(a_{;k-p+1})$$

by Lemma 6.9, whence $E_{\alpha_{k-p}}^{c_{k-p+1}} \sqsubseteq E_{\alpha_{k-p}}^{\sharp_{\alpha_{k-p}}(a_{;k-p+1})}$ by Lemma 5.25. □

For $i \in \mathbb{N}$, $g \in \mathcal{E}_{\alpha_i}$ and $a \in \mathbf{Ad}$, we have $\varphi_i + \varepsilon_i \mathrm{e}^{\psi_i} g(E_{\alpha_i} a_{;i+1})^{\iota_i} \in \mathbf{Ad}_{\nearrow i}$ by Lemma 6.16. We may thus consider the strictly increasing bijection

$$\Psi_{i,g} \ := \ \mathbf{Ad} \longrightarrow \mathbf{Ad}; a \longmapsto (\varphi_i + \varepsilon_i \mathrm{e}^{\psi_i} g(E_{\alpha_i} a_{;i+1})^{\iota_i})_{i;}.$$

We will prove Theorem 1.2 by proving that the function group $\mathcal{G} := \{\Psi_{i,g} : i \in \mathbb{N}, g \in \mathcal{E}_{\alpha_i}\}$ on $\mathbf{Ad}$ generates the class $\mathbf{Ne}$, i.e. that we have $\mathbf{Ne} = \mathbf{Smp}_{\mathcal{G}}$. We first need the following inequality:

LEMMA 6.18. *Assume that Σ is nested. Let $i, j \in \mathbb{N}$ with $i < j$ and let $g \in \mathcal{E}_{\alpha_i}$. On $\mathbf{Ad}$, we have $\Psi_{i,g} < \Psi_{j,H_2}$ if $\sigma_{j+1;i+1} = 1$ and $\Psi_{i,g} < \Psi_{j,H_{1/2}}$ if $\sigma_{j+1;i+1} = -1$.*

Proof. It is enough to prove the result for $j = i+1$. Assume that $\sigma_{i+2;i+1} = 1$. Let $a \in \mathbf{Ad}$ and set $a' := (\Psi_{i+1,H_2}(a))_{i+1;}$, so that

$$\begin{aligned} a_{;i+1} &= \varphi_{i+1} + \varepsilon_{i+1} \mathrm{e}^{\psi_{i+1}} (E_{\alpha_{i+1}} a_{;i+2})^{\iota_{i+1}} \\ a' &= \varphi_{i+1} + \varepsilon_{i+1} \mathrm{e}^{\psi_{i+1}} (2 E_{\alpha_{i+1}} a_{;i+2})^{\iota_{i+1}}. \end{aligned}$$

Note that

$$(\Psi_{i,g}(a))_{i+1;} \ \in \ \mathcal{T}_{\alpha_i}[a_{;i+1}].$$

If $\sigma_{;i+1} = 1$, then $\varepsilon_{i+1} \iota_{i+1} = \sigma_{;i+2} / \sigma_{;i+1} = 1$ and $\Psi_{;i+1}$ is strictly increasing. So we only need to prove that $\mathcal{T}_{\alpha_i}[a_{;i+1}] < a'$, which reduces to proving that $\sharp_{\alpha_i}(a_{;i+1}) < \sharp_{\alpha_i}(a')$. Let τ be the dominant term of $E_{\alpha_{i+1}} a_{;i+2}$. Our assumption that Σ is nested gives $\varphi_i + \varepsilon_i \mathrm{e}^{\psi_i} (E_{\alpha_i} a')^{\iota_i} \in \mathbf{Ad}_{\nearrow i}$, whence $\varphi_{i+1} \triangleleft \sharp_{\alpha_i}(a')$. We deduce that $\varphi_{i+1} + \varepsilon_{i+1} \mathrm{e}^{\psi_{i+1}} (2\tau)^{\iota_{i+1}} \trianglelefteq \sharp_{\alpha_i}(a')$. Lemma 5.27 implies that $\varphi_{i+1} + \varepsilon_{i+1} \mathrm{e}^{\psi_{i+1}} (2\tau)^{\iota_{i+1}}$ is α_i-truncated.

$$\begin{aligned} \sharp_{\alpha_i}(a_{;i+1}) - \varphi_{i+1} &\sim \varepsilon_{i+1} \mathrm{e}^{\psi_{i+1}} \tau^{\iota_{i+1}}, \\ \sharp_{\alpha_i}(a') - \varphi_{i+1} &\sim \varepsilon_{i+1} \mathrm{e}^{\psi_{i+1}} (2\tau)^{\iota_{i+1}} \end{aligned}$$

and $\varepsilon_{i+1} \iota_{i+1} = 1$ implies that

$$\varepsilon_{i+1} \mathrm{e}^{\psi_{i+1}} (2\tau)^{\iota_{i+1}} - \varepsilon_{i+1} \mathrm{e}^{\psi_{i+1}} \tau^{\iota_{i+1}}$$

is a strictly positive term. We deduce that $\sharp_{\alpha_i}(a_{;i+1}) - \varphi_{i+1} < \sharp_{\alpha_i}(a') - \varphi_{i+1}$, whence $\sharp_{\alpha_i}(a_{;i+1}) < \sharp_{\alpha_i}(a')$. The other cases when $\sigma_{;i+1} = -1$ or when $\sigma_{i+2;i+1} = -1$ are proved similarly, using symmetric arguments. □

We are now in a position to prove the following refinement of Theorem 1.2.

THEOREM 6.19. *If Σ is nested, then $\mathbf{Ne}$ is a surreal substructure with $\mathbf{Ne} = \mathbf{Smp}_{\mathcal{G}}$.*

Proof. By Proposition 4.1, the class $\mathbf{Smp}_{\mathcal{G}}$ is a surreal substructure, so it is enough to prove the equality. We first prove that $\mathbf{Smp}_{\mathcal{G}} \subseteq \mathbf{Ne}$.

Assume for contradiction that there are an $a \in \mathbf{Smp}_{\mathcal{G}}$ and a $k \in \mathbb{N}$, which we choose minimal, such that $a_{;k}$ cannot be written as $a_{;k} = \varphi_k +\!\!\!\!+ \varepsilon_k \mathfrak{m}_k$ where $\mathfrak{m}_k = \mathrm{e}^{\psi_k} (E_{\alpha_k}^{a_{;k+1}})^{\iota_k}$ is a hyperserial expansion. Set $\mathfrak{m} := \mathfrak{d}_{a_{;k} - \varphi_k}$, $r := (a_{;k})_{\mathfrak{m}}$ and $\delta := (a_{;k})_{\succ \mathfrak{m}}$.

