Computation of the monodromy of generalized polylogarithms

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1 Introduction

For each multi–index $s = (s_1, s_2, \ldots, s_k)$ of positive integers, one defines the generalized *polylogarithms*

$$\mathrm{Li}_s(z) = \sum_{n_1 > n_2 > \cdots > n_k > 0} \frac{z^{n_1}}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}. \tag{1}$$

This series in $z \in \mathbb{C}$ converges at the interior of the open unit disk. In $z = 1$, these polylogarithms yield the generalized Riemann ζ function [10]

$$\zeta(s) = \mathrm{Li}_s(1) = \sum_{n_1 > n_2 > \cdots > n_k > 0} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}, \tag{2}$$

which converges for $s_1 > 1$.

Let X be the alphabet on two letters x_0 and x_1. Any multi–index $s = (s_1, s_2, \ldots, s_k)$ can be encoded by a unique word $w \in X^* x_1$

$$w = x_0^{s_1 - 1} x_1 x_0^{s_2 - 1} x_1 \cdots x_0^{s_k - 1} x_1 \tag{3}$$

Now each function $\mathrm{Li}_s(z)$, which is also denoted by $L_w(z)$, can be obtained by an *iterated integral* as follows [10]:

$$L_{x_1}(z) = \int_0^z \frac{dt}{1-t} = -\log(1-z)$$

and

$$\begin{cases} L_{x_0 w}(z) &= \displaystyle\int_0^z L_w(t) \frac{dt}{t}, \\ L_{x_1 w}(z) &= \displaystyle\int_0^z L_w(t) \frac{dt}{1-t}, \end{cases} \tag{4}$$

for any $w \in X^* x_1$. These integrals are functions defined on the universal Riemann surface $\mathcal{R}$ above $\mathbb{C} \setminus \{0, 1\}$. The real number $\zeta(s)$ is also denoted by $\zeta_w = L_w(1)$ for all $x \in x_0 X^* x_1$.

It is useful to extend the above definition of L_w to the case when $w \in X^*$. For each $n \geq 0$, we take

$$L_{x_0^n}(z) = \frac{1}{n!} \log^n(z), \tag{5}$$

and we extend the definition to $w \in X^*$ using (4). These generalized polylogarithms are again defined on $\mathcal{R}$ and we will prove the important fact that

$$L(z) = \sum_{w \in X^*} L_w(z) w \tag{6}$$

is a Lie exponential for all $z \in \mathcal{R}$.

The monodromy [1, 9, 3] of the *classical* polylogarithms $\mathrm{Li}_k(z) = L_{x_0^{k-1} x_1}$, when turning around the point $z = 1$ has been computed in [8]

$$\mathcal{M}_1 \,\mathrm{Li}_k(z) = \mathrm{Li}_k(z) - 2i\pi \frac{\log^{k-1}(z)}{(k-1)!}, \quad k > 0. \tag{7}$$

From a theoretical point the monodromy of the series $L(z)$ can be computed using tools developed by J. Écalle in [3]. Notice that the monodromy of $L(z)$ in particular yields the monodromy of each $L_w(z)$ for $w \in X^*$.

In this paper, we give an explicit method to compute the monodromy. Our algorithm has been implemented in the Axiom system, using modules developed in [5] and we have given the output of the algorithm for $l = 6$ in appendix B. Our methods rely on the theory of non commutative power series [2, 1] and the factorization of Lie exponentials [7]. Our formulas for the monodromy of L involve only convergent $\zeta(s)$ defined by (2). In [4] an algorithm was given to compute algebraic relations between the $\zeta(s)$.

