A differential intermediate value theorem

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Abstract

Let $\mathbb{T}$ be the field of grid-based transseries or the field of transseries with finite logarithmic depths. In our PhD. we announced that given a differential polynomial P with coefficients in $\mathbb{T}$ and transseries $\varphi<\psi$ with $P(\varphi)<0$ and $P(\psi)>0$, there exists an $f\in(\varphi,\psi)$, such that $P(f)=0$. In this note, we will prove this theorem.

1 Introduction

1.1 Statement of the results

Let C be a totally ordered exp-log field. In chapter 2 of [vdH97], we introduced the field $\mathbb{T}=C[[[x]]]$ of transseries in x of finite logarithmic and exponential depths. In chapter 5, we then gave an (at least theoretical) algorithm to solve algebraic differential equations with coefficients in $\mathbb{T}$. By that time, the following theorem was already known to us (and stated in the conclusion), but due to lack of time, we had not been able to include the proof.

Theorem 1. *Let P be a differential polynomial with coefficients in $\mathbb{T}$. Given $\varphi<\psi$ in $\mathbb{T}$, such that $P(\varphi)\,P(\psi)<0$, there exists an $f\in(\varphi,\psi)$ with $P(f)=0$.*

In the theorem, (φ,ψ) stands for the open interval between φ and ψ. The proof that we will present in this note will be based on the differential Newton polygon method as described in chapter 5 of [vdH97]. We will freely use any results from there. We recall (and renew) some notations in section 2.

In chapter 1 of [vdH97], we also introduced the field of grid-based $C[\![\![x]\!]\!]\subseteq C[[[x]]]$ transseries in x. In chapter 12, we have shown that our algorithm for solving algebraic differential equations preserves the grid-based property. Therefore, it is easily checked that theorem 1 also holds for $\mathbb{T}=C[\![\![x]\!]\!]$. Similarly, it may be checked that the theorem holds if we take for $\mathbb{T}$ the field of transseries of finite logarithmic depths (and possibly countable exponential depths).

1.2 Proof strategy

Assume that P is a differential polynomial with coefficients in $\mathbb{T}$, which admits a sign change on a non empty interval (φ,ψ) of transseries. The idea behind the proof of theorem 1 is very simple: using the differential Newton polygon method, we shrink the interval (φ,ψ) further and further while preserving the sign change property. Ultimately, we end up with an interval which is reduced to a point, which will then be seen to be a zero of P.

However, in order to apply the above idea, we will need to allow non standard intervals (φ, ψ) in the proof. More precisely, φ and ψ may generally be taken in the compactification of $\mathbb{T}$, as constructed in section 2.6 of [vdH97]. In this paper we will consider non standard φ (resp. ψ) of the following forms:

- $\varphi = \xi \pm \text{Ǝ}$, with $\xi \in \mathbb{T}$;
- $\varphi = \xi \pm \mho$, with $\xi \in \mathbb{T}$;
- $\varphi = \xi \pm \backepsilon \mathfrak{m}$, with $\xi \in \mathbb{T}$ and where $\mathfrak{m}$ is a transmonomial.
- $\varphi = \xi \pm \text{ꝏ} \mathfrak{m}$, with $\xi \in \mathbb{T}$ and where $\mathfrak{m}$ is a transmonomial.
- $\varphi = \xi \pm \gamma$, with $\xi \in \mathbb{T}$ and $\gamma = (x \log x \log \log x \cdots)^{-1}$.

Here Ǝ and $\mho$ respectively designate the infinitely small and large constants $\infty_{\mathbb{T}}^{-1}$ and $\infty_{\mathbb{T}}$ in the compactification of $\mathbb{T}$. Similarly, $\backepsilon$ and ꝏ designate the infinitely small and large constants ∞_C^{-1} and ∞_C in the compactification of C. We may then interpret φ as a cut of the transline $\mathbb{T}$ into two pieces $\mathbb{T} = \{f \in \mathbb{T} | f < \varphi\} \amalg \{f \in \mathbb{T} | f > \varphi\}$. Notice that

$$\begin{aligned}
\{f \in \mathbb{T}^+ | f < \gamma\} &= \{f \in \mathbb{T}^+ | \exists g \in \mathbb{T}^+ \colon g \prec 1 \wedge f = g'\};\\
\{f \in \mathbb{T}^+ | f > \gamma\} &= \{f \in \mathbb{T}^+ | \exists g \in \mathbb{T}^+ \colon g \succ 1 \wedge f = g'\}.
\end{aligned}$$

Remark 2. Actually, the notations $\xi \pm \mho$, $\xi \pm \backepsilon \mathfrak{m}$, and so on are redundant. Indeed, $\xi \pm \mho$ does not depend on ξ, we have $\xi + \backepsilon \mathfrak{m} = \chi + \backepsilon \mathfrak{m}$ whenever $\xi - \chi \prec \mathfrak{m}$, etc.

