LOGARITHMIC HYPERSERIES

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Abstract. We define the field $\mathbb{L}$ of logarithmic hyperseries, construct on $\mathbb{L}$ natural operations of differentiation, integration, and composition, establish the basic properties of these operations, and characterize these operations uniquely by such properties.

1. Introduction

The field of transseries $\mathbb{T}$ was introduced independently by Dahn and Göring [8] in model theory and by Écalle [9] in his proof of the Dulac conjecture. Roughly speaking, transseries are constructed from the real numbers and a variable $x > \mathbb{R}$ using the field operations, exponentiation, and taking logarithms and infinite sums. Here is an example of a transseries:

$$7\mathrm{e}^{\mathrm{e}^{x}+\mathrm{e}^{x/2}+\mathrm{e}^{x/3}+\cdots} + \frac{\mathrm{e}^{\mathrm{e}^{x}}}{\log\log x} + \sqrt{2} + \frac{1}{x} + \frac{2}{x^2} + \frac{6}{x^3} + \frac{24}{x^4} + \cdots + \frac{\mathrm{e}^{-x}}{x} + \frac{\mathrm{e}^{-x}}{x^2} + \cdots .$$

The sign of a transseries is defined to be the sign of its leading coefficient: sign $7 > 0$ in our example; $\mathbb{T}$ is real closed for the corresponding ordering. See [2, Appendix A] for a detailed construction. The field $\mathbb{T}$ can also be equipped with natural 'calculus' operations: differentiation, integration, composition, and functional inversion. The theory of $\mathbb{T}$ as a valued differential field was determined in [2]. In particular, it was shown there that this theory is model complete. Remarkably, $\mathbb{T}$ also satisfies the intermediate value property for differential polynomials: this was first proven in [12] for the ordered differential subfield of $\mathbb{T}$ consisting of the grid-based transseries, and extends to $\mathbb{T}$ itself by model completeness.

Transseries describe 'regular' orders of growth of real functions. Despite its remarkable closure properties, however, $\mathbb{T}$ cannot account for all regular orders of growth. For instance, Kneser [13] constructs a real analytic function e_ω that satisfies the functional equation $\mathrm{e}_\omega(x+1) = \exp \mathrm{e}_\omega(x)$ and that grows regularly—its germ at $+\infty$ lies in a Hardy field—but faster than any iterated exponential. Its functional inverse is infinitely large, but grows slower than any iterated logarithm.

Accordingly, we wish to enlarge the field $\mathbb{T}$ of transseries to a field $\mathbb{H}$ of hyperseries with transfinite iterates e_α and ℓ_α of e^x and $\log x$ for all ordinals α, and with natural operations of exponentiation, differentiation, integration, and composition. These operations should extend the corresponding operations on $\mathbb{T}$. In this paper we achieve this for the purely logarithmic part $\mathbb{L}$ of the intended $\mathbb{H}$ by direct recursive constructions, and with exponentiation replaced by taking logarithms. We also indicate how the natural embedding of $\mathbb{T}_{\log}$ into the field $\mathbf{No}$ of surreal numbers extends naturally to an embedding of $\mathbb{L}$ into $\mathbf{No}$. As indicated in [4], this is part of a plan to eventually construct a canonical exponential field isomorphism $\mathbb{H} \cong \mathbf{No}$ via which $\mathbf{No}$ can be equipped with the 'correct' derivation and composition. Realizing this plan would vindicate the idea that $\mathbb{H}$ covers all regular orders of growth at infinity, as **No** does in a different way.

A first step in the above direction is due to Schmeling [14] and his thesis advisor van der Hoeven. They constructed a field of hyperseries that contains e_α and ℓ_α for $\alpha < \omega^\omega$, but they did not construct a derivation or composition on it. The purely logarithmic part of it will be recovered here as the subfield $\mathbb{L}_{<\omega^\omega}$ of our $\mathbb{L}$.

On a related topic, van der Hoeven's thesis [10] (with more details in [14]) shows how to extend the derivation and composition on $\mathbb{T}$ to larger fields of transseries that contain elements such as $\mathrm{e}^{\sqrt{x}+\sqrt{\log x}+\sqrt{\log\log x}+\cdots}$. The recent paper [6] by Berarducci and Mantova shows how such generalized transseries naturally act on positive infinitely large surreal numbers so as to be compatible with composition and with the derivation on **No** constructed in [5]. While this line of work has some connection to the present paper, it goes into another direction.

In the rest of this introduction we give canonical and precise descriptions of $\mathbb{L}$ with its 'calculus' operations and state its main properties. To prove existence and uniqueness of the operations having these properties is not easy, and makes up the bulk of this paper. First we define $\mathbb{L}$ as an increasing union of Hahn fields over $\mathbb{R}$. Throughout we let α, β, γ range over ordinals, an ordinal is identified with the set of smaller ordinals, and $\alpha + \beta$ denotes the ordinal sum, to be thought of as α followed by β. Moreover, m, n, sometimes subscripted, range over $\mathbb{N} = \{0, 1, 2, \dots\} = \omega$. By convention, a *differential field* has characteristic 0; given its derivation ∂ and an element y in the field we also denote $\partial(y)$ by y', and y'/y by $y^\dagger$.

The monomial group $\mathfrak{L}$. We fix once and for all symbols ℓ_α, one for each α, with $\ell_\alpha \neq \ell_\beta$ whenever $\alpha \neq \beta$. The intended meaning of ℓ_α is as the αth iterated logarithm of $x := \ell_0$ in $\mathbb{L}$, and accordingly we refer to these ℓ_α as *hyperlogarithms*. (The totality of hyperlogarithms is too large to be a set; it is a proper class. We shall freely use classes rather than sets when necessary: our set theory here is von Neumann-Gödel-Bernays set theory with Global Choice (NBG), a conservative extension of ZFC in which all proper classes are in bijective correspondence with the class of all ordinals. Those who find these matters unpalatable may read *ordinal* as meaning *countable ordinal*. Everything goes through with that restriction.)

An *exponent sequence* is a family (r_β) of real numbers r_β, with β ranging over all ordinals, such that for some α we have $r_\beta = 0$ for all $\beta \geqslant \alpha$. To each exponent sequence $r = (r_\beta)$ we associate the formal monomial

$$\ell^r \ := \ \prod_\beta \ell_\beta^{r_\beta},$$

a *logarithmic hypermonomial*. We make the class of logarithmic hypermonomials into an abelian (multiplicatively written) group $\mathfrak{L}$ with the obvious group operation: for exponent sequences $r = (r_\beta)$ and $s = (s_\beta)$ with corresponding logarithmic hypermonomials $\ell^r := \prod_\beta \ell_\beta^{r_\beta}$ and $\ell^s := \prod_\beta \ell_\beta^{s_\beta}$ we set $r + s := (r_\beta + s_\beta)$ and

$$\ell^r \cdot \ell^s \ := \ \ell^{r+s} \ = \ \prod_\beta \ell_\beta^{r_\beta + s_\beta}.$$

The identity of $\mathfrak{L}$ is $1 := \ell^0$ with 0 denoting the exponent sequence (r_β) with $r_\beta = 0$ for all β. We make $\mathfrak{L}$ into a totally ordered abelian group by $\ell^r \prec \ell^s$ iff r is lexicographically less than s, that is, $r \neq s$ and $r_\beta < s_\beta$ for the least β with $r_\beta \neq s_\beta$. We identify ℓ_α with ℓ^r where $r_\alpha = 1$ and $r_\beta = 0$ for all $\beta \neq \alpha$; so $\ell_\alpha \succ 1$.

In this introduction we let $\mathfrak{m}, \mathfrak{n}$ range over logarithmic hypermonomials. We make $\mathbb{R}$ act on $\mathfrak{L}$: for $\mathfrak{m} = \prod_\beta \ell_\beta^{r_\beta}$ and $t \in \mathbb{R}$ we set

$$\mathfrak{m}^t \ := \ \prod_\beta \ell_\beta^{tr_\beta} \in \mathfrak{L}.$$

Thus we have the subgroup $\mathfrak{m}^{\mathbb{R}} := \{\mathfrak{m}^t : \ t \in \mathbb{R}\}$ of $\mathfrak{L}$. For $\mathfrak{m} = \ell^r$ we define

$$\sigma(\mathfrak{m}) \ := \ \{\beta : \ r_\beta \neq 0\},$$

a *set* of ordinals (not just a class); we think of it as the support of $\mathfrak{m}$. The set

$$\mathfrak{L}_{<\alpha} \ := \ \{\mathfrak{m} : \ \sigma(\mathfrak{m}) \subseteq \alpha\}$$

underlies an ordered subgroup of $\mathfrak{L}$. Note that

$$\mathfrak{L}_{<0} \ = \ \{1\}, \qquad \mathfrak{L}_{<1} \ = \ \ell_0^{\mathbb{R}}, \quad \ldots \quad , \mathfrak{L}_{<n+1} \ = \ \ell_0^{\mathbb{R}} \cdots \ell_n^{\mathbb{R}}.$$

Given reals $r(\beta)$ for $\beta < \alpha$ we let $\prod_{\beta<\alpha} \ell_\beta^{r(\beta)}$ denote the logarithmic hypermonomial ℓ^r where $r_\beta = r(\beta)$ for $\beta < \alpha$ and $r_\beta = 0$ for $\beta \geqslant \alpha$.

The Hahn fields $\mathbb{L}_{<\alpha}$. The monomial group $\mathfrak{L}_{<\alpha}$ yields the ordered Hahn field

$$\mathbb{L}_{<\alpha} \ := \ \mathbb{R}[[\mathfrak{L}_{<\alpha}]]$$

consisting of the well-based series over $\mathbb{R}$ with monomials in $\mathfrak{L}_{<\alpha}$. In particular, $\mathbb{L}_{<0} = \mathbb{R}$ and $\mathbb{L}_{<1} = \mathbb{R}[[\ell_0^{\mathbb{R}}]]$. For $\beta \leqslant \alpha$, we have $\mathfrak{L}_{<\beta} \subseteq \mathfrak{L}_{<\alpha}$, as ordered groups, and so $\mathbb{L}_{<\beta} \subseteq \mathbb{L}_{<\alpha}$, as ordered and valued fields. We also set

$$\mathfrak{L}_{\leqslant\alpha} \ := \ \mathfrak{L}_{<\alpha+1}, \qquad \mathbb{L}_{\leqslant\alpha} \ := \ \mathbb{L}_{<\alpha+1} \ = \ \mathbb{R}[[\mathfrak{L}_{\leqslant\alpha}]].$$

Now $\mathbb{L} := \bigcup_\alpha \mathbb{L}_{<\alpha}$ is an ordered and valued field extension of each $\mathbb{L}_{<\alpha}$. It does not have an underlying set, but it has an underlying proper class. We shall use the notations and conventions introduced in [2, Section 3.1 and Appendix A] to discuss these Hahn fields and their union $\mathbb{L}$. (Section 2 below includes a summary of that material.) Thus for $f \in \mathbb{L}^\times$ we have its dominant monomial $\mathfrak{d}(f) \in \mathfrak{L} \subseteq \mathbb{L}$, with $f = c\mathfrak{d}(f)(1+\varepsilon)$ for unique $c \in \mathbb{R}^\times$ and $\varepsilon \prec 1$ (and $\mathfrak{d}(0) := 0 \in \mathbb{L}$ by convention), and $\mathbb{R}$ is viewed as an ordered subfield of $\mathbb{L}$ and $\mathfrak{L}$ as an ordered subgroup of $\mathbb{L}^>$.

The logarithmic field $\mathbb{L}$. We define the logarithm $\log \mathfrak{m}$ of $\mathfrak{m} = \ell^r$ by

$$\log \mathfrak{m} \ := \ \sum_\beta r_\beta \ell_{\beta+1} \in \mathbb{L}.$$

Thus $\log \ell_\alpha = \ell_{\alpha+1}$, $\log \mathfrak{m}\mathfrak{n} = \log \mathfrak{m} + \log \mathfrak{n}$, and $\log \mathfrak{m}^t = t \log \mathfrak{m}$ for real t. For $f \in \mathbb{L}^>$ we have $f = c\mathfrak{d}(f)(1+\varepsilon)$ with $c \in \mathbb{R}^>$ and $\varepsilon \prec 1$, and we set

$$\log f \ := \ \log \mathfrak{d}(f) + \log c + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \varepsilon^n,$$

where $\log c$ is the usual real logarithm of c. The map $f \mapsto \log f : \mathbb{L}^> \to \mathbb{L}$ is a strictly increasing morphism of the multiplicative ordered group $\mathbb{L}^>$ into the ordered additive group of $\mathbb{L}$. Note that if α is an infinite limit ordinal, then $\log \mathbb{L}_{<\alpha}^> \subseteq \mathbb{L}_{<\alpha}$.

The derivation on $\mathbb{L}$. The intended derivation is 'derivative with respect to x' where $x := \ell_0$. This derivation should respect logarithms and commute with infinite sums. To *respect logarithms* will be interpreted to mean that the derivative of ℓ_α is $\prod_{\beta<\alpha} \ell_\beta^{-1}$. (Recall in this connection that the usual derivative of the n-times iterated real logarithm function $\log_n$ is $\prod_{m<n}(\log_m)^{-1}$.) These requirements determine the derivation uniquely:

Proposition 1.1. *There is a unique $\mathbb{R}$-linear derivation ∂ on $\mathbb{L}$ such that:*

(i) $\partial\ell_\alpha = \prod_{\beta<\alpha} \ell_\beta^{-1}$ *for all α;*

(ii) *for every set I and summable family $(f_i)_{i\in I}$ in $\mathbb{L}$ the family (∂f_i) is summable as well and $\partial \sum_i f_i = \sum_i \partial f_i$.*

The summability of a family (f_i) in $\mathbb{L}$ indexed by a set I as in (ii) means: for some α all f_i are in $\mathbb{L}_{<\alpha} = \mathbb{R}[[\mathfrak{L}_{<\alpha}]]$ and $\sum_i f_i$ exists in this Hahn field. For $\alpha = 0$, condition (i) says $\partial x = 1$. It is easy to see that the derivation of Proposition 1.1 must also respect logarithms in the sense that $\partial \log f = \partial f/f$ for all $f \in \mathbb{L}^{>}$. We establish Proposition 1.1 in Section 3, where we show in addition that the derivation ∂ of that proposition has the following properties:

Theorem 1.2. $\{f \in \mathbb{L} : \partial f = 0\} = \mathbb{R}$*, $(\mathbb{L}, \partial)$ is an H-field, and* $\partial\mathbb{L} = \mathbb{L}$.

Here $(\mathbb{L}, \partial)$ denotes the ordered field $\mathbb{L}$ equipped with the derivation ∂. Recall from [2, Chapter 10] that an H-field is an ordered differential field K such that for the constant field C of K and all $f \in K$ we have: if $f > C$, then $f' > 0$, and, with $\mathcal{O}$ the convex hull of C in K, if $f \in \mathcal{O}$, then $f = c + \varepsilon$ for some $c \in C$ and $\varepsilon \in K$ with $|\varepsilon| < C^{>}$. Such an H-field K is viewed as a *valued* field with valuation ring $\mathcal{O}$.

In the rest of this introduction $\mathbb{L}$ is equipped with the above derivation ∂. We also set $f' := \partial f$, $f^{(n)} = \partial^n f$ for $f \in \mathbb{L}$ and introduce the distinguished integration operator $f \mapsto \int f : \mathbb{L} \to \mathbb{L}$ that assigns to $f \in \mathbb{L}$ the unique $g \in \mathbb{L}$ with $g' = f$ and $1 \notin \operatorname{supp} g$; so the constant term of $\int f$ is 0. For example, $\ell_\alpha = \int \prod_{\beta<\alpha} \ell_\beta^{-1}$.

Composition. A good composition should reflect the composition of functions. To construct the 'correct' composition on $\mathbb{L}$ and show it has the desired properties takes considerable effort. Let us define a *composition on* $\mathbb{L}$ to be an operation

$$(f, g) \mapsto f \circ g \ : \ \mathbb{L} \times \mathbb{L}^{>\mathbb{R}} \to \mathbb{L}$$

that has the following properties:

(CL1) for any $g \in \mathbb{L}^{>\mathbb{R}}$ the map $f \mapsto f \circ g : \mathbb{L} \to \mathbb{L}$ is an $\mathbb{R}$-algebra endomorphism;

(CL2) $f \circ x = f$ for all $f \in \mathbb{L}$ and $x \circ g = g$ for all $g \in \mathbb{L}^{>\mathbb{R}}$;

(CL3) $\log(f \circ g) = (\log f) \circ g$ for all $f \in \mathbb{L}^{>}$ and $g \in \mathbb{L}^{>\mathbb{R}}$;

(CL4) for any summable family (f_i) in $\mathbb{L}$ and $g \in \mathbb{L}^{>\mathbb{R}}$ the family $(f_i \circ g)$ is summable and $(\sum_i f_i) \circ g = \sum_i f_i \circ g$;

(CL5) for all $f \in \mathbb{L}$ and $g, h \in \mathbb{L}^{>\mathbb{R}}$ we have $(f \circ g) \circ h = f \circ (g \circ h)$.

Note that (CL1) alone (and the fact that $\mathbb{L}$ is real closed) gives that for fixed $g \in \mathbb{L}^{>\mathbb{R}}$ the map $f \mapsto f \circ g : \mathbb{L} \to \mathbb{L}$ is an embedding of ordered fields sending $\mathbb{L}^{>\mathbb{R}}$ into itself. Thus (CL3) and (CL5) make sense, assuming (CL1).

Thinking of ℓ_α as the αth iterated logarithm of $\log x$ suggests $\ell_\alpha \circ \ell_\beta = \ell_{\beta+\alpha}$, but in view of $1 + \omega = \omega$ this would give $\ell_\omega \circ \ell_1 = \ell_\omega$ as a special case. Since (CL2) gives $\ell_\omega \circ \ell_0 = \ell_\omega$, this would be unreasonable, and in fact the composition we shall construct satisfies $\ell_\omega \circ \ell_1 = \ell_\omega - 1$ instead. Our main result is the following characterization of this composition:

Theorem 1.3. *There is a unique composition* $\circ$ *on* $\mathbb{L}$ *such that for all* $f, g, h \in \mathbb{L}$ *with* $g > \mathbb{R}$ *and* $g \succ h$ *the sum* $\sum_{n=0}^{\infty} \frac{f^{(n)} \circ g}{n!} h^n$ *exists and*

$$f \circ (g+h) \ = \ \sum_{n=0}^{\infty} \frac{f^{(n)} \circ g}{n!} h^n \qquad (\textit{Taylor expansion}),$$

and such that for all β, γ*:*

- $\ell_\gamma \circ \ell_{\omega^\beta} \ = \ \ell_{\omega^\beta + \gamma}$ *if* $\gamma < \omega^{\beta+1}$*;*
- $\ell_{\omega^{\beta+1}} \circ \ell_{\omega^\beta} \ = \ \ell_{\omega^{\beta+1}} - 1$*;*
- *the constant term of* $\ell_{\omega^\gamma} \circ \ell_{\omega^\beta}$ *is* 0 *if* $\gamma > \beta$ *is a limit ordinal.*

We construct this composition in Sections 5 and 6, and use Sections 7 and 8 to prove the more subtle results about it: (CL5) (that is, associativity) and Taylor expansion. In obtaining associativity we also establish the Chain Rule (Proposition 7.8):

$$(f \circ g)' \ = \ (f' \circ g) \cdot g' \text{ for all } f \in \mathbb{L} \text{ and } g \in \mathbb{L}^{>\mathbb{R}}.$$

All this concerns only the *existence* part of Theorem 1.3. The *at most one* part is taken care of in the final Section 9. In the remainder of this introduction we let $\circ$ denote the composition on $\mathbb{L}$ defined by Theorem 1.3.

The $g \in \mathbb{L}^{>\mathbb{R}}$ form a monoid under composition with x as identity, and the invertible elements of this monoid are the g with $\min \sigma(\mathfrak{d}(g)) = 0$: Proposition 8.5.

To construct our composition we work inside Hahn fields $\mathbb{L}_{<\alpha}$ where $\alpha = \omega^\lambda$ and λ is an infinite limit ordinal, and in fact, for such α we have $f \circ g \in \mathbb{L}_{<\alpha}$ for $f, g \in \mathbb{L}_{<\alpha}$ with $g > \mathbb{R}$; so the least α in this setting is ω^ω.

Finally, we indicate in Section 9 the natural ordered and valued field embedding of $\mathbb{L}$ into **No** that is the identity on $\mathbb{R}$, sends $x := \ell_0$ to ω, and respects logarithms and infinite sums: Proposition 9.5. This is also a differential field embedding where **No** is equipped with the derivation ∂_{BM} constructed by Berarducci and Mantova [5].

2. Preliminaries

We summarize here some conventions, notations, and results concerning monomial groups and Hahn fields and refer to [11] and [2, Section 3.1 and Appendix A] for proofs omitted here. We also consider some notions that are particularly useful in the present paper and a planned sequel: multipliability, the support of linear operators on Hahn fields, Taylor deformations, and monomial groups with real powers. In addition we include some miscellaneous facts needed later.

Monomial sets. A *monomial set* is a totally ordered set; we think of its elements as monomials. Let $\mathfrak{M}$ be a monomial set and let $\mathfrak{m}, \mathfrak{n}$ range over elements of $\mathfrak{M}$. Then $\mathfrak{m} \prec \mathfrak{n}$ indicates that $\mathfrak{m}$ is less than $\mathfrak{n}$ in the ordering of $\mathfrak{M}$, and we use the notations $\mathfrak{m} \preccurlyeq \mathfrak{n}$, $\mathfrak{m} \succ \mathfrak{n}$, $\mathfrak{m} \succcurlyeq \mathfrak{n}$ likewise; for example, $\mathfrak{m} \preccurlyeq \mathfrak{n} \Leftrightarrow \mathfrak{m} \prec \mathfrak{n}$ or $\mathfrak{m} = \mathfrak{n}$. A set $\mathfrak{S} \subseteq \mathfrak{M}$ is said to be *well-based* if it is well-ordered in the reverse ordering, that is, there is no infinite strictly increasing sequence $\mathfrak{m}_0 \prec \mathfrak{m}_1 \prec \mathfrak{m}_2 \prec \cdots$ in $\mathfrak{S}$.

Let $\boldsymbol{k}$ be a field. Then $\boldsymbol{k}[[\mathfrak{M}]]$ consists of the formal series $f = \sum_{\mathfrak{m}} f_{\mathfrak{m}} \mathfrak{m}$ with coefficients $f_{\mathfrak{m}} \in \boldsymbol{k}$ whose support

$$\operatorname{supp} f := \{\mathfrak{m} : \ f_{\mathfrak{m}} \neq 0\}$$

is well-based. We construe $\boldsymbol{k}[[\mathfrak{M}]]$ as a vector space over $\boldsymbol{k}$ as suggested by the series notation and identify $\mathfrak{M}$ with a subset of $\boldsymbol{k}[[\mathfrak{M}]]$ via $\mathfrak{m} \mapsto 1\mathfrak{m}$.

