On the computation of limsups

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In the last years, several asymptotic expansion algorithms have appeared, which have the property that they can deal with very general types of singularities, such as singularities arising in the study of algebraic differential equations. However, attention has been restricted so far to functions with "strongly monotonic" asymptotic behaviour: formally speaking, the functions lie in a common Hardy field, or, alternatively, they are determined by transseries.

In this article, we make a first step towards the treatment of functions involving oscillatory behaviour. More precisely, let ϕ be an algebraic function defined on $[-1,1]^q$, let $F_1(x), \ldots, F_q(x)$ be exp-log functions at infinity in x, and let

$$\psi(x) = \phi(\sin(F_1(x)), \ldots, \sin(F_q(x))).$$

We give a method to compute $\limsup_{x\to\infty} \psi(x)$. Moreover, the techniques we use are stronger than this result might suggest, and we outline further applications.

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1. Introduction

In the last years, several asymptotic expansion algorithms have appeared [Sha90, Sha91, GG92, RSSH96, Hoe96a]. These algorithms are have the property that they can deal with very general types of singularities, such as singularities arising in the study of certain algebraic differential equations. However, attention has been restricted so far to functions with "strongly monotonic" asymptotic behaviour. This means that the functions lie in a common Hardy field, or, alternatively, that they are determined by transseries. In this article, we make a first step to the treatment of functions involving oscillatory behaviour. We also notice that Grigoriev obtained some very interesting related results in [Gri94, Gri95] although his more probabilistic point of view is different (even complementary) from ours.

The structure of this paper is as follows: in section 2, we recall a classical density theorem for linear curves on the n-dimensional torus (see for example [Kok36, KN74]). In section 3, this theorem is generalized to more general classes of curves on the torus.

In section 4, we study exp-log functions at infinity: an exp-log function is a function which is built up from the rationals $\mathbb{Q}$ and x, using the field operations, exponentiation and logarithm. An exp-log function at infinity is an exp-log function which is defined in a neighbourhood of infinity. We present a more compact version of an expansion algorithm of exp-log functions at infinity, originally due to Shackell [Sha91] (see also [RSSH96]). For this, we assume the existence of an oracle for deciding whether an exp-log function is zero in a neighbourhood of infinity. This problem has been reduced to the corresponding problem for exp-log constants in [Hoe96b, Hoe96a]. A solution to the constant problem was given by Richardson in [Ric94], modulo Schanuel's conjecture:

Conjecture 1. (Schanuel) *If $\alpha_1, \ldots, \alpha_n$ are $\mathbb{Q}$-linearly independent complex numbers, then the transcendence degree of $\mathbb{Q}[\alpha_1, \ldots, \alpha_n, \mathrm{e}^{\alpha_1}, \ldots, \mathrm{e}^{\alpha_n}]$ over $\mathbb{Q}$ is at least n.*

In section 5, we are given an algebraic function ϕ defined on $[-1, 1]^q$, and exp-log functions at infinity $F_1(x), \ldots, F_q(x)$ in x. We show how to compute

$$\limsup_{x \to \infty} \phi(\sin(F_1(x)), \ldots, \sin(F_q(x))).$$

In section 5, we will assume the existence of an oracle for checking the $\mathbb{Q}$-linear dependence of exp-log constants. Actually, Richardson's algorithm can easily be adapted to yield an algorithm for doing this modulo Schanuel's conjecture.

2. A density theorem on the n-dimensional torus

Let $\lambda_1, \ldots, \lambda_n$ be $\mathbb{Q}$-linearly independent numbers: we will use vector notation, and denote the vector $(\lambda_1, \ldots, \lambda_n)$ by $\boldsymbol{\lambda}$. In this section, we prove that the image of $x \mapsto \boldsymbol{\lambda} x$, from $\mathbb{R}$ in the n-dimensional torus $T^n = \mathbb{R}^n / \mathbb{Z}^n$ is dense. Notice that we use the same notation for $\boldsymbol{\lambda} x$ and its class modulo $\mathbb{Z}^n$. Moreover, we show that the "density" of the image is uniform is a sense that will be made precise. The following theorem is classical:

Theorem 2. (Kronecker) *Let $\lambda_1, \ldots, \lambda_n$ be $\mathbb{Q}$-linearly independent real numbers. Let $e_1, \ldots, e_n$ be the canonical base of $\mathbb{R}^n$. Then $\lambda_1 e_1 \mathbb{Z} + \cdots + \lambda_n e_n \mathbb{Z} + \mathbb{R}(e_1 + \cdots + e_n)$ is dense in $\mathbb{R}^n$.*

Now let X be a measurable subset of T^n, and let I be some interval of $\mathbb{R}$. Denoting the Lebesgue measure by μ, we define

$$\rho(I, X) = \frac{\mu(\{x \in I \mid \boldsymbol{\lambda} x \in X\})}{\mu(I)}. \tag{1}$$

Let us also denote by d the Euclidean distance on T^n. Let S_d, resp. $S_{\boldsymbol{d}}$ denote the shift operator on $\mathbb{R}$ (resp. $\mathbb{R}^n$ or T^n): $S_d(x) = x + d$ and $S_{\boldsymbol{d}}(\boldsymbol{x}) = S_{(d_1, \ldots, d_n)}(x_1, \ldots, x_n) = (x_1 + d_1, \ldots, x_n + d_n) = \boldsymbol{x} + \boldsymbol{d}$. The following are immediate consequences of the definition of ρ:

Proposition 3. *We have*

a) $\rho(I, X) = \sum_{i \in \mathbb{N}} \rho(I, X_i)$, *if* $X = \coprod_{i \in \mathbb{N}} X_i$.

b) $|\rho(I, X) - \rho(S_d I, X)| \leqslant |d| / \mu(I)$, *for all* d.

c) $\rho(I, X) = \rho(S_{-d} I, S_{\boldsymbol{\lambda} d} X)$, *for all* d. □

It will be convenient to adopt some conventions for intervals $I=[a,b]$ (resp. $I=[a,b[$, $I=]a,b]$ or $I=]a,b[$) whose lengths $b-a$ tend to infinity: we say that a property P holds uniformly in I, if the property holds uniformly in a:

$$\exists l_0, \forall a, \forall l > l_0, \ P([a,a+l]).$$

We say that P holds for all I sufficiently close to infinity, if P holds for all sufficiently large a.