Our goal is to prove that there is a number $m \in \{k, k+1\}$ and $\mathfrak{p} \in \mathbf{Mo}^{\succ}$ with

$$\begin{array}{lll} \mathfrak{p} & \in & \mathcal{E}_{\alpha_m}[E_{\alpha_m} a_{;m+1}] \\ \mathfrak{p} & \sqsubseteq & E_{\alpha_m} a_{;m+1} \\ \mathfrak{p} & \sqsubset & E_{\alpha_m} a_{;m+1}, \quad \text{whenever } \delta = 0 \text{ and } r \in \{-1, 1\}. \end{array} \tag{6.7}$$

Assume that this is proved and set $c_m := \varphi_m + \varepsilon_m \mathrm{e}^{\psi_m} \mathfrak{p}^{\iota_m}$. The first condition and Lemma 6.16 yield $c_m \in \mathbf{Ad}_{\nearrow m}$ and the relations $\log \mathfrak{p} \prec \operatorname{supp} \psi_m$ and $\mathrm{e}^{\psi_m} \mathfrak{p}^{\iota_m} \prec \operatorname{supp} \varphi_m$. The second and third condition, together with Lemma 6.17, imply $c := (c_m)_{m;} \sqsubset a$. The first condition also implies that $c \in \mathcal{G}[a]$: a contradiction. Proving the existence of m and $\mathfrak{p}$ is therefore sufficient.

If $\mathfrak{m} \neq \min \operatorname{supp} a_{;k}$ or $\mathfrak{m} = \min \operatorname{supp} a_{;k}$ and $r \notin \{-1, 1\}$, then $m := k$ and $\mathfrak{p} := \mathfrak{d}_{E_{\alpha_k} a_{;k+1}}$ satisfy (6.7). Assume now that $\mathfrak{m} = \min \operatorname{supp} a_{;k}$ and that $r \in \{-1, 1\}$, whence $r = \varepsilon_k$. If $a_{;k+1} \notin \mathbf{No}_{>, \alpha_k}$ then $m := k$ and $\mathfrak{p} := E_{\alpha_k}^{\sharp_{\alpha_k}(a_{;k+1})}$ satisfy (6.7). Assume therefore that $a_{;k+1} \in \mathbf{No}_{>, \alpha_k}$. This implies that there exist $\gamma < \alpha_k$ and $\mathfrak{a} \in \mathbf{Mo}_{\alpha_k \omega}$ with $E_{\alpha_k}^{a_{;k+1}} = L_\gamma \mathfrak{a}$. By the definition of coding sequences, there is a least index $j > k$ with $\varphi_j \neq 0$ or $\psi_j \neq 0$, so

$$E_{\alpha_k}^{a_{;k+1}} = E_{\alpha_k + \cdots + \alpha_{j-1}}(\varphi_j \mathbin{+\!\!\!+} \varepsilon_j \mathrm{e}^{\psi_j} (E_{\alpha_j}^{a_{;j+1}})^{\iota_j}) \notin \mathbf{Mo}_{\alpha_k \omega}.$$

We have $\mathfrak{a} \in \mathbf{Mo}_{\alpha_k \omega}$ and $L_\gamma \mathfrak{a} \in \mathbf{Mo}_{\alpha_k} \setminus \mathbf{Mo}_{\alpha_k \omega}$. So by Corollary 5.6, we must have $\alpha_k = \omega^{\mu+1}$ for a certain $\mu \in \mathbf{On}$ and $\gamma = (\alpha_k)_{/\omega} n$ for a certain $n \in \mathbb{N}^{>}$. Note that $a_{;k+1} = L_{\alpha_k} \mathfrak{a} - n$. Recall that $\varphi_{k+1} \trianglelefteq a_{;k+1}$ and $L_{\alpha_k} \mathfrak{a} \in \mathbf{Mo}^{\succ}$, so $\varphi_{k+1} \in \{L_{\alpha_k} \mathfrak{a}, 0\}$. The case $\varphi_{k+1} = L_{\alpha_k} \mathfrak{a}$ cannot occur for otherwise

$$a_{;k+2} = \left(\frac{a_{;k+1} - \varphi_{k+1}}{\varepsilon_{k+1} \mathrm{e}^{\psi_{k+1}}} \right)^{\iota_{k+1}} = \frac{n^{\iota_{k+1}}}{\varepsilon_{k+1} \mathrm{e}^{\psi_{k+1}}}$$

would not lie in $\mathbf{No}^{>, \succ}$. So $\varphi_{k+1} = 0$. Let $m := k+1$ and

$$\mathfrak{p} := \left(\frac{L_{\alpha_k} \mathfrak{a}}{\mathrm{e}^{\psi_{k+1}}} \right)^{\iota_{k+1}} = \left(\frac{\mathfrak{d}_{a_{;k+1}}}{\mathrm{e}^{\psi_{k+1}}} \right)^{\iota_{k+1}} = \mathfrak{d}_{E_{\alpha_{k+1}} a_{;k+2}}.$$

We have $\mathfrak{p} \in \mathcal{E}_{\alpha_{k+1}}[E_{\alpha_{k+1}} a_{;k+2}]$ and $\mathfrak{p} \sqsubset E_{\alpha_{k+1}} a_{;k+2}$, so m and $\mathfrak{p}$ satisfy (6.7). We deduce that $\mathbf{Smp}_{\mathcal{G}}$ is a subclass of $\mathbf{Ne}$.

Conversely, consider $b \in \mathbf{Ne}$ and set $c := \pi_{\mathcal{G}}[b]$. So there are $i_1, i_2 \in \mathbb{N}$ and $(g, h) \in \mathcal{E}'_{\alpha_{i_1}} \times \mathcal{E}'_{\alpha_{i_2}}$ with $\Psi_{i_1, g_1}(b) < c < \Psi_{i_2, g_2}(b)$. Let $i := \max(i_1 + 1, i_2 + 1)$. By Lemma 6.18, there exist $d_1, d_2 \in \{1/2, 2\}$ with $\Psi_{i_1, g_1} < \Psi_{i, H_{d_1}}$ and $\Psi_{i_2, g_2} < \Psi_{i, H_{d_2}}$, whence $\Psi_{i, H_{d_1^{-1}}}(b) < c < \Psi_{i, H_{d_2}}(b)$. Since $\Phi_{;i}$ is strictly monotonous, we get $c_{;i} - \varphi_i \asymp b_{;i} - \varphi_i$. The numbers $\varepsilon_i (c_{;i} - \varphi_i)$ and $\varepsilon_i (b_{;i} - \varphi_i)$ are monomials, so $c_{;i} - \varphi_i = b_{;i} - \varphi_i$. Therefore $b = c \in \mathbf{Smp}_{\mathcal{G}}$. □

In view of Theorem 6.19, Lemma 6.18, and Proposition 4.1, we have the following parametrization of $\mathbf{Ne}$:

$$\forall z \in \mathbf{No}, \quad \Xi_{\mathbf{Ne}} z = \{L, \Psi_{\mathbb{N}, \mathcal{H}} \Xi_{\mathbf{Ne}} z_L \mid \Psi_{\mathbb{N}, \mathcal{H}} \Xi_{\mathbf{Ne}} z_R, R\}.$$

We conclude this section with a few remarkable identities for $\Xi_{\mathbf{Ne}}$.