Let $(f_i)_{i\in I}$ be a family in $\boldsymbol{k}[[\mathfrak{M}]]$. We say that (f_i) is **summable** if $\bigcup_i \operatorname{supp} f_i$ is well-based and for each $\mathfrak{m} \in \mathfrak{M}$ there are only finitely many $i \in I$ such that $\mathfrak{m} \in \operatorname{supp} f_i$; in that case we define its sum $\sum_i f_i$ to be the series $f \in \boldsymbol{k}[[\mathfrak{M}]]$ such that $f_\mathfrak{m} = \sum_i f_{i,\mathfrak{m}}$ for each $\mathfrak{m} \in \mathfrak{M}$. (This agrees with the usual notation for elements of $\boldsymbol{k}[[\mathfrak{M}]]$: for a series $f = \sum_\mathfrak{m} f_\mathfrak{m}\mathfrak{m} \in \boldsymbol{k}[[\mathfrak{M}]]$ as above the family $(f_\mathfrak{m}\mathfrak{m})$ is indeed summable with sum f; conversely, every summable family $(f_\mathfrak{m}\mathfrak{m})$ with coefficients $f_\mathfrak{m} \in \boldsymbol{k}$ yields a series $f = \sum_\mathfrak{m} f_\mathfrak{m}\mathfrak{m} \in \boldsymbol{k}[[\mathfrak{M}]]$.) Instead of "$(f_i)$ is summable" we also say that $\sum_i f_i$ exists. Sometimes the following equivalence is useful: (f_i) is not summable if and only if there is a sequence (i_n) of distinct indices and an increasing sequence $(\mathfrak{m}_n)$ in $\mathfrak{M}$ with $\mathfrak{m}_n \in \operatorname{supp}(f_{i_n})$ for all n.

The *dominant monomial* $\mathfrak{d}(f) \in \mathfrak{M}$ of a nonzero $f \in \boldsymbol{k}[[\mathfrak{M}]]$ is defined by

$$\mathfrak{d}(f) \ := \ \max \operatorname{supp} f.$$

We also set $\mathfrak{d}(0) := 0 \in \boldsymbol{k}[[\mathfrak{M}]]$ and extend the ordering of $\mathfrak{M}$ to a total ordering on the disjoint union $\mathfrak{M} \cup \{0\}$ by $0 \prec \mathfrak{m}$ for all $\mathfrak{m} \in \mathfrak{M}$. The binary relations $\prec$ and $\preccurlyeq$ on $\mathfrak{M} \cup \{0\}$ are extended to binary relations $\prec$ and $\preccurlyeq$ on $\boldsymbol{k}[[\mathfrak{M}]]$ as follows:

$$f \prec g :\Leftrightarrow \ \mathfrak{d}(f) \prec \mathfrak{d}(g), \qquad f \preccurlyeq g :\Leftrightarrow \ \mathfrak{d}(f) \preccurlyeq \mathfrak{d}(g).$$

Let $\mathfrak{N}$ also be a monomial set and $\Phi : \boldsymbol{k}[[\mathfrak{M}]] \to \boldsymbol{k}[[\mathfrak{N}]]$ a map. We call Φ **strongly additive** if it is additive and for every summable family (f_i) in $\boldsymbol{k}[[\mathfrak{M}]]$ the family $(\Phi(f_i))$ is summable in $\boldsymbol{k}[[\mathfrak{N}]]$ and $\Phi(\sum_i f_i) = \sum_i \Phi(f_i)$. If Φ is strongly additive and $\Theta : \boldsymbol{k}[[\mathfrak{M}]] \to \boldsymbol{k}[[\mathfrak{N}]]$ is strongly additive, then so is

$$\Phi + \Theta \ : \ \boldsymbol{k}[[\mathfrak{M}]] \to \boldsymbol{k}[[\mathfrak{N}]], \qquad f \mapsto \Phi(f) + \Theta(f).$$

If Φ is strongly additive, $\mathfrak{G}$ is a monomial set, and $\Theta : \boldsymbol{k}[[\mathfrak{G}]] \to \boldsymbol{k}[[\mathfrak{M}]]$ is strongly additive, then so is $\Phi \circ \Theta : \boldsymbol{k}[[\mathfrak{G}]] \to \boldsymbol{k}[[\mathfrak{N}]]$. We call Φ **strongly $\boldsymbol{k}$-linear** if it is $\boldsymbol{k}$-linear and strongly additive; note that then for any $f = \sum_\mathfrak{m} f_\mathfrak{m}\mathfrak{m} \in \boldsymbol{k}[[\mathfrak{M}]]$ the sum $\sum_\mathfrak{m} f_\mathfrak{m}\Phi(\mathfrak{m})$ exists in $\boldsymbol{k}[[\mathfrak{N}]]$ and equals $\Phi(f)$. Thus a strongly $\boldsymbol{k}$-linear map $\boldsymbol{k}[[\mathfrak{M}]] \to \boldsymbol{k}[[\mathfrak{N}]]$ is determined by its restriction to $\mathfrak{M}$. The following converse is the "totally ordered" case of [11, Proposition 3.5]:

Lemma 2.1. *Let $\Phi : \mathfrak{M} \to \boldsymbol{k}[[\mathfrak{N}]]$ be such that for every well-based $\mathfrak{S} \subseteq \mathfrak{M}$ the family $(\Phi(\mathfrak{m}))_{\mathfrak{m}\in\mathfrak{S}}$ is summable. Then Φ extends (uniquely) to a strongly $\boldsymbol{k}$-linear map $\boldsymbol{k}[[\mathfrak{M}]] \to \boldsymbol{k}[[\mathfrak{N}]]$.*

The next result on inverting strongly linear maps is almost the "totally ordered" case of [1, Corollary 1.4], which in turn follows from [11, Theorems 6.1, 6.3].

Lemma 2.2. *Let $\Phi : \boldsymbol{k}[[\mathfrak{M}]] \to \boldsymbol{k}[[\mathfrak{M}]]$ be a strongly $\boldsymbol{k}$-linear map with $\Phi(\mathfrak{m}) \prec \mathfrak{m}$ for all $\mathfrak{m}$. Let I be the identity map on $\boldsymbol{k}[[\mathfrak{M}]]$. Then $I+\Phi : \boldsymbol{k}[[\mathfrak{M}]] \to \boldsymbol{k}[[\mathfrak{M}]]$ is bijective with strongly $\boldsymbol{k}$-linear inverse $(I+\Phi)^{-1}$ given by $(I+\Phi)^{-1}(f) = \sum_{n=0}^{\infty}(-1)^n\Phi^n(f)$, where the last sum always exists.*

Proof. For infinite $\boldsymbol{k}$ this is clear from [1, Corollary 1.4]. For finite $\boldsymbol{k}$ we reduce to the previous case by extending $\boldsymbol{k}$ to an infinite field K and using Lemma 2.1 to extend Φ to a strongly K-linear map $K[[\mathfrak{M}]] \to K[[\mathfrak{M}]]$. □

We only include the case of finite $\boldsymbol{k}$ for the sake of completeness, since the results above only get applied in later sections of this paper for $\boldsymbol{k}$ of characteristic 0.

Monomial groups and Hahn fields. A *monomial group* is a monomial set $\mathfrak{M}$ equipped with a (multiplicatively written) group operation $\mathfrak{M}\times\mathfrak{M}\to\mathfrak{M}$ that makes $\mathfrak{M}$ into an ordered commutative group. Let $\mathfrak{M}$ be a monomial group. We indicate its identity by 1 (or $1_{\mathfrak{M}}$ if we wish to specify $\mathfrak{M}$). For sets $\mathfrak{S}_1,\mathfrak{S}_2\subseteq\mathfrak{M}$ we set

$$\mathfrak{S}_1\mathfrak{S}_2 \;:=\; \{\mathfrak{m}\mathfrak{n}:\ \mathfrak{m}\in\mathfrak{S}_1,\ \mathfrak{n}\in\mathfrak{S}_2\},$$

and recall that if $\mathfrak{S}_1,\mathfrak{S}_2$ are well-based, then so is $\mathfrak{S}_1\mathfrak{S}_2$, and for every $\mathfrak{g}\in\mathfrak{S}_1\mathfrak{S}_2$ there are only finitely many pairs $(\mathfrak{m},\mathfrak{n})\in\mathfrak{S}_1\times\mathfrak{S}_2$ with $\mathfrak{g}=\mathfrak{m}\mathfrak{n}$. For $\mathfrak{S}\subseteq\mathfrak{M}$ we define $\mathfrak{S}^n\subseteq\mathfrak{M}$ by recursion on n by $\mathfrak{S}^0=\{1\}$, $\mathfrak{S}^{n+1}=\mathfrak{S}^n\mathfrak{S}$, and we also set $\mathfrak{S}^\infty:=\bigcup_n\mathfrak{S}^n$, the submonoid of $\mathfrak{M}$ generated by $\mathfrak{S}$. Recall Neumann's Lemma: if $\mathfrak{S}\subseteq\mathfrak{M}^{\preccurlyeq 1}$ is well-based, then so is $\mathfrak{S}^\infty$; if $\mathfrak{S}\subseteq\mathfrak{M}^{\prec 1}$ and $\mathfrak{g}\in\mathfrak{S}^\infty$, then there are only finitely many tuples $(n,\mathfrak{m}_1,\dots,\mathfrak{m}_n)$ with $\mathfrak{m}_1,\dots,\mathfrak{m}_n\in\mathfrak{S}$ and $\mathfrak{g}=\mathfrak{m}_1\cdots\mathfrak{m}_n$.

Let $\boldsymbol{k}$ be a field. Recall from [2, Section 3.1] how $\boldsymbol{k}[[\mathfrak{M}]]$ is then construed as a field extension of $\boldsymbol{k}$ with $\mathfrak{M}$ a subgroup of its multiplicative group.

Corollary 2.3. *Suppose $(\varepsilon_i)_{i\in I}$ is a summable family in $\boldsymbol{k}[[\mathfrak{M}]]^{\prec 1}=\boldsymbol{k}[[\mathfrak{M}^{\prec 1}]]$. Then the family $(\varepsilon_i^n)_{i\in I,n\geqslant 1}$ is summable, and so is the family $(\sum_{n=1}^\infty c_{in}\varepsilon^n)_{i\in I}$ for any family $(c_{in})_{i\in I,n\geqslant 1}$ of coefficients in $\boldsymbol{k}$.*

Proof. The first part is an easy consequence of Neumann's Lemma, and the second part follows from the first part. □

We shall often use the following result whose proof is routine:

Lemma 2.4. *Suppose (f_i) and (g_j) are summable families in $\boldsymbol{k}[[\mathfrak{M}]]$. Then (f_ig_j) is summable and $\sum_{i,j}f_ig_j=(\sum_i f_i)(\sum_j g_j)$.*

Thus for $f\in\boldsymbol{k}[[\mathfrak{M}]]$ the map $g\mapsto fg:\boldsymbol{k}[[\mathfrak{M}]]\to\boldsymbol{k}[[\mathfrak{M}]]$ is strongly $\boldsymbol{k}$-linear. Given also a monomial group $\mathfrak{N}$ we have:

Corollary 2.5. *Let $\Phi:\boldsymbol{k}[[\mathfrak{M}]]\to\boldsymbol{k}[[\mathfrak{N}]]$ be strongly additive, (f_n) a summable family in $\boldsymbol{k}[[\mathfrak{M}]$ and $\varepsilon\in\boldsymbol{k}[[\mathfrak{N}]]^{\prec 1}$. Then $\sum_n\Phi(f_n)\varepsilon^n$ exists.*

Proof. Use Lemma 2.4 and the summability of $\big(\Phi(f_n)\big)$ and (ε^n). □

We call $\boldsymbol{k}[[\mathfrak{M}]]$ a Hahn field over $\boldsymbol{k}$; it is a valued field with valuation ring

$$\mathcal{O}\;=\;\{f\in\boldsymbol{k}[[\mathfrak{M}]]:\ f\preccurlyeq 1\}$$

and maximal ideal $\smallo=\{f\in\boldsymbol{k}[[\mathfrak{M}]]:\ f\prec 1\}$ of $\mathcal{O}$. For the corresponding valuation v on $\boldsymbol{k}[[\mathfrak{M}]]$ and $f,g\in\boldsymbol{k}[[\mathfrak{M}]]$ we have

$$f\preccurlyeq g\Leftrightarrow v(f)\geqslant v(g),\quad f\prec g\Leftrightarrow v(f)>v(g).$$

For $f\in\boldsymbol{k}[[\mathfrak{M}]]$ we have the decomposition $f=f_\succ+f_1+f_\prec$ where $f_\succ:=\sum_{\mathfrak{m}\succ 1}f_{\mathfrak{m}}\mathfrak{m}$ is the *purely infinite part of* f and $f_\prec:=\sum_{\mathfrak{m}\prec 1}f_{\mathfrak{m}}\mathfrak{m}$ is the infinitesimal part of f. We also set $f_{\preccurlyeq 1}:=f_1+f_{\prec 1}$.

If $\boldsymbol{k}$ is given as an ordered field (for example when $\boldsymbol{k}=\mathbb{R}$), then we equip $\boldsymbol{k}[[\mathfrak{M}]]$ with the field ordering such that $f>0\Leftrightarrow f_{\mathfrak{d}(f)}>0$ (for $f\in\boldsymbol{k}[[\mathfrak{M}]]^{\neq}$) and refer to the resulting ordered field extension $\boldsymbol{k}[[\mathfrak{M}]]$ of $\boldsymbol{k}$ as an *ordered Hahn field.*

Substitution in ordinary power series. Let $\boldsymbol{k}$ be a field and $\mathfrak{M}$ a monomial group. Let $t = (t_1, \dots, t_n)$ be a tuple of distinct variables and let

$$F \ = \ F(t) \ = \ \sum_{\nu} c_\nu t^\nu \in \boldsymbol{k}[[t]] \ := \ \boldsymbol{k}[[t_1, \dots, t_n]]$$

be a formal power series over $\boldsymbol{k}$, the sum ranging over all $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n$, and $c_\nu \in \boldsymbol{k}$, $t^\nu := t_1^{\nu_1} \cdots t_n^{\nu_n}$. For any tuple $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ of elements of $\scriptstyle\mathcal{O} = \boldsymbol{k}[[\mathfrak{M}]]^{\prec 1}$ the family $(c_\nu \varepsilon^\nu)$ is summable, where $\varepsilon^\nu := \varepsilon_1^{\nu_1} \cdots \varepsilon_n^{\nu_n}$ (Neumann's Lemma). Put

$$F(\varepsilon) \ := \ \sum_{\nu} c_\nu \varepsilon^\nu \in \mathcal{O} \ = \ \boldsymbol{k}[[\mathfrak{M}]]^{\preccurlyeq 1} \ = \ \boldsymbol{k}[[\mathfrak{M}^{\preccurlyeq 1}]].$$

Fixing ε and varying F we obtain a $\boldsymbol{k}$-algebra morphism

$$F \mapsto F(\varepsilon) \ : \ \boldsymbol{k}[[t]] \to \boldsymbol{k}[[\mathfrak{M}]].$$

In the rest of this subsection we assume that $\boldsymbol{k}$ *characteristic* 0 *and identify* $\mathbb{Q}$ *with a subfield of* $\boldsymbol{k}$ *in the usual way.* Then we have the formal power series

$$\exp(t) \ := \ \sum_{i=0}^{\infty} t^i/i! \in \mathbb{Q}[[t]], \qquad \log(1+t) \ := \ \sum_{j=1}^{\infty} (-1)^{j-1} t^j/j \in \mathbb{Q}[[t]]$$

in a single variable t. In $\mathbb{Q}[[t_1, t_2]] \subseteq \boldsymbol{k}[[t_1, t_2]]$ we have the identities

$$\exp(t_1 + t_2) \ = \ \exp(t_1)\exp(t_2), \quad \log(1 + t_1 + t_2 + t_1 t_2) \ = \ \log(1+t_1) + \log(1+t_2).$$

Also $\log\big(\exp(t)\big) = t$ and $\exp\big(\log(1+t)\big) = 1 + t$ in $\mathbb{Q}[[t]] \subseteq \boldsymbol{k}[[t]]$. Substituting elements of $\boldsymbol{k}[[\mathfrak{M}]]^{\prec 1}$ in these identities yields that

$$h \ \mapsto \ \exp(h) \ = \ \sum_{i=0}^{\infty} h^i/i! \ : \ \boldsymbol{k}[[\mathfrak{M}]]^{\prec 1} \to 1 + \boldsymbol{k}[[\mathfrak{M}]]^{\prec 1},$$

is an isomorphism of the additive subgroup $\boldsymbol{k}[[\mathfrak{M}]]^{\prec 1}$ of $\boldsymbol{k}[[\mathfrak{M}]]$ onto the multiplicative subgroup $1 + \boldsymbol{k}[[\mathfrak{M}]]^{\prec 1}$ of $\boldsymbol{k}[[\mathfrak{M}]]^\times$, with inverse

$$1 + \varepsilon \ \mapsto \ \log(1+\varepsilon) \ = \ \sum_{j=1}^{\infty} (-1)^{j-1} \varepsilon^j/j \ : \ 1 + \boldsymbol{k}[[\mathfrak{M}]]^{\prec 1} \to \boldsymbol{k}[[\mathfrak{M}]]^{\prec 1}.$$

Corollary 2.6. *Let* $(\varepsilon_i)_{i \in I}$ *be a family in* $\boldsymbol{k}[[\mathfrak{M}]]^{\prec 1}$. *Then*

$$(\varepsilon_i) \text{ is summable} \iff \big(\log(1+\varepsilon_i)\big) \text{ is summable.}$$

Proof. The direction $\Rightarrow$ is a special case of Corollary 2.3. For $\Leftarrow$, apply that corollary to the case $c_{in} := 1/n!$ using $-1 + \exp(\log(1+\varepsilon_i)) = \varepsilon_i$. □

Multipliability. Let $\boldsymbol{k}$ be a field of characteristic 0 and $\mathfrak{M}$ a monomial group. Let $(\varepsilon_i)_{i \in I}$ be a family of elements in $\boldsymbol{k}[[\mathfrak{M}]]^{\prec 1}$. We declare E to range over the finite subsets of I and would like to define $\prod_i (1 + \varepsilon_i)$ as the sum over all E of the products $\prod_{i \in E} \varepsilon_i$. This would require the family $\big(\prod_{i \in E} \varepsilon_i\big)_E$ to be summable, and thus in particular its subfamily $(\varepsilon_i)_{i \in I}$ to be summable. By Corollary 2.6 the summability of $(\varepsilon_i)_i$ is equivalent to that of $\big(\log(1+\varepsilon_i)\big)_i$. Moreover:

Lemma 2.7. *Suppose* $(\varepsilon_i)_{i \in I}$ *is summable. Then the family* $\big(\prod_{i \in E} \varepsilon_i\big)_E$ *is also summable and* $\exp\big(\sum_i \log(1+\varepsilon_i)\big) = \sum_E \prod_{i \in E} \varepsilon_i$.

Proof. The summability of $\big(\prod_{i\in E}\varepsilon_i\big)_E$ follows from Neumann's Lemma: use that for $|E| = n$ we have $\operatorname{supp}\prod_{i\in E}\varepsilon_i \subseteq \big(\bigcup_{i\in I}\operatorname{supp}\varepsilon_i\big)^n$. Next, the desired identity holds for finite I, and then follows easily for arbitrary I using similar reasoning as needed for summability of $\big(\prod_{i\in E}\varepsilon_i\big)_E$. □

Accordingly we say that the family $(1+\varepsilon_i)$ is **multipliable** if (ε_i) is summable (equivalently, $\big(\log(1+\varepsilon_i)\big)_i$ is summable), and in that case we set

$$\prod_i(1+\varepsilon_i) \ :=\ \sum_E\prod_{i\in E}\varepsilon_i \ =\ 1+\sum_{E\neq\emptyset}\prod_{i\in E}\varepsilon_i \ \in\ 1+\boldsymbol{k}[[\mathfrak{M}]]^{\prec 1},$$

with $\log\prod_i(1+\varepsilon_i) = \sum_i\log(1+\varepsilon_i)$. Instead of calling $(1+\varepsilon_i)$ multipliable we also say that $\prod_i(1+\varepsilon_i)$ exists. The basic facts about these infinite products follow easily from corresponding facts about infinite sums by taking logarithms.

A useful identity. It is routine to check that for any elements $g_1, g_2, g_3, \dots$ in a field K of characteristic 0 we have an identity

$$\log\big(1+\sum_{n=1}^{\infty}\frac{g_n}{n!}t^n\big) \ =\ \sum_{n=1}^{\infty}\frac{L_n(g_1,\dots,g_n)}{n!}t^n$$

in the ring $K[[t]]$ of formal power series over K, where the $L_n \in \mathbb{Q}[X_1,\dots,X_n]$ are polynomials independent of the sequence $g_1, g_2, g_3, \dots$. The L_n are the *logarithmic polynomials* from [7, p. 140], but we don't need further details given there about them. In the later subsection on Taylor deformations we shall use the following:

Lemma 2.8. *Let K be a differential field, $y \in K^\times$, and $n \geqslant 1$. Then*

$$(y^\dagger)^{(n-1)} \ =\ L_n\left(\frac{y'}{y},\dots,\frac{y^{(n)}}{y}\right).$$

Proof. If these identities hold for some y that is differentially transcendental (over $\mathbb{Q}$), then they hold for all y as in the lemma. Take a real analytic function $f : I \to \mathbb{R}$ on a nonempty open interval $I \subseteq \mathbb{R}$ such that f is differentially transcendental and everywhere positive. (Thus f lies in the differential fraction field of the differential domain of real analytic functions on I.) For $a \in I$ the Taylor series of f at a is the formal series $\sum_n \frac{1}{n!}f^{(n)}(a)t^n \in \mathbb{R}[[t]]$. Likewise, the Taylor series of $\log f$ at a is

$$\sum_{n=0}^{\infty}\frac{1}{n!}(\log f)^{(n)}(a)t^n \ =\ \log f(a)+\sum_{n=1}^{\infty}\frac{1}{n!}(f^\dagger)^{(n-1)}(a)t^n.$$

Now $f = f(a)\cdot\big(1+\frac{f-f(a)}{f(a)}\big)$, so $\log f = \log f(a)+\log\big(1+\frac{f-f(a)}{f(a)}\big)$, the Taylor series of $\frac{f-f(a)}{f(a)}$ at a is $\sum_{n=1}^{\infty}\frac{1}{n!}\frac{f^{(n)}(a)}{f(a)}t^n$, so the Taylor series of $\log f$ at a also equals

$$\log f(a)+\sum_{n=1}^{\infty}\frac{1}{n!}L_n\big(\frac{f'(a)}{f(a)},\dots,\frac{f^{(n)}(a)}{f(a)}\big)t^n.$$

This yields $(f^\dagger)^{(n-1)}(a) = L_n\big(\frac{f'(a)}{f(a)},\dots,\frac{f^{(n)}(a)}{f(a)}\big)$ for all $a \in I$, that is, $(f^\dagger)^{(n-1)} = L_n\big(\frac{f'}{f},\dots,\frac{f^{(n)}}{f}\big)$, which gives the desired result. □

This lemma can also be proved more formally by expressing the L_n in terms of the Bell polynomials as in [7, p.140], but the details would take up considerable space.

The support of a linear operator. This notion will play a role similar to that of the norm of a linear operator on a Banach space. Let $\boldsymbol{k}$ be a field, $\mathfrak{M}$ a monomial group, $\mathfrak{G}$ a subset of $\mathfrak{M}$, and let a map $S : \mathfrak{G} \to \boldsymbol{k}[[\mathfrak{M}]]$ be given. Then we define the (operator) support of S, denoted by $\operatorname{supp} S$, to be the smallest set $\mathfrak{S} \subseteq \mathfrak{M}$ such that $\operatorname{supp} S(\mathfrak{g}) \subseteq \mathfrak{S}\mathfrak{g}$ for all $\mathfrak{g} \in \mathfrak{G}$. The proof of the next lemma is routine.

Lemma 2.9. *Suppose* $\operatorname{supp} S$ *is well-based. Then* S *extends uniquely to a strongly* $\boldsymbol{k}$*-linear operator* $\boldsymbol{k}[[\mathfrak{G}]] \to \boldsymbol{k}[[\mathfrak{M}]]$*. Denoting this extension also by* S*, we have* $\operatorname{supp} S(f) \subseteq (\operatorname{supp} S)(\operatorname{supp} f)$ *for all* $f \in \boldsymbol{k}[[\mathfrak{G}]]$.