The next theorem is also classical, but for convenience of the reader we present a proof, since similar techniques will be used in the next section:

THEOREM 4. (BOHR, SIERPIŃSKI, WEYL) *Let $\lambda_1,\ldots,\lambda_n$ be $\mathbb{Q}$-linearly independent real numbers and let ρ be given by (1). Let*

$$X=[a_1,b_1[\times\cdots\times[a_n,b_n[\subseteq T^n$$

be an n-dimensional block, with $0\leqslant a_i\leqslant b_i\leqslant 1$ for all i. Then

$$\lim_{\mu(I)\to\infty}\rho(I,X)=\mu(X),$$

uniformly in I.

Proof. The theorem trivially holds if $a_i=0$ and $b_i=1$ for all but one $1\leqslant i\leqslant n$. Hence, it suffices to prove the theorem when the a_i's and the b_i's are rational numbers. Indeed, let $a_1',b_1',\ldots,a_n',b_n'$ be rational numbers with $|a_1'-a_1|+|b_1'-b_1|+\cdots+|a_n'-a_n|+|b_n'-b_n|\leqslant\delta$, and denote $X'=[a_1',b_1'[\times\cdots\times[a_n',b_n'[$. Then $|\rho(I,X')-\rho(I,X)|\leqslant 2\,\delta$ for $\mu(I)$ sufficiently large, uniformly in I.

Because of Proposition 3(a) and (b), it suffices to prove the theorem for fixed $\boldsymbol{p}=(p_1,\ldots,p_n)\in(\mathbb{N}^*)^n$ and for all

$$X=X_{\boldsymbol{k}}=\left[\frac{k_1}{p_1},\frac{k_1+1}{p_1}\right[\times\cdots\times\left[\frac{k_n}{p_n},\frac{k_n+1}{p_n}\right[,$$

with $0\leqslant k_1<p_1,\ldots,0\leqslant k_n<p_n$. We remark that $[0,1[^n=\coprod_{\boldsymbol{k}}X_{\boldsymbol{k}}$, so that $\sum_{\boldsymbol{k}}\rho(I,X_{\boldsymbol{k}})=1$.

Now let $\varepsilon>0$. For each $\boldsymbol{k}$, we can find $x_{\boldsymbol{k}}$, with $\mathrm{d}(\boldsymbol{\lambda}x,\boldsymbol{k})<\varepsilon/n$, by Proposition 2. Consequently, we have $\mu(S_{-\boldsymbol{\lambda}x_{\boldsymbol{k}}}X_{\boldsymbol{k}}\triangle X_{\boldsymbol{0}})<\varepsilon$, where $A\triangle B$ denotes the symmetric difference of A and B. Hence, $\mu(X_{\boldsymbol{l}}\triangle S_{\boldsymbol{\lambda}(x_{\boldsymbol{l}}-x_{\boldsymbol{k}})}X_{\boldsymbol{k}})<2\,\varepsilon$, for each $\boldsymbol{l}$ with $l_1<p_1,\ldots,l_n<p_n$. Using Proposition 3, we can now estimate

$$\begin{aligned}|\rho(I,X_{\boldsymbol{l}})-\rho(I,X_{\boldsymbol{k}})| &\leqslant |\rho(I,S_{\boldsymbol{\lambda}(x_{\boldsymbol{l}}-x_{\boldsymbol{k}})})-\rho(I,X_{\boldsymbol{k}})|+\mu(X_{\boldsymbol{l}}\triangle S_{\boldsymbol{\lambda}(x_{\boldsymbol{l}}-x_{\boldsymbol{k}})}X_{\boldsymbol{k}})\\ &\leqslant |\rho(S_{x_{\boldsymbol{k}}-x_{\boldsymbol{l}}}I,X_{\boldsymbol{k}})-\rho(I,X_{\boldsymbol{k}})|+2\,\varepsilon\leqslant\frac{|x_{\boldsymbol{k}}-x_{\boldsymbol{l}}|}{\mu(I)}+2\,\varepsilon.\end{aligned}$$

Taking $\mu(I)>|x_{\boldsymbol{k}}-x_{\boldsymbol{l}}|/\varepsilon$, for any $\boldsymbol{k}$ and $\boldsymbol{l}$, we get

$$\left|\rho(I,X_{\boldsymbol{k}})-\frac{1}{p_1\cdots p_n}\right|\leqslant\frac{1}{p_1\cdots p_n}\sum_{\boldsymbol{k}}|\rho(I,X_{\boldsymbol{k}})-\rho(I,X_{\boldsymbol{l}})|<3\,\varepsilon.$$

Hence $|\rho(I,X_{\boldsymbol{k}})-\mu(X_{\boldsymbol{k}})|<3\,\varepsilon$, for sufficiently large $\mu(I)$, uniformly in I. This completes our proof. □

3. A more general density theorem

In this section we will obtain a more general uniform density theorem on the torus, when the application $x \mapsto \boldsymbol{\lambda} x$ from section 2 is replaced by a non linear mapping, which satisfies suitable regularity conditions. Before coming to this generalization, we will need some definitions and lemmas. We say that a function f defined in a neighbourhood of infinity is *steadily dominated* by x, if f has a continuous second derivative, f tends to infinity, f' decreases strictly towards zero, and f'' / f' tends to zero. We remark that such functions admit functional inverses in a neighbourhood of infinity.