LEMMA 6.20. *If Σ is nested, then for $i \in \mathbb{N}$ and $a, b \in \mathbf{Ne}$, we have $a \sqsubseteq b \Longleftrightarrow a_{;i} \sqsubseteq b_{;i}$.*

Proof. By [5, Lemma 4.5] and since the function $\Phi_{i;}$ is strictly monotonous, it is enough to prove that $\forall a, b \in \mathbf{Ne}, a \sqsubseteq b \Longleftarrow a_{;i} \sqsubseteq b_{;i}$. By induction, we may also restrict to the case when $i = 1$. So assume that $a_{;1} \sqsubseteq b_{;1}$. Recall that $L_{\alpha_0} a_{;1} \prec \mathcal{E}_{\alpha_0} b_{;1}$ by Lemma 6.9. Since $a_{;1}, b_{;1} \in \mathbf{No}_{>, \alpha_0}$, we deduce with Lemma 5.25 that $E_{\alpha_0}^{a_{;1}} \sqsubseteq E_{\alpha_0}^{b_{;1}}$. It follows using the decomposition lemmas that $a \sqsubseteq b$. □

PROPOSITION 6.21. *If Σ is nested, then we have* $\mathbf{Ne} = (\mathbf{Ne}_{\nearrow 1})_{1;} = \varphi_0 + \varepsilon_0 \mathrm{e}^{\psi_0} (E_{\alpha_0}^{\mathbf{Ne}_{\nearrow 1}})^{\iota_0}$.

Proof. We have $\mathbf{Ne} \subseteq (\mathbf{Ne}_{\nearrow 1})_{1;}$ by definition of $\mathbf{Ne}$. So we only need to prove that $(\mathbf{Ne}_{\nearrow 1})_{1;} \subseteq \mathbf{Ne}$. Consider $b \in \mathbf{Ne}_{\nearrow 1}$. Since Σ is nested, the number $a := \varphi_0 + \varepsilon_0 \mathrm{e}^{\psi_0} (E_{\alpha_0} b)$ is Σ-admissible, so we need only justify that $E_{\alpha_0} b \in \mathbf{Mo}_{\alpha_0} \setminus L_{<\alpha_0} \mathbf{Mo}_{\alpha_0 \omega}$. Since a is Σ-admissible, we have $\varphi_1 \lhd \sharp_{\alpha_0}(b)$. But b is $\Sigma_{\nearrow 1}$-nested, so $b = \varphi_1 \mathbin{+\!\!\!+} \tau$ for a certain term τ. We deduce that $b = \sharp_{\alpha_0}(b) \in \mathbf{No}_{>,\alpha_0}$, whence $E_{\alpha_0} b \in \mathbf{Mo}_{\alpha_0}$.

Assume for contradiction that $E_{\alpha_0}^b \in L_{<\alpha_0} \mathbf{Mo}_{\alpha_0 \omega}$ and write $E_{\alpha_0}^b = L_\gamma \mathfrak{a}$ where $\mathfrak{a} \in \mathbf{Mo}_{\alpha_0 \omega}$ and $\gamma < \alpha_0$. Note that $\gamma \neq 0$: otherwise φ_i and ψ_i would be zero for all $i \geqslant 1$, thereby contradicting Definition 6.1(e). By Corollary 5.6, we must have $\alpha_0 = \omega^{\mu+1}$ for a certain ordinal μ and $\gamma = \omega^\mu n$ for a certain $n \in \mathbb{N}^>$. Consequently, $b = L_{\alpha_0} \mathfrak{a} - n \in \mathbf{Mo} - n$. If $\varphi_1 \neq 0$, then the condition $\varphi_1 \lhd \sharp_{\alpha_0}(b)$ implies $\varphi_1 = b$, which leads to the contradiction that $b_{1;2} = 0 \notin \mathbf{No}^{>,>}$. If $\varphi_1 = 0$, then $\mathbf{Ne}_{\nearrow 1} \subseteq \varepsilon_1 \mathbf{Mo}$, whence $n = 0$: a contradiction. □

COROLLARY 6.22. *If Σ is nested, then for $z \in \mathbf{No}$, we have*

$$\Xi_{\mathbf{Ne}} z \;=\; \varphi_0 + \varepsilon_0 \mathrm{e}^{\psi_0} (E_{\alpha_0}^{\Xi_{\mathbf{Ne}_{\nearrow 1}} \sigma_{;1} z})^{\iota_0}.$$

COROLLARY 6.23. *If Σ is nested and $k \in \mathbb{N}$, then*

$$\Xi_{\mathbf{Ne}} \;=\; \Phi_{k;} \circ \Xi_{\mathbf{Ne}_{\nearrow k}} \circ H_{\sigma_{;k}}.$$

PROPOSITION 6.24. *Assume that Σ is nested with $(\varphi_0, \varepsilon_0, \psi_0, \iota_0) = (0, 1, 0, 1)$, assume that $\alpha_0 \in \omega^{\mathbf{On}+1}$ and write $\beta := (\alpha_0)_{/\omega}$. Consider the coding sequence Σ' with $(\varphi_i', \varepsilon_i', \psi_i', \iota_i', \alpha_i') = (\varphi_i, \varepsilon_i, \psi_i, \iota_i, \alpha_i)$ for all $i \in \mathbb{N}$, with the only exception that*

$$\varphi_1' \;=\; \varphi_1 - n.$$

If $\psi_1 < 0$, or $\psi_1 = 0$ and $\iota_1 = -1$, then Σ' is nested and we have

$$\Xi_{\mathbf{Ne}'} \;=\; L_{\beta n} \circ \Xi_{\mathbf{Ne}},$$

where $\mathbf{Ne}'$ is the class of $\Sigma^{[n]}$-nested numbers.