Now consider a generalized interval $I = (\varphi, \psi)$, where φ and ψ may be as above. We have to give a precise meaning to the statement that P admits a sign change on I. This will be the main object of sections 3 and 4. We will show there that, given a cut φ of the above type, the function $\sigma_P(f) = \operatorname{sign} P(f)$ may be prolongated by continuity into φ from at least one direction:

- If $\varphi = \xi + \text{Ǝ}$, then σ_P is constant on $(\varphi, \chi) = (\xi, \chi)$ for some $\chi > \varphi$.
- If $\varphi = \xi + \mho$, then σ_P is constant on (χ, φ) for some $\chi < \varphi$.
- If $\varphi = \xi + \backepsilon \mathfrak{m}$, then σ_P is constant on (χ, φ) for some $\chi < \varphi$.
- If $\varphi = \xi + \text{ꝏ} \mathfrak{m}$, then σ_P is constant on (φ, χ) for some $\chi > \varphi$.
- If $\varphi = \xi + \gamma$, then σ_P is constant on (φ, χ) for some $\chi > \varphi$.

(In the cases $\varphi = \xi - \text{Ǝ}$, $\varphi = -\mho$ and so on, one has to interchange left and right continuity in the above list.) Now we understand that P admits a sign change on a generalized interval (φ, ψ) if $\sigma_P(\varphi)\, \sigma_P(\psi) < 0$.

2 List of notations

Asymptotic relations.

$$\begin{aligned}
f \prec g &\Leftrightarrow f = o(g);\\
f \preccurlyeq g &\Leftrightarrow f = O(g);\\
f \prec\!\!\prec g &\Leftrightarrow \log|f| \prec \log|g|;\\
f \preccurlyeq\!\!\!\preccurlyeq g &\Leftrightarrow \log|f| \preccurlyeq \log|g|.
\end{aligned}$$

Logarithmic derivatives.

$$\begin{aligned} f^{\dagger} &= f'/f; \\ f^{\langle i \rangle} &= f^{\dagger \cdots \dagger} \qquad (i \text{ times}). \end{aligned}$$

Natural decomposition of P.

$$P(f) = \sum_{\boldsymbol{i}} P_{\boldsymbol{i}} f^{(\boldsymbol{i})} \tag{1}$$

Here we use vector notation for tuples $\boldsymbol{i} = (i_0, \ldots, i_r)$ and $\boldsymbol{j} = (j_0, \ldots, j_r)$ of integers:

$$\begin{aligned} |\boldsymbol{i}| &= r; \\ \boldsymbol{i} \leqslant \boldsymbol{j} &\Leftrightarrow i_0 \leqslant j_0 \wedge \cdots \wedge i_r \leqslant i_r; \\ f^{\boldsymbol{i}} &= f^{i_0} (f')^{i_1} \cdots (f^{(r)})^{i_r}; \\ \binom{\boldsymbol{j}}{\boldsymbol{i}} &= \binom{j_1}{i_1} \cdots \binom{j_r}{i_r}. \end{aligned}$$

Decomposition of P along orders.

$$P(f) = \sum_{\boldsymbol{\omega}} P_{[\boldsymbol{\omega}]} f^{[\boldsymbol{\omega}]} \tag{2}$$

In this notation, $\boldsymbol{\omega}$ runs through tuples $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_l)$ of integers in $\{0, \ldots, r\}$ of length l at most d, and $P_{[\boldsymbol{\omega}]} = P_{[\omega_{\sigma(1)}, \ldots, \omega_{\sigma(l)}]}$ for all permutations of integers. We again use vector notation for such tuples

$$\begin{aligned} |\boldsymbol{\omega}| &= l; \\ \|\boldsymbol{\omega}\| &= \omega_1 + \cdots + \omega_{|\boldsymbol{\omega}|}; \\ \boldsymbol{\omega} \leqslant \boldsymbol{\tau} &\Leftrightarrow |\boldsymbol{\omega}| = |\boldsymbol{\tau}| \wedge \omega_1 \leqslant \tau_1 \wedge \cdots \wedge \omega_{|\boldsymbol{\omega}|} \leqslant \tau_{|\boldsymbol{\tau}|}; \\ f^{[\boldsymbol{\omega}]} &= f^{(\omega_1)} \cdots f^{(\omega_{|\boldsymbol{\omega}|})}; \\ \binom{\boldsymbol{\tau}}{\boldsymbol{\omega}} &= \binom{\tau_1}{\omega_1} \cdots \binom{\tau_{|\boldsymbol{\tau}|}}{\omega_{|\boldsymbol{\omega}|}}. \end{aligned}$$

We call $\|\boldsymbol{\omega}\|$ the *weight* of $\boldsymbol{\omega}$ and

$$\|P\| = \max_{\boldsymbol{\omega} | P_{[\boldsymbol{\omega}]} \neq 0} \|\boldsymbol{\omega}\|$$

the *weight* of P.