More generally, we say that if f and g are functions in a neighbourhood of infinity, such that g is invertible, then f is steadily dominated by g, if $f \circ g^{\text{inv}}$ is steadily dominated by x. In this case, we write $f \prec\!\!\prec_s g$. It is easily verified that if $f \prec\!\!\prec_s x$ and $g \prec\!\!\prec_s x$, then $f \circ g \prec\!\!\prec_s x$, so that $\prec\!\!\prec_s$ is transitive. We also remark that if $f \prec\!\!\prec_s g$ and if h is a function, which has a continuous second derivative and tends to infinity, then $f \circ h \prec\!\!\prec_s g \circ h$. We finally have the following property of steady domination:

Lemma 5. *Let h be steadily dominated by x and let $l > 0$ and $\varepsilon > 0$ be given. Then for all sufficiently large x we have $|h'(x + d) - h'(x)| < \varepsilon\, h'(x)$, for all d with $|d| < l$.*

Proof. Let x_0 be such that $|h''(x) / h'(x)| < \varepsilon\, h'(x)$, for all $x \geqslant x_0 - l$. We have $|h'(x + d) - h'(x)| \leqslant |d\, h''(\xi)| < \varepsilon\, h'(\xi)$, for some ξ between x and $x + d$. If d is positive, then $h'(\xi) \leqslant h'(x)$, and we are done. In the other case, we have $|h'(x + d) - h'(x)| \leqslant \varepsilon\, h'(x) - \varepsilon\, |h'(x + d) - h'(x)|$, whence $|h'(x + d) - h'(x)| < (\varepsilon / (1 - \varepsilon))\, h'(x)$. □

Now let X be a measurable subset of $\mathbb{R}$. For each interval I, we define:

$$\rho(I, X) = \frac{\mu(I \cap X)}{\mu(I)}.$$

We say that X admits an *asymptotic density* $\rho(X)$ if

$$\lim_{\mu(I) \rightarrow \infty} \rho(I, X) = \rho(X),$$

uniformly in I, for I sufficiently close to infinity. More generally, if h is steadily dominated by x, then we say that X admits h-*asymptotic density* $\rho_h(X)$ if

$$\lim_{\mu(h^{\text{inv}}(I)) \rightarrow \infty} \rho(I, X) = \rho(X),$$

uniformly in I, for I sufficiently close to infinity.

Lemma 6. *Let X be a measurable subset of $\mathbb{R}$ and let h be steadily dominated by x. If $\rho(X)$ exists, then so does $\rho_h(h(X))$ and we have $\rho_h(h(X)) = \rho(X)$.*

Proof. Let $\varepsilon > 0$. Let $l \in \mathbb{R}$ be such that $|\rho(I, X) - \rho| < \varepsilon$, whenever $\mu(I) > l$. Let $I = [\alpha, \beta[$ with $\mu(h^{\text{inv}}(I)) > l$ and subdivide $h^{\text{inv}}(I)$ in $q = \lfloor (h^{\text{inv}}(\beta) - h^{\text{inv}}(\alpha)) / l \rfloor \geqslant 1$ parts of equal length $l' > l$:

$$[(h^{\text{inv}}(\alpha), h^{\text{inv}}(\beta))[= [a_1, b_1[\amalg \cdots \amalg [a_q, b_q[,$$

with $b_i = a_{i+1}$ for $1 \leqslant i < q$. Then we have

$$\begin{aligned}(\rho-\varepsilon)\sum_{i=1}^{q}\int_{a_i}^{b_i} h'(b_i)\,\mathrm{d}x &\leqslant \sum_{i=1}^{q}\int_{a_i}^{b_i}\chi_X(x)\,h'(b_i)\,\mathrm{d}x\\ &\leqslant \mu(h^{\mathrm{inv}}(X)\cap I)\\ &\leqslant \sum_{i=1}^{q}\int_{a_i}^{b_i}\chi_X(x)\,h'(a_i)\,\mathrm{d}x \;\leqslant\; (\rho+\varepsilon)\sum_{i=1}^{q}\int_{a_i}^{b_i} h'(a_i)\,\mathrm{d}x.\end{aligned}$$

By Lemma 5, for all sufficiently large x, we have $|h'(x+d) - h'(x)| \leqslant \varepsilon\, h'(x)$, for all d with $|d| \leqslant l'$. Hence,

$$\begin{aligned}\left|\sum_{i=1}^{q}\int_{a_i}^{b_i} h'(x)\,\mathrm{d}x - \sum_{i=1}^{q}\int_{a_i}^{b_i} h'(b_i)\,\mathrm{d}x\right| &\leqslant \sum_{i=1}^{q}\int_{a_i}^{b_i}|h'(x)-h'(b_i)|\,\mathrm{d}x\\ &\leqslant \varepsilon\sum_{i=1}^{q}\int_{a_i}^{b_i} h'(x)\,\mathrm{d}x \;=\; \varepsilon\,\mu(I),\end{aligned}$$

and we have a similar estimation, when replacing b_i by a_i. Consequently,

$$(\rho-\varepsilon)\,(1-\varepsilon)\,\mu(I) \leqslant \mu(h(X)\cap I) \leqslant (\rho+\varepsilon)\,(1+\varepsilon)\,\mu(I).$$

This completes our proof. □

Let $f_1 \prec\!\!\prec_s \cdots \prec\!\!\prec_s f_p$ be continuous functions defined in a neighbourhood of infinity, which strictly increase towards infinity. Let $\lambda_{i,j} > 0$ $(1 \leqslant j \leqslant n_i)$ be such that $\lambda_{i,1}, \ldots, \lambda_{i,n_i}$ are $\mathbb{Q}$-linearly independent for each i. Now consider the curve

$$g(x) = (f_1(\lambda_{1,1}\,x), \ldots, f_1(\lambda_{1,n_1}\,x), \ldots, f_p(\lambda_{p,1}\,x), \ldots, f_p(\lambda_{p,n_p}\,x))$$

on T^n $(n = n_1 + \cdots + n_p)$, which is defined for sufficiently large x. By analogy with the preceding section, we define

$$\rho_{f,g}(I,X) = \frac{\mu(\{x \in I \mid g(f_1^{\mathrm{inv}}(x)) \in X\})}{\mu(I)}, \qquad (2)$$

for intervals I sufficiently close to infinity, and measurable subsets X of T^n.