For a strongly $\boldsymbol{k}$-linear map $T : \boldsymbol{k}[[\mathfrak{G}]] \to \boldsymbol{k}[[\mathfrak{M}]]$ we define $\operatorname{supp} T$ as the support of its restriction to $\mathfrak{G}$. If $\mathfrak{G} = \mathfrak{M}$ and $\operatorname{supp} S$ is well-based, then we have for each n the strongly $\boldsymbol{k}$-linear operator $S^n : \boldsymbol{k}[[\mathfrak{M}]] \to \boldsymbol{k}[[\mathfrak{M}]]$ with $\operatorname{supp} S^n \subseteq (\operatorname{supp} S)^n$. Simple applications of Neumann's Lemma give:

Lemma 2.10. *Suppose* $\mathfrak{G} = \mathfrak{M}$ *and* $\operatorname{supp} S$ *is well-based. Let* $h \in \boldsymbol{k}[[\mathfrak{M}]]$ *be such that* $(\operatorname{supp} S)(\operatorname{supp} h) \prec 1$*, and let* (s_n) *be any sequence in* $\boldsymbol{k}$*. Then* $\sum_{n=0}^{\infty} s_n S^n(\mathfrak{m}) h^n$ *exists for all* $\mathfrak{m}$*, and the map* $P : \mathfrak{M} \to \boldsymbol{k}[[\mathfrak{M}]]$ *given by* $P(\mathfrak{m}) := \sum_{n=0}^{\infty} s_n S^n(\mathfrak{m}) h^n$ *has well-based support* $\operatorname{supp} P \subseteq \big((\operatorname{supp} S)(\operatorname{supp} h)\big)^{\infty}$.

Lemma 2.11. *If* $T : \boldsymbol{k}[[\mathfrak{G}]] \to \boldsymbol{k}[[\mathfrak{M}]]$ *is* $\boldsymbol{k}$*-linear,* $\mathfrak{S} \subseteq \mathfrak{M}$ *is well-based, and* $\operatorname{supp} T(f) \subseteq \mathfrak{S} \cdot \operatorname{supp} f$ *for all* $f \in \boldsymbol{k}[[\mathfrak{G}]]$*, then* T *is strongly* $\boldsymbol{k}$*-linear.*

Thus with the hypothesis and notation of Lemma 2.10 the sum $\sum_{n=0}^{\infty} s_n S^n(f) h^n$ exists for all $f \in \boldsymbol{k}[[\mathfrak{M}]]$ and the map $T : \boldsymbol{k}[[\mathfrak{M}]] \to \boldsymbol{k}[[\mathfrak{M}]]$ given by

$$T(f) := \sum_{n=0}^{\infty} s_n S^n(f) h^n$$

is the unique strongly $\boldsymbol{k}$-linear operator $\boldsymbol{k}[[\mathfrak{M}]] \to \boldsymbol{k}[[\mathfrak{M}]]$ that extends P. Moreover, $\operatorname{supp} T(f) \subseteq \big((\operatorname{supp} S)(\operatorname{supp} h)\big)^{\infty} \cdot \operatorname{supp} f$ for $f \in \boldsymbol{k}[[\mathfrak{M}]]$. In the next lemma, an easy variant of Lemma 2.2, we let I be the identity map on $\boldsymbol{k}[[\mathfrak{M}]]$.

Lemma 2.12. *Suppose* $D : \boldsymbol{k}[[\mathfrak{M}]] \to \boldsymbol{k}[[\mathfrak{M}]]$ *is strongly* $\boldsymbol{k}$*-linear and* $\operatorname{supp} D$ *is well-based and* $\operatorname{supp} D \prec 1$*. Then* $I + D : \boldsymbol{k}[[\mathfrak{M}]] \to \boldsymbol{k}[[\mathfrak{M}]]$ *is bijective with strongly* $\boldsymbol{k}$*-linear inverse* $(I+D)^{-1} = I + E$*, where* $E : \boldsymbol{k}[[\mathfrak{M}]] \to \boldsymbol{k}[[\mathfrak{M}]]$ *is strongly* $\boldsymbol{k}$*-linear,* $\operatorname{supp} E \subseteq \bigcup_{n=1}^{\infty} (\operatorname{supp} D)^n \prec 1$*, and* $E(f) = \sum_{n=1}^{\infty} (-1)^n D^n(f)$ *for* $f \in \boldsymbol{k}[[\mathfrak{M}]]$.

Taylor Deformations. Let $\boldsymbol{k}$ be a field of characteristic 0 and $\mathfrak{M}$ a subgroup of the monomial group $\mathfrak{N}$, so $\boldsymbol{k}[[\mathfrak{M}]]$ is a subfield of $\boldsymbol{k}[[\mathfrak{N}]]$. Let there be given a $\boldsymbol{k}$-linear derivation ∂ on $\boldsymbol{k}[[\mathfrak{M}]]$ with well-based support $\operatorname{supp} \partial \prec 1$ and a strongly $\boldsymbol{k}$-linear field embedding $\Phi : \boldsymbol{k}[[\mathfrak{M}]] \to \boldsymbol{k}[[\mathfrak{N}]]$. Let $\varepsilon \in \boldsymbol{k}[[\mathfrak{N}]]^{\prec 1}$. Then for $f \in \boldsymbol{k}[[\mathfrak{M}]]$ the sum $\sum_n \frac{\partial^n f}{n!}$ exists in $\boldsymbol{k}[[\mathfrak{M}]]$ by the remark following Lemma 2.11, hence

$$\sum_{n=0}^{\infty} \frac{\Phi(\partial^n f)}{n!} \varepsilon^n = \Phi(f) + \Phi(\partial f)\varepsilon + \frac{\Phi(\partial^2 f)}{2} \varepsilon^2 + \cdots$$

exists in $\boldsymbol{k}[[\mathfrak{N}]]$ by Corollary 2.5. This yields a $\boldsymbol{k}$-linear (Taylor) map

$$T : \boldsymbol{k}[[\mathfrak{M}]] \to \boldsymbol{k}[[\mathfrak{N}]], \qquad T(f) := \sum_{n=0}^{\infty} \frac{\Phi(\partial^n f)}{n!} \varepsilon^n.$$

For $\varepsilon = 0$ we have $T = \Phi$; in general we view T as a *deformation* of Φ.

Lemma 2.13. $T : \boldsymbol{k}[[\mathfrak{M}]] \to \boldsymbol{k}[[\mathfrak{N}]]$ *is a strongly* $\boldsymbol{k}$*-linear field embedding.*

Proof. It is routine to check that $T(1) = 1$ and $T(fg) = T(f)T(g)$ for $f, g \in \boldsymbol{k}[[\mathfrak{M}]]$. Let the family (f_i) in $\boldsymbol{k}[[\mathfrak{M}]]$ be summable. Then so is the family $(\partial^n f_i)_{i,n}$ as is easily verified. Hence $\sum_{i,n} \frac{\Phi(\partial^n f_i)}{n!}$ exists. To derive that $\sum_i T(f_i)$ exists and equals $T(\sum_i f_i)$, use Lemma 2.4 and regroup terms. $\square$

To show that suitable logarithm maps on $\boldsymbol{k}[[\mathfrak{N}]]$ that commute with Φ also commute with its deformation T we assume for the next lemma that $\boldsymbol{k}$ is an *ordered* field (so $\boldsymbol{k}[[\mathfrak{M}]]$ and $\boldsymbol{k}[[\mathfrak{N}]]$ are ordered Hahn fields over $\boldsymbol{k}$), and that Φ is an embedding of *ordered* and *valued* fields. In addition we assume that $\boldsymbol{k}[[\mathfrak{N}]]$ is equipped with a map $\log : \boldsymbol{k}[[\mathfrak{N}]]^{>} \to \boldsymbol{k}[[\mathfrak{N}]]$ such that $\log(1+h) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} h^n$ for $h \in \boldsymbol{k}[[\mathfrak{N}]]^{\prec 1}$, $\log(fg) = \log(f) + \log(g)$ for $f, g \in \boldsymbol{k}[[\mathfrak{N}]]^{>}$, and $\log \boldsymbol{k}[[\mathfrak{M}]]^{>} \subseteq \boldsymbol{k}[[\mathfrak{M}]]$.

Lemma 2.14. *Suppose $f \in \mathbb{R}[[\mathfrak{M}]]^{>}$, $(\log f)' = f^{\dagger}$, and $\log \Phi(f) = \Phi(\log f)$. Then*

$$\log T(f) \;=\; T(\log f).$$

Proof. From $T(f) = \Phi(f) \cdot \left(1 + \sum_{n=1}^{\infty} \frac{1}{n!} \Phi\left(\frac{f^{(n)}}{f}\right) \varepsilon^n\right)$ we obtain

$$\log T(f) \;=\; \log \Phi(f) + \log\left(1 + \sum_{n=1}^{\infty} \frac{1}{n!} \Phi\left(\frac{f^{(n)}}{f}\right) \varepsilon^n\right).$$

Using $\log \Phi(f) = \Phi(\log f)$ and $f^{(n)} \preccurlyeq f$ for all n, this yields

$$\begin{aligned}
\log T(f) \;&=\; \Phi(\log f) + \sum_{n=1}^{\infty} \frac{1}{n!} L_n\left(\Phi\left(\frac{f'}{f}\right), \ldots, \Phi\left(\frac{f^{(n)}}{f}\right)\right) \varepsilon^n \\
&=\; \Phi(\log f) + \sum_{n=1}^{\infty} \frac{1}{n!} \Phi\left(L_n\left(\frac{f'}{f}, \ldots, \frac{f^{(n)}}{f}\right)\right) \varepsilon^n.
\end{aligned}$$

Lemma 2.8 gives $L_n\left(\frac{f'}{f}, \ldots, \frac{f^{(n)}}{f}\right) = (\log f)^{(n)}$ for $n \geqslant 1$, so

$$\log T(f) \;=\; \Phi(\log f) + \sum_{n=1}^{\infty} \frac{\Phi\left(\log(f)^{(n)}\right)}{n!} \varepsilon^n \;=\; T(\log f). \qquad \square$$

Suppose next that ∂ comes with a strongly $\boldsymbol{k}$-linear extension to a derivation on $\boldsymbol{k}[[\mathfrak{N}]]$, also denoted by ∂, and that the embedding Φ obeys a 'chain rule' in the sense that we are given an element $\phi \in \boldsymbol{k}[[\mathfrak{N}]]$ such that $\partial \Phi(f) = \Phi(\partial f) \cdot \phi$ for all $f \in \boldsymbol{k}[[\mathfrak{M}]]$. Then a routine computation yields also a chain rule for T:

Lemma 2.15. $\partial(Tf) \;=\; T(\partial f) \cdot (\phi + \partial \varepsilon)$ *for all* $f \in \boldsymbol{k}[[\mathfrak{M}]]$.

Monomial groups with real powers. Let the monomial group $\mathfrak{M}$ have real powers, that is, it is equipped with an operation $(s, \mathfrak{m}) \mapsto \mathfrak{m}^s \;:\; \mathbb{R} \times \mathfrak{M} \to \mathfrak{M}$ such that for all $s, t \in \mathbb{R}$ and all $\mathfrak{m}, \mathfrak{n}$ we have

$$\mathfrak{m}^1 \;=\; \mathfrak{m}, \quad \mathfrak{m}^{s+t} \;=\; \mathfrak{m}^s \mathfrak{m}^t, \quad (\mathfrak{m}^s)^t \;=\; \mathfrak{m}^{st}, \quad (\mathfrak{m}\mathfrak{n})^s \;=\; \mathfrak{m}^s \mathfrak{n}^s \quad (\text{so } \mathfrak{m}^0 = 1).$$

Then we extend this operation to a power operation

$$(s, f) \mapsto f^s \;:\; \mathbb{R} \times \mathbb{R}[[\mathfrak{M}]]^{>} \to \mathbb{R}[[\mathfrak{M}]]^{>}$$

as follows: first, if $f = 1 + \varepsilon$ with $\varepsilon \prec 1$, then we set

$$f^s \;:=\; \exp(s \log f) \;=\; \sum_{n=0}^{\infty} \binom{s}{n} \varepsilon^n \;\in\; 1 + \mathbb{R}[[\mathfrak{M}]]^{\prec 1},$$

and for $f = c\mathfrak{d}(f)(1+\varepsilon)$ ($c \in \mathbb{R}^>$, $\varepsilon \prec 1$), we set

$$f^s \ := \ c^s \mathfrak{d}(f)^s (1+\varepsilon)^s \ \in \ \mathbb{R}[[\mathfrak{M}]]^>$$

where c^s has the usual value in $\mathbb{R}^>$. It is easy to verify that then for all $s, t \in \mathbb{R}$ and $f, g \in \mathbb{R}[[\mathfrak{M}]]^>$ we have

$$f^0 \ = \ 1, \quad f^1 \ = \ f, \quad f^{s+t} = f^s f^t, \quad (f^s)^t = f^{st}, \quad (fg)^s = f^s g^s.$$

In the introduction we introduced the $\mathfrak{L}_{<\alpha}$ as monomial groups with real powers, and also defined a logarithm map on $\mathbb{L}^>$. Note that the definitions given there lead to $\log f^t = t \log f$ for $f \in \mathbb{L}^>$ and $t \in \mathbb{R}$.

A useful well-ordering. Let $\mathbb{N}^{\mathbb{N}}$ be lexicographically ordered and consider the set D of all sequences $(d_0, d_1, d_2, \dots) \in \mathbb{N}^{\mathbb{N}}$ such that $d_0 \geqslant d_1 \geqslant d_2 \geqslant \cdots$ and $d_n = 0$ for all sufficiently large n.

Lemma 2.16. *D is a well-ordered subset of $\mathbb{N}^{\mathbb{N}}$.*

Proof. Consider the map that assigns to any sequence $(d_0, d_1, d_2, \dots) \in D$ the ordinal $\omega^{d_0} + \omega^{d_1} + \cdots + \omega^{d_m}$ if m is such that $d_m \neq 0$ and $d_n = 0$ for all $n > m$, and assigns to the sequence $(0, 0, 0, \dots)$ the ordinal 0. Observe that this map is injective and order preserving. □

Next, let D_∞ be the larger set of all sequences $(d_0, d_1, d_2, \dots) \in \mathbb{N}^{\mathbb{N}}$ such that

$$d_0 \ \geqslant \ d_1 \ \geqslant \ d_2 \ \geqslant \ \cdots.$$

Corollary 2.17. *D_∞ is a well-ordered subset of $\mathbb{N}^{\mathbb{N}}$.*

Proof. Any strictly decreasing infinite sequence in D_∞ would be a sequence in $D_m := \{(d_0, d_1, d_2, \dots) \in D_\infty : \ d_0 \leqslant m\}$ for some m, so it is enough to show that D_m is well-ordered. Now D_m is the disjoint union of its subsets $D_{m,i}$, $i = 0, \dots, m$, where $D_{m,i}$ consists of the $(d_0, d_1, \dots) \in D_m$ with $d_n = i$ for all sufficiently large n, and it follows easily from Lemma 2.16 that each of the sets $D_{m,i}$ is well-ordered. □

Some more notation. For ordinals $\alpha < \gamma$ we let $\mathfrak{L}_{[\alpha,\gamma)}$ be the convex subgroup of $\mathfrak{L}_{<\gamma}$ whose elements are the hypermonomials $\prod_{\alpha \leqslant \beta < \gamma} \ell_\beta^{r_\beta}$. This gives the Hahn subfield $\mathbb{L}_{[\alpha,\gamma)} := \mathbb{R}[[\mathfrak{L}_{[\alpha,\gamma)}]]$ of $\mathbb{L}_{<\gamma}$. Note that

$$\mathfrak{L}_{<\gamma} \ = \ \mathfrak{L}_{[\alpha,\gamma)} \cdot \mathfrak{L}_{<\alpha} \ \text{ with } \ \mathfrak{L}_{[\alpha,\gamma)} \cap \mathfrak{L}_{<\alpha} \ = \ \{1\}.$$

As in [2, p. 713] this yields an identification of ordered fields

$$\mathbb{L}_{<\gamma} \ = \ \mathbb{L}_{[\alpha,\gamma)}[[\mathfrak{L}_{<\alpha}]],$$

that we shall use for certain $\alpha < \gamma$.

3. Differentiating and Integrating in $\mathbb{L}$

In the Introduction we defined the Hahn fields $\mathbb{L}_{<\alpha} = \mathbb{R}[[\mathfrak{L}_{<\alpha}]]$ over $\mathbb{R}$ and their union $\mathbb{L}$. Using the preliminary section it is easy to verify the results stated in the Introduction up to (but not including) the subsection on the derivation of $\mathbb{L}$. Towards Proposition 1.1 we shall construct for every α a strongly $\mathbb{R}$-linear derivation ∂_α on $\mathbb{L}_{<\alpha}$; the derivation ∂ on $\mathbb{L}$ will be the common extension of these ∂_α. The main work in this section is then to show that $\partial\mathbb{L} = \mathbb{L}$.

The derivation. We set

$$\ell'_\alpha \ := \ \prod_{\beta<\alpha} \ell_\beta^{-1} \in \mathfrak{L}_{<\alpha}, \qquad \ell^\dagger_\alpha \ := \ \prod_{\beta\leqslant\alpha} \ell_\beta^{-1} \ = \ \ell'_{\alpha+1} \in \mathfrak{L}_{\leqslant\alpha}.$$

Note that $\ell'_\alpha \preccurlyeq 1$ and $\ell^\dagger_\alpha \preccurlyeq x^{-1} \prec 1$, and that for $\alpha > \beta$ we have

$$\ell'_\alpha \ \prec \ \ell'_\beta \ \text{ in } \ \mathfrak{L}_{<\alpha}, \qquad \ell^\dagger_\alpha \prec \ \ell^\dagger_\beta \ \text{ in } \ \mathfrak{L}_{\leqslant\alpha}.$$

Next, we extend the above to any logarithmic hypermonomial $\mathfrak{m} = \ell^r$ by

$$\mathfrak{m}^\dagger \ := \ \sum_\beta r_\beta \ell^\dagger_\beta, \qquad \mathfrak{m}' \ := \ \mathfrak{m}\mathfrak{m}^\dagger \ = \ \sum_\beta r_\beta \mathfrak{m} \ell^\dagger_\beta.$$

Thus if $\mathfrak{m} \in \mathfrak{L}_{<\alpha}$, then $\mathfrak{m}^\dagger, \mathfrak{m}' \in \mathbb{L}_{<\alpha}$. As in the Introduction we let $\mathfrak{m}, \mathfrak{n}$ range over $\mathfrak{L} = \bigcup_\alpha \mathfrak{L}_{<\alpha}$.

Lemma 3.1. *The following hold for all* $\mathfrak{m}, \mathfrak{n}$*:*

- (i) $(\mathfrak{m}\mathfrak{n})^\dagger = \mathfrak{m}^\dagger + \mathfrak{n}^\dagger$, *and* $(\mathfrak{m}^t)^\dagger = t\mathfrak{m}^\dagger$ *for* $t \in \mathbb{R}$*;*
- (ii) $(\mathfrak{m}\mathfrak{n})' = \mathfrak{m}'\mathfrak{n} + \mathfrak{m}\mathfrak{n}'$*;*
- (iii) $\mathfrak{m} \neq 1 \ \Rightarrow \ \mathfrak{m}', \mathfrak{m}^\dagger \neq 0$*;*
- (iv) $\mathfrak{m} \prec 1,\ \mathfrak{n} \neq 1 \ \Rightarrow \ \mathfrak{m}' \prec \mathfrak{n}^\dagger$*;*
- (v) $\mathfrak{m} \prec \mathfrak{n} \neq 1 \ \Rightarrow \ \mathfrak{m}' \prec \mathfrak{n}'$*;*
- (vi) $\mathfrak{m} \in \mathfrak{L}_{<\alpha} \ \Rightarrow \ \operatorname{supp} \mathfrak{m}' \ \subseteq \ \{\ell^\dagger_\beta : \ \beta < \alpha\}\mathfrak{m}$.

Proof. This is mostly routine, and we only prove here (iv) and (v). So assume $\mathfrak{m} \prec 1$ and $\mathfrak{n} \neq 1$. For $\beta = \min \sigma(\mathfrak{m})$ we have $\mathfrak{m} = \ell_\beta^{r_\beta} \prod_{\beta<\rho<\alpha} \ell_\rho^{r_\rho}$, $r_\beta < 0$, so

$$\mathfrak{d}(\mathfrak{m}') \ = \ \mathfrak{m}\ell^\dagger_\beta \ = \ \big(\prod_{\rho<\beta} \ell_\rho^{-1}\big) \cdot \ell_\beta^{r_\beta - 1} \cdot \prod_{\beta<\rho<\alpha} \ell_\rho^{r_\rho}.$$

Also, for $\gamma = \min \sigma(\mathfrak{n})$ we have $\mathfrak{d}(\mathfrak{n}^\dagger) = \prod_{\rho\leqslant\gamma} \ell_\rho^{-1}$. By distinguishing the cases $\gamma < \beta$ and $\gamma \geqslant \beta$ and recalling that $r_\beta < 0$ we get $\mathfrak{d}(\mathfrak{m}') \prec \mathfrak{d}(\mathfrak{n}^\dagger)$, so $\mathfrak{m}' \prec \mathfrak{n}^\dagger$.

As to (v), assume $\mathfrak{m} \prec \mathfrak{n} \neq 1$. Then $\mathfrak{m} = \mathfrak{n}\mathfrak{v}$ with $\mathfrak{v} \prec 1$, so $\mathfrak{m}' = \mathfrak{n}'(\mathfrak{v} + \mathfrak{v}'/\mathfrak{n}^\dagger)$. It remains to note that $\mathfrak{v}' \prec \mathfrak{n}^\dagger$ by (iv). □

Item (vi) and Lemma 2.9 yield a unique strongly $\mathbb{R}$-linear derivation ∂_α on $\mathbb{L}_{<\alpha}$ such that $\partial_\alpha(\mathfrak{m}) = \mathfrak{m}'$ for all $\mathfrak{m} \in \mathfrak{L}_{<\alpha}$. Note that (vi) and that lemma also gives

$$\operatorname{supp} \partial_\alpha(f) \ \subseteq \ \{\ell^\dagger_\beta : \ \beta < \alpha\} \cdot \operatorname{supp} f \quad \text{ for } \ f \in \mathbb{L}_{<\alpha}$$

and that $\{\ell^\dagger_\beta : \ \beta < \alpha\}$ is a well-based subset of $\mathfrak{L}_{<\alpha}$ with largest element $\ell^\dagger_0 = x^{-1}$. In particular, $\operatorname{supp} \partial_\alpha = \{\ell^\dagger_\beta : \ \beta < \alpha\} \preccurlyeq x^{-1}$, so $\operatorname{supp} \partial_\alpha$ is well-based.

It is clear that for $\alpha > \beta$ the derivation ∂_α extends ∂_β. Thus we have a common extension of the ∂_α to a derivation ∂ on $\mathbb{L}$. This is the derivation of Proposition 1.1, which is thereby established. We set $f' := \partial f$ and $f^{(n)} := \partial^n f$ for $f \in \mathbb{L}$ and $g^\dagger := g'/g$ for $g \in \mathbb{L}^\times$; this creates no notational conflict, since for $f = \mathfrak{m}$ or $g = \mathfrak{m}$ this agrees with the previously defined $\mathfrak{m}'$ and $\mathfrak{m}^\dagger$. It is also easy to check that $\mathfrak{m}^\dagger = (\log \mathfrak{m})'$ and $(1+\varepsilon)^\dagger = [\log(1+\varepsilon)]'$ for $\varepsilon \in \mathbb{L}^{\prec 1}$, from which it follows that $g^\dagger = (\log |g|)'$ for $g \in \mathbb{L}^{\neq}$ and $(f^t)^\dagger = tf^\dagger$ for $f \in \mathbb{L}^>$.