Additive, multiplicative and compositional conjugations or upward shifting.

$$\begin{aligned} P_{+h}(f) &= P(h + f); \\ P_{\times h}(f) &= P(h f); \\ P{\uparrow}(f{\uparrow}) &= P(f){\uparrow}. \end{aligned}$$

Additive conjugation:

$$P_{+h, \boldsymbol{i}} = \sum_{\boldsymbol{j} \geqslant \boldsymbol{i}} \binom{\boldsymbol{j}}{\boldsymbol{i}} h^{\boldsymbol{j} - \boldsymbol{i}} P_{\boldsymbol{j}}. \tag{3}$$

Multiplicative conjugation:

$$P_{\times h, [\boldsymbol{\omega}]} = \sum_{\boldsymbol{\tau} \geqslant \boldsymbol{\omega}} \binom{\boldsymbol{\tau}}{\boldsymbol{\omega}} h^{[\boldsymbol{\tau} - \boldsymbol{\omega}]} P_{[\boldsymbol{\tau}]}. \tag{4}$$

Upward shifting (compositional conjugation):

$$(P{\uparrow})_{[\boldsymbol{\omega}]} = \sum_{\boldsymbol{\tau} \geqslant \boldsymbol{\omega}} s_{\boldsymbol{\tau}, \boldsymbol{\omega}} e^{-\|\boldsymbol{\tau}\| x} (P_{[\boldsymbol{\tau}]}{\uparrow}), \tag{5}$$

where the $s_{\boldsymbol{\tau},\boldsymbol{\omega}}$ are generalized Stirling numbers of the first kind:

$$\begin{aligned} s_{\boldsymbol{\tau},\boldsymbol{\omega}} &= s_{\tau_1,\omega_1} \cdots s_{\tau_{|\tau|},\omega_{|\omega|}}; \\ (f(\log x))^{(j)} &= \sum_{i=0}^{j} s_{j,i}\, x^{-j} f^{(i)}(\log x). \end{aligned}$$

3 Behaviour of $\boldsymbol{\sigma_P}$ near zero and infinity

3.1 Behaviour of $\boldsymbol{\sigma_P}$ near infinity

Lemma 3. *Let P be a differential polynomial with coefficients in $\mathbb{T}$. Then $P(\pm f)$ has constant sign for all sufficiently large $f \in \mathbb{T}$.*

Proof. If $P=0$, then the lemma is clear, so assume that $P \neq 0$. Using the rules

$$\begin{aligned} f &= f; \\ f' &= f^{\dagger} f; \\ f'' &= (f^{\dagger})^2 f + f^{\dagger\dagger} f^{\dagger} f; \\ f''' &= (f^{\dagger})^3 f + 3 f^{\dagger\dagger} (f^{\dagger})^2 f + (f^{\dagger\dagger})^2 f^{\dagger} f + f^{\dagger\dagger\dagger} f^{\dagger\dagger} f^{\dagger} f; \\ &\vdots \end{aligned}$$

we may rewrite $P(f)$ as an expression of the form

$$P(f) = \sum_{\boldsymbol{i}=(i_0,\ldots,i_r)} P_{\langle \boldsymbol{i} \rangle} f^{\langle \boldsymbol{i} \rangle}, \tag{6}$$

where $P_{\langle \boldsymbol{i} \rangle} \in \mathbb{T}$ and $f^{\langle \boldsymbol{i} \rangle} = f^{i_0} (f^{\dagger})^{i_1} \cdots (f^{\langle r \rangle})^{i_r}$ for each $\boldsymbol{i}$. Now consider the lexicographical ordering $\leqslant^{\text{lex}}$ on $\mathbb{N}^{r+1}$, defined by

$$\begin{aligned} \boldsymbol{i} <^{\text{lex}} \boldsymbol{j} \iff & (i_0 < j_0) \vee \\ & (i_0 = j_0 \wedge i_1 < j_0) \vee \\ & \vdots \\ & (i_0 = j_0 \wedge \cdots \wedge i_{r-1} = j_{r-1} \wedge i_r < j_r). \end{aligned}$$

This ordering is total, so there exists a maximal $\boldsymbol{i}$ for $\leqslant^{\text{lex}}$, such that $P_{\langle \boldsymbol{i} \rangle} \neq 0$. Now let $k \geqslant 1$ be sufficiently large such that $P_{\langle \boldsymbol{j} \rangle} \prec\!\!\prec \exp_k x$ for all $\boldsymbol{j}$. Then

$$\sigma_P(\pm f) = (\pm 1)^{i_0} \operatorname{sign} P_{\langle \boldsymbol{i} \rangle} \tag{7}$$

for all postive, infinitely large $f \succ\!\!\succ \exp_{k+r} x$, since $\exp_k x \prec\!\!\prec f^{\langle r \rangle} \prec\!\!\prec \cdots \prec\!\!\prec f^{\dagger} \prec\!\!\prec f$ for all such f. □

3.2 Behaviour of $\boldsymbol{\sigma_P}$ near zero

Lemma 4. *Let P be a differential polynomial with coefficients in $\mathbb{T}$. Then $P(\pm \varepsilon)$ has constant sign for all sufficiently small $\varepsilon \in \mathbb{T}^{+}_{*}$.*