Proof. Assume that $\psi_1 < 0$, or $\psi_1 = 0$ and $\iota_1 = -1$. In particular, if a is Σ-admissible, then $a_{;1} - \varphi_1 \prec 1$, so $a_{;1} - \varphi_1 \prec \operatorname{supp} \varphi_1'$. For $b \in \mathbf{No}^{>,>}$, it follows that $E_{\alpha_0}(b - n)$ is $\Sigma^{[n]}$-admissible if and only if $E_{\alpha_0} b$ is Σ-admissible. Let $\mathbf{Ad}'_{\nearrow i}$ be the class of $\Sigma'_{\nearrow i}$-admissible numbers, for each $i \in \mathbb{N}$. We have $L_{\beta n} \mathbf{Ad} = \mathbf{Ad}'$ by the previous remarks, and Σ' is admissible. For $i > 1$, we have $\Sigma'_{\nearrow i} = \Sigma_{\nearrow i}$, so

$$\mathbf{Ad}'_{\nearrow i} \;=\; \mathbf{Ad}_{\nearrow i} \;\supseteq\; \varphi_i' + \varepsilon_i' \mathrm{e}^{\psi_i'} (E_{\alpha_i'} \mathbf{Ad}'_{\nearrow i+1})^{\iota_i'}.$$

Moreover, $\mathbf{Ad}'_{\nearrow 1} = \mathbf{Ad}_{\nearrow 1} - n$, so

$$\begin{aligned} \mathbf{Ad}' \;&\supseteq\; L_{\beta n} \mathbf{Ad} \;\supseteq\; L_{\beta n} E_{\alpha_0} \mathbf{Ad}_{\nearrow 1} \;=\; L_{\beta n} E_{\alpha_0} (\mathbf{Ad}'_{\nearrow 1} + n) \;=\; E_{\alpha_0^{[n]}} \mathbf{Ad}'_{\nearrow 1} \\ \mathbf{Ad}'_{\nearrow 1} \;&\supseteq\; \varphi_1 - n + \varepsilon_1 \mathrm{e}^{\psi_1} (E_{\alpha_1} \mathbf{Ad}_{\nearrow 2})^{\iota_1} \;=\; \varphi_1' + \varepsilon_1' \mathrm{e}^{\psi_1'} (E_{\alpha_1^{[n]}} \mathbf{Ad}'_{\nearrow 2})^{\iota_1'}. \end{aligned}$$

So Σ' is nested. We deduce that $L_{\beta n} \mathbf{Ne} = \mathbf{Ne}'$, that is, we have a strictly increasing bijection $L_{\beta n}: \mathbf{Ne} \longrightarrow \mathbf{Ne}'$. It is enough to prove that for $a, b \in \mathbf{Ne}$ with $a \sqsubseteq b$, we have $L_{\beta n} a \sqsubseteq L_{\beta n} b$. Proceeding by induction on n, we may assume without loss of generality that $n = 1$. By [6, identity (6.3)], the function L_β has the following equation on $\mathbf{Mo}_{\alpha_0}$:

$$\forall \mathfrak{a} \in \mathbf{Mo}_{\alpha_0}, \quad L_\beta \mathfrak{a} \;=\; \{L_\beta \mathfrak{a}_L^{\mathbf{Mo}_{\alpha_0}} \mid L_\beta \mathfrak{a}_R^{\mathbf{Mo}_{\alpha_0}}, \mathfrak{a}\}_{\mathbf{Mo}_{\alpha_0}}.$$

So it is enough to prove that $L_\beta b < a$. Note that $L_\beta b = E_{\alpha_0}^{b_{;1} - 1}$ and $a = E_{\alpha_0}^{a_{;1}}$ where $b_{;1} - \varphi_1$, $a_{;1} - \varphi_1 \prec 1$. So $b_{;1} - a_{;1} \prec 1$, whence $b_{;1} - 1 < a_{;1}$. This concludes the proof. □

6.4. Pre-nested and nested numbers

Let $a \in \mathbf{No}$ be a number. We say that a is *pre-nested* if there exists an infinite path P in a without any bad index for a. In that case, Lemma 6.2 yields a coding sequence Σ_P which is admissible due to the fact that $a \in (L \mid R)$ with the notations from Section 6. By Theorem 6.15, we get a smallest $k \in \mathbb{N}$ such that $(\Sigma_P)_{\nearrow k}$ is nested. If $k=0$, then we say that a is *nested*. In that case, Theorem 6.19 ensures that the class $\mathbf{Ne}$ of Σ_P-nested numbers forms a surreal substructure, so a can uniquely be written as $a = \Xi_{\mathbf{Ne}}(c)$ for some surreal parameter $c \in \mathbf{No}$.

One may wonder whether it could happen that $k>0$. In other words: do there exist pre-nested numbers that are not nested? For this, let us now describe an example of an admissible sequence Σ^* such that the class $\mathbf{Ne}_{\Sigma^*}$ of Σ^*-nested numbers contains a smallest element b. This number b is pre-nested, but cannot be nested by Theorem 6.19. Note that our example is "transserial" in the sense that it does not involve any hyperexponentials.

Example 6.25. Let $\Sigma = (\varphi_i, \varepsilon_i, 0, 1, 1)_{i \in \mathbb{N}}$ be a nested sequence with $\varepsilon_1 = -1$. Let a be the simplest Σ-nested number. We define a coding sequence $\Sigma^* = (\varphi_i^*, \varepsilon_i^*, 0, 1, 1)_{i \in \mathbb{N}}$ by

$$\begin{aligned} \varepsilon_0^* &:= -1 \\ \varphi_0^* &:= \mathrm{e}^{\varphi_1 - \frac{1}{2}\mathrm{e}^{a_{;2}}} \\ (\varphi_i^*, \varepsilon_i^*) &:= (\varphi_i, \varepsilon_i) \qquad \text{for all } i>0. \end{aligned}$$

Note that

$$a_{;1} = \varphi_1 - \mathrm{e}^{a_{;2}} = \varphi_1 + \varepsilon_1 \mathrm{e}^{a_{;2}},$$

where $\mathrm{e}^{a_{;2}}$ is an infinite monomial, so $b := \varphi_0^* - \mathrm{e}^{a_{;1}}$ is Σ^*-nested. In particular, the sequence Σ^* is admissible.

Assume for contradiction that there is a Σ^*-nested number c with $c<b$. Since $\varepsilon_0^* = \varepsilon_1^* = -1$, we have $c_{;2} < b_{;2}$. Recall that $c_{;2}$ and $b_{;2}$ are purely large, so $\mathrm{e}^{c_{;2}} \prec \mathrm{e}^{b_{;2}} = \mathrm{e}^{a_{;2}}$. In particular

$$\mathrm{e}^{c_{;1}} = \mathrm{e}^{\varphi_1 - \mathrm{e}^{c_{;2}}} \succcurlyeq \mathrm{e}^{\varphi_1 - \frac{1}{2}\mathrm{e}^{a_{;2}}} = \varphi_0^*,$$

which contradicts the assumption that c is Σ^*-nested. We deduce that b is the minimum of the class $\mathbf{Ne}_{\Sigma^*}$ of Σ^*-nested numbers. In view of Theorem 6.19, the sequence Σ^* cannot be nested.

The above examples shows that there exist admissible sequences that are not nested. Let us now construct an admissible sequence $\Sigma^{\emptyset}$ such that the class $\mathbf{Ne}_{\Sigma^{\emptyset}}$ of $\Sigma^{\emptyset}$-nested numbers is actually empty.