Below we consider $\mathbb{L}_{<\alpha}$ as a differential field with derivation ∂_α, and also as an ordered and valued field. For the rest of this section we assume familiarity with the basic facts on H-fields and their asymptotic couples from [2].

Lemma 3.2. $\mathbb{L}_{<\alpha}$ *is an* H*-field with constant field* $\mathbb{R}$.

Proof. Note that if $\mathfrak{m} \succ 1$, then $\mathfrak{m}' > 0$. Let $f \in \mathbb{L}_{<\alpha}$.

Suppose $f > 0$ and $f \succ 1$. Then $\mathfrak{d}(f) \succ 1$, so $\mathfrak{d}(f)' > 0$. Also $\mathfrak{d}(f)' \succ \mathfrak{m}'$ for all $\mathfrak{m} \in \operatorname{supp}(f) \setminus \{\mathfrak{d}(f)\}$ by Lemma 3.1(v), and thus $f' > 0$.

Next, assume $f \notin \mathbb{R}$; we claim that $f' \neq 0$. By subtracting a real number from f we arrange $1 \notin \operatorname{supp}(f)$. Then the same item (v) yields $f' \neq 0$. □

We make the additive group Γ of exponent sequences into an ordered abelian group by $r < s :\Leftrightarrow \ell^r \prec \ell^s$. We define the valuation $v : \mathbb{L}^\times \to \Gamma$ by $v(f) = -r$ if $\mathfrak{d}(f) = \ell^r$; thus $v(f) > v(g) \Leftrightarrow f \prec g$ for all $f, g \in \mathbb{L}$. We have

$$\Gamma_{<\alpha} \ := \ v(\mathbb{L}^\times_{<\alpha}) \ = \ \{r \in \Gamma : \ r_\beta = 0 \text{ for all } \beta \geqslant \alpha\}.$$

Note that if $\beta < \alpha$, then $\mathbb{L}_{<\beta}$ is an H-subfield of $\mathbb{L}_{<\alpha}$. Next we consider the asymptotic couple $(\Gamma_{<\alpha}, \psi_{<\alpha})$ of $\mathbb{L}_{<\alpha}$. We have an order-preserving bijection

$$\beta \mapsto v(\ell_\beta^\dagger) \ : \ \alpha \to \Psi_{<\alpha}$$

from α onto the Ψ-set $\Psi_{<\alpha}$ of $\mathbb{L}_{<\alpha}$. In particular, if $\alpha \neq 0$, then $\Psi_{<\alpha}$ has least element $v(\ell_0^{-1})$, and this element is positive and is the unique fixed point of ψ.

Lemma 3.3. *If $\alpha = \beta + 1$, then $v(\ell_\alpha') = v(\ell_\beta^\dagger) = \max \Psi_{<\alpha} > 0$. If $\alpha \neq 0$ is a limit ordinal, then $v(\ell_\alpha') > 0$ is a gap in $\mathbb{L}_{<\alpha}$.*

Proof. The first claim follows from the above order-preserving bijection $\alpha \to \Psi_{<\alpha}$. Suppose $\alpha \neq 0$ is a limit. Then in $\Gamma_{<\alpha}$,

$$\Psi_{<\alpha} \ < \ v(\ell_\alpha') \ < \ (\Gamma^>_{<\alpha})',$$

so $v(\ell_\alpha') > 0$ is indeed a gap in $\mathbb{L}_{<\alpha}$. □

Integration. *In this subsection* $\mathfrak{m}$ *and* $\mathfrak{n}$ *range over* $\mathfrak{L}_{<\alpha}$. We use the modified derivation $\mathfrak{d} := \frac{1}{\ell_\alpha'}\partial_\alpha$ on $\mathbb{L}_{<\alpha}$, which is strongly $\mathbb{R}$-linear. It follows from Lemma 3.3 that for the Ψ-set $\Psi_{\mathfrak{d}}$ of the asymptotic couple of the H-field $(\mathbb{L}_{<\alpha}, \mathfrak{d})$ we have $\max \Psi_{\mathfrak{d}} = 0$ if α is a successor ordinal, and $\sup \Psi_{\mathfrak{d}} = 0 \notin \Psi$, otherwise. Thus $\mathfrak{d}$ is small, but $\mathfrak{d}(\mathfrak{m}) \succcurlyeq \mathfrak{m}$ for $\mathfrak{m} \neq 1$. Moreover:

Lemma 3.4. *If $\mathfrak{m} \succ 1$, then $\operatorname{supp} \mathfrak{d}(\mathfrak{m}) \succ 1$. If $\mathfrak{m} \prec 1$, then $\operatorname{supp} \mathfrak{d}(\mathfrak{m}) \prec 1$.*

Proof. Let $\mathfrak{m} = \ell^r$. Then

$$\mathfrak{d}(\mathfrak{m}) \ = \ \Big(\prod_{\beta<\alpha} \ell_\beta\Big) \cdot \Big(\sum_{\gamma \in \sigma(\mathfrak{m})} r_\gamma \mathfrak{m} \ell_\gamma^\dagger\Big) \ = \ \sum_{\gamma \in \sigma(\mathfrak{m})} r_\gamma \Big(\mathfrak{m} \prod_{\gamma<\beta<\alpha} \ell_\beta\Big).$$

It remains to note that for $\gamma \in \sigma(\mathfrak{m})$ and $\mathfrak{n} := \mathfrak{m} \prod_{\gamma<\beta<\alpha} \ell_\beta$ we have $\min \sigma(\mathfrak{n}) = \min \sigma(\mathfrak{m})$ and $r_{\min \sigma(\mathfrak{n})} = r_{\min \sigma(\mathfrak{m})}$. □

Thus $\mathfrak{d}$ maps $\mathbb{R}[[\mathfrak{L}^{\neq 1}_{<\alpha}]]$ into itself. For $\xi \neq 0$ in the asymptotic couple of $(\mathbb{L}_{<\alpha}, \mathfrak{d})$ we set $\xi^\dagger := \psi_{\mathfrak{d}}(\xi) \leqslant 0$ and $\xi' := \xi + \xi^\dagger$, so $\xi^\dagger = o(\xi)$ by [2, Lemma 9.2.10(iv)], hence $(\xi - \xi^\dagger)^\dagger = \xi^\dagger$, and thus $(\xi - \xi^\dagger)' = \xi$. It follows that for any $\mathfrak{m} \neq 1$ there is a unique $\mathfrak{n} \neq 1$ with $\mathfrak{d}(\mathfrak{n}) \asymp \mathfrak{m}$, namely $\mathfrak{n} = \mathfrak{d}\big(\frac{\mathfrak{m}}{\mathfrak{d}(\mathfrak{m})/\mathfrak{m}}\big)$, and the map that assigns to any $\mathfrak{m} \neq 1$ the unique $\mathfrak{n} \neq 1$ with $\mathfrak{d}(\mathfrak{n}) \asymp \mathfrak{m}$ is an automorphism of the ordered set $\mathfrak{L}^{\neq 1}_{<\alpha}$. We define $T \ : \ \mathfrak{L}^{\neq 1}_{<\alpha} \ \to \ \mathbb{R}^\times \cdot \big(\mathfrak{L}^{\neq 1}_{<\alpha}\big)$ by

$$T(\mathfrak{m}) \ := \ c\,\mathfrak{n}, \quad \text{with } c \in \mathbb{R}^\times \text{ and } \mathfrak{n} \in \mathfrak{L}^{\neq 1}_{<\alpha} \text{ such that } \ c\,\mathfrak{d}(\mathfrak{n}) \sim \mathfrak{m}.$$

Using Lemma 2.1 we note that T extends uniquely to a strongly $\mathbb{R}$-linear bijection $\mathbb{R}[[\mathfrak{L}^{\neq 1}_{<\alpha}]] \to \mathbb{R}[[\mathfrak{L}^{\neq 1}_{<\alpha}]]$, also denoted by T, with a strongly $\mathbb{R}$-linear inverse T^{-1}. By virtue of the definition of T we have for nonzero $g \in \mathbb{R}[[\mathfrak{L}^{\neq}_{<\alpha}]]$:

$$\delta(Tg) \;=\; g + E(g) \text{ with } E(g) \prec g.$$

This determines the strongly $\mathbb{R}$-linear selfmap $E := \delta \circ T - I$ on $\mathbb{R}[[\mathfrak{L}^{\neq 1}_{<\alpha}]]$, where I is the identity on $\mathbb{R}[[\mathfrak{L}^{\neq}_{<\alpha}]]$. Since $E(g) \prec g$ for all nonzero $g \in \mathbb{R}[[\mathfrak{L}^{\neq 1}_{<\alpha}]]$, it follows from Lemma 2.2 that the strongly $\mathbb{R}$-linear selfmap $I + E$ on $\mathbb{R}[[\mathfrak{L}^{\neq 1}_{<\alpha}]]$ is bijective with strongly $\mathbb{R}$-linear inverse $(I+E)^{-1}$. From $\delta \circ T = I + E$ we get $\delta \circ T \circ (I+E)^{-1} = I$, that is, $\delta^{-1} := T \circ (I+E)^{-1}$ is a strongly $\mathbb{R}$-linear right inverse to δ on $\mathbb{R}[[\mathfrak{L}^{\neq 1}_{<\alpha}]]$. In terms of the original derivation ∂, this yields a distinguished strongly $\mathbb{R}$-linear bijective integration operator

$$\int \;:\; \mathbb{R}[[\mathfrak{L}_{<\alpha} \setminus \{\ell'_\alpha\}]] \;\to\; \mathbb{R}[[\mathfrak{L}^{\neq 1}_{<\alpha}]], \qquad \int f \;:=\; \delta^{-1}\big(f/\ell'_\alpha\big).$$

We call it an integration operator because $\partial\big(\int f\big) = f$ for $f \in \mathbb{R}[[\mathfrak{L}_{<\alpha} \setminus \{\ell'_\alpha\}]]$.

Integration, continued. The domain of the above integration operator depends on α, but it assigns to each f in its domain the unique $g \in \mathbb{L}$ with $g' = f$ and $1 \notin \operatorname{supp} g$. It follows that these operators for the various α have a common extension to an operator $\int : \mathbb{L} \to \mathbb{L}$ that assigns to each $f \in \mathbb{L}$ the unique $g \in \mathbb{L}$ with $g' = f$ and $1 \notin \operatorname{supp} g$. Thus we have now fully established Theorem 1.2 and the rest of the subsection "The derivation on $\mathbb{L}$" in the Introduction. Note also that $\int$ maps $\mathbb{L}_{<\alpha}$ into $\mathbb{L}_{<\alpha} + \mathbb{R}\ell_\alpha$, more precisely, bijectively onto $\mathbb{R}[[\mathfrak{L}^{\neq 1}_{<\alpha}]] + \mathbb{R}\ell_\alpha$.

The remainder of this section will not be used, but relates the above to material in [2]. We assume now that α is an infinite limit ordinal, and set

$$\mathbb{L}^{\cup}_{<\alpha} \;:=\; \bigcup_{\beta<\alpha} \mathbb{L}_{<\beta}.$$

We saw that $\mathbb{L}_{<\alpha}$ is not closed under $\int$, but we now observe that its H-subfield $\mathbb{L}^{\cup}_{<\alpha}$ is closed under $\int$ and is the union of its chain of spherically complete H-subfields $\mathbb{L}_{<\beta}$ with $\beta < \alpha$, and if such β is a successor ordinal, then $\mathbb{L}_{<\beta}$ is grounded. Thus by [2, Corollary 11.7.15, Theorem 15.0.1]:

Corollary 3.5. *The H-field $\mathbb{L}^{\cup}_{<\alpha}$ is ω-free and newtonian.*

Of course the H-field $\mathbb{L}$ is likewise ω-free and newtonian. As to the case $\alpha = \omega$, we recall from [2, Appendix A] that $\mathbb{T}$ has distinguished elements ℓ_n. We have a unique field embedding $\mathbb{L}^{\cup}_{<\omega} \to \mathbb{T}$ that is the identity on $\mathbb{R}$, sends $\ell_n^r \in \mathbb{L}^{\cup}_{<\omega}$ to $\ell_n^r \in \mathbb{T}$ for all n and all $r \in \mathbb{R}$, and respects infinite sums. This embedding also respects the natural logarithm maps on the multiplicative groups of positive elements of $\mathbb{L}^{\cup}_{<\omega}$ and $\mathbb{T}$, and the natural derivations on these fields. The image of this embedding is the H-subfield $\mathbb{T}_{\log}$ of $\mathbb{T}$; we identify $\mathbb{L}^{\cup}_{<\omega}$ with $\mathbb{T}_{\log}$ via this embedding.

4. Preliminaries on Composition

In this section $\mathfrak{N}$ is a monomial group with real powers. We fix $\mathbb{R}[[\mathfrak{N}]]$ as an ambient Hahn field equipped with its natural ordering and valuation. Let $\mathfrak{M}$ be a power closed monomial subgroup of $\mathfrak{N}$ with a distinguished element $x \in \mathfrak{M}^{\succ 1}$. Then we have the (ordered valued) Hahn subfield $K := \mathbb{R}[[\mathfrak{M}]]$ of $\mathbb{R}[[\mathfrak{N}]]$. We let h range over the elements of $\mathbb{R}[[\mathfrak{N}]]^{>\mathbb{R}}$.

Composition on Hahn Fields. A **K-composition with** h is a map

$$f \mapsto f \circ h \ : \ K \to \mathbb{R}[[\mathfrak{N}]]$$

such that the following conditions are satisfied:

(1) $1 \circ h = 1$ and $x \circ h = h$;
(2) for all $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathfrak{M}$, $(\mathfrak{m}_1 \mathfrak{m}_2) \circ h = (\mathfrak{m}_1 \circ h) \cdot (\mathfrak{m}_2 \circ h)$;
(3) for all $f \in K$, $(\mathfrak{m} \circ h)_{\mathfrak{m} \in \operatorname{supp}(f)}$ is summable and $\sum_{\mathfrak{m}} f_{\mathfrak{m}} (\mathfrak{m} \circ h) = f \circ h$.

Lemma 4.1. *Any K-composition with h is an ordered field embedding $K \to \mathbb{R}[[\mathfrak{N}]]$ and is strongly $\mathbb{R}$-linear.*

Proof. Let a K-composition with h be given. Since K and $\mathbb{R}[[\mathfrak{N}]]$ are real closed fields, the map $f \mapsto f \circ h : K \to \mathbb{R}[[\mathfrak{N}]]$ will be an ordered field embedding if it is a ring morphism. Let $f, g \in K$. Then

$$\begin{aligned} (f+g) \circ h \ &= \ \left(\sum_{\mathfrak{m}} (f_{\mathfrak{m}} + g_{\mathfrak{m}}) \mathfrak{m} \right) \circ h \ = \ \sum_{\mathfrak{m}} (f_{\mathfrak{m}} + g_{\mathfrak{m}})(\mathfrak{m} \circ h) \\ &= \ \sum_{\mathfrak{m}} \big(f_{\mathfrak{m}}(\mathfrak{m} \circ h) + g_{\mathfrak{m}}(\mathfrak{m} \circ h)\big) \ = \ \sum_{\mathfrak{m}} f_{\mathfrak{m}}(\mathfrak{m} \circ h) + \sum_{\mathfrak{m}} g_{\mathfrak{m}}(\mathfrak{m} \circ h) \\ &= \ (f \circ h) + (g \circ h). \end{aligned}$$

Similarly, using Lemma 2.4,

$$\begin{aligned} (fg) \circ h \ &= \ \sum_{\mathfrak{m}} \left(\sum_{\mathfrak{m}_1 \mathfrak{m}_2 = \mathfrak{m}} f_{\mathfrak{m}_1} g_{\mathfrak{m}_2} \right) \mathfrak{m} \circ h \ = \ \sum_{\mathfrak{m}} \left(\sum_{\mathfrak{m}_1 \mathfrak{m}_2 = \mathfrak{m}} f_{\mathfrak{m}_1} (\mathfrak{m}_1 \circ h) g_{\mathfrak{m}_2} (\mathfrak{m}_2 \circ h) \right) \\ &= \ \left(\sum_{\mathfrak{m}_1} f_{\mathfrak{m}_1} (\mathfrak{m}_1 \circ h) \right) \left(\sum_{\mathfrak{m}_2} g_{\mathfrak{m}_2} (\mathfrak{m}_2 \circ h) \right) \ = \ (f \circ h)(g \circ h). \end{aligned}$$

Strong linearity follows from Lemma 2.1 and clause (3) above. □

Here are some consequences of Lemma 4.1 for a K-composition $f \mapsto f \circ h$ with h: $\mathfrak{m} \circ h > 0$ for $\mathfrak{m} \in \mathfrak{M}$, so $(\mathfrak{m} \circ h)^t$ is defined for all real t, and for $f, f_1, f_2 \in K$,

$$f > \mathbb{R} \ \Leftrightarrow \ f \circ h > \mathbb{R}, \qquad f_1 \preccurlyeq f_2 \ \Leftrightarrow \ f_1 \circ h \preccurlyeq f_2 \circ h.$$

Thus for $f \prec 1$ in K we have $f \circ h \prec 1$ in $\mathbb{R}[[\mathfrak{N}]]$, and

$$\exp(f) \circ h = \exp(f \circ h), \qquad \big(\log(1+f)\big) \circ h = \log\big((1+f) \circ h\big).$$

Lemma 4.2. *Let a K-composition with h be given such that $\mathfrak{m}^t \circ h = (\mathfrak{m} \circ h)^t$ for all $\mathfrak{m} \in \mathfrak{M}$ and $t \in \mathbb{R}$. Then $f^t \circ h = (f \circ h)^t$ for all $f \in K^{>}$ and $t \in \mathbb{R}$.*

Proof. Let $f \in K^{>}$. Then $f = c\mathfrak{m}(1+\varepsilon)$ where $c \in \mathbb{R}^{>}$, $\mathfrak{m} \in \mathfrak{M}$ and $\varepsilon \in K^{\prec 1}$, so

$$\begin{aligned} f^t \circ h \ &= \ (c\mathfrak{m}(1+\varepsilon))^t \circ h \ = \ c^t (\mathfrak{m}^t \circ h) \left(\left(\sum_n \binom{t}{n} \varepsilon^n \right) \circ h \right) \\ &= \ c^t (\mathfrak{m} \circ h)^t \sum_n \binom{t}{n} (\varepsilon \circ h)^n \ = \ (f \circ h)^t \qquad (t \in \mathbb{R}). \end{aligned}$$

□

A **K-composition** is a map $\circ : K \times \mathbb{R}[[\mathfrak{N}]]^{>\mathbb{R}} \to \mathbb{R}[[\mathfrak{N}]]$ such that for all h the map $f \mapsto f \circ h : K \to \mathbb{R}[[\mathfrak{N}]]$ is a K-composition with h.

Lemma 4.3. *Let $\circ$ be a K-composition, and let $g \in K^{>\mathbb{R}}$, $h \in \mathbb{R}[[\mathfrak{N}]]^{>\mathbb{R}}$ be such that $\mathfrak{m} \circ g \in K$ and $(\mathfrak{m} \circ g) \circ h = \mathfrak{m} \circ (g \circ h)$ for all $\mathfrak{m} \in \mathfrak{M}$. Then $f \circ g \in K$ and $(f \circ g) \circ h = f \circ (g \circ h)$ for all $f \in K$.*

Proof. Let $f \in K$. It is clear that then $f \circ g = \sum_{\mathfrak{m}} f_{\mathfrak{m}}(\mathfrak{m} \circ g) \in K$, and

$$(f \circ g) \circ h \;=\; \sum_{\mathfrak{m}} f_{\mathfrak{m}}(\mathfrak{m} \circ g) \circ h \;=\; \sum_{\mathfrak{m}} f_{\mathfrak{m}} \mathfrak{m} \circ (g \circ h) \;=\; f \circ (g \circ h). \qquad \square$$

We now consider the case $\mathfrak{M} = \mathfrak{N}$, and define a **composition on** K for $K = \mathbb{R}[[\mathfrak{N}]]$ to be a map $\circ : K \times K^{>\mathbb{R}} \to K$ such that:

(1) $\circ$ is a K-composition;
(2) $\mathfrak{m}^t \circ g = (\mathfrak{m} \circ g)^t$ for all $\mathfrak{m} \in \mathfrak{N}$, $t \in \mathbb{R}$, and $g \in K^{>\mathbb{R}}$;
(3) $(\mathfrak{m} \circ g) \circ h = \mathfrak{m} \circ (g \circ h)$ for all $g, h \in K^{>\mathbb{R}}$.

Thus given a composition $\circ$ on K it follows from Lemmas 4.2 and 4.3 that clauses (2) and (3) hold in a more general form: $f^t \circ g = (f \circ g)^t$ for all $f \in K^{>}$, $t \in \mathbb{R}$, and $g \in K^{>\mathbb{R}}$, and $(f \circ g) \circ h = f \circ (g \circ h)$ for all $f \in K$ and $g, h \in K^{>\mathbb{R}}$.

Taylor Expansion. Let there be given a K-composition $f \mapsto f \circ h : K \to \mathbb{R}[[\mathfrak{N}]]$ with h and an $\mathbb{R}$-linear derivation ∂ on K with well-based support $\operatorname{supp} \partial \prec 1$ and $\partial x = 1$, and an element $\varepsilon \in \mathbb{R}[[\mathfrak{N}]]^{\prec 1}$. Then we set $g := h + \varepsilon$ and 'deform' the above K-composition with h to a K-composition $f \mapsto f \circ g : K \to \mathbb{R}[[\mathfrak{N}]]$ with g as follows: with Φ the above K-composition with h we apply the subsection on Taylor deformations in Section 2 to obtain a strongly $\mathbb{R}$-linear operator

$$\circ_g \;:\; K \;\to\; \mathbb{R}[[\mathfrak{N}]], \qquad f \mapsto \sum_{n=0}^{\infty} \frac{\partial^n(f) \circ h}{n!} \varepsilon^n.$$

We think of $\circ_g$ as composition with g on the right, which explains the notation. In this subsection we set $f \circ g := \circ_g(f)$ for $f \in K$. Note that $1 \circ g = 1$ and $x \circ g = g$. Then by Lemma 2.13:

Lemma 4.4. *The map $f \mapsto f \circ g : K \to \mathbb{R}[[\mathfrak{N}]]$ is a K-composition with g.*

Assume in addition that ∂ extends to a strongly $\boldsymbol{k}$-linear derivation on $\boldsymbol{k}[[\mathfrak{N}]]$, denoted also by ∂. Then by Lemma 2.15 the 'chain rule' is inherited:

Lemma 4.5. *If $\partial(f \circ h) = \big((\partial f) \circ h\big) \cdot \partial h$ for all $f \in \boldsymbol{k}[[\mathfrak{M}]]$, then $\partial(f \circ g) = \big((\partial f) \circ g\big) \cdot \partial g$ for all $f \in \boldsymbol{k}[[\mathfrak{M}]]$.*

Revisiting multipliability. Let $(f_i)_{i \in I}$ be a family in $\mathbb{L}^{>}$, where I is a set. We call (f_i) *multipliable* if the family $(\log(f_i))$ is summable. Note that if $f_i = 1 + \varepsilon_i$, with all $\varepsilon_i \in \mathbb{R}[[\mathfrak{L}_{<\alpha}]]^{\prec 1}$ for a fixed α, then this agrees with $(1 + \varepsilon_i)$ being multipliable as defined in Section 2. In general we have α such that $f_i = c_i \mathfrak{m}_i (1 + \varepsilon_i)$, $c_i \in \mathbb{R}^{>}$, $\mathfrak{m}_i \in \mathfrak{L}_{<\alpha}$, $\varepsilon_i \in \mathbb{R}[[\mathfrak{L}_{<\alpha}]]^{\prec 1}$ for all i. Then (f_i) is multipliable if and only if $(1 + \varepsilon_i)$ is multipliable, $c_i = 1$ for all but finitely many i, and, with $\mathfrak{m}_i = \prod_{\beta < \alpha} \ell_\beta^{r_{\beta i}}$, there are for every $\beta < \alpha$ only finitely many i with $r_{\beta i} \neq 0$.