2 Polylogarithms and Chen series

Let $R\langle X \rangle$ and $R\langle\langle X \rangle\rangle$ be the algebras of non commutative polynomials resp. power series in x_0 and x_1 over a ring R. The coefficient of $w \in X^*$ in a series $S \in R\langle\langle X\rangle\rangle$ is denoted by $(S|w)$. We recursively define a *shuffle product* $ш$ on $R\langle X\rangle$ as follows:

$$\begin{cases} \forall w \in X^*, \quad 1 ш w = w ш 1 = w, \\ \forall u,v \in X^*, xu ш yv = x(u ш yv) + y(xu ш v). \end{cases} \tag{8}$$

Here 1 denotes the empty word. In this paper, we will always have $R = \mathbb{C}$, or $R = \mathcal{F}(\mathcal{R})$, the ring of analytic functions on $\mathcal{R}$.

A series $S \in R\langle\langle X\rangle\rangle$ is called a *Lie exponential*, if it satisfies one of the following, equivalent conditions [6]:

1. There exists a *Lie series* $L \in \mathcal{L}ie_R\langle\langle X\rangle\rangle$ with $S = e^L$.
2. $\Delta(S) = S \otimes S$, where $\Delta(S)$ denotes the usual coproduct defined on letters $x \in X$ by $\Delta(x) = x \otimes 1 + 1 \otimes x$.
3. $\forall u,v \in X^*$, $(S|u ш v) = (S|u)(S|v)$.

For a differentiable path $\gamma : [0,1] \to \mathbb{C}\backslash\{0,1\}$ between a and b, let S_γ be the evaluation in $z = b$ of the solution to the differential equation

$$\frac{d}{dz}S(z) = \left(\frac{x_0}{z} + \frac{x_1}{1-z}\right) S(z) \tag{9}$$

with initial condition $S(a) = 1$. This series $S_\gamma \in \mathbb{C}\langle X\rangle$ is called [2] the *Chen series* along γ (and associated to the differential forms $\omega_0 = dz/z$ et $\omega_1 = dz/1-z$).

It is classical [2] that S_γ is a Lie exponential, which only depends on the homotopy class of γ. Moreover, for the concatenation $\gamma_1\gamma_2$ of two paths γ_1 and γ_2, one has

$$S_{\gamma_1\gamma_2} = S_{\gamma_2}S_{\gamma_1}, \tag{10}$$

In particular, $S_{\gamma^{-1}} = S_\gamma^{-1}$.

The polylogarithms L_w induce a series $L \in \mathcal{F}(\mathcal{R})\langle X\rangle$ by:

$$L(z) = \sum_{w \in X^*} L_w(z) w, \quad \forall z \in \mathcal{F}\mathcal{R}.$$

Theorem 1 *$L(z)$ is a Lie exponential for all $z \in \mathcal{R}$. In particular, one has the* shuffle relations

$$L_{u ш v}(z) = L_u(z)L_v(z), \quad \forall z \in \mathcal{R}, \forall u,v \in X^*. \tag{11}$$

PROOF – L satisfies the differential equation

$$\frac{d}{dz}L(z) = \left(\frac{x_0}{z} + \frac{x_1}{1-z}\right) L(z). \tag{12}$$

If $\varepsilon \to 0$, one has

$$L(\varepsilon) = e^{x_0 \log \varepsilon} + O(\sqrt{\varepsilon}). \tag{13}$$

Let $T(z) = \Delta L(z) - L(z) \otimes L(z)$ and $V(z) = \frac{x_0}{z} + \frac{x_1}{1-z}$. We observe that $T(z)$ satisfies the differential equation

$$\frac{d}{dz}T(z) = (V(z) \otimes 1 + 1 \otimes V(z))\, T(z), \tag{14}$$

with initial condition $T(0) = 0$. Therefore $T(z) = 0, \forall z \in \mathcal{R}$. □

REMARK – The shuffle relation $L_{x_0}L_{x_1} = L_{x_0x_1} + L_{x_1x_0}$ is equivalent to Euler's relation [8]

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = \zeta(2) - \log(z)\log(1-z).$$

Let z_0 be a point of $\mathcal{R}$, which we identify with its projection on $\mathbb{C}$ and let $z_0 \rightsquigarrow z$ be a differentiable path on $\mathbb{C}\backslash\{0,1\}$. Then L admits an analytic continuation along this path. The series $L(z)$ and $S_{z_0 \rightsquigarrow z}L(z)$ both satisfy the differential equation (12) and take the same value in $z = z_0$. This proves that

$$L(z) = S_{z_0 \rightsquigarrow z}L(z_0), \tag{15}$$

for all paths $z_0 \rightsquigarrow z$ in $\mathbb{C}\backslash\{0,1\}$.