Example 6.26. We use the same notations as in Example 6.25. Define $\left(\varphi_0^{\emptyset}, \varepsilon_0^{\emptyset}\right) := (\mathrm{e}^b, 1)$ and set $\left(\varphi_i^{\emptyset}, \varepsilon_i^{\emptyset}\right) := (\varphi_{i-1}^*, \varepsilon_{i-1}^*)$ for all $i>0$. We claim that the coding sequence $\Sigma^{\emptyset} := \left(\varphi_i^{\emptyset}, \varepsilon_i^{\emptyset}, 0, 1, 1\right)_{i \in \mathbb{N}}$ is admissible. In order to see this, let $\psi := {}^1\!/_2\, \mathrm{e}^{b_{;1}}$. Then

$$\mathrm{e}^{\varphi_1^{\emptyset} + \varepsilon_1 \psi} = \mathrm{e}^{\varphi_0^* + \varepsilon_0^* \psi} \prec \mathrm{e}^{\varphi_0^* + \varepsilon_0^* \mathrm{e}^{b_{;1}}} = \mathrm{e}^b.$$

Since $\varphi_1^{\emptyset} + \varepsilon_1 \psi$ is $(\Sigma^{\emptyset})_{\nearrow 1}$-admissible (i.e. Σ^*-admissible), we deduce that $\mathrm{e}^b + \mathrm{e}^{\varphi_1^{\emptyset} + \varepsilon_1 \psi}$ is $\Sigma^{\emptyset}$-admissible, whence $\Sigma^{\emptyset}$ is admissible. Assume for contradiction that $\mathbf{Ne}_{\Sigma^{\emptyset}}$ is non-empty, and let $\mathrm{e}^b + \mathfrak{m} \in \mathbf{Ne}_{\Sigma^{\emptyset}}$. Then $\log \mathfrak{m}$ is Σ^*-nested, so $\log \mathfrak{m} \geqslant b$, whence $\mathfrak{m} \succcurlyeq \mathrm{e}^b$: a contradiction.

7. NUMBERS AS HYPERSERIES

Traditional transseries in x can be regarded as infinite expressions that involve x, real constants, infinite summation, exponentiation and logarithms. It is convenient to regard such expressions as infinite labeled trees. In this section, we show that surreal numbers can be represented similarly as infinite expressions in ω that also involve hyperexponentials and hyperlogarithms. One technical difficulty is that the most straightforward way to do this leads to ambiguities in the case of nested numbers. These ambiguities can be resolved by associating a surreal number to every infinite path in the tree. In view of the results from Section 6, this will enable us to regard any surreal number as a unique hyperseries in ω.

Remark 7.1. In the case of ordinary transseries, our notion of tree expansions below is slightly different from the notion of tree representations that was used in [30, 38]. Nevertheless, both notions coincide modulo straightforward rewritings.

7.1. Introductory example

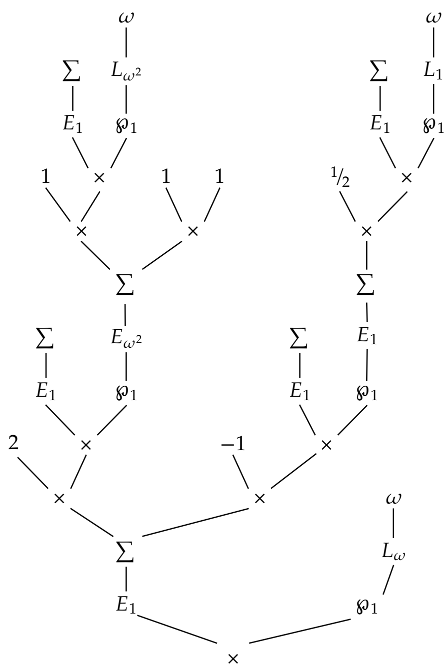
Let us consider the monomial $\mathfrak{m} = \exp(2 E_{\omega} \omega - \sqrt{\omega} + L_{\omega+1} \omega)$ from Example 5.3. We may recursively expand $\mathfrak{m}$ as

$$\mathfrak{m} \;=\; \mathrm{e}^{2 E_{\omega^2}^{L_{\omega^2} \omega + 1} - E_1^{\frac{1}{2} L_1 \omega}} (L_{\omega} \omega).$$

In order to formalize the general recursive expansion process, it is more convenient to work with the unsimplified version of this expression

$$\mathfrak{m} \;=\; \mathrm{e}^{2 \cdot \mathrm{e}^0 \cdot \left(E_{\omega^2}^{1 \cdot \mathrm{e}^0 \cdot (L_{\omega^2} \omega)^1 + 1 \cdot 1} \right)^1 + (-1) \cdot \mathrm{e}^0 \cdot \left(E_1^{1/2 \cdot \mathrm{e}^0 \cdot (L_1 \omega)^1} \right)^1} (L_{\omega} \omega)^1.$$

Introducing $\wp_c : x \longmapsto x^c$ as a notation for the "power" operator, the above expression may naturally be rewritten as a tree:



In the next subsection, we will describe a general procedure to expand surreal monomials and numbers as trees.

7.2. Tree expansions

In what follows, a *tree* T is a set of nodes N_T together with a function that associates to each node $\nu \in N_T$ an *arity* $\ell_\nu \in \mathbf{On}$ and a sequence $(\nu[\alpha])_{\alpha<\ell_\nu} \in N_T^{\ell_\nu}$ of *children*; we write $C_\nu := \{\nu[\alpha] : \alpha < \ell_\nu\}$ for the set of children of ν. Moreover, we assume that N_T contains a special element ρ_T, called the *root* of T, such that for any $\nu \in N_T$ there exist a unique h (called the *height* of ν and also denoted by h_ν) and unique nodes $\nu_0, \ldots, \nu_h$ with $\nu_0 = \rho_T$, $\nu_h = \nu$, and $\nu_i \in C_{\nu_{i-1}}$ for $i = 1, \ldots, h$. The height h_T of the tree T is the maximum of the heights of all nodes; we set $h_T := \omega$ if there exist nodes of arbitrarily large heights.

Given a class $\mathbf{\Sigma}$, an $\mathbf{\Sigma}$*-labeled tree* is a tree together with a map $\lambda : N_T \longrightarrow \mathbf{\Sigma}; \nu \longmapsto \lambda_\nu$, called the *labeling*. Our final objective is to express numbers using $\mathbf{\Sigma}$-labeled trees, where

$$\mathbf{\Sigma} \; := \; \mathbb{R}^{\neq} \cup \{\omega, \textstyle\sum, \times, \wp_{-1}, \wp_1\} \cup L_{\omega^{\mathbf{On}}} \cup E_{\omega^{\mathbf{On}}}.$$

Instead of computing such expressions in a top-down manner (from the leaves until the root), we will compute them in a bottom-up fashion (from the root until the leaves). For this purpose, it is convenient to introduce a separate formal symbol $?_c$ for every $c \in \mathbf{On}$, together with the extended signature

$$\mathbf{\Sigma}^\sharp \; := \; \mathbf{\Sigma} \cup \{?_c : c \in \mathbf{No}\}.$$

We use $?_c$ as a placeholder for a tree expression for c whose determination is postponed to a later stage.