Theorem 7. *Let $f_1, \ldots, f_p$, g and ρ be given as above and let*

$$X = [a_1, b_1[\times \cdots \times [a_n, b_n[\subseteq T^n$$

be an n-dimensional block. Then

$$\lim_{\mu(I)\to\infty} \rho_{f,g}(I,X) = \mu(X),$$

uniformly, for intervals sufficiently close to infinity.

Proof. We proceed by induction over p. If $p=0$, we have nothing to prove. As before, it suffices to prove the theorem for multidimensional blocks $X=X_1\times\tilde{X}$, with $X_1\subseteq T^{n_1}$ and $\tilde{X}\subseteq T^{\tilde{n}}$, where $\tilde{n}=n_2+\cdots+n_p$. We denote by $g_1(x)$ resp. $\tilde{g}(x)$ the projections of $g(x)$ on T^{n_1} resp. $T^{\tilde{n}}$, when considering T^n as the product of T^{n_1} and $T^{\tilde{n}}$. Without loss of generality, we may assume that $f_1=x$.

Given a subset A of $\mathbb{R}$ or T^n and its frontier ∂A, we denote for any $\varepsilon>0$

$$\Omega_\varepsilon A=\{x\in A\,|\,\mathrm{d}(x,\partial A)>\varepsilon\}.$$

Let $\varepsilon>0$. If $g_1(x)\in\Omega_\varepsilon X_1$, then $g_1(x+d)\in X_1$ for all d with $|d|<l$, where $l=\max(1/\lambda_{1,1},\ldots,1/\lambda_{1,n_1})\,\varepsilon$. Hence, for I sufficiently close to infinity,

$$I\cap g_1^{\mathrm{inv}}(\Omega_\varepsilon X_1)\;\subseteq\;(I\cap\Omega_l\,g_1^{\mathrm{inv}}(X_1))+]-l,l[\;\subseteq\;I\cap g_1^{\mathrm{inv}}(X_1).$$

Therefore, Theorem 4 implies that for I sufficiently close to infinity

$$\left|\frac{\mu(I\cap g_1^{\mathrm{inv}}(X_1))}{\mu(I)}-\mu(X_1)\right|<\varepsilon \tag{3}$$

and (using that $\mu(\Omega_\varepsilon X_1\bigtriangleup X_1)<2\,n_1$)

$$\left|\frac{\mu((I\cap\Omega_l\,g_1^{\mathrm{inv}}(X_1))+]-l,l[)}{\mu(I)}-\frac{\mu(I\cap g_1^{\mathrm{inv}}(X_1))}{\mu(I)}\right|\leqslant(2\,n_1+1)\,\varepsilon. \tag{4}$$

Now $(I\cap\Omega_l\,g_1^{\mathrm{inv}}(X_1))+]-l,l[$ is a finite union of intervals, say

$$I\cap\Omega_l\,g_1^{\mathrm{inv}}(X_1)\cap]-l,l[=I_0\amalg\cdots\amalg I_{q+1},$$

where $I_1,\ldots,I_q$ have length at least $2\,l$, and where I_0 and I_{q+1} have length at most $2\,l$.

By the induction hypothesis, we have

$$\lim_{\mu(J)\to\infty}\frac{\mu(J\cap f_2(\tilde{g}^{\mathrm{inv}}(\tilde{X})))}{\mu(J)}=\mu(\tilde{X}),$$

uniformly, for J sufficiently close to infinity. Using Lemma 6 for $h=f_2^{\mathrm{inv}}$, this gives us

$$\lim_{\mu(f_2(J))\to\infty}\frac{\mu(J\cap\tilde{g}^{\mathrm{inv}}(\tilde{X}))}{\mu(J)}=\mu(\tilde{X}),$$

uniformly, for J sufficiently close to infinity. In particular, we have

$$\left|\frac{\mu(J\cap\tilde{g}^{\mathrm{inv}}(\tilde{X}))}{\mu(J)}-\mu(\tilde{X})\right|<\varepsilon,$$

for all J sufficiently close to infinity, with $\mu(J)>l$. Thus, choosing I sufficiently close to infinity, we have

$$\left|\frac{\mu(I_i\cap\tilde{g}^{\mathrm{inv}}(\tilde{X}))}{\mu(I_i)}-\mu(\tilde{X})\right|<\varepsilon,$$

for all $1\leqslant i\leqslant q$.

Taking $\mu(I) > 2\,l/\varepsilon$, and using (3) and (4), this gives us

$$\begin{aligned} |\rho_{f,g}(I,X) - \mu(X)| &\leqslant \left|\frac{\mu(\coprod_{i=0}^{q+1} I_i \cap \tilde{g}^{\mathrm{inv}}(\tilde{X}))}{\mu(I)} - \mu(X)\right| + \\ &\quad \left|\frac{\mu(I \cap g^{\mathrm{inv}}(X))}{\mu(I)} - \frac{\mu(\coprod_{i=0}^{q+1} I_i \cap \tilde{g}^{\mathrm{inv}}(\tilde{X}))}{\mu(I)}\right| \\ &\leqslant \left|\sum_{i=0}^{q+1} \frac{\mu(I_i \cap \tilde{g}^{\mathrm{inv}}(\tilde{X})) - \mu(\tilde{X})\,\mu(I_i)}{\mu(I)}\right| + \\ &\quad \left|\frac{\mu(\tilde{X})\,\mu(\coprod_{i=0}^{q+1} I_i)}{\mu(I)} - \mu(X)\right| + (2\,n_1 + 1)\,\varepsilon \\ &\leqslant \sum_{i=1}^{q} \frac{\mu(I_i)\,\varepsilon}{\mu(I)} + (4\,n_1 + 5)\,\varepsilon \leqslant (4\,n_1 + 6)\,\varepsilon. \end{aligned}$$