If (f_i) is multipliable, then so is $(f_i^{r_i})$ for any family (r_i) of real numbers. Suppose the family (f_i) in $\mathbb{L}^{>}_{<\alpha}$ is multipliable. Then

$$\sum_i \log(f_i) \;=\; \sum_{\beta < \alpha} s_\beta \ell_{\beta+1} + c + \varepsilon \tag{4.1}$$

where the s_β and c are real numbers and $\varepsilon \in \mathbb{L}_{<\alpha}^{\prec 1}$. Thus we may define

$$\prod_{i\in I} f_i \ := \ \left(\prod_{\beta<\alpha} \ell_\beta^{s_\beta}\right) \mathrm{e}^c \sum_{n=0}^{\infty} \frac{1}{n!}\varepsilon^n \ \in \mathbb{L}_{<\alpha}^{>}.$$

Then $\log \prod_i f_i = \sum_i \log f_i$, so $(\prod_i f_i)^\dagger = \sum_i f_i^\dagger$ and $\prod_i f_i^t = (\prod_i f_i)^t$ for $t \in \mathbb{R}$. If also the family (g_i) in $\mathbb{L}_{<\alpha}^{>}$ is multipliable, then $(f_i g_i)$ is multipliable, and

$$\prod_i f_i \cdot \prod_i g_i \ = \ \prod_i f_i g_i.$$

Any family $(g_j)_{j=1,\dots,n}$ in $\mathbb{L}^{>}$ is multipliable with $\prod_j g_j = g_1 \cdots g_n$. Also, for any family $(r_\beta)_{\beta<\alpha}$ of real numbers the family $(\ell_\beta^{r_\beta})_{\beta<\alpha}$ is multipliable, and $\prod_{\beta<\alpha} \ell_\beta^{r_\beta}$ is the logarithmic hypermonomial that we expressed this way earlier. Retracing the definitions gives:

Lemma 4.6. *Suppose the family (f_i) in $\mathbb{L}_{<\alpha}^{>}$ is multipliable. Then the family $(\mathfrak{d}(f_i))$ is multipliable as well and $\mathfrak{d}(\prod_i f_i) = \prod_i \mathfrak{d}(f_i)$.*

We define the function $\log_n : \mathbb{L}^{>\mathbb{R}} \to \mathbb{L}^{>\mathbb{R}}$ by recursion on n:

$$\log_0(g) \ := \ g, \qquad \log_{n+1}(g) \ := \ \log(\log_n(g)).$$

Thus $\log_n$ maps $\mathbb{L}_{<\alpha}^{>\mathbb{R}}$ into itself if α is an infinite limit ordinal. For $g \in \mathbb{L}^{>\mathbb{R}}$ and $\lambda := \min \sigma(\mathfrak{d}g)$ we have $\mathfrak{d}(\log g) = \ell_{\lambda+1}$, and an easy induction on n gives

$$\log_n(g)_{\succ} \ = \ \ell_{\lambda+n} \ \text{ for } n \geqslant 2, \qquad \log_n(g)_{\succcurlyeq} \ = \ \ell_{\lambda+n} \ \text{ for } n \geqslant 3.$$

Here is a useful lemma regarding the functions $\log_n$:

Lemma 4.7. *Let $g \in \mathbb{L}^{>\mathbb{R}}$. Then the family $(\log_n(g))_n$ is multipliable.*

Proof. By the above remarks, we have for $n \geqslant 2$ that $\log_n(g) = \ell_{\lambda+n} + \varepsilon_n$ where $\lambda = \lambda_g$ and $\varepsilon_n \preccurlyeq 1$. Thus, for $\big(\log_n(g)\big)$ to be multipliable, it suffices that $(\varepsilon_n)_{n\geqslant 2}$ is summable. For $n \geqslant 2$, we have

$$\begin{aligned}
\log_{n+1}(g) \ &= \ \log(\ell_{\lambda+n} + \varepsilon_n) \ = \ \log\left(\ell_{\lambda+n}\big(1 + \frac{\varepsilon_n}{\ell_{\lambda+n}}\big)\right) \\
&= \ \ell_{\lambda+n+1} + \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i}\left(\frac{\varepsilon_n}{\ell_{\lambda+n}}\right)^i, \quad \text{so} \\
\varepsilon_{n+1} \ &= \ \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i}\left(\frac{\varepsilon_n}{\ell_{\lambda+n}}\right)^i.
\end{aligned}$$

Using this equality for ε_{n+1}, a straightforward induction on n shows that every $\mathfrak{m} \in \operatorname{supp} \varepsilon_n$ with $n \geqslant 2$ is of the form

$$\mathfrak{n}_1 \cdots \mathfrak{n}_{d_1} \ell_{\lambda+2}^{-d_2} \cdots \ell_{\lambda+n-1}^{-d_{n-1}} \ \in (\operatorname{supp} \varepsilon_2)^\infty \cdot \mathfrak{S}_n,$$

where $\mathfrak{n}_1, \dots, \mathfrak{n}_{d_1} \in \operatorname{supp} \varepsilon_2$, $d_1, \dots, d_{n-1} \in \mathbb{N}^{\geqslant 1}$, $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_{n-1}$, and

$$\mathfrak{S}_n \ := \ \left\{ \prod_{2\leqslant j<n} \ell_{\lambda+j}^{-d_j} : \ d_2, \dots, d_{n-1} \in \mathbb{N}^{\geqslant 1}, \ d_2 \geqslant d_3 \geqslant \cdots \geqslant d_{n-1} \right\}$$

The set $(\operatorname{supp} \varepsilon_2)^\infty$ is well-based by Neumann's Lemma. The (disjoint) union $\mathfrak{S} := \bigcup_{n\geqslant 2} \mathfrak{S}_n$ is well-based by Lemma 2.16. Thus the family $(\varepsilon_n)_{n\geqslant 2}$ is summable. □

Lemma 4.8. *Let $\circ$ be a composition on $\mathbb{L}$ as defined in the Introduction and let $f, g \in \mathbb{L}$, $f > 0$, $g > \mathbb{R}$. Then $f^t \circ g = (f \circ g)^t$ for $t \in \mathbb{R}$. If in addition the family (f_i) in $\mathbb{L}^>$ is multipliable, then the family $(f_i \circ g)$ is multipliable, and*

$$\Big(\prod_i f_i\Big) \circ g \;=\; \prod_i (f_i \circ g).$$

Proof. By (CL1) and (CL3) we have

$$\log(f^t \circ g) \;=\; (\log f^t) \circ g \;=\; (t \log f) \circ g \;=\; t \log(f \circ g) \;=\; \log[(f \circ g)^t],$$

so $f^t \circ g = (f \circ g)^t$. The second part follows likewise by taking logarithms. □

5. Composing with Hyperlogarithms

Recall that our goal is to construct a 'good' composition operation on $\mathbb{L}$. In this section we only compose on the right with hyperlogarithms.

We fix an ordinal $\alpha = \omega^\lambda$ where λ is an infinite limit ordinal. Then $\xi + \eta < \alpha$ for all ordinals $\xi, \eta < \alpha$. We work in the Hahn field $\mathbb{L}_{<\alpha} = \mathbb{R}[[\mathfrak{L}_{<\alpha}]]$ over $\mathbb{R}$. Let $\beta < \lambda$ be given and set $\mu := \omega^{\beta+1} < \alpha$, so $\omega^\beta + \gamma < \mu$ for $\gamma < \mu$.

Composing with ℓ_{ω^β}. We shall use the modified derivation $\mathfrak{d} := \frac{1}{\ell'_\mu}\partial_\alpha$ on $\mathbb{L}_{<\alpha}$. Note that $\frac{1}{\ell'_\mu} = \prod_{\rho<\mu} \ell_\rho$. Hence for $\mathfrak{m} \in \mathfrak{L}_{[\mu,\alpha)}$ we have

$$\operatorname{supp}(\mathfrak{d}\mathfrak{m}) \;\subseteq\; \Big\{ \prod_{\mu \leqslant \rho \leqslant \gamma} \ell_\rho^{-1} :\; \mu \leqslant \gamma < \alpha \Big\} \cdot \mathfrak{m}.$$

Thus the strongly $\mathbb{R}$-linear operator $\mathfrak{d}$ on $\mathbb{L}_{<\alpha}$ maps $\mathbb{L}_{[\mu,\alpha)}$ into itself.

To explain and motivate the role of $\mathfrak{d}$ in defining $f \circ \ell_{\omega^\beta}$ for $f \in \mathbb{L}_{[\mu,\alpha)}$ we include the following remark; it is important for understanding what is going on, but is of a purely heuristic nature and can be skipped.

Remark. The composition $\circ$ on $\mathbb{L}$ to be constructed will be such that the map $f \mapsto f \circ \ell_\mu : \mathbb{L}_{[\mu,\alpha)} \to \mathbb{L}_{[\mu,\alpha)}$ is bijective, which gives an inverse map

$$f \mapsto f^{\uparrow_\mu} \;:\; \mathbb{L}_{[\mu,\alpha)} \to \mathbb{L}_{[\mu,\alpha)}, \qquad f^{\uparrow_\mu} \circ \ell_\mu = f \;\text{ for }\; f \in \mathbb{L}_{[\mu,\alpha)}.$$

We also want $\circ$ to obey the Chain Rule and admit Taylor expansion, and to satisfy $\ell_\mu \circ \ell_{\omega^\beta} = \ell_\mu - 1$. Then for $f \in \mathbb{L}_{[\mu,\alpha)}$ we have $(f^{\uparrow_\mu} \circ \ell_\mu)' = \big((f^{\uparrow_\mu})' \circ \ell_\mu\big) \cdot \ell'_\mu = f'$, so $(f^{\uparrow_\mu})' \circ \ell_\mu = f'/\ell'_\mu = \mathfrak{d}(f)$, and thus by induction on n,

$$(f^{\uparrow_\mu})^{(n)} \circ \ell_\mu \;=\; \mathfrak{d}^n(f).$$

Using Taylor expansion this leads for such f to

$$\begin{aligned} f \circ \ell_{\omega^\beta} \;&=\; (f^{\uparrow_\mu} \circ \ell_\mu) \circ \ell_{\omega^\beta} \;=\; f^{\uparrow_\mu} \circ (\ell_\mu \circ \ell_{\omega^\beta}) \;=\; f^{\uparrow_\mu} \circ (\ell_\mu - 1) \\ &=\; \sum_{n=0}^{\infty} \frac{(f^{\uparrow_\mu})^{(n)} \circ \ell_\mu}{n!}(-1)^n \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\mathfrak{d}^n(f). \end{aligned}$$

After this remark we now resume the formal exposition. The restriction of $\mathfrak{d}$ to an operator on $\mathbb{L}_{[\mu,\alpha)}$ has support contained in the well-based set

$$\mathfrak{S} \;:=\; \Big\{ \prod_{\mu \leqslant \rho \leqslant \gamma} \ell_\rho^{-1} :\; \mu \leqslant \gamma < \alpha \Big\} \;\prec\; 1.$$

Thus by Lemma 2.10 and the remark following Lemma 2.11 the sum $\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\delta^n f$ exists in $\mathbb{L}_{[\mu,\alpha)}$ for all $f\in\mathbb{L}_{[\mu,\alpha)}$, and we have a strongly $\mathbb{R}$-linear operator

$$f\mapsto f\circ\ell_{\omega^\beta}\ :\ \mathbb{L}_{[\mu,\alpha)}\to\mathbb{L}_{[\mu,\alpha)},\qquad f\circ\ell_{\omega^\beta}\ :=\ \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\delta^n f, \tag{5.1}$$

with $\operatorname{supp}(f\circ\ell_{\omega^\beta})\subseteq\mathfrak{S}^\infty\cdot\operatorname{supp}f$ for $f\in\mathbb{L}_{[\mu,\alpha)}$. A routine computation gives

$$(fg)\circ\ell_{\omega^\beta}\ =\ (f\circ\ell_{\omega^\beta})\cdot(g\circ\ell_{\omega^\beta})\qquad (f,g\in\mathbb{L}_{[\mu,\alpha)}). \tag{5.2}$$

Lemma 5.1. *We have $f\sim f\circ\ell_{\omega^\beta}$ for nonzero $f\in\mathbb{L}_{[\mu,\alpha)}$, and the map $f\mapsto f\circ\ell_{\omega^\beta}$ is an automorphism of the field $\mathbb{L}_{[\mu,\alpha)}$.*

Proof. Let f range over $\mathbb{L}_{[\mu,\alpha)}$ and set $\Phi(f)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\delta^n f$. Then the map $\Phi:\mathbb{L}_{[\mu,\alpha)}\to\mathbb{L}_{[\mu,\alpha)}$ is $\mathbb{R}$-linear with well-based support $\operatorname{supp}\Phi\subseteq\bigcup_{n=1}^{\infty}\mathfrak{S}^n\prec 1$, and thus $\Phi(f)\prec f$ if $f\neq 0$. Now $f\circ\ell_{\omega^\beta}=f+\Phi(f)$, so $f\sim f\circ\ell_{\omega^\beta}$ for $f\neq 0$, and $f\mapsto f\circ\ell_{\omega^\beta}:\mathbb{L}_{[\mu,\alpha)}\to\mathbb{L}_{[\mu,\alpha)}$ is bijective by Lemma 2.2. □

Remark. It is natural to denote the operator

$$f\ \mapsto\ f\circ\ell_{\omega^\beta}\ =\ \left[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\delta^n\right](f)$$

on $\mathbb{L}_{[\mu,\alpha)}$ by $\mathrm{e}^{-\delta}$. More generally, any $s\in\mathbb{R}$ yields an operator

$$\mathrm{e}^{s\delta}\ :\ \mathbb{L}_{[\mu,\alpha)}\to\mathbb{L}_{[\mu,\alpha)},\quad f\mapsto\sum_{n=0}^{\infty}\frac{s^n}{n!}\delta^n f,$$

and $\mathrm{e}^{s\delta}\circ\mathrm{e}^{t\delta}=\mathrm{e}^{(s+t)\delta}$ for $s,t\in\mathbb{R}$, so we have a group $\mathrm{e}^{\mathbb{R}\delta}$ of such operators.

Next we define for a monomial $\prod_{\gamma<\mu}\ell_\gamma^{r_\gamma}\in\mathfrak{L}_{<\mu}$,

$$\left(\prod_{\gamma<\mu}\ell_\gamma^{r_\gamma}\right)\circ\ell_{\omega^\beta}\ :=\ \prod_{\gamma<\mu}\ell_{\omega^\beta+\gamma}^{r_\gamma}\in\mathfrak{L}_{<\mu}. \tag{5.3}$$

Note that $\mathfrak{m}\mapsto\mathfrak{m}\circ\ell_{\omega^\beta}:\mathfrak{L}_{<\mu}\to\mathfrak{L}_{<\mu}$ is an embedding of ordered groups, and that this map is contractive: $\mathfrak{m}\circ\ell_{\omega^\beta}\succ\mathfrak{m}$ if $\mathfrak{m}\in\mathfrak{L}_{<\mu}^{\prec 1}$, and $\mathfrak{m}\circ\ell_{\omega^\beta}\prec\mathfrak{m}$ if $\mathfrak{m}\in\mathfrak{L}_{<\mu}^{\succ 1}$.

Finally, using $\mathbb{L}_{<\alpha}=\mathbb{L}_{[\mu,\alpha)}[[\mathfrak{L}_{<\mu}]]$ and representing $f\in\mathbb{L}_{<\alpha}$ as

$$f\ =\ \sum_{\mathfrak{m}\in\mathfrak{L}_{<\mu}}f_{[\mathfrak{m}]}\mathfrak{m}$$

where all $f_{[\mathfrak{m}]}\in\mathbb{L}_{[\mu,\alpha)}$ and $\{\mathfrak{m}\in\mathfrak{L}_{<\mu}:f_{\mathfrak{m}}\neq 0\}$ is well-based, we note that

$$\sum_{\mathfrak{m}\in\mathfrak{L}_{<\mu}}\left(f_{[\mathfrak{m}]}\circ\ell_{\omega^\beta}\right)(\mathfrak{m}\circ\ell_{\omega^\beta})$$

exists in the Hahn field $\mathbb{L}_{[\mu,\alpha)}[[\mathfrak{L}_{<\mu}]]$ over $\mathbb{L}_{[\mu,\alpha)}$, since all $f_{[\mathfrak{m}]}\circ\ell_{\omega^\beta}\in\mathbb{L}_{[\mu,\alpha)}$ and $\mathfrak{m}\circ\ell_{\omega^\beta}\prec\mathfrak{n}\circ\ell_{\omega^\beta}$ for all $\mathfrak{m}\prec\mathfrak{n}$ in $\mathfrak{L}_{<\mu}$. Thus we may define the operation

$$f\mapsto f\circ\ell_{\omega^\beta}\ :\ \mathbb{L}_{<\alpha}\to\mathbb{L}_{<\alpha},\qquad f\circ\ell_{\omega^\beta}\ :=\ \sum_{\mathfrak{m}\in\mathfrak{L}_{<\mu}}\left(f_{[\mathfrak{m}]}\circ\ell_{\omega^\beta}\right)(\mathfrak{m}\circ\ell_{\omega^\beta}). \tag{5.4}$$

We do not create here a conflict of notation: if $f\in\mathbb{L}_{[\mu,\alpha)}$ or $f\in\mathfrak{L}_{<\mu}$, then this agrees with the previously defined $f\circ\ell_{\omega^\beta}$. In particular, $\ell_0\circ\ell_{\omega^\beta}=\ell_{\omega^\beta}$.

Lemma 5.2. *The map* $f \mapsto f \circ \ell_{\omega^\beta} : \mathbb{L}_{<\alpha} \to \mathbb{L}_{<\alpha}$ *is an* $\mathbb{L}_{<\alpha}$*-composition with* ℓ_{ω^β}*, where we take* $x := \ell_0$ *as the distinguished element of* $\mathfrak{L}_{<\alpha}^{\succ 1}$*.*

Proof. To see that clause (2) in the definition of "K-composition with h" holds, let $\mathfrak{m}, \mathfrak{n} \in \mathfrak{L}_{<\alpha}$. Then $\mathfrak{m} = \mathfrak{m}_{<\mu}\mathfrak{m}_{\geqslant\mu}$, $\mathfrak{n} = \mathfrak{n}_{<\mu}\mathfrak{n}_{\geqslant\mu}$, $\mathfrak{m}_{<\mu}, \mathfrak{n}_{<\mu} \in \mathfrak{L}_{<\mu}$, $\mathfrak{m}_{\geqslant\mu}, \mathfrak{n}_{\geqslant\mu} \in \mathfrak{L}_{[\mu,\alpha)}$. Using (5.3), $(\mathfrak{m}_{<\mu}\mathfrak{n}_{<\mu}) \circ \ell_{\omega^\beta} = (\mathfrak{m}_{<\mu} \circ \ell_{\omega^\beta})(\mathfrak{n}_{<\mu} \circ \ell_{\omega^\beta})$, and by (5.2),

$$(\mathfrak{m}_{\geqslant\mu}\mathfrak{n}_{\geqslant\mu}) \circ \ell_{\omega^\beta} \;=\; (\mathfrak{m}_{\geqslant\mu} \circ \ell_{\omega^\beta})(\mathfrak{n}_{\geqslant\mu} \circ \ell_{\omega^\beta}).$$

Now using also (5.4), we have

$$(\mathfrak{m}\mathfrak{n}) \circ \ell_{\omega^\beta} \;=\; \big((\mathfrak{m}_{<\mu}\mathfrak{n}_{<\mu}) \circ \ell_{\omega^\beta}\big) \cdot \big((\mathfrak{m}_{\geqslant\mu}\mathfrak{n}_{\geqslant\mu}) \circ \ell_{\omega^\beta}\big) \;=\; (\mathfrak{m} \circ \ell_{\omega^\beta})(\mathfrak{n} \circ \ell_{\omega^\beta}).$$

That clause (3) is satisfied follows easily from the strong linearity of the map in (5.1) and the existence of the sums in (5.4). □

Corollary 5.3. *Let* $0 \neq f \in \mathbb{L}_{<\alpha}^{\preccurlyeq 1}$*. Then* $f \circ \ell_{\omega^\beta} \prec \ell_\mu^{-n}$ *for all* n*, or* $f \circ \ell_{\omega^\beta} \sim f$*.*

Proof. With $f = \sum_{\mathfrak{m} \in \mathfrak{L}_{<\mu}} f_{\mathfrak{m}}\mathfrak{m}$ as above, set $\mathfrak{d} := \max\{\mathfrak{m} \in \mathfrak{L}_{<\mu} : \; f_{\mathfrak{m}} \neq 0\}$. Then either $\mathfrak{d} \prec 1$ and $f \circ \ell_{\omega^\beta} \prec \ell_\mu^{-n}$ for all n, or $\mathfrak{d} = 1$ and $f \circ \ell_{\omega^\beta} \sim f$ by Lemma 5.1. □

Lemma 5.4. *Let* $\nu \leqslant \mu$*. Then* $\mathbb{L}_{[\nu,\alpha)} \circ \ell_{\omega^\beta} = \mathbb{L}_{[\omega^\beta+\nu,\alpha)}$*.*

Proof. For $\nu = \mu$ this follows from Lemma 5.1 and $\omega^\beta + \mu = \mu$. Let $\nu < \mu$. Then $\mathbb{L}_{[\nu,\alpha)} = \mathbb{L}_{[\mu,\alpha)}[[\mathfrak{L}_{[\nu,\mu)}]]$. Accordingly, for $f \in \mathbb{L}_{[\nu,\alpha)}$ we have $f = \sum_{\mathfrak{m}} f_{[\mathfrak{m}]}\mathfrak{m}$ with $\mathfrak{m}$ ranging over $\mathfrak{L}_{[\nu,\mu)}$ and all $f_{[\mathfrak{m}]} \in \mathbb{L}_{[\mu,\alpha)}$. Then

$$f \circ \ell_{\omega^\beta} \;=\; \sum_{\mathfrak{m}} (f_{[\mathfrak{m}]} \circ \ell_{\omega^\beta})(\mathfrak{m} \circ \ell_{\omega^\beta}).$$

Now $\mathbb{L}_{[\mu,\alpha)} \circ \ell_{\omega^\beta} = \mathbb{L}_{[\mu,\alpha)}$, and $\mathfrak{L}_{[\nu,\mu)} \circ \ell_{\omega^\beta} = \mathfrak{L}_{[\omega^\beta+\nu,\omega^\beta+\mu)} = \mathfrak{L}_{[\omega^\beta+\nu,\mu)}$ by (5.3). It remains to note that $\mathbb{L}_{[\omega^\beta+\nu,\alpha)} = \mathbb{L}_{[\mu,\alpha)}[[\mathfrak{L}_{[\omega^\beta+\nu,\mu)}]]$. □

For $\nu = 0$ this gives $\mathbb{L}_{<\alpha} \circ \ell_{\omega^\beta} = \mathbb{L}_{[\omega^\beta,\alpha)}$.