Consider a $\mathbf{\Sigma}^\sharp$-labeled tree T and a map $v : N_T \longrightarrow \mathbf{On}$. We say that v is an *evaluation* of T if for each node $\nu \in N_T$ one of the following statements holds:

E1. $\lambda_\nu \in \mathbb{R}^{\neq} \cup \{\omega\}$, $\ell_\nu = 0$, and $v(\nu) = \lambda_\nu$;

E2. $\lambda_\nu = \sum$, the family $(v(\nu[\alpha]))_{\alpha<\ell_\nu}$ is well based and $v(\nu) = \sum_{\alpha<\ell_v} v(\nu[\alpha])$;

E3. $\lambda_\nu = \times$, $\ell_\nu = 2$, and $v(\nu) = v(\nu[0])\, v(\nu[1])$;

E4. $\lambda_\nu = \wp_\iota$, $\iota \in \{-1, 1\}$, $\ell_\nu = 1$, and $v(\nu) = v(\nu[0])^\iota$;

E5. $\lambda_\nu = L_{\omega^\mu}$, $\ell_\nu = 1$, and $v(\nu) = L_{\omega^\mu} v(\nu[0])$;

E6. $\lambda_\nu = E_{\omega^\mu}$, $\ell_\nu = 1$, and $v(\nu) = E_{\omega^\mu} v(\nu[0])$;

E7. $\lambda_\nu = ?_\alpha$, $\ell_\nu = 0$, and $v(\nu) = \alpha$.

We call $v(\rho_T)$ the *value* of T via v. We say that $a \in \mathbf{No}$ is *a value* of T if there exists an evaluation of T with $a = v(\rho_T)$.

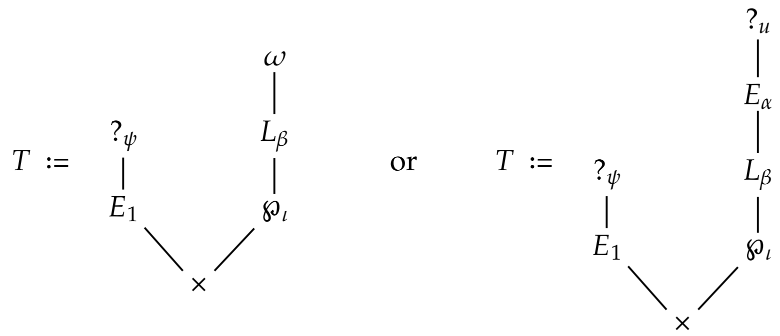
LEMMA 7.2.

a) *If T has finite height, then there exists at most one evaluation of T.*

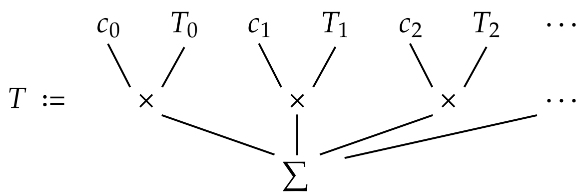
b) *Let v and v' be evaluations of T with $v(\rho_T) = v'(\rho_T)$. Then $v = v'$.*

Proof. This is straightforward, by applying the rules E1–E7 recursively (from the leaves to the root in the case of (*a*) and the other way around for (*b*)). □

Although evaluations with a given end-value are unique for a fixed tree T, different trees may produce the same value. Our next aim is to describe a standard way to expand numbers using trees. Let us first consider the case of a monomial $\mathfrak{m} \in \mathbf{Mo}$. If $\mathfrak{m} = 1$, then the *standard monomial expansion* of $\mathfrak{m}$ is the $\mathbf{\Sigma}^{\#}$-labeled tree T with $N_T = \{\rho_T\}$ and $\lambda_{\rho_T} = 1$. Otherwise, we may write $\mathfrak{m} = \mathrm{e}^{\psi} (L_\beta g)^\iota$ with $g = \omega$ or $g = E_\alpha^u$. Depending on whether $g = \omega$ or $g = E_\alpha^u$, we respectively take



and call T the *standard monomial expansion* of $\mathfrak{m}$. Let us next consider a general number $a \in \mathbf{No}$ and let $\ell \in \mathbf{On}$ be the ordinal size of its support. Then we may write $a = \sum_{\alpha < \ell} c_\alpha \mathfrak{m}_\alpha$ for a sequence $(c_\alpha)_{\alpha < \ell} \in (\mathbb{R}^{\neq})^\ell$ and a $\prec$-decreasing sequence $(\mathfrak{m}_\alpha)_{\alpha < \ell} \in \mathbf{Mo}^\ell$. For each $\alpha < \ell$, let T_α be the standard monomial expansion of $\mathfrak{m}_\alpha$. Then we define the $\mathbf{\Sigma}^{\#}$-labeled tree



and call it the *standard expansion* of a. Note that the height of T is at most 6, there exists a unique evaluation $v: N_T \longrightarrow \mathbf{No}$ of T, and $v(\rho_T) = a$.

Now consider two trees T and T' with respective labelings $\lambda: N_T \longrightarrow \mathbf{\Sigma}^{\#}$ and $\lambda': N_{T'} \longrightarrow \mathbf{\Sigma}^{\#}$. We say that T' *refines* T if $N_{T'} \supseteq N_T$ and there exist evaluations $v: N_T \longrightarrow \mathbf{No}$ and $v': N_{T'} \longrightarrow \mathbf{No}$ such that $v(\nu) = v'(\nu)$ for all $\nu \in N_T$ and $\lambda_\nu = \lambda'_\nu$ whenever $\lambda_\nu \notin ?_{\mathbf{No}}$. Now assume that $v(\rho_T) = a$ for some evaluation $v: N_T \longrightarrow \mathbf{No}$. Then we say that T is a *tree expansion* of a if for every $\nu \in N_T$ with $\lambda_\nu = \sum$, the subtree T' of T with root ν refines the standard expansion of $v(\nu)$. In particular, a tree expansion T of a number $a \in \mathbf{No}$ with $\lambda_{\rho_T} \notin ?_{\mathbf{No}}$ always refines the standard expansion of a.

LEMMA 7.3. *Any $a \in \mathbf{No}$ has a unique tree expansion with labels in $\mathbf{\Sigma}$.*

Proof. Given $n \in \mathbb{N}$, we say that an $\mathbf{\Sigma}^{\#}$-labeled tree T is *n-settled* if $\lambda_\nu \notin ?_{\mathbf{No}}$ for all nodes $\nu \in N_T$ of height $< n$. Let us show how to construct a sequence $(T_n)_{n \in \mathbb{N}}$ of $\mathbf{\Sigma}^{\#}$-labeled tree expansions of a such that the following statements hold for each $n \in \mathbb{N}$:

S1. T_n is an n-settled and of finite height;

S2. $v_n(\rho_{T_n}) = a$ for some (necessarily unique) evaluation $v_n: N_{T_n} \longrightarrow \mathbf{No}$ of T_n;

S3. If $n > 0$, then T_n refines T_{n-1};

S4. If T is a tree expansion of a with labels in $\mathbf{\Sigma}$, then T refines T_n.