Proof. If $P=0$, then the lemma is clear. Assume that $P\neq 0$ and rewrite $P(f)$ as in (6). Now consider the twisted lexicographical ordering $\leqslant^{\mathrm{tl}}$ on $\mathbb{N}^{r+1}$, defined by

$$\begin{aligned}\boldsymbol{i}<^{\mathrm{tl}}\boldsymbol{j} \iff & (i_0>j_0)\vee\\ & (i_0=j_0\wedge i_1<j_0)\vee\\ & \quad\vdots\\ & (i_0=j_0\wedge\cdots\wedge i_{r-1}=j_{r-1}\wedge i_r<j_r).\end{aligned}$$

This ordering is total, so there exists a maximal $\boldsymbol{i}$ for $\leqslant^{\mathrm{tl}}$, such that $P_{\langle \boldsymbol{i}\rangle}\neq 0$. If $k\geqslant 1$ is sufficiently large such that $P_{\langle \boldsymbol{j}\rangle}\prec\!\!\prec \exp_k x$ for all $\boldsymbol{j}$, then

$$\sigma_P(\pm\varepsilon)=(\pm 1)^{i_0}\operatorname{sign}P_{\boldsymbol{i}} \tag{8}$$

for all postive infinitesimal $\varepsilon\succ\!\!\succ\exp_{k+r}x$. □

3.3 Canonical form of differential Newton polynomials

Assume that P has purely exponential coefficients. In what follows, we will denote by $N_{P,\mathfrak{m}}$ the *purely exponential differential Newton polynomial* associated to a monomial $\mathfrak{m}$, i.e.

$$N_{P,\mathfrak{m}}(c)=\sum_{\boldsymbol{i}} P_{\times\mathfrak{m},\boldsymbol{i},\mathfrak{d}(P_{\times\mathfrak{m}})}\,c^{\boldsymbol{i}}, \tag{9}$$

where

$$\mathfrak{d}_P=\max_{\boldsymbol{i},\preccurlyeq}\mathfrak{d}_{P_{\boldsymbol{i}}}. \tag{10}$$

The following theorem shows how $N_P=N_{P,1}$ looks like after sufficiently many upward shiftings:

Theorem 5. *Let P be a differential polynomial with purely exponential coefficients. Then there exists a polynomial $Q\in C[c]$ and an integer ν, such that for all $i\geqslant\|P\|$, we have $N_{P\uparrow_i}=Q\,(c')^{\nu}$.*

Proof. Let ν be minimal, such that there exists an $\boldsymbol{\omega}$ with $\|\boldsymbol{\omega}\|=\nu$ and $(N_P\uparrow)_{[\boldsymbol{\omega}]}\neq 0$. Then we have $\mathfrak{d}(N_P\uparrow)=e^{-\nu x}$ and

$$N_{P\uparrow}(c)=\sum_{\|\boldsymbol{\omega}\|=\mu}\left(\sum_{\boldsymbol{\tau}\geqslant\boldsymbol{\omega}} s_{\boldsymbol{\tau},\boldsymbol{\omega}}\,N_{P,[\boldsymbol{\tau}]}\right)c^{[\boldsymbol{\omega}]}, \tag{11}$$

by formula (5). Since $N_{P\uparrow}\neq 0$, we must have $\nu\leqslant\|N_P\|$. Consequently, $\|N_P\|\geqslant\nu=\|N_{P\uparrow}\|\geqslant\|N_{P\uparrow\uparrow}\|\geqslant\cdots$. Hence, for some $i\leqslant\|P\|$, we have $\|N_{P\uparrow_{i+1}}\|=\|N_{P\uparrow_i}\|$. But then (11) applied on $P\uparrow_i$ instead of P yields $N_{P\uparrow_{i+1}}=N_{P\uparrow_i}$. This shows that $N_{P\uparrow_i}$ is independent of i, for $i\geqslant\|P\|$.

In order to prove the theorem, it now suffices to show that $N_{P\uparrow}=N_P$ implies $N_{P\uparrow}=Q\,(c')^{\nu}$ for some polynomial $Q\in C[c]$. For all differential polynomials R of homogeneous weight ν, let

$$R^*=\sum_j\,([c^j\,(c')^{\nu}]\,R)\,c^j\,(c')^{\nu}. \tag{12}$$

Since $N_{P\uparrow}^*=N_P^*$, it suffices to show that $P=0$ whenever $N_P^*=0$. Now $N_P^*=0$ implies that $N_P(x)=0$. Furthermore, (5) yields

$$N_P\uparrow=e^{-\nu x}\,N_P. \tag{13}$$

Consequently, we also have $N_P(e^x) = e^{\nu x}\,(N_P{\uparrow})(e^x) = e^{\nu x}\,(N_P(x)){\uparrow} = 0$. By induction, it follows that $N_P(\exp_i x) = 0$ for any iterated exponential of x. We conclude that $N_P = P = 0$, by the lemma 3. □

Remark 6. Given any differential polynomial P with coefficients in $\mathbb{T}$, this polynomial becomes purely exponential after sufficiently many upward shiftings. After at most $\|P\|$ more upward shiftings, the purely exponential Newton polynomial stabilizes. The resulting purely exponential differential Newton polynomial, which is in $C[c]\,(c')^{\mathbb{N}}$, is called the *differential Newton polynomial* of P.