Let $R \in]0,1[$ and denote by $\gamma_0(R)$ (resp. $\gamma_1(R)$) the circular paths of radius R and turning around 0 (resp. 1) in the trigonometric sense, starting in $z = R$ (resp. $z = 1 - R$). One easily proves the estimate

$$(S_{\gamma_0(R)}|w) < \frac{1}{|w|!}(2\pi)^{|w|}(2R)^{|w|_{x_1}}, \tag{16}$$

for the coefficients of the Chen series along $\gamma_0(R)$ (for $R < 1/2$), where $|w|$ denotes the length of the word w and $|w|_{x_1}$ the number of occurrences of x_1 in w. For $R \to 0$, this estimate yields

$$\begin{cases} \lim_{R\to 0} S_{\gamma_0(R)} &= e^{2i\pi x_0}, \\ \lim_{R\to 0} S_{\gamma_1(R)} &= e^{-2i\pi x_1}. \end{cases} \tag{17}$$

For each $t \in]0,1[$, let $\mathcal{M}_0L(t)$ (resp. $\mathcal{M}_1L(t)$), be the analytic continuation of $L(t)$ along $\gamma_0(t)$, (resp. $\gamma_1(t)$). From (15), we get $\mathcal{M}_iL(t) = S_{\gamma_i(t)}L(t)$ for $i = 0,1$. We will now show how to compute two Lie exponentials $M_0, M_1 \in \mathbb{C}\langle X\rangle$, which do not depend on t, such that $\forall t \in]0,1[$

$$\mathcal{M}_iL(t) = L(t)M_i, \quad \text{pour } i = 0 \text{ where } 1. \tag{18}$$

2.1 Monodromy of L around 0

Since a Chen series only depends on the homotopy class of its underlying path, we deduce from (15) that

$$\begin{aligned} \mathcal{M}_0L(t) &= S_{\varepsilon\rightsquigarrow t}S_{\gamma_0(\varepsilon)}S_{t\rightsquigarrow\varepsilon}L(t), \\ &= L(t)\ L^{-1}(\varepsilon)S_{\gamma_0(\varepsilon)}L(\varepsilon). \end{aligned}$$

This yields

$$M_0 = \lim_{\varepsilon\to 0^+} L^{-1}(\varepsilon)S_{\gamma_0(\varepsilon)}L(\varepsilon) = e^{2i\pi x_0}. \tag{19}$$

Indeed, for $\varepsilon \to 0$, the series $S_{\gamma_0(\varepsilon)} = e^{2i\pi x_0} + O(\varepsilon)$ and $L(\varepsilon) = e^{(\log\varepsilon)x_0} + O(\sqrt{\varepsilon})$ commute. In particular, (18) implies

$$\begin{cases} \mathcal{M}_0L_{x_0} &= L_{x_0} + (2i\pi), \\ \forall w \in X^*, \quad \mathcal{M}_0L_{wx_1} &= L_{wx_1}. \end{cases} \tag{20}$$

2.2 Monodromy of L around 1

Similarly,

$$\begin{aligned}\mathcal{M}_1 L(t) &= S_{1-\varepsilon\rightsquigarrow t} S_{\gamma_1(\varepsilon)} S_{t\rightsquigarrow 1-\varepsilon} L(t),\\ &= L(t)\ L^{-1}(1-\varepsilon) S_{\gamma_1(\varepsilon)} L(1-\varepsilon),\end{aligned}$$

whence

$$M_1 = \lim_{\varepsilon\to 0^+} L^{-1}(1-\varepsilon) e^{-2i\pi x_1} L(1-\varepsilon). \qquad (21)$$

In order to isolate the divergent terms of $L(1-\varepsilon)$, we will show how to factor $L(z)$ as an infinite product of Lie exponentials, by using the dual Poincaré–Birkoff–Witt basis.