We will write $\lambda_n: N_{T_n} \longrightarrow \mathbf{\Sigma}^\sharp$ for the labeling of T_n.

We take T_0 such that $N_{T_0} = \{\rho_{T_0}\}$ and $\lambda_{\rho_{T_0}} = ?_a$. Setting $v_0(\rho_{T_0}) := a$, the conditions **S1**, **S2**, **S3**, and **S4** are naturally satisfied.

Assume now that T_n has been constructed and let us show how to construct T_{n+1}. Let S be the subset of N_{T_n} of nodes ν of level n with $v_n(\nu) \in ?_{\mathbf{No}}$. Given $\nu \in S$, let T_ν be the standard expansion of $v_n(\nu)$ and let v_ν be the unique evaluation of T_ν. We define T_{n+1} to be the tree that is obtained from T_n when replacing each node $\nu \in S$ by the tree T_ν.

Since each tree T_ν is of height at most 6, the height of T_{n+1} is finite. Since T_{n+1} is clearly $(n+1)$-settled, this proves **S1**. We define an evaluation $v_{n+1}: N_{T_{n+1}} \longrightarrow \mathbf{\Sigma}^\sharp$ by setting $v_{n+1}(\sigma) = v_n(\sigma)$ for any $\sigma \in N_{T_n}$ and $v_{n+1}(\sigma) = v_\nu(\sigma)$ for any $\nu \in S$ and $\sigma \in N_{T_\nu}$ (note that v_{n+1} is well defined since $v_\nu(\rho_{T_\nu}) = (\lambda_n)_\nu = v_n(\nu)$ for all $\nu \in S$). We have $v_{n+1}(\rho_{T_{n+1}}) = v_n(\rho_{T_n}) = a$, so **S2** holds for v_{n+1}. By construction, $N_{T_{n+1}} \supseteq N_{T_n}$ and the evaluations v_n and v_{n+1} coincide on N_{T_n}; this proves **S3**. Finally, let T be a tree expansion of a with labels in $\mathbf{\Sigma}$ and let v be the unique evaluation of T with $v(\rho_T) = a$. Then T refines T_n, so v coincides with v_n on N_{T_n}. Let $\nu \in S$. Since T is a tree expansion of a, the subtree T' of T with root ν refines T_ν, whence $N_T \supseteq N_{T_\nu}$. Moreover, $v(\nu) = v_{n+1}(\nu)$, so v coincides with v_ν on T_ν. Altogether, this shows that T refines T_{n+1}.

Having completed the construction of our sequence, we next define a $\mathbf{\Sigma}$-labeled tree T_∞ and a map $v_\infty: N_{T_\infty} \longrightarrow \mathbf{No}$ by taking $N_{T_\infty} = \bigcup_{n \in \mathbb{N}} N_{T_n}$ and by setting $(\lambda_\infty)_\nu := (\lambda_n)_\nu$ and $v_\infty(\nu) = v_n(\nu)$ for any $n \in \mathbb{N}$ and $\nu \in N_{T_n}$ such that $(\lambda_n)_\nu \notin ?_{\mathbf{No}}$. By construction, we have $v_\infty(\rho_{T_\infty}) = a$ and T_∞ refines T_n for every $n \in \mathbb{N}$.

We claim that T_∞ is a tree expansion of a. Indeed, consider a node $\nu \in N_{T_\infty}$ of height n with $\lambda_\nu = \sum$. Then $\nu \in N_{T_{n+1}}$ and $(\lambda_{n+1})_\nu = \sum$, since T_{n+1} is $(n+1)$-settled. Consequently, the subtree of T_{n+1} with root ν refines the standard expansion of $v_{n+1}(\nu)$. Since T_∞ refines T_{n+1}, it follows that the subtree of T_∞ with root ν also refines the standard expansion of $v_\infty(\nu) = v_{n+1}(\nu)$. This completes the proof of our claim.

It remains to show that T_∞ is the unique tree expansion of a with labels in $\mathbf{\Sigma}$. So let T be any tree expansion of a with labeling $\lambda: N_T \longrightarrow \mathbf{\Sigma}$. For every $n \in \mathbb{N}$, it follows from **S4** that $N_T \supseteq N_{T_n}$. Moreover, since T_n is n-settled, λ coincides with both λ_n and λ_∞ on those nodes in N_{T_n} that are of height $<n$. Consequently, $N_T \supseteq N_{T_\infty}$ and λ coincides with λ_∞ on N_{T_∞}. Since every node in N_T has finite height, we conclude that $T = T_\infty$. □

7.3. Hyperserial descriptions

From now on, we only consider tree expansions with labels in $\mathbf{\Sigma}$, as in Lemma 7.3. Given a class **Ne** of nested numbers as in Section 6, it can be verified that every element in **Ne** has the same tree expansion. We still need a notational way to distinguish numbers with the same expansion.

Let $a \in \mathbf{No}$ be a pre-nested number. By Theorem 6.15, we get a smallest $k \in \mathbb{N}$ such that $(\Sigma_P)_{\nearrow k}$ is nested. Hence $a_{P,k} \in \mathbf{Ne}$ for the class **Ne** of $(\Sigma_P)_{\nearrow k}$-nested numbers. Theorem 6.19 implies that there exists a unique number c with $a_{P,k} = \Xi_{\mathbf{Ne}}(\sigma_{;k} c)$. We call c the *nested rank* of a and write $\zeta_a := c$. By Corollary 6.23, we note that $\zeta_{u_{P,i}} = \sigma_{;i} \zeta_a$ for all $i \in \mathbb{N}$. Given an arbitrary infinite path P in a number $a \in \mathbf{No}$, there exists a $k > 0$ such that $P_{\nearrow k}$ has no bad indices for $a_{P,k}$ (modulo a further increase of k, we may even assume $a_{P,k}$ to be nested). Let $\sigma_{P,k} := \operatorname{sign}(r_{P,0} \cdots r_{P,k-1}) \iota_{P,0} \cdots \iota_{P,k-1} \in \{-1, 1\}$. We call $\zeta_P := \sigma_{P,k} \zeta_{u_{P,k}}$ the *nested rank* of P, where we note that the value of $\sigma_{P,k} \zeta_{u_{P,k}}$ does not depend on the choice of k.