4 Behaviour of $\boldsymbol{\sigma_P}$ near constants

In the previous section, we have seen how to compute $P(\xi \pm \exists)$ and $P(\xi \pm \mho)$ for all $\xi \in \mathbb{T}$. In this section, we show how to compute $P(\xi \pm \backepsilon\,\mathfrak{m})$ and $P(\xi \pm \frown\mathfrak{m})$ for all $\xi \in \mathbb{T}$ and all transmonomials $\mathfrak{m}$. Modulo an additive and a multiplicative conjugation with ξ resp. $\mathfrak{m}$, we may assume without loss of generality that $\xi = 0$ and $\mathfrak{m} = 1$. Hence it will suffice to study the behaviour of $\sigma_P(c \pm \varepsilon)$ for $c \in C$ and positive infinitesimal (but sufficiently large) ε, as well as the behaviour of $\sigma_P(f)$ for positive infinitely large (but sufficiently small) f.

Modulo suffiently upward shiftings (we have $\sigma_P(c+\varepsilon) = \sigma_{P\uparrow}(c+\varepsilon{\uparrow})$ and $\sigma_P(f) = \sigma_{P\uparrow}(f{\uparrow})$), we may assume that P has purely exponential coefficients. By theorem 5 and modulo at most $\|P\|$ more upward shiftings, we may also assume that

$$N_P(c) = Q(c)\,(c')^{\nu}, \tag{14}$$

for some polynomial $Q \in C[c]$ and $k \in \mathbb{N}$. We will denote by μ the multiplicity of c as a root of Q. Finally, modulo division of P by its dominant monomial (this does not alter σ_P), we may assume without loss of generality that $\mathfrak{d}_P = 1$.

4.1 Behaviour of $\boldsymbol{\sigma_P}$ in between constants

Lemma 7. *For all $0 < \varepsilon \prec 1$ with $\varepsilon \lll e^x$, the signs of $P(c-\varepsilon)$ and $P(c+\varepsilon)$ are independent of ε and given by*

$$(-1)^{\mu}\,\sigma_P(c - \backepsilon) = (-1)^{\nu}\,\sigma_P(c + \backepsilon) = \sigma_{Q^{(\mu)}}(c). \tag{15}$$

Proof. Since P is purely exponential and $\mathfrak{d}_P = 1$, there exists an $\alpha > 0$ such that

$$P(c+\varepsilon) - N_P(c+\varepsilon) \prec e^{-\alpha x} \tag{16}$$

for all $\varepsilon \prec 1$. Let $\varepsilon > 0$ be such that $e^{-\beta x} \prec \varepsilon \prec 1$, where $\beta = \alpha/(\mu + \nu)$. Then $Q(c \pm \varepsilon) \sim \frac{1}{\mu!}\,Q^{(\mu)}(c)\,(\pm\varepsilon)^{\mu}$, whence

$$e^{-\mu\beta x} \preccurlyeq Q(c+\varepsilon) \preccurlyeq 1. \tag{17}$$

Furthermore, $-\beta e^{-\beta x} \prec \varepsilon' \prec -\gamma$, whence

$$e^{-\nu\beta x} \prec (\varepsilon')^{\nu} \prec \gamma^{\nu}. \tag{18}$$

Put together, (17) and (18) imply that $N_P(c) \succ e^{-\alpha x}$. Hence $\sigma_P(c+\varepsilon) = \sigma_{N_P}(c+\varepsilon)$, by (16). Now

$$\sigma_P(c \pm \varepsilon) = \sigma_Q(c \pm \varepsilon)\,\operatorname{sign}\,((c \pm \varepsilon)')^{\nu} = (\pm 1)^{\mu}\,\sigma_{Q^{(\mu)}}(c)\,(\mp 1)^{\nu}, \tag{19}$$

since $\varepsilon' < 0$ for all positive infinitesimal ε. □

Corollary 8. *If P is homogeneous of degree i, then*

$$\sigma_P(\ni) = \sigma_P(\varepsilon) = \sigma_{R_{P,i}}(\varepsilon^\dagger) = \sigma_{R_{p,i}}(-\gamma), \tag{20}$$

for all $0 < \varepsilon \prec 1$ with $\varepsilon \prec\!\!\!\prec e^x$.

Corollary 9. *Let $c_1 < c_2$ be constants such that $\sigma_P(c_1 + \ni)\, \sigma_P(c_2 - \ni) < 0$. Then there exists a constant $c \in (c_1, c_2)$ with $\sigma_P(c - \ni)\, \sigma_P(c + \ni) < 0$.*

Proof. In the case when ν is odd, then $\sigma_P(c - \ni)\, \sigma_P(c + \ni) < 0$ holds for any $c > c_1$ with $Q(c) \neq 0$, by (15). Assume therefore that ν is even and let μ_1, μ_2 denote the multiplicities of c_1, c_2 as roots of Q. From (15) we deduce that

$$(-1)^{\mu_2}\, \sigma_{Q^{(\mu_1)}}(c_1)\, \sigma_{Q^{(\mu_2)}}(c_2) < 0. \tag{21}$$