Lemma 5.5. $(f \circ \ell_{\omega^\beta})' = (f' \circ \ell_{\omega^\beta}) \cdot \ell'_{\omega^\beta}$ *for* $f \in \mathbb{L}_{<\alpha}$*.*

Proof. For a monomial $\mathfrak{m} = \prod_{\gamma<\mu} \ell_\gamma^{r_\gamma} \in \mathfrak{L}_{<\mu}$, we have by (5.3):

$$\begin{aligned}
(\mathfrak{m} \circ \ell_{\omega^\beta})' &= \left(\prod_{\gamma<\mu} \ell_{\omega^\beta+\gamma}^{r_\gamma}\right)' = (\mathfrak{m} \circ \ell_{\omega^\beta}) \sum_{\gamma<\mu} r_\gamma \ell_{\omega^\beta+\gamma}^\dagger \\
&= (\mathfrak{m} \circ \ell_{\omega^\beta}) \sum_{\gamma<\mu} r_\gamma \left(\prod_{\rho\leqslant\omega^\beta+\gamma} \ell_\rho^{-1}\right) \\
&= (\mathfrak{m} \circ \ell_{\omega^\beta}) \sum_{\gamma<\mu} r_\gamma \left(\prod_{\rho<\omega^\beta} \ell_\rho^{-1}\right) \left(\prod_{\rho\leqslant\gamma} \ell_{\omega^\beta+\rho}^{-1}\right) \\
&= \left(\prod_{\rho<\omega^\beta} \ell_\rho^{-1}\right) (\mathfrak{m} \circ \ell_{\omega^\beta}) \sum_{\gamma<\mu} r_\gamma \left(\prod_{\rho\leqslant\gamma} \ell_\rho^{-1}\right) \circ \ell_{\omega^\beta} \\
&= \left(\prod_{\rho<\omega^\beta} \ell_\rho^{-1}\right) \sum_{\gamma<\mu} r_\gamma (\mathfrak{m} \circ \ell_{\omega^\beta})(\ell_\gamma^\dagger \circ \ell_{\omega^\beta}) \;=\; \ell'_{\omega^\beta}\,(\mathfrak{m}' \circ \ell_{\omega^\beta})\,.
\end{aligned}$$

For $g \in \mathbb{L}_{\geqslant\mu,<\alpha}$ we have $g \circ \ell_{\omega^\beta} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(\delta^n g)$ by (5.1), so

$$\begin{aligned}(g \circ \ell_{\omega^\beta})' &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(\delta^n g)' = \ell_\mu' \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(\delta^{n+1} g) \\ &= \ell_\mu' \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left(\delta^n\left(\frac{g'}{\ell_\mu'}\right)\right) = \ell_\mu'\left(\left(\frac{g'}{\ell_\mu'}\right)\circ \ell_{\omega^\beta}\right) \\ &= \left(\prod_{\rho<\mu} \ell_\rho^{-1}\right)\left(\prod_{\rho<\mu} \ell_{\omega^\beta+\rho}\right)(g' \circ \ell_{\omega^\beta}) = \left(\prod_{\rho<\omega^\beta} \ell_\rho^{-1}\right)(g'\circ \ell_{\omega^\beta}) \\ &= \ell_{\omega^\beta}' \cdot (g' \circ \ell_{\omega^\beta}).\end{aligned}$$

Finally, for $f \in \mathbb{L}_{<\alpha}$ we have $f = \sum_{\mathfrak{m}\in\mathfrak{L}_{<\mu}} f_\mathfrak{m}\mathfrak{m}$ where all $f_\mathfrak{m} \in \mathbb{L}_{\geqslant\mu,<\alpha}$, so

$$\begin{aligned}(f \circ \ell_{\omega^\beta})' &= \sum_{\mathfrak{m}\in\mathfrak{L}_{<\mu}} \big((f_\mathfrak{m} \circ \ell_{\omega^\beta})(\mathfrak{m}\circ \ell_{\omega^\beta})\big)' \\ &= \sum_{\mathfrak{m}\in\mathfrak{L}_{<\mu}} (f_\mathfrak{m} \circ \ell_{\omega^\beta})'(\mathfrak{m}\circ \ell_{\omega^\beta}) + (f_\mathfrak{m} \circ \ell_{\omega^\beta})(\mathfrak{m}\circ \ell_{\omega^\beta})' \\ &= \ell_{\omega^\beta}' \cdot \sum_{\mathfrak{m}\in\mathfrak{L}_{<\mu}} (f_\mathfrak{m}' \circ \ell_{\omega^\beta})(\mathfrak{m}\circ \ell_{\omega^\beta}) + (f_\mathfrak{m} \circ \ell_{\omega^\beta})(\mathfrak{m}'\circ \ell_{\omega^\beta}) \\ &= \ell_{\omega^\beta}' \cdot \sum_{\mathfrak{m}\in\mathfrak{L}_{<\mu}} \big((f_\mathfrak{m}\mathfrak{m})' \circ \ell_{\omega^\beta}\big) = \ell_{\omega^\beta}' \cdot (f' \circ \ell_{\omega^\beta}).\end{aligned}$$ □

Note that by (5.3) we have $\ell_\gamma \circ \ell_{\omega^\beta} = \ell_{\omega^\beta+\gamma}$ for $\gamma < \mu$. The next lemma gives more information about $\ell_\gamma \circ \ell_{\omega^\beta}$ for $\gamma \geqslant \mu$.

Lemma 5.6. *We have $\ell_\mu \circ \ell_{\omega^\beta} = \ell_\mu - 1$. If $\mu < \gamma < \alpha$, then*

$$\ell_\gamma \circ \ell_{\omega^\beta} = \ell_\gamma - \varepsilon_\gamma, \qquad \text{with } 0 < \varepsilon_\gamma \preccurlyeq \ell_\mu^{-1} \prec \ell_\gamma^{-1} \prec 1.$$

Proof. Let $\mu \leqslant \gamma < \alpha$. Then $\ell_\gamma \in \mathfrak{L}_{[\mu,\alpha)}$ and $\delta\ell_\gamma = \prod_{\mu\leqslant\rho<\gamma} \ell_\rho^{-1}$, so $\delta\ell_\mu = 1$ and $\delta\ell_\gamma \preccurlyeq \ell_\mu^{-1} \prec 1$ if $\gamma > \mu$. The derivation δ on $\mathbb{L}_{[\mu,\alpha)}$ has support $\preccurlyeq \ell_\mu^{-1}$, so $\delta^n\ell_\gamma \preccurlyeq \ell_\mu^{-2}$ if $n \geqslant 2$ and $\gamma > \mu$. Therefore $\ell_\gamma \circ \ell_{\omega^\beta} = \ell_\gamma - \delta\ell_\gamma + \frac{1}{2}\delta^2\ell_\gamma - \cdots$ is as described in the lemma. □

Composing with arbitrary hyperlogarithms. *In this subsection we assume that $\gamma < \alpha$.* We have

$$\gamma = \omega^{\beta_1} + \omega^{\beta_2} + \cdots + \omega^{\beta_k} \qquad (k \in \mathbb{N}) \tag{5.5}$$

where $\lambda > \beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant \beta_k$; this is essentially the Cantor normal form of γ, but we allow the exponents to be repeated and require all coefficients to be 1. For $f \in \mathbb{L}_{<\alpha}$, we set

$$f \circ \ell_\gamma := \Big(\big(\cdots\big((f \circ \ell_{\omega^{\beta_k}}) \circ \ell_{\omega^{\beta_{k-1}}}\big)\circ\cdots\big)\circ \ell_{\omega^{\beta_2}}\Big)\circ \ell_{\omega^{\beta_1}}. \tag{5.6}$$

For $\gamma = 0$ (so $k = 0$), this means $f \circ \ell_0 := f$, by convention. Several of the results below are proved by induction on the length k of the representation in (5.5), using also $\ell_\nu \circ \ell_{\omega^\beta} = \ell_{\omega^\beta+\nu}$ for $\nu < \omega^{\beta+1}$, which holds by definition according to (5.3). For example, recalling that $x := \ell_0$, such an induction easily gives $x \circ \ell_\gamma = \ell_\gamma$. As a consequence of Lemma 5.2 we obtain in this way:

Corollary 5.7. *The map $f \mapsto f \circ \ell_\gamma : \mathbb{L}_{<\alpha} \to \mathbb{L}_{<\alpha}$ is an $\mathbb{L}_{<\alpha}$-composition with ℓ_γ.*

We use Lemma 5.5 likewise to obtain:

Corollary 5.8. $(f\circ\ell_\gamma)' = (f'\circ\ell_\gamma)\cdot\ell_\gamma'$ *for* $f\in\mathbb{L}_{<\alpha}$.

Corollary 5.9. *Suppose* $\gamma\neq 0$ *and* $\nu<\omega^{\beta_k+1}$. *Then*

$$\ell_\nu\circ\ell_\gamma \;=\; \ell_{\gamma+\nu}, \qquad (f\circ\ell_\nu)\circ\ell_\gamma \;=\; f\circ\ell_{\gamma+\nu} \;\text{ for }\; f\in\mathbb{L}_{<\alpha}.$$

For later use we also record the following variants:

Lemma 5.10. *Let* $\gamma\neq 0$ *and* $\nu\leqslant\omega^{\beta_k+1}$. *Then for* $\mathfrak{m}=\prod_{\rho<\nu}\ell_\rho^{r_\rho}\in\mathfrak{L}_{<\nu}$ *we have* $\mathfrak{m}\circ\ell_\gamma=\prod_{\rho<\nu}\ell_{\gamma+\rho}^{r_\rho}\in\mathfrak{L}_{[\gamma,\gamma+\nu)}$, *and so the map*

$$\mathfrak{m}\mapsto\mathfrak{m}\circ\ell_\gamma \;:\; \mathfrak{L}_{<\nu}\to\mathfrak{L}_{[\gamma,\gamma+\nu)}$$

is an isomorphism of ordered groups. In particular, we have an isomorphism

$$\mathfrak{m}\mapsto\mathfrak{m}\circ\ell_\gamma \;:\; \mathfrak{L}_{<\omega}\to\mathfrak{L}_{[\gamma,\gamma+\omega)}$$

of ordered groups.

Lemma 5.11. *Let* $\gamma\neq 0$, $\nu\leqslant\omega^{\beta_k+1}$. *Then* $\mathbb{L}_{[\nu,\alpha)}\circ\ell_\gamma=\mathbb{L}_{[\gamma+\nu,\alpha)}$.

Proof. By induction on $k\geqslant 1$. The case $k=1$ is Lemma 5.4. The inductive step from $k-1$ to k uses that for $k>1$ we have $\omega^{\beta_k}+\nu\leqslant\omega^{\beta_{k-1}+1}$. □

The proof of associativity in Section 7 will depend on the next two lemmas. In the first one we assume $\beta<\lambda$ and set $\mu:=\omega^{\beta+1}$.

Lemma 5.12. *Let* $n\geqslant 1$. *Then* $\ell_\gamma\circ\ell_{\omega^\beta n}$ *takes the following values:*

$$\ell_{\omega^\beta n+\gamma} \text{ for } \gamma<\mu, \quad \ell_\mu - n \text{ for } \gamma=\mu, \quad \ell_\gamma-\varepsilon \text{ with } 0<\varepsilon\preccurlyeq\ell_\mu^{-1} \text{ for } \gamma>\mu.$$

Proof. For $n=1$ this is Lemma 5.6. Assuming inductively that the lemma holds for a certain n we use

$$\ell_\gamma\circ\ell_{\omega^\beta(n+1)} \;=\; (\ell_\gamma\circ\ell_{\omega^\beta n})\circ\ell_{\omega^\beta}$$

and Corollary 5.3 to show it holds for $n+1$ instead of n. □

Let $g\in\mathbb{L}^{>\mathbb{R}}$. Set $\lambda_g:=\min\sigma(\mathfrak{d}g)$ (an ordinal) and call it the *logarithmicity of* g. Thus $\lambda_{\ell_\nu}=\nu$ for any ordinal ν. Consider the Cantor normal form of λ_g:

$$\lambda_g \;=\; \omega^{\beta_1}n_1+\cdots+\omega^{\beta_k}n_k \qquad (k\in\mathbb{N},\ \beta_1>\cdots>\beta_k,\ n_1,\dots,n_k\in\mathbb{N}^{\geqslant 1}).$$

For any ordinal ν we set

$$\lambda_{g;\nu} \;:=\; n_i \;\text{ if }\; \nu=\omega^{\beta_i+1}, \qquad \lambda_{g;\nu} \;:=\; 0 \;\text{ if }\; \nu\notin\{\omega^{\beta_1+1},\dots,\omega^{\beta_k+1}\}.$$

Lemma 5.13. *Let* $\nu<\alpha$. *Then* $\ell_\nu\circ\ell_\gamma=\ell_{\gamma+\nu}-\lambda_{\ell_\gamma;\nu}-\varepsilon$ *with* $0\leqslant\varepsilon\prec 1$.

Proof. This is clear for $\gamma=0$. Assume $\gamma>0$ has Cantor normal form

$$\gamma \;=\; \omega^{\beta_1}n_1+\cdots+\omega^{\beta_k}n_k \qquad (\beta_1>\cdots>\beta_k,\ k,n_1,\dots,n_k\geqslant 1).$$

We first show by induction on k:

(1) if $\nu=\omega^{\beta_1+1}$, then $\ell_\nu\circ\ell_\gamma=\ell_\nu-n_1-\varepsilon$ with $0\leqslant\varepsilon\prec 1$;
(2) if $\nu>\omega^{\beta_1+1}$, then $\ell_\nu\circ\ell_\gamma=\ell_\nu-\varepsilon$ with $0\leqslant\varepsilon\prec 1$.

The previous lemma gives this for $k=1$. Assume it holds for a certain γ as above. Then with $\beta_k > \beta_{k+1}$ and $n_{k+1} \geqslant 1$ the definitions easily yield

$$\ell_\nu \circ \ell_{\gamma+\omega^{\beta_{k+1}} n_{k+1}} \;=\; (\ell_\nu \circ \ell_{\omega^{\beta_{k+1}} n_{k+1}}) \circ \ell_\gamma.$$

Let $\nu \geqslant \omega^{\beta_1+1}$. Then $\ell_\nu \circ \ell_{\omega^{\beta_{k+1}} n_{k+1}} = \ell_\nu - \varepsilon$ with $0 < \varepsilon \prec 1$ by the previous lemma. If $\nu = \omega^{\beta_1+1}$, we get $\ell_\nu \circ \ell_{\gamma+\omega^{\beta_{k+1}} n_{k+1}} = (\ell_\nu - \varepsilon) \circ \ell_\gamma = \ell_\nu - n_1 - \varepsilon^*$ with $0 < \varepsilon^* \prec 1$ by (1). If $\nu > \omega^{\beta_1+1}$, then we get $\ell_\nu \circ \ell_{\gamma+\omega^{\beta_{k+1}} n_{k+1}} = (\ell_\nu - \varepsilon) \circ \ell_\gamma = \ell_\nu - \varepsilon^*$ with $0 < \varepsilon^* \prec 1$ by (2). This concludes the proof of (1) and (2) and shows that the lemma holds for $\nu \geqslant \omega^{\beta_1+1}$.

Now assume that $\nu < \omega^{\beta_1+1}$. We first consider the subcase that $\nu = \omega^{\beta_i+1}$ where $1 < i \leqslant k$. Then $\gamma = \gamma_1 + \gamma_2$ with $\gamma_1 = \omega^{\beta_1} n_1 + \cdots + \omega^{\beta_{i-1}} n_{i-1}$ and $\gamma_2 = \omega^{\beta_i} n_i + \cdots + \omega^{\beta_k} n_k$, hence $\ell_\nu \circ \ell_\gamma = (\ell_\nu \circ \ell_{\gamma_2}) \circ \ell_{\gamma_1}$. By (1) above with γ_2 in the role of γ we have $\ell_\nu \circ \ell_{\gamma_2} = \ell_\nu - n_i - \varepsilon$ with $0 \leqslant \varepsilon \prec 1$, so

$$\ell_\nu \circ \ell_\gamma \;=\; (\ell_\nu - n_i - \varepsilon) \circ \ell_{\gamma_1} \;=\; \ell_\nu \circ \ell_{\gamma_1} - n_i - \varepsilon^*, \qquad 0 \leqslant \varepsilon^* \prec 1,$$

and $\ell_\nu \circ \ell_{\gamma_1} = \ell_{\gamma_1+\nu}$ by Corollary 5.9. Now $\gamma_2 + \nu = \nu$, so $\gamma + \nu = \gamma_1 + \nu$, and thus $\ell_\nu \circ \ell_\gamma = \ell_{\gamma+\nu} - n_i - \varepsilon^* = \ell_{\gamma+\nu} - \lambda_{\ell_\gamma;\nu} - \varepsilon^*$, so the lemma holds in this case. Next assume we are in the subcase $\omega^{\beta_{i-1}+1} > \nu > \omega^{\beta_i+1}$ where $1 < i \leqslant k$. With $\gamma = \gamma_1 + \gamma_2$ as before we argue as in the previous subcase, using (2) with γ_2 in the role of γ, and obtain that the lemma holds in this case as well. The remaining subcase $\nu < \omega^{\beta_k+1}$ is taken care of by Corollary 5.9. □

6. Composition with Arbitrary Elements

As in the previous section, $\alpha = \omega^\lambda$, where λ is an infinite limit ordinal. We now fix $g \in \mathbb{L}^{>\mathbb{R}}_{<\alpha}$. We shall define $\mathfrak{m} \circ g$ for $\mathfrak{m} \in \mathfrak{L}_{<\omega}$ and $f \circ g$ for $f \in \mathbb{L}_{[\omega,\alpha)}$ and then use this to define the map $f \mapsto f \circ g : \mathbb{L}_{<\alpha} \to \mathbb{L}_{<\alpha}$.

Note that for $\mathfrak{m} = \prod_n \ell_n^{r_n} \in \mathfrak{L}_{<\omega}$ we have $\log_n(g)^{r_n} \in \mathbb{L}_{<\alpha}$ for all n and that the family $\big(\log_n(g)^{r_n}\big)$ is multipliable by Lemma 4.7. Therefore we may define

$$\mathfrak{m} \circ g \;:=\; \prod_n \log_n(g)^{r_n} \;\in\; \mathbb{L}_{<\alpha}. \tag{6.1}$$

Thus $1 \circ g = 1$ and $\ell_n \circ g = \log_n(g)$. Also for $\mathfrak{m}, \mathfrak{n} \in \mathfrak{L}_{<\omega}$, $t \in \mathbb{R}$,

$$(\mathfrak{m}\mathfrak{n}) \circ g \;=\; (\mathfrak{m} \circ g)(\mathfrak{n} \circ g), \quad \mathfrak{m}^t \circ g \;=\; (\mathfrak{m} \circ g)^t.$$

Lemma 6.1. *There exists a well-based set $\mathfrak{S} = \mathfrak{S}(g) \subseteq \mathfrak{L}_{<\alpha}$ such that for all $\mathfrak{m} = \prod_n \ell_n^{r_n} \in \mathfrak{L}_{<\omega}$ we have*

$$\operatorname{supp}(\mathfrak{m} \circ g) \;\subseteq\; \mathfrak{d}(g)^{r_0} \left(\prod_{n \geqslant 1} \ell_{\lambda_g+n}^{r_n} \right) \cdot \mathfrak{S}.$$

Proof. Set $\mathfrak{S} := \Big(\bigcup_{n\geqslant 1} (\operatorname{supp} \log_n(g))^{\prec 1}\Big)^\infty$. By Lemma 4.7, $\bigcup_{n\geqslant 1} (\operatorname{supp} \log_n(g))^{\prec 1}$ is well-based, and so is $\mathfrak{S}$ by Neumann's Lemma. By (4.1), we have

$$\sum_{n=0}^{\infty} r_n \log_{n+1}(g) \;=\; \sum_{\beta<\alpha} s_\beta \ell_{\beta+1} + c + \varepsilon$$

where the s_β and c are reals and $\varepsilon \prec 1$, so $\operatorname{supp}\varepsilon \subseteq \bigcup_{n\geqslant 1}(\operatorname{supp}\log_n(g))^{\prec 1}$. Now

$$\mathfrak{m}\circ g \;=\; \mathfrak{d}(\mathfrak{m}\circ g)\cdot e^c \cdot \sum_{m=0}^{\infty}\frac{1}{m!}\varepsilon^m,$$

and so $\operatorname{supp}(\mathfrak{m}\circ g) \subseteq \mathfrak{d}(\mathfrak{m}\circ g)\cdot\mathfrak{S}$. It remains to note that by Lemma 4.6,

$$\mathfrak{d}\left(\mathfrak{m}\circ g\right) \;=\; \mathfrak{d}(g)^{r_0}\left(\prod_{n\geqslant 1}\ell_{\lambda_g+n}^{r_n}\right). \qquad \square$$

Let $f \in \mathbb{L}_{[\omega,\alpha)}$; we shall define $f\circ g$ by introducing $f^{\uparrow_3} \in \mathbb{L}_{[\omega,\alpha)}$, to be thought of as $f\circ(\exp\circ\exp\circ\exp)$, and then setting $f\circ g := f^{\uparrow_3}\circ\log_3(g)$, exploiting that $\log_3(g)_{\succcurlyeq}$ is a hyperlogarithm. Lemma 5.1 for $\beta = 0$ gives $f\circ\ell_1 \sim f$ if $f\neq 0$, so

$$f\circ\ell_3 \;=\; ((f\circ\ell_1)\circ\ell_1)\circ\ell_1 \;=\; f + R(f),$$

where $R(f)\prec f$ for $f\neq 0$ and $R(0)=0$. The map $f\mapsto f\circ\ell_1$ is a strongly $\mathbb{R}$-linear field automorphism of $\mathbb{L}_{[\omega,\alpha)}$ by Lemma 5.1, and so is $f\mapsto f\circ\ell_3$, and the latter has inverse $f\mapsto f^{\uparrow_3} := \sum_{n=0}^{\infty}(-1)^n R^n(f)$ by Lemma 2.2. Thus

$$f^{\uparrow_3}\circ\ell_3 \;=\; (f\circ\ell_3)^{\uparrow_3} \;=\; f. \tag{6.2}$$

Now $\log_3(g) = \ell_{\lambda_g+3}+\varepsilon$ where $\varepsilon \preccurlyeq \ell_{\lambda_g+2}^{-1} \prec 1$. By Corollary 5.7 we have the $\mathbb{L}_{<\alpha}$-composition $\phi\mapsto\phi\circ\ell_{\lambda_g+3} : \mathbb{L}_{<\alpha}\to\mathbb{L}_{<\alpha}$ with ℓ_{λ_g+3}. Then Lemma 4.4 yields a deformation of it to an $\mathbb{L}_{<\alpha}$-composition $T_g : \mathbb{L}_{<\alpha}\to\mathbb{L}_{<\alpha}$ with $\log_3(g)$ by

$$T_g(\phi) \;:=\; \sum_{n=0}^{\infty}\frac{\phi^{(n)}\circ\ell_{\lambda_g+3}}{n!}\varepsilon^n. \tag{6.3}$$

Thus $T_g(\ell_0) = \log_3(g)$, and Lemma 4.5 gives for $\phi\in\mathbb{L}_{<\alpha}$,

$$T_g(\phi) \;=\; T_g(\phi')\log_3(g)'. \tag{6.4}$$

Composing T_g with $f\mapsto f^{\uparrow_3}$ yields the strongly $\mathbb{R}$-linear embedding

$$f \;\mapsto\; f\circ g \;:=\; \sum_{n=0}^{\infty}\frac{(f^{\uparrow_3})^{(n)}\circ\ell_{\lambda_g+3}}{n!}\varepsilon^n \;:\; \mathbb{L}_{[\omega,\alpha)}\to\mathbb{L}_{<\alpha} \tag{6.5}$$

of ordered and valued fields. Towards extending this to $\mathbb{L}_{<\alpha}$, let $f\in\mathbb{L}_{<\alpha}$. Using $\mathbb{L}_{<\alpha} = \mathbb{L}_{[\omega,\alpha)}[[\mathfrak{L}_{<\omega}]]$ we have $f = \sum_{\mathfrak{m}\in\mathfrak{L}_{<\omega}} f_{[\mathfrak{m}]}\mathfrak{m}$ where all $f_{[\mathfrak{m}]}\in\mathbb{L}_{[\omega,\alpha)}$ and the set $\{\mathfrak{m}\in\mathfrak{L}_{<\omega} : f_{[\mathfrak{m}]}\neq 0\}$ is well-based. To justify defining

$$f\circ g \;:=\; \sum_{\mathfrak{m}\in\mathfrak{L}_{<\omega}}\left(f_{[\mathfrak{m}]}\circ g\right)(\mathfrak{m}\circ g)$$

we need the following:

Proposition 6.2. *The sum* $\sum_{\mathfrak{m}\in\mathfrak{L}_{<\omega}}\left(f_{[\mathfrak{m}]}\circ g\right)(\mathfrak{m}\circ g)$ *exists.*

Towards establishing this proposition we define "β-summability" and prove some lemmas about it. Let $\beta<\alpha$. Then for $h\in\mathbb{L}_{<\alpha}$ we have $h = \sum_{\mathfrak{n}\in\mathfrak{L}_{<\beta}} h_{[\mathfrak{n}]}\mathfrak{n}$ with all $h_{[\mathfrak{n}]}\in\mathbb{L}_{[\beta,\alpha)}$, and well-based $\operatorname{supp}_\beta(h) := \{\mathfrak{n}\in\mathfrak{L}_{<\beta} :\; h_{[\mathfrak{n}]}\neq 0\}$.