This completes the proof. □

4. Expansions of exp-log functions at infinity

Let $\mathfrak{T}$ denote the field of germs at infinity of exp-log functions and $\mathfrak{C}$ the subfield of exp-log constants. Elements of $\mathfrak{T}$ can be represented by exp-log expressions — i.e. finite trees whose internal nodes are labeled by $+, -, \cdot, /, \exp$ or $\log$, and whose leaves are labeled by x or rational numbers. The set of exp-log expressions which can be evaluated in a neighbourhood of infinity is denoted by $\mathfrak{T}^{\mathrm{expr}}$. We have a natural projection $f \mapsto \bar{f}$ from $\mathfrak{T}^{\mathrm{expr}}$ onto $\mathfrak{T}$. We make the assumption that we have at our disposal an oracle which can decide whether a given exp-log expression in $\mathfrak{T}^{\mathrm{expr}}$ is zero in a neighbourhood of infinity. In view of [Hoe96b, Hoe96a] it actually suffices to assume the existence of an oracle to decide whether a given exp-log constant is zero.

4.1. Grid-based series

Let us first recall some basic concepts. An *effective asymptotic basis* is an ordered finite set $\{b_1, \ldots, b_n\}$ of positive infinitesimal exp-log expressions in $\mathfrak{T}^{\mathrm{expr}}$, such that $\log b_i \prec\!\!\prec \log b_{i+1}$ (i.e. $\log b_i = o(\log b_{i+1})$) for $1 \leqslant i \leqslant n-1$. For instance, the set $B = \{\log^{-1} x, x^{-1}, \mathrm{e}^{-x^2}\}$ is an effective asymptotic basis. An effective asymptotic basis B generates an *effective asymptotic scale*, namely the set S_B of all products $b_1^{\alpha_1} \cdots b_n^{\alpha_n}$ of powers of the b_i's, with the α_i's in $\mathfrak{C}$. Elements of S_B are also called *monomials*.

Given an effective asymptotic basis B, let $\mathfrak{G}_B^{\mathrm{expr}}$ denote the set of expressions which are built up from $\mathfrak{C}, S_B, +, -, \cdot, /$ and the operations $\varepsilon \mapsto \exp \varepsilon$, resp. $\varepsilon \mapsto \log(1+\varepsilon)$, for infinitesimal ε. We observe that f can be expanded as a series in b_n with coefficients in $\mathfrak{G}_{\{b_1, \ldots, b_{n-1}\}}^{\mathrm{expr}}$. Moreover, these coefficients can recursively be expanded in $b_{n-1}, \ldots, b_1$:

$$\begin{aligned} f &= \sum_{\alpha_n \in \mathfrak{C}} f_{\alpha_n} b_n^{\alpha_n} \\ &\;\vdots \\ f_{\alpha_n, \ldots, \alpha_2} &= \sum_{\alpha_1 \in \mathfrak{C}} f_{\alpha_n, \ldots, \alpha_1} b_1^{\alpha_1}. \end{aligned}$$

The exp-log expressions of the form $f_{\alpha_n, \ldots, \alpha_i}$ are called *iterated coefficients* of f. In particular, the iterated coefficients of the form $f_{\alpha_n, \ldots, \alpha_1}$ are exp-log constants.

The above expansions of f have an important property [Hoe96a]: the support of f as a series in b_n is included in a set of the form $\lambda_1 \mathbb{N} + \cdots + \lambda_p \mathbb{N} + \nu$, where the λ_i's and ν are constants in $\mathfrak{C}$ — we say that f is a *grid-based series*. From this property, it follows that the support of f is well-ordered.

Another important property of the expansion of f in b_n and the expansions of its iterated coefficients is that they can be computed automatically. By this we mean that for each integer i, we can compute the first i terms of the expansion of f and so can we for its iterated coefficients. In particular, we can compute the sign of f, test whether f is infinitesimal, etc.

For the computation of the expansions of f in b_n, we use the usual Taylor series formulas. In the case of division $1/f$, we compute the first term $f_\mu b_n^\mu$ of f and then use the formula $1/f = (1/f_\mu)\, b_n^{-\mu}\, (1/(1+\varepsilon))$, where $\varepsilon = (f/f_\mu b_n^\mu) - 1$. The only problem when applying these formulas is that we have to avoid indefinite cancelation: note that indefinite cancelation only occurs if after having computed the first i terms of the expansion, f is actually equal to the sum of these terms. But this can be tested using the oracle, and we stop the expansion in this case.

4.2. Automatic expansions of exp-log expressions

The asymptotic expansion algorithm takes an exp-log expression $f \in \mathfrak{T}^{\text{expr}}$ on input, computes a suitable effective asymptotic basis B and rewrites f into an element of $\mathfrak{G}_B^{\text{expr}}$. The main idea of the algorithm lies in imposing some suitable conditions on B: we say that a linearly ordered set $B = \{b_1, \ldots, b_n\}$ is an *effective normal basis* if

NB1. B is an effective asymptotic basis.

NB2. $\log b_i \in \mathfrak{G}^{\text{expr}}_{\{b_1, \ldots, b_{i^*}\}}$ for all $i > 1$, where $\log \log b_i^{-1} \asymp \log b_{i^*}$.

NB+. $b_1 = \log_l^{-1} x$ for some $l \in \mathbb{N}$, where $\log_l x \overset{\text{def}}{=} \log \overset{l \text{ times}}{\cdots} \log$.