3 Computation of the monodromy around 1

3.1 Duality in the space of polynomials

We define a pairing on $R\langle X\rangle$ by $(u|v) = \delta_u^v$, for words $u, v \in X^*$ (δ denotes the Kronecker symbol), and extending by bilinearity. Let $B \subset R\langle X\rangle$ and $B^* \subset R\langle X\rangle$ be two bases of $R\langle X\rangle$ related by a bijection

$$* : B \ni b \to b^* \in B^*.$$

These bases are said to be *dual* if

$$\forall a \in B, b^* \in B^*, \quad (a|b^*) = \delta_a^b \qquad (22)$$

For instance, the basis $R\langle X\rangle$ of all words $w \in X^*$ is self-dual.

It is well known that the algebra $R\langle X\rangle$ with the usual concatenation product is an enveloping algebra of the free Lie algebra $\mathcal{L}ie_R\langle X\rangle$. Let B_1 be an ordered basis of the free Lie algebra. Let B the associated Poincaré–Birkoff–Witt basis, that is, the set of elements of the form $b_1 b_2 \cdots b_k$, with $k \geq 0$ and $b_1 \geq b_2 \geq \cdots \geq b_k$, with the b_i in B_1. The commutative R-algebra $R\langle X\rangle$ is freely generated by the elements of $B_1^* = \{b^*;\ b \in B_1\}$, for the shuffle product $ш$.

3.2 Factorization of the double series

Consider the R–algebra $R\langle X\rangle \otimes R\langle X\rangle$, with the shuffle product on the first factor and the concatenation product on the second. By definition,

$$(x\otimes y)(x'\otimes y') = x \,ш\, x' \otimes y\cdot y', \qquad (23)$$

for $x, y, x', y' \in R\langle X\rangle$. The following factorization is classical:

$$\sum_{w\in X^*} w\otimes w = \sum_{b\in B} b^*\otimes b = \prod_{b\in B_1} e^{b^*\otimes b}. \qquad (24)$$

This yields the following factorization of $L(z)$ as a product of decreasing exponentials:

$$L(z) = \sum_{w\in X^*} L_w(z)\ w = \sum_{b\in B} L_{b^*}(z)\ b = \prod_{b\in B_1} e^{L_{b^*}(z)\ b}. \qquad (25)$$

3.3 Construction of the dual Lyndon basis

By definition, a *Lyndon word* is a non empty word $l \in X^*$, which is inferior to each of its strict right factors (for the lexicographical ordering):

$$\forall u, v \in X^+,\ l = uv \Rightarrow l < v \qquad (26)$$

The set of Lyndon words is denoted by $\mathcal{L}yndon(X)$.

Example – For $X = \{x_0, x_1\}$ with $x_0 < x_1$, the Lyndon words of length $\leqslant 5$ on X^* are the following (in increasing order):

$$\{x_0, x_0^4x_1, x_0^3x_1, x_0^3x_1^2, x_0^2x_1, x_0^2x_1x_0x_1, x_0^2x_1^2,\\ x_0^2x_1^3, x_0x_1, x_0x_1x_0x_1^2, x_0x_1^2, x_0x_1^3, x_0x_1^4, x_1\}.$$

The *bracket form* $P(l) \in \mathcal{L}ie_R\langle X\rangle$ of a Lyndon word $l = uv$ with $l, u, v \in \mathcal{L}yndon(X)$ (v being as long as possible) is defined recursively by

$$\left\{\begin{array}{lcl} P(l) &=& [P(u), P(v)]\\ P(x) &=& x \quad \text{for each letter } x\in X,\end{array}\right. \qquad (27)$$