Let $(h_i)_{i\in I}$ be a family of elements in $\mathbb{L}_{<\alpha}$. We say that (h_i) is *β-summable* if $\bigcup_{i\in I}\operatorname{supp}_\beta(h_i)$ is well-based and $\{i\in I : \mathfrak{n}\in\operatorname{supp}_\beta(h_i)\}$ is finite for every $\mathfrak{n}\in\mathfrak{L}_{<\beta}$. If (h_i) is a β-summable, then $\sum_{i\in I} h_i$ exists as an element of the Hahn field $\mathbb{L}_{[\beta,\alpha)}[[\mathfrak{L}_{<\beta}]]$ over $\mathbb{L}_{[\beta,\alpha)}$, and therefore also as an element of the Hahn field $\mathbb{L}_{<\alpha}$

over $\mathbb{R}$ with the same value under the usual identification of $\mathbb{L}_{[\beta,\alpha)}[[\mathfrak{L}_{<\beta}]]$ with $\mathbb{L}_{<\alpha}$. If (h_i) is summable, then $\bigcup_{i\in I}\operatorname{supp}_\beta(h_i)$ is well-based, but (h_i) is not necessarily β-summable. If (h_i) is β-summable, then $(c_i h_i)$ is also β-summable for any family (c_i) from $\mathbb{L}_{[\beta,\alpha)}$. For $\mathfrak{m}\in\mathfrak{L}_{<\alpha}$ we have $\mathfrak{m}=\mathfrak{m}_{<\beta}\mathfrak{m}_{\geqslant\beta}$ with $\mathfrak{m}_{<\beta}\in\mathfrak{L}_{<\beta}$, $\mathfrak{m}_{\geqslant\beta}\in\mathfrak{L}_{[\beta,\alpha)}$, and then $\operatorname{supp}_\beta(\mathfrak{m})=\{\mathfrak{m}_{<\beta}\}$. Here is a consequence of Lemma 6.1:

Corollary 6.3. *Suppose the family $(\mathfrak{m}_i)_{i\in I}$ in $\mathfrak{L}_{<\omega}$ is summable. Then the family $(\mathfrak{m}_i\circ g)$ is $(\lambda_g+\omega)$-summable.*

Proof. Set $\beta:=\lambda_g+\omega$. Lemma 6.1 gives a well-based set $\mathfrak{S}\subseteq\mathfrak{L}_{<\alpha}$ such that for every monomial $\mathfrak{m}=\prod_n \ell_n^{r_n}\in\mathfrak{L}_{<\omega}$, we have

$$\operatorname{supp}(\mathfrak{m}\circ g)\ \subseteq\ \mathfrak{d}(g)^{r_0}\left(\prod_{n\geqslant 1}\ell_{\lambda_g+n}^{r_n}\right)\cdot\mathfrak{S}.$$

Set $\mathfrak{S}_{<\beta}:=\{\mathfrak{n}_{<\beta}:\ \mathfrak{n}\in\mathfrak{S}\}$. This set is still well-based and we have for $\mathfrak{m}\in\mathfrak{L}_{<\omega}$:

$$\operatorname{supp}_\beta(\mathfrak{m}\circ g)\ \subseteq\ \mathfrak{g}^{r_0}\left(\prod_{n\geqslant 1}\ell_{\lambda_g+n}^{r_n}\right)\cdot\mathfrak{S}_{<\beta},\qquad \mathfrak{g}\ :=\ \mathfrak{d}(g)_{<\beta}.$$

Now $\mathfrak{g}=\prod_n \ell_{\lambda_g+n}^{s_n}$ with reals s_n and $s_0>0$, so we have an embedding

$$\Phi\ :\ \mathfrak{L}_{<\omega}\to\mathfrak{L}_{[\lambda_g,\beta)},\qquad \mathfrak{m}=\prod_n \ell_n^{r_n}\ \mapsto\ \mathfrak{g}^{r_0}\prod_{n\geqslant 1}\ell_{\lambda_g+n}^{r_n},$$

of ordered groups. Suppose towards a contradiction that $(\mathfrak{m}_i\circ g)$ is not β-summable. Then we have a sequence (i_n) of distinct indices and an increasing sequence $(\mathfrak{n}_n)$ in $\mathfrak{L}_{<\beta}$ with $\mathfrak{n}_n\in\operatorname{supp}_\beta(\mathfrak{m}_{i_n}\circ g)$ for all n. By passing to a subsequence we arrange that $(\mathfrak{m}_{i_n})$ is strictly decreasing. Now $\mathfrak{n}_n=\Phi(\mathfrak{m}_{i_n})\mathfrak{v}_n$ with $\mathfrak{v}_n\in\mathfrak{S}_{<\beta}$, and Φ is order-preserving, so $(\mathfrak{v}_n)$ is strictly increasing, contradicting that $\mathfrak{S}_{<\beta}$ is well-based. $\square$

Lemma 6.4. *Let $0<\gamma<\alpha$ and $f\in\mathbb{L}_{<\alpha}$. Then*

$$\operatorname{supp}_{\gamma+\omega}(f\circ\ell_\gamma)\ =\ \{\mathfrak{m}\circ\ell_\gamma:\ \mathfrak{m}\in\operatorname{supp}_\omega(f)\}.$$

Proof. We have $f=\sum_{\mathfrak{m}\in\operatorname{supp}_\omega(f)} f_{[\mathfrak{m}]}\mathfrak{m}$ where all $f_{[\mathfrak{m}]}\in\mathbb{L}_{[\omega,\alpha)}$. Then

$$f\circ\ell_\gamma\ =\ \sum_{\mathfrak{m}\in\operatorname{supp}_\omega(f)}(f_{[\mathfrak{m}]}\circ\ell_\gamma)(\mathfrak{m}\circ\ell_\gamma).$$

It remains to note that by Lemmas 5.11 and 5.10 we have $f_{[\mathfrak{m}]}\circ\ell_\gamma\in\mathbb{L}_{[\gamma+\omega,\alpha)}$ and $\mathfrak{m}\circ\ell_\gamma\in\mathfrak{L}_{<\gamma+\omega}$, for all $\mathfrak{m}\in\operatorname{supp}_\omega(f)$. $\square$

Next a result about the map T_g introduced in (6.3). It involves the set

$$\mathfrak{D}\ :=\ \left\{\prod_m \ell_m^{-d_m}:\ d_0,d_1,d_2,\dots\in\mathbb{N},\ d_0\geqslant d_1\geqslant d_2\geqslant\cdots\right\}\ \subseteq\ \mathfrak{L}_{<\omega}.$$

Note that $\mathfrak{D}$ is well-based by Corollary 2.17.

Lemma 6.5. *Let $\phi\in\mathbb{L}_{[\omega,\alpha)}$ and set $\beta:=\lambda_g+\omega$. Then*

$$\operatorname{supp}_\beta T_g(\phi)\ \subseteq\ (\mathfrak{D}\circ\ell_{\lambda_g+3})\cdot(\operatorname{supp}_\beta\varepsilon)^\infty,$$

and the right hand side is a well-based subset of $\mathfrak{L}_{<\beta}$ independent of ϕ.

Proof. The operator support of the derivation ∂_α on $\mathbb{L}_{<\alpha}$ and $\operatorname{supp}_\omega(\phi) \subseteq \{1\}$ give

$$\operatorname{supp}_\omega(\phi^{(n)}) \;\subseteq\; \left\{ \prod_m \ell_m^{-d_m} \,:\, d_0, d_1, d_2, \dots \in \mathbb{N},\; n = d_0 \geqslant d_1 \geqslant d_2 \geqslant \cdots \right\}.$$

Thus by Lemmas 5.10 and 6.4, and $\lambda_g + 3 + \omega = \lambda_g + \omega = \beta$,

$$\bigcup_n \operatorname{supp}_\beta \left(\phi^{(n)} \circ \ell_{\lambda_g+3}\right) \;\subseteq\; \mathfrak{D} \circ \ell_{\lambda_g+3} \;\subseteq\; \mathfrak{L}_{<\beta}.$$

Now $\mathfrak{D} \circ \ell_{\lambda_g+3}$ is well-based by Lemma 5.10, and $\varepsilon \preccurlyeq \ell_{\lambda_g+2}^{-1}$ gives $\operatorname{supp}_\beta \varepsilon \prec 1$. □

Proof of Proposition 6.2. We shall prove the stronger result that the family

$$\left((f_{[\mathfrak{m}]} \circ g)(\mathfrak{m} \circ g)\right)_{\mathfrak{m} \in \operatorname{supp}_\omega(f)}$$

is β-summable for $\beta := \lambda_g + \omega$. Suppose towards a contradiction that $(\mathfrak{m}_i)_{i<\omega}$ is a sequence of distinct elements of $\operatorname{supp}_\omega(f)$ and $(\mathfrak{n}_i)_{i<\omega}$ is an increasing sequence in $\mathfrak{L}_{<\beta}$ such that $\mathfrak{n}_i \in \operatorname{supp}_\beta\left((f_{[\mathfrak{m}_i]} \circ g)(\mathfrak{m}_i \circ g)\right)$ for all i. We have $\mathfrak{n}_i = \mathfrak{p}_i \mathfrak{q}_i$ where $\mathfrak{p}_i \in \operatorname{supp}_\beta(f_{[\mathfrak{m}_i]} \circ g)$ and $\mathfrak{q}_i \in \operatorname{supp}_\beta(\mathfrak{m}_i \circ g)$. By Corollary 6.3, the family $(\mathfrak{m}_i \circ g)$ is β-summable and so $(\mathfrak{q}_i)$ has a strictly decreasing subsequence. Thus $(\mathfrak{p}_i)$ has a strictly increasing subsequence. This contradicts the fact that Lemma 6.5 gives a well-based set $\mathfrak{S}_g \subseteq \mathfrak{L}_{<\beta}$ such that $\operatorname{supp}_\beta \phi \circ g \subseteq \mathfrak{S}_g$ for all $\phi \in \mathbb{L}_{[\omega,\alpha)}$. □

We can now define

$$f \circ g \;:=\; \sum_{\mathfrak{m} \in \mathfrak{L}_{<\omega}} \left(f_{[\mathfrak{m}]} \circ g\right)(\mathfrak{m} \circ g)\,. \tag{6.6}$$

This agrees for $f = \mathfrak{m} \in \mathfrak{L}_{<\omega}$ and $f \in \mathbb{L}_{[\omega,\alpha)}$ with $f \circ g$ as defined in (6.1) and (6.5). It also agrees for $g = \ell_\gamma$ ($\gamma < \alpha$) with $f \circ \ell_\gamma$ as defined at the end of the previous section, but this requires an argument: For such g we have $\lambda_g = \gamma$ and $\log_3(g) = \ell_{\gamma+3}$, $\varepsilon = 0$, and so for $f \in \mathbb{L}_{[\omega,\alpha)}$ and with $f \circ g$ as defined by (6.5),

$$f \circ g \;=\; f^{\uparrow_3} \circ \ell_{\gamma+3} \;=\; (f^{\uparrow_3} \circ \ell_3) \circ \ell_\gamma \;=\; f \circ \ell_\gamma$$

where the second equality uses Lemma 5.9. For $f = \mathfrak{m} = \prod_n \ell_n^{r_n} \in \mathfrak{L}_{<\omega}$, we have

$$f \circ g \;=\; \prod_n \log_n(g)^{r_n} \;=\; \prod_n \ell_{\gamma+n}^{r_n} \;=\; \mathfrak{m} \circ \ell_\gamma$$

where for the last equality we use the first part of Lemma 5.10. For arbitrary $f \in \mathbb{L}_{<\alpha}$ we have $f = \sum_{\mathfrak{m} \in \operatorname{supp}_\omega(f)} f_{[\mathfrak{m}]}\mathfrak{m}$ with all $f_{[\mathfrak{m}]} \in \mathbb{L}_{[\omega,\alpha)}$, and then

$$f \circ g \;=\; \sum_{\mathfrak{m}} (f_{[\mathfrak{m}]} \circ g)(\mathfrak{m} \circ g) \;=\; \sum_{\mathfrak{m}} (f_{[\mathfrak{m}]} \circ \ell_\gamma)(\mathfrak{m} \circ \ell_\gamma) \;=\; f \circ \ell_\gamma$$

where the last equality uses Corollary 5.7 and $f \circ \ell_\gamma$ is defined as in (5.6).

Lemma 6.6. *The map $f \mapsto f \circ g : \mathbb{L}_{<\alpha} \to \mathbb{L}_{<\alpha}$ is an $\mathbb{L}_{<\alpha}$-composition with g.*

Proof. Let $\mathfrak{m}, \mathfrak{n} \in \mathfrak{L}_{<\alpha}$; to get $(\mathfrak{m}\mathfrak{n}) \circ g = (\mathfrak{m} \circ g) \cdot (\mathfrak{n} \circ g)$, we use the decomposition $\mathfrak{m} = \mathfrak{m}_{<\omega}\mathfrak{m}_{\geqslant\omega}$ where $\mathfrak{m}_{<\omega} \in \mathfrak{L}_{<\omega}$ and $\mathfrak{m}_{\geqslant\omega} \in \mathfrak{L}_{[\omega,\alpha)}$, and likewise for $\mathfrak{n}$. The desired equality then follows from the relevant definitions and properties we already stated.

To verify that clause (3) in the definition of "K-composition with h" is satisfied, with $\mathfrak{M} = \mathfrak{N} = \mathfrak{L}_{<\alpha}$ and g in the role of h, let $f \in \mathbb{L}_{<\alpha}$ and let P be the set of pairs $(\mathfrak{m}, \mathfrak{n})$ with $\mathfrak{m} \in \mathfrak{L}_{<\omega}$, $\mathfrak{n} \in \mathfrak{L}_{[\omega,\alpha)}$, and $\mathfrak{m}\mathfrak{n} \in \operatorname{supp}(f)$. Then $f = \sum_{(\mathfrak{m},\mathfrak{n}) \in P} f_{\mathfrak{m}\mathfrak{n}}\mathfrak{m}\mathfrak{n}$, with all $f_{\mathfrak{mn}} \in \mathbb{R}^\times$; our job is to show that then $\big((\mathfrak{mn}) \circ g\big)_{(\mathfrak{m},\mathfrak{n})\in P}$ is summable and $\sum_{(\mathfrak{m},\mathfrak{n})\in P} f_{\mathfrak{mn}}\big((\mathfrak{mn}) \circ g\big) = f \circ g$. Note that

$$\operatorname{supp}_\omega f \ = \ \{\mathfrak{m} \in \mathfrak{L}_{<\omega} : \ (\mathfrak{m},\mathfrak{n}) \in P \text{ for some } \mathfrak{n} \in \mathfrak{L}_{[\omega,\alpha)}\}.$$

For $\mathfrak{m} \in \operatorname{supp}_\omega f$ we set $P(\mathfrak{m}) := \{\mathfrak{n} \in \mathfrak{L}_{[\omega,\alpha)} : \ (\mathfrak{m},\mathfrak{n}) \in P\}$ and

$$f_{[\mathfrak{m}]} \ := \sum_{\mathfrak{n}\in P(\mathfrak{m})} f_{\mathfrak{mn}}\mathfrak{n} \ \in \ \mathbb{L}_{[\omega,\alpha)}.$$

Then $f = \sum_{\mathfrak{m}\in\operatorname{supp}_\omega f} f_{[\mathfrak{m}]}\mathfrak{m}$, so $f \circ g = \sum_{\mathfrak{m}\in\operatorname{supp}_\omega f}(f_{[\mathfrak{m}]} \circ g)(\mathfrak{m} \circ g)$; we shall need this equality at the end of the proof and first address the summability issue.

Suppose towards a contradiction that $\big((\mathfrak{mn}) \circ g\big)_{(\mathfrak{m},\mathfrak{n})\in P}$ is not summable. Then we have a sequence $(\mathfrak{m}_i, \mathfrak{n}_i)_{i<\omega}$ of distinct elements in P and an increasing sequence $(\mathfrak{g}_i)$ in $\mathfrak{L}_{<\alpha}$ with $\mathfrak{g}_i \in \operatorname{supp}\big((\mathfrak{m}_i\mathfrak{n}_i) \circ g\big)$ for all i. Now $\mathfrak{m}_i \in \operatorname{supp}_\omega f$ for all i; by passing to a subsequence we arrange that either $(\mathfrak{m}_i)$ is constant, or $(\mathfrak{m}_i)$ is strictly decreasing, and so we now consider these two cases.

Case 1: $(\mathfrak{m}_i)$ is constant, say $\mathfrak{m}_i = \mathfrak{m}$ for all i. Then $\mathfrak{n}_i \in P(\mathfrak{m})$ for all i, and

$$f_{[\mathfrak{m}]} \circ g \ = \sum_{\mathfrak{n}\in P(\mathfrak{m})} f_{\mathfrak{mn}}(\mathfrak{n} \circ g), \qquad (\mathfrak{m} \circ g)(f_{[\mathfrak{m}]} \circ g) \ = \sum_{\mathfrak{n}\in P(\mathfrak{m})} f_{\mathfrak{mn}}\big((\mathfrak{mn}) \circ g\big),$$

using the first part of the proof. In particular, the sum $\sum_{\mathfrak{n}\in P(\mathfrak{m})} f_{\mathfrak{mn}}((\mathfrak{mn}) \circ g)$ exists, contradicting that $(\mathfrak{g}_i)$ is increasing. So we must be in the next case:

Case 2: $(\mathfrak{m}_i)$ is strictly decreasing. Now

$$\mathfrak{g}_i \in \operatorname{supp}\big((\mathfrak{m}_i\mathfrak{n}_i) \circ g\big) \ \subseteq \ \operatorname{supp}\big((\mathfrak{m}_i \circ g)(\mathfrak{n}_i \circ g)\big),$$

so $\mathfrak{g}_i = \mathfrak{p}_i\mathfrak{q}_i$ with $\mathfrak{p}_i \in \operatorname{supp}(\mathfrak{m}_i \circ g)$ and $\mathfrak{q}_i \in \operatorname{supp}(\mathfrak{n}_i \circ g)$. Set $\beta := \lambda_g + \omega$. Since $(\mathfrak{m}_i)$ is strictly decreasing, we can arrange by passing to a subsequence that $((\mathfrak{p}_i)_{<\beta})$ is strictly decreasing, in view of Lemma 6.1 and the proof of Corollary 6.3. Since $(\mathfrak{g}_i)$ is increasing, $((\mathfrak{q}_i)_{<\beta})$ must be strictly increasing, contradicting Lemma 6.5. We have now shown that $\big((\mathfrak{mn}) \circ g\big)_{(\mathfrak{m},\mathfrak{n})\in P}$ is summable. Its sum is $f \circ g$:

$$\begin{aligned}
\sum_{(\mathfrak{m},\mathfrak{n})\in P} f_{\mathfrak{mn}}\big((\mathfrak{mn}) \circ g\big) \ &= \sum_{\mathfrak{m}\in\operatorname{supp}_\omega f} \Big(\sum_{\mathfrak{n}\in P(\mathfrak{m})} f_{\mathfrak{mn}}(\mathfrak{mn}) \circ g\Big) \\
&= \sum_{\mathfrak{m}\in\operatorname{supp}_\omega f} \Big(\sum_{\mathfrak{n}\in P(\mathfrak{m})} f_{\mathfrak{mn}}(\mathfrak{n} \circ g)(\mathfrak{m} \circ g)\Big) \\
&= \sum_{\mathfrak{m}\in\operatorname{supp}_\omega f} \Big(\sum_{\mathfrak{n}\in P(\mathfrak{m})} f_{\mathfrak{mn}}(\mathfrak{n} \circ g)\Big) \cdot \mathfrak{m} \circ g \\
&= \sum_{\mathfrak{m}\in\operatorname{supp}_\omega f} (f_{[\mathfrak{m}]} \circ g)(\mathfrak{m} \circ g) \ = \ f \circ g. \qquad \square
\end{aligned}$$

Extending to $\mathbb{L}$. For $f, g \in \mathbb{L}$ with $g > \mathbb{R}$ there are of course many α for which $f, g \in \mathbb{L}_{<\alpha}$, and for each of those we have a value $f \circ g \in \mathbb{L}_{<\alpha}$; it is easy to check that this value $f \circ g$ is independent of α. Thus we have constructed a map

$$(f, g) \mapsto f \circ g : \mathbb{L} \times \mathbb{L}^{>\mathbb{R}} \to \mathbb{L}.$$

In the next section we show that this map is a composition on $\mathbb{L}$ as defined in the introduction. **Given $f, g \in \mathbb{L}$ with $g > \mathbb{R}$ we let from now on $f \circ g$ denote**

the value of the above map at (f,g). We continue nevertheless our work in the setting of a Hahn field $\mathbb{L}_{<\alpha} = \mathbb{R}[[\mathfrak{L}_{<\alpha}]]$ with α as before.

7. Properties of Composition

As before, $\alpha = \omega^\lambda$, where λ is an infinite limit ordinal. Our job is to show that the map $(f,g) \mapsto f \circ g : \mathbb{L}_{<\alpha} \times \mathbb{L}_{<\alpha}^{>\mathbb{R}} \to \mathbb{L}_{<\alpha}$ defined in the previous section is a composition on $\mathbb{L}_{<\alpha}$ in the sense of Section 4. We first prove the chain rule, and then use this to derive associativity. But our proof of the chain rule requires a special case of associativity, namely $\log(f \circ g) = (\log f) \circ g$ for $f \in \mathbb{L}_{<\alpha}^{>}$ and $g \in \mathbb{L}_{<\alpha}^{>\mathbb{R}}$. This is our starting point:

Compatibility of taking logarithms and composition. The first lemma treats the case that g in $f \circ g$ is a hyperlogarithm.