Such a basis is constructed gradually during the algorithm — i.e. B is a global variable in which we insert new elements during the execution of the algorithm, while maintaining the property that B is an effective normal basis. We also say that B is a *dynamic effective normal basis*. Let us now explicitly give the algorithm, using a PASCAL-like notation:

Algorithm `expand`(f). The algorithm takes an exp-log expression $f \in \mathfrak{T}^{\text{expr}}$ on input and rewrites it into a grid-based series in $\mathfrak{G}_B^{\text{expr}}$, where the global variable B contains an effective normal basis which is initialized by $B := \{x^{-1}\}$.

case $f \in \mathbb{Q}$: **return** f

case $f = x$: **return** $(x^{-1})^{-1}$

case $f = g \top h$, where $\top \in \{+, -, \cdot, /\}$:
  if $\top = /$ **and** $\bar{h} = 0$ **then error** "division by zero"
  return `expand`$(g) \top$ `expand`(h)

case $f = \log g$:
  $g :=$ `expand`(g)
  • Denote $B = \{b_1 = \log_l^{-1} x, b_2, \ldots, b_n\}$.
  if $g \leqslant 0$ **then error** "invalid logarithm"
  • Rewrite $g = c\, b_1^{\alpha_1} \cdots b_n^{\alpha_n} (1 + \varepsilon)$, with infinitesimal ε in $\mathfrak{G}_B^{\text{expr}}$ and $c \in \mathfrak{C}$.
  if $\alpha_1 \neq 0$ **then** $B := B \cup \{\log_{l+1}^{-1} x\}$
  return $\log c + \alpha_1 \log b_1 + \cdots + \alpha_n \log b_n + \log (1 + \varepsilon)$

case $f = \mathrm{e}^g$:
  $g := \mathtt{expand}(g)$
  • Denote $B = \{b_1, \ldots, b_n\}$.
  if $g = O(1)$ **then return** $\mathrm{e}^c \, \mathrm{e}^{g-c}$, where $c := g_{0, \overset{n \text{times}}{\cdots}, 0}$
  if $\exists 1 < i \leqslant n, \ g \asymp \log b_i$ **then**
    $\alpha := \lim g / \log b_i$
    return $b_i^{\alpha} \, \mathtt{expand}(\mathrm{e}^{g - \alpha \log b_i})$
  • Let i^* be such that $\log |g| \asymp \log b_{i^*}$.
  $g^+ := g_{0, \overset{n - i^* \text{times}}{\cdots}, 0}$
  $g^- := g - g^+$
  $B := B \cup \{\mathrm{e}^{-|g^+|}\}$
  return $(\mathrm{e}^{-|g^+|})^{-\mathbf{sign}\, g^+} \mathrm{e}^{g^-}$

Let us comment the algorithm. The first three cases do not need explanation. In the case $f = \log g$, the fact that B is an effective normal basis is used at the end: $\alpha_1 \log b_1 + \cdots + \alpha_n \log b_n$ is indeed an expression in $\mathfrak{G}_B^{\mathrm{expr}}$. The expansion of the exponential of a bounded series g is done by a straightforward Taylor series expansion. If g is unbounded, then we test whether g is asymptotic to the logarithm of an element in B — i.e. we test whether $\alpha := \lim g / \log b_i$ is a non zero finite number for some i. If this is so, then $f = b_i^{\alpha} \mathrm{e}^{g - \alpha \log b_i}$ and $\mathrm{e}^{g - \alpha \log b_i}$ is expanded recursively. We remark that no infinite loop can arise from this, because successive values of g in such a loop would be asymptotic to the logarithms of smaller and smaller elements of B, while B remains unchanged. Finally, if g is not asymptotic to the logarithm of an element in B, then B has to be extended with an element of the order of growth of f. The decomposition $g = g^+ + g^-$ is computed in order to ensure that B remains an effective normal basis.

5. On the automatic computation of limsups

In this section we show how Theorem 7 can be applied to compute limsups (or liminfs) of certain bounded functions, involving trigonometric functions. The idea is based on the following consequence of Theorem 7.

THEOREM 8. *Let $1 \prec\!\!\prec f_1 \prec\!\!\prec \cdots \prec\!\!\prec f_p$ be exp-log functions at infinity. Let $\lambda_{i,j} > 0$ $(1 \leqslant j \leqslant n_i)$ be such that $\lambda_{i,1}, \ldots, \lambda_{i,n_i}$ are $\mathbb{Q}$-linearly independent for each i. Denote $U = \{x + \sqrt{-1}\, y \in \mathbb{C} \mid x^2 + y^2 = 1\}$ and $n = n_1 + \cdots + n_p$. Let ϕ be a continuous function from U^n into $\mathbb{R}$ and let*

$$\psi(x) = \phi(\mathrm{e}^{\sqrt{-1}\, \lambda_{1,1} f_1(x)}, \ldots, \mathrm{e}^{\sqrt{-1}\, \lambda_{p,n_p} f_p(x)}).$$

Then

$$\limsup_{x \to \infty} \psi(x) = \sup_{\boldsymbol{x} \in U^n} \phi(\boldsymbol{x}).$$

Proof. We first notice that we will be able to apply Theorem 7 on our input data: by a well known theorem, which goes back to Hardy [Har11], the germs at infinity of $f_1, \ldots, f_p$ lie in a common Hardy field. Consequently, $f_1 \prec\!\!\prec_s \cdots \prec\!\!\prec_s f_p$, and $f_1, \ldots, f_p$ are strictly increasing in a suitable neighbourhood of infinity.

The mapping ψ is defined in a neighbourhood V of infinity, and can be factored $V \xrightarrow{\psi} \mathbb{R} = V \xrightarrow{\psi_1} T^n \xrightarrow{\psi_2} \mathbb{R}$, with

$$\psi_1(x) = \left(\frac{\lambda_{1,1} f_1(x)}{2\pi}, \ldots, \frac{\lambda_{p,n_p} f_p(x)}{2\pi} \right)$$

and

$$\psi_2(x_1, \ldots, x_n) = \phi(\mathrm{e}^{2\pi\sqrt{-1}\,x_1}, \ldots, \mathrm{e}^{2\pi\sqrt{-1}\,x_n}),$$

where ψ_1 and ψ_2 are both continuous. Since T^n is compact, there exists a point $\boldsymbol{x}$ in which ψ_2 attains its maximum. Let $\varepsilon > 0$. There exists a neighbourhood V of $\boldsymbol{x}$, such that $|\psi_2(\boldsymbol{y}) - \psi_2(\boldsymbol{x})| < \varepsilon$, for any $\boldsymbol{y}$ in V. By Theorem 7, there exist x, with $\psi_1(x) \in V$ as close to infinity as we wish. For such x, we have $|\psi(x) - \sup_{\boldsymbol{x}\in U^n} \phi(\boldsymbol{x})| < \varepsilon$. □

We now turn to the computation of this limit.