It is classical that the set $B_1 = \{P(l);\ l \in \mathcal{L}yndon(X)\}$, ordered lexicographically, is a basis for the free Lie algebra. Moreover, each word $w \in X^*$ can be expressed uniquely as an increasing product of Lyndon words:

$$w = l_1^{\alpha_1} l_2^{\alpha_2} \ldots l_k^{\alpha_k}, \quad l_1 > l_2 > \cdots > l_k, \quad k \geq 0. \quad (28)$$

The Poincaré–Birkoff–Witt basis $B = \{P(w);\ w \in X^*\}$ and its dual basis $B^* = \{P^*(w);\ w \in X^*\}$ are obtained by setting [7]

$$\left\{\begin{array}{lcl} P(w) &=& P(l_1)^{\alpha_1} P(l_2)^{\alpha_2} \ldots P(l_k)^{\alpha_k},\\ P^*(w) &=& C P^*(l_1)^{ш\,\alpha_1} \,ш \ldots ш\, P^*(l_k)^{ш\,\alpha_k},\\ && \text{where } C = (\alpha_1!\alpha_2!\ldots\alpha_k!)^{-1}\\ P^*(l) &=& xP^*(w), \quad \forall l \in \mathcal{L}yndon(X),\\ && \text{where } l = xw,\ x\in X,\ w\in X^*.\end{array}\right. \qquad (29)$$

Lemma 1

$$\forall l \in \mathcal{L}yndon(X), \quad P^*(l) = l + \sum_{w, |w|<|l|} \lambda_w w$$

See table **??**.

Lemma 2 *One has* $P^*(w) \in x_0\mathbb{Z}\langle X\rangle x_1$ *for all* $w \in x_0X^*x_1$.

Proof – The Lyndon words involved in the decomposition (28) of a word $w \in X^*x_1$ (resp. $w \in x_0X^*x_1$) all belong to X^*x_1 (resp. $x_0X^*x_1$). □

3.4 Formulas for the monodromy around 1

3.4.1 Formulas up to order 4

Let us discuss our method in detail for the computation of the monodromy of L around 1 at order 4. We start from the factorization (25) of $L(z)$:

$$L = \prod_{l \in \mathcal{L}yndon(X)} \exp\left(L_{P^*(l)} P(l)\right) \tag{30}$$

We will kill all Lie brackets of length superior to 4, which is equivalent to working in the *free nilpotent Lie algebra* of order 4. The factorization (30) yields

$$\begin{aligned} L(z) &= e^{L_{x_1}(z) x_1} e^{L_{x_0x_1x_1}(z)[[x_0,x_1],x_1]} e^{L_{x_0x_1}(z)[x_0,x_1]} \\ &\times\ e^{L_{x_0x_0x_1}(z)[x_0,[x_0,x_1]]} e^{L_{x_0}(z) x_0}. \end{aligned}$$

Evaluated in $z = 1-\varepsilon$, this factorization becomes

$$\begin{aligned} L(1-\varepsilon) &= e^{L_{x_1}(1-\varepsilon)x_1} e^{\zeta_{x_0x_1x_1}[[x_0,x_1],x_1]} \\ &\times\ e^{\zeta_{x_0x_1}[x_0,x_1]} e^{\zeta_{x_0x_0x_1}[x_0,[x_0,x_1]]}. \end{aligned}$$

The important point is that the divergent terms only appear in the exponential $\exp(L_{x_1}(1-\varepsilon)x_1)$. Furthermore, $\exp(L_{x_0}(1-\varepsilon)x_0) \sim 1$, since $L_{x_0}(1-\varepsilon) = \log(1-\varepsilon) \sim -\varepsilon$. This observation hold in general, by lemma 2.