Let T be the tree expansion of a number $a \in \mathbf{No}$ and let $v: N_T \longrightarrow \mathbf{No}$ be the evaluation with $a = v(\rho_T)$. An *infinite path* in T is a sequence $\nu_0, \nu_1, \ldots$ of nodes in N_T with $\nu_0 = \rho_T$ and $\nu_{i+1} \in C_{\nu_i}$ for all $i \in \mathbb{N}$. Such a path induces an infinite path P in a: let $i_1 < i_2 < \cdots$ be the indices with $\lambda_{\nu_{i_k}} = \sum$; then we take $\tau_{P,k} = v(\nu_{i_k+1})$ for each $k \in \mathbb{N}$. It is easily verified that this induces a one-to-one correspondence between the infinite paths in T and the infinite paths in a. We call $\xi_\nu := \xi_P$ the *nested rank* of the infinite path $\nu = (\nu_n)_{n \in \mathbb{N}}$ in T. Denoting by I_T the set of all infinite paths in T, we thus have a map $\xi: I_T \longrightarrow \mathbf{No}; \nu \longmapsto \xi_\nu$. We call (T, ξ) the *hyperserial description* of a.

We are now in a position to prove the final theorem of this paper.

Proof of Theorem 1.3. Consider two numbers $a, a' \in \mathbf{No}$ with the same hyperserial description (T, ξ) and let $v, v': N_T \longrightarrow \mathbf{No}$ be the evaluations of T with $v(\rho_T) = a$ and $v'(\rho_T) = a'$. We need to prove that $a = a'$. Assume for contradiction that $a \neq a'$. We define an infinite path $\nu_0, \nu_1, \ldots$ in T with $v(\nu_n) \neq v'(\nu_n)$ for all n by setting $\nu_0 := \rho_T$ and $\nu_{n+1} := \nu_n[m]$, where $m \in \mathbb{N}$ is minimal such that $v(\nu_n[m]) \neq v'(\nu_n[m])$. (Note that such a number m indeed exists, since otherwise $v(\nu_n) = v'(\nu_n)$ using the rules E1–E7.) This infinite path also induces infinite paths P and P' in a and a' with $a_{P,n} = v(\nu_{i_n})$ and $a_{P',n} = v'(\nu_{i_n})$ for a certain sequence $i_1 < i_2 < \cdots$ and all $n \in \mathbb{N}$. Let $n > 0$ be such that $P_{\nearrow n}$ and $P'_{\nearrow n}$ have no bad indices for $a_{P,n}$ and $a_{P',n}$. The way we chose $\nu_0, \nu_1, \ldots$ ensures that the coding sequences associated to the paths $P_{\nearrow n}$ and $P'_{\nearrow n}$ coincide, so they induce the same nested surreal substructure $\mathbf{Ne}$. It follows that $v(\nu_{i_n}) = a_{P,n} = \Xi_{\mathbf{Ne}}(\sigma_{;n} \xi_\nu) = a_{P',n} = v'(\nu_{i_n})$, which contradicts our assumptions. We conclude that a and a' must be equal. □

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INDEX

GLOSSARY

$\mathbb{R}[[\mathfrak{M}]]$	field of well-based series with real coefficients over $\mathfrak{M}$	7
$\operatorname{supp} f$	support of a series	7
$\operatorname{term} f$	set of terms of a series	7
$\mathfrak{d}_f$	$\max \operatorname{supp} f$	7
$f_{\succ \mathfrak{m}}$	truncation $\sum_{\mathfrak{n} \succ \mathfrak{m}} f_{\mathfrak{n}} \mathfrak{n}$ of f	7
$f_{\succ}$	$f_{\succ 1}$	7
$h = f \mathbin{+\!\!\!+} g$	$h = f + g$ and $\operatorname{supp} f \succ g$	7
$f \trianglelefteq g$	$\operatorname{supp} f \succ g - f$	7
$f \prec g$	$\mathbb{R}^{>} \lvert f \rvert < \lvert g \rvert$	8
$f \preccurlyeq g$	$\exists r \in \mathbb{R}^{>}, \lvert f \rvert < r \lvert g \rvert$	8
$f \asymp g$	$f \preccurlyeq g$ and $g \preccurlyeq f$	8
$\mathbb{S}_{\succ}$	series $f \in \mathbb{S}$ with $\operatorname{supp} f \succ 1$	8
$\mathbb{S}^{\prec}$	series $f \in \mathbb{S}$ with $f \prec 1$	8
$\mathbb{S}^{>,\succ}$	series $f \in \mathbb{S}$ with $f \geqslant 0$ and $f \succ 1$	8
$\mathbb{L}$	field of logarithmic hyperseries	8
$\mathfrak{L}_{<\alpha}$	group of logarithmic hypermonomials of force $<\alpha$	8
$\mathbb{L}_{<\alpha}$	field of logarithmic hyperseries of force $<\alpha$	9
$g^{\uparrow\gamma}$	unique series in $\mathbb{L}$ with $g = (g^{\uparrow\gamma}) \circ \ell_{\gamma}$	9
On	class of ordinals	9
$\sqsubseteq$	simplicity relation	9
$\alpha \dotplus \beta$	ordinal sum of α and β	10
$a \dot{\times} \beta$	ordinal product of α and β	10
ω^{γ}	ordinal exponentiation with base ω at γ	10
$\rho \ll \sigma$	$\rho \prec \omega^{\eta}$ for each exponent η of σ	10
$\rho \underline{\ll} \sigma$	$\rho \preccurlyeq \omega^{\eta}$ for each exponent η of σ	10
$\gamma < \beta$	$\gamma n < \beta$ for all $n \in \mathbb{N}$	10
$\gamma \preccurlyeq \beta$	$\exists n \in \mathbb{N}, \gamma \leqslant \beta n$	10
$\gamma \asymp \beta$	$\gamma \preccurlyeq \beta \preccurlyeq \gamma$	10
μ_{-}	$\mu = \mu_{-} + 1$ if μ is a successor and $\mu_{-} = \mu$ if μ is a limit	10
$\alpha_{/\omega}$	$\omega^{\mu_{-}}$ for $\alpha = \omega^{\mu}$	10
$\circ$	composition law $\circ \colon \mathbb{L} \times \mathbf{No}^{>,\succ} \longrightarrow \mathbf{No}$	10
$\mathbb{T}_{\succ,\beta}$	class of β-truncated series	12
$\sharp_{\beta}(s)$	$\trianglelefteq$-maximal β-truncated truncation of s	12
$\mathcal{G}[a]$	class of numbers b with $\exists g, h \in \mathcal{G}, ga \leqslant b \leqslant ha$	14
$\mathbf{Smp}_{\mathcal{G}}$	class of $\mathcal{G}$-simple elements	14
$\pi_{\mathcal{G}}$	projection $\mathbf{S} \longrightarrow \mathbf{Smp}_{\mathcal{G}}$	14
$\underline{\leqslant}$	comparison between sets of strictly increasing bijections	14
$X \lesseqgtr Y$	X and Y are mutually pointwise cofinal	14
$\langle X \rangle$	function group generated by X	14