Theorem 9. *Let $F_1, \ldots, F_q$ be exp-log functions at infinity. Let $\phi: U^q \to \mathbb{R}$ a real algebraic function, where we consider U^q as a real algebraic variety. Assume that we have an oracle to test the $\mathbb{Q}$-linear dependence of exp-log constants. Then there exists an algorithm to compute the limsup of $\psi(x) = \phi(\mathrm{e}^{\sqrt{-1}\,F_1(x)}, \ldots, \mathrm{e}^{\sqrt{-1}\,F_q(x)})$.*

Proof. Using the identity $\mathrm{e}^{-x} = 1/\mathrm{e}^x$, we may always assume without loss of generality, that the F_i's are all positive. Now the algorithm consists of the following steps:

Step 1. Compute a common effective normal basis for $F_1, \ldots, F_p$, using the algorithm from section 4. Order the F_i's w.r.t. $\prec\!\!\prec$; that is, $F_i \asymp F_j$ or $F_i \prec\!\!\prec F_j$, whenever $i < j$.

Step 2. Simultaneously modify the F_i's and the algebraic function ϕ in the $\mathrm{e}^{\sqrt{-1}\,F_i}$'s, until we either have $F_i \prec\!\!\prec F_j$, or $F_i = \lambda\, F_j$, for some λ, whenever $i < j$. As long as this is not the case, we take j maximal, such that the above does not hold, and do the following:

First compute the limit λ of F_i / F_j. Next insert $F_i' := F_i - \lambda\, F_j$ and $F_j' := \lambda\, F_j$ into the set of F_i's and remove F_i. The new expression for ϕ is obtained by replacing each $\mathrm{e}^{\sqrt{-1}\,F_i}$ by $\mathrm{e}^{\sqrt{-1}\,F_i'}\,\mathrm{e}^{\sqrt{-1}\,F_j'}$.

Step 3. Compute exp-log functions $f_1 \prec\!\!\prec \cdots \prec\!\!\prec f_p$, and constants $\lambda_{i,j}$ $(1 \leqslant j \leqslant n_i)$, such that each F_l can be written as $F_l = \lambda_{i,j}\, f_i$, for some i and j. Replacing $\mathrm{e}^{\sqrt{-1}\,F_i}$ by its limit for each bounded F_i, we reduce the general case to the case when $1 \prec\!\!\prec f_1$.

Step 4. This step consists in making the $\lambda_{i,j}$'s $\mathbb{Q}$-linearly independent for each fixed i. Whenever there exists a non trivial $\mathbb{Q}$-linear relation between the $\lambda_{i,j}$'s (for fixed i), we may assume without loss of generality that this relation is given by

$$a_{n_i}\, \lambda_{i,n_i} = a_1\, \lambda_{i,1} + \cdots + a_{n_i-1}\, \lambda_{i,n_i-1},$$

for $a_1, \ldots, a_{n_i}$ in $\mathbb{Z}$ and $a_{n_i} > 0$. As long as we can find such a relation, we do the following:

For all $j < n_i$, replace $\lambda_{i,j}$ by $\lambda_{i,j}' := \lambda_{i,j} / a_{n_i}$ and $\mathrm{e}^{\sqrt{-1}\,\lambda_{i,j} f_i}$ by $(\mathrm{e}^{\sqrt{-1}\,\lambda_{i,j}' f_i})^{a_{n_i}}$ in the expression for ϕ. Next, replace $\mathrm{e}^{\sqrt{-1}\,\lambda_{i,n_i} f_i}$ by $(\mathrm{e}^{\sqrt{-1}\,\lambda_{i,1}'})^{a_1} \cdots (\mathrm{e}^{\sqrt{-1}\,\lambda_{i,n_i-1}'})^{a_{n_i-1}}$ in the expression for i.

Step 5. By Theorem 8, the limsup of ψ is the maximum of ϕ on U^n, where $n = n_1 + \cdots + n_p$. To compute this maximum, we determine the set of zeros of the gradient of ϕ on U^n. Then ϕ is constant on each connected component and the maximum of these constant values yields $\max_{U^n} \phi$. To compute the zero set of the gradient of ϕ and its connected components, one may for instance use cylindrical decomposition (see [Col75]). Of course, other algorithms from effective real algebraic geometry can be used instead.

The correctness of our algorithm is clear. The termination of the loop in step 2 follows from the fact that the new F_i' is asymptotically smaller then F_j, so that either the $\asymp$-class of F_j strictly decreases, or the number of i's with $F_i \asymp F_j$, but not $F_i = \lambda\, F_j$ for some λ. The number of $\asymp$-classes which can be attained is bounded by the initial value of q. □

Corollary 10. *Let $F_1, \ldots, F_q$ be exp-log functions at infinity and ϕ be an algebraic function in q variables, defined on $[-1,1]^q$. Assume that we have an oracle to test the $\mathbb{Q}$-linear dependence of exp-log constants. Then there exists an algorithm to compute the limsup of $\psi(x) = \phi(\sin(F_1(x)), \ldots, \sin(F_q(x)))$.* □

Example 11. Consider the function

$$\psi(x) = \frac{2\sin x^2 - \sin(x^3/(x-1))}{3 + \sin \mathrm{e}\, x^2 - \sin(\mathrm{e}\, x^2 + 1)}.$$