In other words, the signs of $Q(c)$ for $c \downarrow c_1$ and $c \uparrow c_2$ are different. Hence, there exists a root c of Q between c_1 and c_2 which has odd multiplicity μ. For this root c, (15) again implies that $\sigma_P(c - \ni)\, \sigma_P(c + \ni) < 0$. □

4.2 Behaviour of σ_P before and after the constants

Lemma 10. *For all $0 < f \succ 1$ with $f \prec\!\!\!\prec e^x$, the signs of $P(-f)$ and $P(f)$ are independent of f and given by*

$$(-1)^{\deg Q + \nu}\, \sigma_P(-\infty) = \sigma_P(\infty) = \operatorname{sign} Q_{\deg Q}. \tag{22}$$

Proof. Since P is purely exponential and $\mathfrak{d}_P = 1$, there exists an $\alpha > 0$ such that

$$P(f) - N_P(f) \prec e^{-\alpha x}, \tag{23}$$

since $f, f', f'', \ldots \prec\!\!\!\prec e^x$. Furthermore $Q(\pm f) \sim Q_{\deg Q}\, (\pm f)^{\deg Q} \prec\!\!\!\prec e^x$ and $(\pm f')^\nu \prec\!\!\!\prec e^x$, whence $N_P(f) \prec\!\!\!\prec e^x$. In particular, $N_P(f) \succ e^{-\alpha x}$, so that $\sigma_P(f) = \sigma_{N_P}(f)$, by (23). Now

$$\sigma_P(\pm f) = \sigma_Q(\pm \varepsilon)\, \operatorname{sign}(\pm f')^\nu = \operatorname{sign} Q_{\deg Q}\, (\pm 1)^{\deg Q + \mu}, \tag{24}$$

since $f' > 0$ for positive infinitely large f. □

Corollary 11. *If P is homogeneous of degree i, then*

$$\sigma_P(\infty) = \sigma_P(f) = \sigma_{R_{P,i}}(f^\dagger) = \sigma_{R_{P,i}}(\gamma), \tag{25}$$

for all $0 < f \succ 1$ with $f \prec\!\!\!\prec e^x$.

Corollary 12. *Let c_1 be a constant such that $\sigma_P(c_1 + \ni)\, \sigma_P(\infty) < 0$. Then there exists a constant $c > c_1$ with $\sigma_P(c - \ni)\, \sigma_P(c + \ni) < 0$.*

Proof. In the case when ν is odd, then $\sigma_P(c - \ni)\, \sigma_P(c + \ni) < 0$ holds for any $c > c_1$ with $Q(c) \neq 0$, by (15). Assume therefore that ν is even and let μ_1 be the multiplicity of c_1 as a root of Q. From (15) and (22) we deduce that

$$\sigma_{Q^{(\mu_1)}}(c_1)\, \operatorname{sign} Q_{\deg Q} < 0. \tag{26}$$

In other words, the signs of $Q(c)$ for $c \downarrow c_1$ and $c \uparrow \infty$ are different. Hence, there exists a root $c > c_1$ of Q which has odd multiplicity μ. For this root c, (15) implies that $\sigma_P(c - \ni)\, \sigma_P(c + \ni) < 0$. □

5 Proof of the intermediate value theorem

It is convenient to prove the following generalizations of theorem 1.

Theorem 13. *Let ξ and $\mathfrak{v}$ be a transseries resp. a transmonomial in $\mathbb{T}$. Assume that P changes sign on an open interval I of one of the following forms:*

a) $I = (\xi, \chi)$*, for some* $\chi > \xi$ *with* $\mathfrak{d}(\chi - \xi) = \mathfrak{v}$.

b) $I = (\xi - \backepsilon\, \mathfrak{v}, \xi)$.

c) $I = (\xi, \xi + \backepsilon\, \mathfrak{v})$.

d) $I = (\xi - \backepsilon\, \mathfrak{v}, \xi + \backepsilon\, \mathfrak{v})$.

Then P changes sign at some $f \in I$.

Theorem 14. *Let ξ and $\mathfrak{v} \succ \gamma$ be a transseries resp. a transmonomial in $\mathbb{T}$. Assume that P changes sign on an open interval I of one of the following forms:*

a) $I = (\xi + \gamma, \chi - \gamma)$*, for some* $\chi > \xi$ *with* $\mathfrak{d}(\chi - \xi) = \mathfrak{v}$.

b) $I = (\xi - \backepsilon\, \mathfrak{v}, \xi - \gamma)$.

c) $I = (\xi + \gamma, \xi + \backepsilon\, \mathfrak{v})$.

d) $I = (\xi - \backepsilon\, \mathfrak{v}, \xi + \backepsilon\, \mathfrak{v})$.

Then P changes sign on $(f - \gamma, f + \gamma)$ for some $f \in I$ with $(f - \gamma, f + \gamma) \subseteq I$.