Because of (21), the monodromy $\mathcal{M}_1 L(t) = L(t) M_1$ is given by the series

$$\begin{aligned} M_1 = &\, e^{-\zeta_{x_0x_0x_1}[x_0,[x_0,x_1]]} e^{-\zeta_{x_0x_1}[x_0,x_1]} e^{-\zeta_{x_0x_1x_1}[[x_0,x_1],x_1]} \\ &\times e^{-L_{x_1}(1-\varepsilon)x_1} e^{-2i\pi x_1} e^{L_{x_1}(1-\varepsilon)x_1} \\ &\times e^{\zeta_{x_0x_1x_1}[[x_0,x_1],x_1]} e^{\zeta_{x_0x_1}[x_0,x_1]} e^{\zeta_{x_0x_0x_1}[x_0,[x_0,x_1]]}. \end{aligned}$$

The exponentials in x_1 commute, whence the divergent terms disappear. On the other hand, the exponentials of brackets of length 3 also disappear, since they commute with the other exponentials (the brackets of order 4 vanish). We now apply the classical formulas for the adjoint representation of Lie groups:

$$e^{\mathrm{ad}\,X} e^{Y} = e^X e^Y e^{-X} = e^{\exp(X) Y \exp(-X)} = e^{\exp(\mathrm{ad}\,X)\ Y}.$$

After some computations, we now obtain

$$\begin{aligned} M_1 &= e^{-\zeta_{x_0x_1}[x_0,x_1]} e^{-2i\pi x_1} e^{\zeta_{x_0x_1}[x_0,x_1]} \\ &= e^{-\zeta_{x_0x_1}\ \mathrm{ad}([x_0,x_1])} e^{-2i\pi x_1} \\ &= e^{\exp(-\zeta_{x_0x_1}\ \mathrm{ad}([x_0,x_1]))\ (-2i\pi x_1)} \\ &= e^{-2i\pi x_1 + 2i\pi \zeta_{x_0x_1}[[x_0,x_1],x_1]}. \end{aligned}$$

3.4.2 The general case

Applied in the general case, the above method yields the following theorem:

Theorem 2 *The monodromy of the series $L(t)$ for $t \in]0,1[$ around $z=1$ is given by $\mathcal{M}_1 L(t) = L(t) M_1$, where M_1 is a Lie exponential given by the formula*

$$M_1 = \left(\prod_{l \in \mathcal{L}yndon(X) \setminus \{x_0,x_1\}} e^{-\zeta_{P^*(l)}\,\mathrm{ad}\,P(l)} \right) e^{-2i\pi x_1} \tag{31}$$

The constants $\zeta_{P^*(l)}$ are finite, since $P^*(l) \in x_0 \mathbb{Q}\langle X \rangle x_1$, by construction (29).

This theorem is effective, since the computation of the monodromy of the L_w for $|w| \leqslant n$ only necessitates the knowledge of M_1 for words of length $< n$.

Corollary 1 *The monodromy of the polylogarithms L_w is given by*

$$\begin{aligned} \forall w \in X^*, \quad \mathcal{M}_0 L_{wx_0} &= L_{wx_0} + 2i\pi L_w + \cdots \\ \mathcal{M}_1 L_{wx_1} &= L_{wx_1} - 2i\pi L_w + \cdots, \end{aligned}$$

where the remaining terms are linear combinations of polylogarithms coded by words of lengths $< w$.

Proof – Consequence of (18), by remarking that (31) implies that

$$\left\{ \begin{aligned} M_0 &= 1 + 2i\pi x_0 + \text{words of length} > 1 \\ M_1 &= 1 - 2i\pi x_1 + \text{words of length} > 1 \end{aligned} \right. \tag{32}$$

See also the results from appendix A. □

Corollary 2 *The monodromy group of the functions L_w for $|w| \leqslant n$ is nilpotent at order $n+1$.*

Proof – We have

$$M_0 = e^{2i\pi x_0} \quad \text{and} \quad M_1 = e^{-2i\pi x_1 + \cdots}.$$

From

$$e^A e^B e^{-A} e^{-B} = e^{[A,B] + \cdots},$$

it follows that the commutator $M_0 M_1 M_0^{-1} M_1^{-1}$ does not contain any Lie brackets of length 1. Iterating this computation, the brackets of lengths 2, next 3, etc. until n disappear. □

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