The first step consists in expanding $x^2 = x^2$, $x^3/(x-1) = x^2 + x + \cdots$, $\mathrm{e}\, x^2 = \mathrm{e}\, x^2$ and $\mathrm{e}\, x^2 + 1 = \mathrm{e}\, x^2 + 1$. All these functions have the same $\asymp$-class, but they are not all homothetic. Therefore, some rewriting needs to be done. First, $x^3/(x-1) = x^2 + x^2/(x-1)$, and we rewrite

$$\mathrm{e}^{\sqrt{-1}\, x^3/(x-1)} = \mathrm{e}^{\sqrt{-1}\, x^2}\, \mathrm{e}^{\sqrt{-1}\, x^2/(x-1)},$$

which corresponds to the rewriting

$$\sin \frac{x^3}{x-1} = \sin x^2 \cos \frac{x^2}{x-1} + \sin \frac{x^2}{x-1} \cos x^2,$$

if we consider real and imaginary parts. Similarly, we rewrite

$$\mathrm{e}^{\sqrt{-1}\, (\mathrm{e} x^2 + 1)} = \mathrm{e}^{\sqrt{-1}\, \mathrm{e} x^2}\, \mathrm{e}^{\sqrt{-1}},$$

which corresponds to the rewriting

$$\sin(\mathrm{e}\, x^2 + 1) = \sin \mathrm{e}\, x^2 \cos 1 + \sin 1 \cos \mathrm{e}\, x^2.$$

In step 4, no $\mathbb{Q}$-linear relations are found, so that we have to determine the maximal value of

$$\phi(a, b, c, \hat{a}, \hat{b}, \hat{c}) = \frac{2a - a\hat{c} - c\hat{a}}{3 + b - b\cos 1 - \hat{b}\sin 1} \tag{5}$$

on U^3. Here we have abbreviated $a = \sin x^2, b = \sin \mathrm{e}\, x^2, c = \sin(x^2(x-1)), \hat{a} = \cos x^2, b = \cos \mathrm{e}\, x^2, c = \cos(x^2(x-1))$ (hence U^3 is the set of points with $a^2 + \hat{a}^2 = b^2 + \hat{b}^2 = c^2 + \hat{c}^2 = 1$). The maximum of ϕ is attained for $a = 1, b = -1/2, c = 0, \hat{a} = 0, \hat{b} = \sqrt{3}/2, \hat{c} = -1$. We deduce that

$$\limsup_{x \to \infty} \psi(x) = \frac{6}{5 + \cos 1 - \sqrt{3}\sin 1} = l.$$

Similarly, exploiting the symmetry of (5), we have

$$\liminf_{x \to \infty} \psi(x) = \frac{-6}{5 + \cos 1 - \sqrt{3}\sin 1} = -l.$$

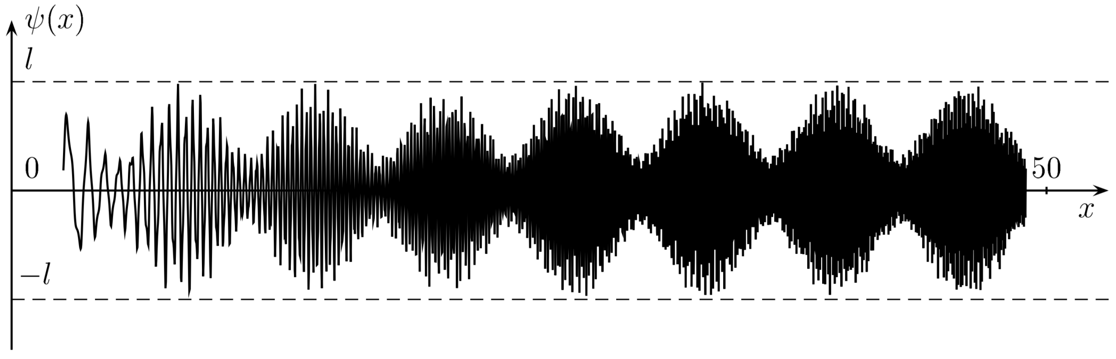

Figure 1. Plot of the function ψ from Example 11.

6. Conclusion

We have shown how to compute limsups of certain functions involving trigonometric functions, exponentiation and logarithm. Actually, the techniques we have used are far more general than Theorem 9 might suggest. Let us now briefly mention some generalizations. For more details, we refer to [Hoe96a].

In Theorem 9, the crucial property of the functions $F_1, \ldots, F_q$ is that they are strongly monotonic and that we have an asymptotic expansion algorithm for them. Consequently, more general functions than exp-log functions can be taken instead, like Liouvillian functions, functions which are determined by systems of real exp-log equations in several variables, etc.

The crucial property of the function ϕ is that it belongs to a class for which a cylindrical decomposition algorithm exists. Again, more general classes of functions can be considered. In particular, modulo suitable oracles, one can consider the class of solutions to real exp-log systems in several variables.

Our techniques can also be used to compute automatic asymptotic expansions of sin-exp-log functions at infinity of trigonometric depth one (i.e. without nested sines). However, some difficult number theoretical phenomena may occur in this case, as the following example illustrates:

$$2 - \sin x - \sin \mathrm{e}\, x \geqslant_{\infty} \frac{1}{\Gamma(x+2)}.$$

This asymptotic inequality follows from the number theoretical properties of e. In general, such inequalities are very hard to obtain (if decidable at all!): a systematic way to obtain them would in particular yield solutions to deep unsolved problems in the field of Diophantine approximation (for a nice survey, see [Lan71]).

Nevertheless, we notice that the above example is “degenerate” in the sense that 2 is precisely equal to the limsup of $\sin x + \sin x^2$. In the generic case, a complete asymptotic expansion for sin-exp-log functions at infinity of trigonometric depth one does exist. In the degenerate case, we need assume the existence of a suitable oracle for Diophantine questions.

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