Proof. Let us first show that cases a, b and d may all be reduced to case c. We will show this in the case of theorem 13; the proof is similar in the case of theorem 14. Let us first show that case a may be reduced to cases b, c and d. Indeed, if P changes sign on (ξ, χ), then P changes sign on $(\xi, \xi + \backepsilon\, \mathfrak{v})$, $(\xi + \backepsilon\, \mathfrak{v}, \chi - \backepsilon\, \mathfrak{v})$ or $(\chi - \backepsilon\, \mathfrak{v}, \chi)$. In the second case, modulo a multiplicative conjugation and upward shifting, corollary 9 implies that there exists a $0 < \lambda < (\chi - \xi)_{\mathfrak{v}}$ such that P admits a sign change on $((\xi + \lambda\, \mathfrak{v}) - \backepsilon\, \mathfrak{v}, (\xi + \lambda\, \mathfrak{v}) + \backepsilon\, \mathfrak{v})$. Similarly, case d may be reduced to cases b and c by splitting the interval in two parts. Finally, cases b and c are symmetric when replacing $P(f)$ by $P(-f)$.

Without loss of generality we may assume that $\xi = 0$, modulo an additive conjugation of P by ξ. We prove the theorem by a triple induction over the order r of P, the Newton degree d of the asymptotic algebraic differential equation

$$P(f) = 0 \qquad (f \prec \mathfrak{v}) \tag{27}$$

and the maximal length l of a sequence of privileged refinements of Newton degree d (we have $l \leqslant (r+1)^d$, by proposition 5.12 in [vdH97]).

Let us show that, modulo upward shiftings, we may assume without loss of generality that P and $\mathfrak{v}$ are purely exponential and that $N_P \in C[c]\,(c')^{\mathbb{N}}$. In the case of theorem 13, we indeed have $\sigma_{P\uparrow}(0) = \sigma_P(0)$ and $\sigma_{P\uparrow}(\backepsilon\, \mathfrak{v}\uparrow) = \sigma_P(\backepsilon\, \mathfrak{v})$. In the case of theorem 14, we also have $\sigma_{P\uparrow_{\times e^{-x}}}(\gamma) = \sigma_{P\uparrow}(\gamma\uparrow) = \sigma_P(\gamma)$. Furthermore, if $f \in (\gamma, \backepsilon\, \mathfrak{v}\uparrow e^x) = I\uparrow e^x$ is such that $P\uparrow e^x$ changes sign on $(f - \gamma, f + \gamma) \subseteq I\uparrow e^x$, then $f\downarrow/x \in (\gamma, \backepsilon\, \mathfrak{v}) = I$ is such that P changes sign on $(f\downarrow/x - \gamma, f\downarrow/x + \gamma) \subseteq I$.

Case 1: (27) is quasi-linear. Let $\mathfrak{m}$ be the potential dominant monomial relative to (27). We may assume without loss of generality that $\mathfrak{m} = 1$, modulo a multiplicative conjugation with $\mathfrak{m}$. Since By $N_P \in C[c]\,(c')^{\mathbb{N}}$, we have $N_P = \alpha\, c + \beta$ or $N_P = \alpha\, c'$ for certain constants $\alpha, \beta \in C$.

In the case when $N_P = \alpha\, c + \beta$, there exists a solution to (27) with $f \sim -\beta/\alpha \neq 0$. Now $\sigma_P(0) = \operatorname{sign} \beta$ and $\sigma_P(\Omega) = \operatorname{sign} \alpha$. We claim that $\sigma_P(\Omega) = \sigma_{R_{P,1}}(\gamma)$ and $\sigma_{R_{P,1}}(\mathfrak{v}^\dagger - \gamma) = \sigma_P(\backepsilon\, \mathfrak{v})$ must be equal. Otherwise $R_{P,1}$ would admit a solution between γ and $\mathfrak{v}^\dagger - \gamma$, by the induction hypothesis. But then the potential dominant monomial relative to (27) should have been $e^{\int \chi}$, if χ is the largest such solution. Our claim implies that $(\operatorname{sign}\alpha)\,(\operatorname{sign}\beta) = \sigma_P(0)\,\sigma_P(\backepsilon\mathfrak{v}) < 0$, so that $f > 0$. Finally, lemma 4 implies that P admits a sign-change at f. Lemma 7 also shows that $\sigma_P(f-\gamma)\,\sigma_P(f+\gamma) = \sigma_P(f-\backepsilon)\,\sigma_P(f+\backepsilon) < 0$.

In the case when $N_P = \alpha\, c'$, then any constant $\lambda \in C$ is a root of N_P. Hence, for each $\lambda > 0$, there exists a solution f to (27) with $f \sim \lambda$. Again by lemmas 4 and 7, it follows that P admits a sign change at f and on $(f-\gamma, f+\gamma)$.

Case 2: $d > 1$. Let $\mathfrak{m}$ be the largest classical potential dominant monomial relative to (27). Since $\sigma_P(0)\,\sigma_P(\backepsilon\,\mathfrak{v}) < 0$ (resp. $\sigma_P(\gamma)\,\sigma_P(\backepsilon\,\mathfrak{v}) < 0$), one of the following always holds:

Case 2a. We have $\sigma_P(0)\,\sigma_P(\backepsilon\,\mathfrak{m}) < 0$ (resp. $\sigma_P(\gamma)\,\sigma_P(\backepsilon\,\mathfrak{m}) < 0$).

Case 2b. We have $\sigma_P(\backepsilon\,\mathfrak{m})\,\sigma_P(\Omega\,\mathfrak{m}) < 0$.

Case 2c. We have $\sigma_P(\Omega\,\mathfrak{m})\,\sigma_P(\backepsilon\,\mathfrak{v}) < 0$.

For the proof of theorem 14, we also assume that $\mathfrak{m} \succ \gamma$ in the above three cases and distinguish a last **case 2d** in which $\mathfrak{m} \prec \gamma$.

Case 2a. We are directly done by the induction hypothesis, since the equation

$$P(f) = 0 \qquad (f \prec \mathfrak{m}). \tag{28}$$

has a strictly smaller Newton degree than (27).

Case 2b. Modulo multiplicative conjugation with $\mathfrak{m}$, we may assume without loss of generality that $\mathfrak{m} = 1$. By corollary 12, there exists a $c > 0$ such that $\sigma_P(c - \backepsilon)\,\sigma_P(c + \backepsilon) < 0$. Actually, for any transseries $\varphi \sim c$ we then have $\sigma_P(\varphi - \backepsilon)\,\sigma_P(\varphi + \backepsilon) < 0$. Take φ such that

$$P_{+\varphi}(\tilde{f}) = 0 \qquad (\tilde{f} \prec 1) \tag{29}$$

is a privileged refinement of (27). Then either the Newton degree of (29) is strictly less than d, or the longest chain of refinements of (29) of Newton degree d is strictly less than l. We conclude by the induction hypothesis.

Case 2c. Since $\mathfrak{m}$ is the largest classical dominant monomial relative to (27), the degree of the Newton polynomial associated to any monomial between $\mathfrak{m}$ and $\mathfrak{v}$ must be d. Consequently,

$$\sigma_P(\Omega\,\mathfrak{m})\,\sigma_P(\backepsilon\,\mathfrak{v}) = \sigma_{P_d}(\Omega\,\mathfrak{m})\,\sigma_{P_d}(\backepsilon\,\mathfrak{v}) = \sigma_{R_{P,d}}(\mathfrak{m}^\dagger + \gamma)\,\sigma_{R_{P,d}}(\mathfrak{v}^\dagger - \gamma) < 0. \tag{30}$$

By the induction hypothesis, there exists a monomial $\mathfrak{n}$ with $\mathfrak{m}^\dagger + \gamma < \mathfrak{n}^\dagger < \mathfrak{v}^\dagger - \gamma$ and

$$\sigma_{R_{P,d}}(\mathfrak{n}^\dagger - \gamma)\,\sigma_{R_{P,d}}(\mathfrak{n}^\dagger + \gamma) < 0. \tag{31}$$

In other words, $\mathfrak{n}$ is a dominant monomial, such that $\mathfrak{m} \prec \mathfrak{n} \prec \mathfrak{v}$ and

$$\sigma_{P_d}(\backepsilon\,\mathfrak{n})\,\sigma_{P_d}(\Omega\,\mathfrak{n}) < 0. \tag{32}$$

We conclude by the same argument as in case 2b, where we let $\mathfrak{n}$ play the role of $\mathfrak{m}$.

Case 2d. Since $\mathfrak{m} \prec \gamma$ is the largest classical dominant monomial relative to (27), the degree of the Newton polynomial associated to any monomial between γ and $\mathfrak{v}$ must be d. Consequently,

$$\sigma_P(\gamma)\,\sigma_P(\backepsilon\,\mathfrak{v}) = \sigma_{P_d}(\gamma)\,\sigma_{P_d}(\backepsilon\,\mathfrak{v}) = \sigma_{R_{P,d}}(x^\dagger + \gamma)\,\sigma_{R_{P,d}}(\mathfrak{v}^\dagger - \gamma) < 0. \tag{33}$$

By the induction hypothesis, there exists a monomial $\mathfrak{n}$ with $x^{\dagger}+\gamma<\mathfrak{n}^{\dagger}<\mathfrak{v}^{\dagger}-\gamma$ and

$$\sigma_{R_{P,d}}(\mathfrak{n}^{\dagger}-\gamma)\,\sigma_{R_{P,d}}(\mathfrak{n}^{\dagger}+\gamma)<0. \tag{34}$$

In other words, $\mathfrak{n}$ is a dominant monomial, such that $\gamma\prec x\prec\mathfrak{n}\prec\mathfrak{v}$ and

$$\sigma_{P_d}(\backepsilon\,\mathfrak{n})\,\sigma_{P_d}(\varpi\,\mathfrak{n})<0. \tag{35}$$

We again conclude by the same argument as in case 2b. □

Corollary 15. *Any differential polynomial of odd degree and with coefficients in $\mathbb{T}$ admits a root in $\mathbb{T}$.*

Proof. Let P be a polynomial of odd degree with coefficients in $\mathbb{T}$. Then formula (7) shows that for sufficiently large $f\in\mathbb{T}_{*}^{+}$ we have $\sigma_P(-f)\,\sigma_P(f)<0$, since i_0 is odd in this formula. We now apply the intermediate value theorem between $-f$ and f. □

Bibliography

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