Lemma 7.1. *Let $f \in \mathbb{L}_{<\alpha}^{>}$ and $\gamma < \alpha$. Then $\log(f \circ \ell_\gamma) = (\log f) \circ \ell_\gamma$.*

Proof. Suppose $\gamma = \omega^\beta$, and set $\mu := \omega^{\beta+1}$. By Lemma 5.5,

$$\big(\log(f \circ \ell_\gamma)\big)' \;=\; \frac{(f \circ \ell_\gamma)'}{f \circ \ell_\gamma} \;=\; \frac{f' \circ \ell_\gamma}{f \circ \ell_\gamma}\ell_\gamma' \;=\; \big((\log f)' \circ \ell_\gamma\big)\ell_\gamma' \;=\; \big((\log f) \circ \ell_\gamma\big)',$$

So it remains to show that $\log(f \circ \ell_\gamma)$ and $(\log f) \circ \ell_\gamma$ have the same constant term. We have $f = c\mathfrak{m}(1+\varepsilon)$, where $c \in \mathbb{R}^{>}$, $\mathfrak{m} = \prod_{\rho<\alpha} \ell_\gamma^{r_\rho}$, and $\varepsilon \prec 1$. Then

$$\log f \;=\; \log c + \sum_{\rho<\alpha} r_\rho \ell_{\rho+1} + \log(1+\varepsilon)$$

As $\log(1+\varepsilon)$ is infinitesimal and $\ell_{\rho+1} \circ \ell_\gamma$ has constant term 0 by Corollary 5.6 and its proof, the constant term of $(\log f) \circ \ell_\gamma$ is $\log c$. Note that $\mathfrak{m} \circ \ell_\gamma$ has leading coefficient 1: if $\mathfrak{m} \in \mathfrak{L}_{<\mu}$, then this follows from (5.3); if $\mathfrak{m} \in \mathbb{L}_{[\mu,\alpha)}$, then it follows from $\mathfrak{m} \circ \ell_\gamma \sim \mathfrak{m}$, which holds by Lemma 5.1. Since

$$f \circ \ell_\gamma \;=\; c(\mathfrak{m} \circ \ell_\gamma)\big(1 + (\varepsilon \circ \ell_\gamma)\big)$$

and $\varepsilon \circ \ell_\gamma \prec 1$, the leading coefficient of $f \circ \ell_\gamma$ is c, so the constant term of $\log(f \circ \ell_\gamma)$ is $\log c$ as well. The general case now follows by induction on k in (5.6). ☐

We now turn our attention to $\mathbb{L}_{[\omega,\alpha)}$ and the map $f \mapsto f^{\uparrow_3}$. Note that if $f \in \mathbb{L}_{[\omega,\alpha)}^{>}$, then also $\log f \in \mathbb{L}_{[\omega,\alpha)}$. Thus the statement of the following lemma makes sense:

Lemma 7.2. *Let $f \in \mathbb{L}_{[\omega,\alpha)}^{>}$. Then $\log(f^{\uparrow_3}) = (\log f)^{\uparrow_3}$.*

Proof. Using Lemma 7.1 we have

$$\log(f^{\uparrow_3}) \circ \ell_3 \;=\; \log(f^{\uparrow_3} \circ \ell_3) \;=\; \log f \;=\; (\log f)^{\uparrow_3} \circ \ell_3$$

and so $\log(f^{\uparrow_3}) = (\log f)^{\uparrow_3}$. ☐

Lemma 7.3. *Let $f \in \mathbb{L}_{[\omega,\alpha)}^{>}$ and $g \in \mathbb{L}_{<\alpha}^{>\mathbb{R}}$. Then $\log(f \circ g) = (\log f) \circ g$.*

Proof. Let T_g be the Taylor deformation in (6.3), so $f \circ g = T_g(f^{\uparrow_3})$. By Lemma 7.1 we have $\log(f^{\uparrow_3} \circ \ell_{\lambda_g+3}) = \log(f^{\uparrow_3}) \circ \ell_{\lambda_g+3}$. Then Lemmas 2.14 and 7.2 give

$$\log(f \circ g) \;=\; \log T_g(f^{\uparrow_3}) \;=\; T_g\big(\log(f^{\uparrow_3})\big) \;=\; T_g\big((\log f)^{\uparrow_3}\big) \;=\; (\log f) \circ g. \quad ☐$$

We can now prove the main result of this subsection:

Proposition 7.4. *Let $f \in \mathbb{L}_{<\alpha}^{>}$ and $g \in \mathbb{L}_{<\alpha}^{>\mathbb{R}}$. Then $\log(f \circ g) = (\log f) \circ g$.*

Proof. We have $f = c\mathfrak{d}(f)(1+\varepsilon)$ where $c \in \mathbb{R}^{>}$ and $\varepsilon \prec 1$. Then

$$(\log f)\circ g \;=\; \log c + \big(\log \mathfrak{d}(f)\big)\circ g + \big(\log(1+\varepsilon)\big)\circ g.$$

The strong linearity of composition with g gives $\big(\log(1+\varepsilon)\big)\circ g = \log\big((1+\varepsilon)\circ g\big)$. Thus, it remains to show that $\big(\log \mathfrak{d}(f)\big)\circ g = \log\big(\mathfrak{d}(f)\circ g\big)$. Now $\mathfrak{d}(f) = \mathfrak{m}\mathfrak{n}$ where $\mathfrak{m} \in \mathfrak{L}_{<\omega}$ and $\mathfrak{n} \in \mathfrak{L}_{[\omega,\alpha)}$, so $\log \mathfrak{d}(f) = \log \mathfrak{m} + \log \mathfrak{n}$. By Lemma 7.3, $(\log \mathfrak{n})\circ g = \log(\mathfrak{n}\circ g)$. We have $\mathfrak{m} = \prod_n \ell_n^{r_n}$, so

$$(\log \mathfrak{m})\circ g \;=\; \left(\sum_n r_n \ell_{n+1}\right)\circ g \;=\; \sum_n r_n \log_{n+1}(g) \;=\; \log(\mathfrak{m}\circ g),$$

where the last equality uses Lemma 4.7. Thus $\big(\log \mathfrak{d}(f)\big)\circ g \;=\; \log\big(\mathfrak{d}(f)\circ g\big)$. □

Corollary 7.5. *Let $f \in \mathbb{L}_{<\alpha}^{>}$, $g \in \mathbb{L}_{<\alpha}^{>\mathbb{R}}$, and $t \in \mathbb{R}$. Then $(f\circ g)^t = f^t \circ g$.*

Proof. Take logarithms and use Proposition 7.4. □

Corollary 7.6. *Suppose the family (f_i) in $\mathbb{L}_{<\alpha}^{>}$ is multipliable and $g \in \mathbb{L}_{<\alpha}^{>\mathbb{R}}$. Then the family $(f_i \circ g)$ is multipliable and $(\prod_i f_i)\circ g = \prod_i f_i \circ g$.*

Proof. Take logarithms and use Proposition 7.4. □

The chain rule. Let $g \in \mathbb{L}_{<\alpha}^{>\mathbb{R}}$. Recall that T_g is a strongly $\mathbb{R}$-linear endomorphism of the ordered field $\mathbb{L}_{<\alpha}$. We show that T_g coincides with $\phi \mapsto (\phi\circ\ell_3)\circ g$:

Lemma 7.7. *For all $\phi \in \mathbb{L}_{<\alpha}$ we have $T_g(\phi) = (\phi\circ\ell_3)\circ g$.*

Proof. The map $\phi \mapsto (\phi\circ\ell_3)\circ g : \mathbb{L}_{<\alpha} \to \mathbb{L}_{<\alpha}$ is also a strongly $\mathbb{R}$-linear endomorphism of the ordered field $\mathbb{L}_{<\alpha}$; it agrees with T_g on $\mathbb{L}_{[\omega,\alpha)}$, since for $\phi \in \mathbb{L}_{[\omega,\alpha)}$ we have $\phi\circ\ell_3 \in \mathbb{L}_{[\omega,\alpha)}$ and $\phi = (\phi\circ\ell_3)^{\uparrow 3}$. By the strong linearity of both maps it is enough to prove the lemma for $\phi \in \mathfrak{L}_{<\alpha}$, and for such ϕ we have $\phi = \mathfrak{m}\mathfrak{n}$ with $\mathfrak{m} \in \mathfrak{L}_{<\omega}$ and $\mathfrak{n} \in \mathfrak{L}_{[\omega,\alpha)}$. Thus it is enough to show that $T_g(\mathfrak{m}) = (\mathfrak{m}\circ\ell_3)\circ g$ for $\mathfrak{m} \in \mathfrak{L}_{<\omega}$. Taking logarithms this reduces to showing for such $\mathfrak{m}$ that $\log T_g(\mathfrak{m}) = \log\big((\mathfrak{m}\circ\ell_3)\circ g\big)$; by Lemma 2.14 and Proposition 7.4, this is equivalent to $T_g(\log \mathfrak{m}) = \big(\log(\mathfrak{m}\circ\ell_3)\big)\circ g$, and thus by Proposition 7.4 to $T_g(\log \mathfrak{m}) = \big((\log \mathfrak{m})\circ\ell_3\big)\circ g$. Since for $\mathfrak{m} \in \mathfrak{L}_{<\omega}$ we have $\log \mathfrak{m} = \sum_n r_n \ell_{n+1}$ this reduces further to showing that $T_g(\ell_n) = (\ell_n\circ\ell_3)\circ g$ for all n. This holds for $n=0$ by earlier remarks, and for arbitrary n it follows by induction on n, using again at each step Lemma 2.14 and Proposition 7.4. □

Proposition 7.8. *Let $f \in \mathbb{L}_{<\alpha}$. Then $(f\circ g)' = (f'\circ g)g'$.*

Proof. Note that for $f \neq 0$ we have: $(f\circ g)' = (f'\circ g)g' \Leftrightarrow (f\circ g)^\dagger = (f^\dagger\circ g)g'$. An easy induction gives $\log_n(g)^\dagger = (\ell_n^\dagger\circ g)g'$. Thus for $\mathfrak{m} = \prod_n \ell_n^{r_n} \in \mathfrak{L}_{<\omega}$,

$$\begin{aligned}(\mathfrak{m}\circ g)^\dagger = \big(\prod_n \log_n(g)^{r_n}\big)^\dagger \;&=\; \sum_n r_n \log_n(g)^\dagger \;=\; \sum_n r_n(\ell_n^\dagger\circ g)g' \\ &=\; \big(\sum_n r_n(\ell_n^\dagger\circ g)\big)g' \;=\; \big((\sum_n r_n\ell_n^\dagger)\circ g\big)g' \;=\; (\mathfrak{m}^\dagger\circ g)g',\end{aligned}$$

so $(\mathfrak{m}\circ g)' = (\mathfrak{m}'\circ g)g'$. Let $f \in \mathbb{L}_{[\omega,\alpha)}$. Then $f\circ g = T_g(f^{\uparrow 3})$, so by (6.4),

$$(f\circ g)' \;=\; \big(T_g(f^{\uparrow 3})\big)' \;=\; T_g\big((f^{\uparrow 3})'\big)\cdot\log_3(g)' \;=\; T_g\big((f^{\uparrow 3})'\big)\cdot(\ell_3'\circ g)\cdot g'. \tag{7.1}$$

We have $f' = (f^{\uparrow_3} \circ \ell_3)' = \big((f^{\uparrow_3})' \circ \ell_3\big)\ell_3'$ by (6.2) and Corollary 5.8, so $(f^{\uparrow_3})' \circ \ell_3 = (f'/\ell_3')$. Applying Lemma 7.7 to $\phi := (f^{\uparrow_3})'$, this gives

$$T_g\big((f^{\uparrow_3})'\big) \;=\; (f'/\ell_3') \circ g \;=\; (f' \circ g)/(\ell_3' \circ g). \tag{7.2}$$

Combining (7.1) and (7.2) gives $(f \circ g)' = (f' \circ g)g'$. Finally, for arbitrary $f \in \mathbb{L}_{<\alpha}$ we have $f = \sum_{\mathfrak{m} \in \mathfrak{L}_{<\omega}} f_{[\mathfrak{m}]}\mathfrak{m}$ where all $f_{[\mathfrak{m}]} \in \mathbb{L}_{[\omega,\alpha)}$. Then

$$\begin{aligned}(f \circ g)' &= \sum_{\mathfrak{m}} \big((f_{[\mathfrak{m}]} \circ g)(\mathfrak{m} \circ g)\big)' \\ &= \sum_{\mathfrak{m}} \big((f_{[\mathfrak{m}]} \circ g)'(\mathfrak{m} \circ g) + (f_{[\mathfrak{m}]} \circ g)(\mathfrak{m} \circ g)'\big) \\ &= g' \sum_{\mathfrak{m}} \big((f_{[\mathfrak{m}]}' \circ g)(\mathfrak{m} \circ g) + (f_{[\mathfrak{m}]} \circ g)(\mathfrak{m}' \circ g)\big) \\ &= g' \sum_{\mathfrak{m}} \big((f_{[\mathfrak{m}]}\mathfrak{m})' \circ g\big) = g'(f' \circ g).\end{aligned}$$
□

Associativity. Towards proving associativity we use the next lemma to get a handle on the infinite part and constant term of $f \circ g$ for various f, g. *Throughout this subsection we fix $g, h \in \mathbb{L}_{<\alpha}^{>\mathbb{R}}$.*

Lemma 7.9. *Assume $\omega \leqslant \gamma < \alpha$. Then $\ell_\gamma \circ g - \ell_\gamma \circ \ell_{\lambda_g} \prec 1$.*

Proof. For $n \geqslant 1$ we have $\ell_\gamma^{\uparrow_3} \circ \ell_3 = \ell_\gamma \prec \ell_{n+3} = \ell_n \circ \ell_3$, hence $\ell_\gamma^{\uparrow_3} \prec \ell_n$, and thus $(\ell_\gamma^{\uparrow_3})' \prec \ell_n' \prec 1$. Thus by (6.5),

$$\ell_\gamma \circ g - \big(\ell_\gamma^{\uparrow_3} \circ \ell_{\lambda_{g+3}}\big) \;=\; \sum_{n=1}^{\infty} \frac{(\ell_\gamma^{\uparrow_3})^{(n)} \circ \ell_{\lambda_g+3}}{n!} \big(\log_3(g) - \ell_{\lambda_{g+3}}\big)^n \;\prec\; 1.$$

Now $\ell_{\lambda_g+3} = \log_3(\ell_{\lambda_g})$, so by (6.5) with ℓ_{λ_g} in the role of g and $\varepsilon = 0$,

$$\ell_\gamma \circ \ell_{\lambda_g} \;=\; \ell_\gamma^{\uparrow_3} \circ \ell_{\lambda_g+3}.$$
□

Combining Lemmas 5.13 and 7.9 gives:

Corollary 7.10. *Assume $\omega \leqslant \gamma < \alpha$. Then*

$$\ell_\gamma \circ g \;=\; \ell_{\lambda_g+\gamma} - \lambda_{g;\gamma} + \varepsilon, \qquad \varepsilon \prec 1.$$

Lemma 7.11. *We have $\lambda_{(g \circ h)} = \lambda_h + \lambda_g$ (ordinal sum).*

Proof. Recall that $\mathfrak{d}(\log f) = \ell_{\lambda_f+1}$ for $f \in \mathbb{L}^{>\mathbb{R}}$, so it suffices to show:

$$\log(g \circ h) \;\asymp\; \ell_{\lambda_h+\lambda_g+1}.$$

By Proposition 7.4, we have

$$\log(g \circ h) \;=\; (\log g) \circ h \;\asymp\; \ell_{\lambda_g+1} \circ h.$$

We end by noting that $\ell_{\lambda_g+1} \circ h \asymp \ell_{\lambda_h+\lambda_g+1}$; this is a consequence of the remarks after Lemma 4.6 if λ_g is finite, and follows by Corollary 7.10 if $\lambda_g \geqslant \omega$. □

To use this lemma we recall that for ordinals μ and ν with Cantor normal forms

$$\mu \;=\; \omega^{\beta_1}m_1 + \cdots + \omega^{\beta_k}m_k, \qquad \nu \;=\; \omega^{\gamma_1}n_1 + \cdots + \omega^{\gamma_l}n_l, \qquad (k, l \geqslant 1),$$

the Cantor normal form of $\nu + \mu$ is as follows:

- if $\gamma_1 < \beta_1$, then $\nu + \mu = \mu$;

- if $1 \leqslant j \leqslant l$ and $\gamma_j = \beta_1$, then
$$\nu + \mu \;=\; \omega^{\gamma_1} n_1 + \cdots + \omega^{\gamma_j}(n_j + m_1) + \omega^{\beta_2} m_2 + \cdots + \omega^{\beta_k} n_k;$$
- if $1 \leqslant j \leqslant l$ and $\gamma_j > \beta_1$, $\gamma_{j'} < \beta_1$ for all j' with $j < j' \leqslant l$, then
$$\nu + \mu \;=\; \omega^{\gamma_1} n_1 + \cdots + \omega^{\gamma_j} n_j + \omega^{\beta_1} m_1 + \cdots + \omega^{\beta_k} n_k.$$

Here $\mu, \nu \geqslant 1$. Below we also need the trivial case where $\mu = 0$ or $\nu = 0$.

Lemma 7.12. *Let $\gamma \geqslant \omega$. Then*
$$\lambda_{(g\circ h);\gamma} \;=\; \lambda_{g;\gamma} + \lambda_{h;\lambda_g+\gamma}.$$

Proof. We express λ_g and λ_h in Cantor normal form (allowing $k = 0$ or $l = 0$):
$$\lambda_g \;=\; \omega^{\beta_1} m_1 + \cdots + \omega^{\beta_k} m_k, \qquad \lambda_h \;=\; \omega^{\gamma_1} n_1 + \cdots + \omega^{\gamma_l} n_l.$$
Using Lemma 7.11 and the above remarks about Cantor normal forms, we have
$$\lambda_{(g\circ h);\gamma} \;=\; \begin{cases} 0 & \text{if } \gamma \notin \{\omega^{\beta_1+1}, \dots, \omega^{\beta_k+1}, \omega^{\gamma_1+1}, \dots, \omega^{\gamma_l+1}\} \\ m_1 & \text{if } k \geqslant 1,\ \gamma = \omega^{\beta_1+1},\ \text{and } \beta_1 \notin \{\gamma_1, \dots, \gamma_l\} \\ m_i & \text{if } k > 1,\ \gamma = \omega^{\beta_i+1},\ 1 < i \leqslant k \\ n_j + m_1 & \text{if } k \geqslant 1,\ \gamma = \omega^{\gamma_j+1},\ 1 \leqslant j \leqslant l,\ \text{and } \gamma_j = \beta_1 \\ n_j & \text{if } k \geqslant 1,\ \gamma = \omega^{\gamma_j+1},\ 1 \leqslant j \leqslant l,\ \text{and } \gamma_j > \beta_1 \\ 0 & \text{if } k \geqslant 1,\ \gamma \notin \{\omega^{\beta_1+1}, \dots, \omega^{\beta_k+1}\},\ \gamma = \omega^{\gamma_j+1}, 1 \leqslant j \leqslant l, \gamma_j < \beta_1 \\ n_j & \text{if } k = 0,\ \gamma = \omega^{\gamma_j+1}, 1 \leqslant j \leqslant l. \end{cases}$$
It remains to calculate the values of $\lambda_{g;\gamma}$ and $\lambda_{h;\lambda_g+\gamma}$:

- If $\gamma \notin \{\omega^{\beta_1+1}, \dots, \omega^{\beta_k+1}, \omega^{\gamma_1+1}, \dots, \omega^{\gamma_l+1}\}$, then $\lambda_{g;\gamma} = 0$ and moreover $\lambda_g + \gamma \notin \{\omega^{\gamma_1+1}, \dots, \omega^{\gamma_l+1}\}$, so $\lambda_{h;\lambda_g+\gamma} = 0$.
- If $k \geqslant 1$, $\gamma = \omega^{\beta_1+1}$ and $\beta_1 \notin \{\gamma_1, \dots, \gamma_l\}$, then $\lambda_{g;\gamma} = m_1$ and $\lambda_g + \gamma = \gamma$, so $\lambda_{h;\lambda_g+\gamma} = 0$.
- If $k > 1$, $\gamma = \omega^{\beta_i+1}$, $1 < i \leqslant k$, then $\lambda_{g;\gamma} = m_i$, $\lambda_g + \gamma \notin \{\omega^{\gamma_1+1}, \dots, \omega^{\gamma_l+1}\}$, so $\lambda_{h;\lambda_g+\gamma} = 0$.
- If $k \geqslant 1$, $\gamma = \omega^{\gamma_j+1}$, $1 \leqslant j \leqslant l$, and $\gamma_j = \beta_1$, then $\lambda_{g;\gamma} = m_1$ and $\lambda_g + \gamma = \gamma$, so $\lambda_{h;\lambda_g+\gamma} = n_j$.
- If $k \geqslant 1$, $\gamma = \omega^{\gamma_j+1}$, $1 \leqslant j \leqslant l$, and $\gamma_j > \beta_1$, then $\lambda_{g;\gamma} = 0$ and $\lambda_g + \gamma = \gamma$, so $\lambda_{h;\lambda_g+\gamma} = n_j$.
- If $k \geqslant 1$, $\gamma \notin \{\omega^{\beta_1+1}, \dots, \omega^{\beta_k+1}\}$, $\gamma = \omega^{\gamma_j+1}$, $\gamma_j < \beta_1$, then $\lambda_{g;\gamma} = 0$ and $\lambda_g + \gamma \notin \{\omega^{\gamma_1+1}, \dots, \omega^{\gamma_l+1}\}$, so $\lambda_{h;\lambda_g+\gamma} = 0$.
- If $k = 0$, $\gamma = \omega^{\gamma_j+1}$, $1 \leqslant j \leqslant l$, then $\lambda_{g;\gamma} = 0$, $\lambda_g + \gamma = \gamma$, so $\lambda_{h;\lambda_g+\gamma} = n_j$.

Thus $\lambda_{(g\circ h);\gamma} \;=\; \lambda_{g;\gamma} + \lambda_{h;\lambda_g+\gamma}$ in all cases. □

Lemma 7.13. *Let $\omega \leqslant \gamma < \alpha$. Then $\big(\ell_\gamma \circ (g \circ h)\big) - \big((\ell_\gamma \circ g) \circ h\big) \prec 1$.*

Proof. Corollary 7.10 gives $\varepsilon, \varepsilon^* \prec 1$ in $\mathbb{L}_{<\alpha}$ such that
$$\begin{aligned}(\ell_\gamma \circ g) \circ h \;&=\; (\ell_{\lambda_g+\gamma} - \lambda_{g;\gamma} + \varepsilon) \circ h \;=\; (\ell_{\lambda_g+\gamma} \circ h) - \lambda_{g;\gamma} + (\varepsilon \circ h) \\ &=\; \big(\ell_{\lambda_h+\lambda_g+\gamma} - \lambda_{h;\lambda_g+\gamma} + \varepsilon^*\big) - \lambda_{g;\gamma} + (\varepsilon \circ h).\end{aligned}$$
Corollary 7.10 and Lemmas 7.11 and 7.12 give $\varepsilon^{**} \prec 1$ in $\mathbb{L}_{<\alpha}$ such that
$$\ell_\gamma \circ (g \circ h) \;=\; \ell_{\lambda_{(g\circ h)}+\gamma} - \lambda_{(g\circ h);\gamma} + \varepsilon^{**} \;=\; \ell_{\lambda_h+\lambda_g+\gamma} - \lambda_{h;\lambda_g+\gamma} - \lambda_{g;\gamma} + \varepsilon^{**}. \quad \square$$

Proposition 7.14. *For all $f \in \mathbb{L}_{<\alpha}$ we have $f \circ (g \circ h) = (f \circ g) \circ h$.*

Proof. By strong linearity this reduces to $\mathfrak{m}\circ(g\circ h) = (\mathfrak{m}\circ g)\circ h$ for $\mathfrak{m} \in \mathfrak{L}_{<\alpha}$. Taking logarithms and using Proposition 7.4 this reduces further to showing $\ell_\gamma \circ (g\circ h) = (\ell_\gamma \circ g)\circ h$ for $\gamma < \alpha$. This goes by induction on γ. The case $\gamma = 0$ is obvious, and for the step from γ to $\gamma+1$ we take logarithms and use Proposition 7.4. Let now $\gamma < \alpha$ be an infinite limit ordinal. Proposition 7.8 (and an inductive assumption for the second equality below) give

$$\begin{aligned}\big(\ell_\gamma \circ (g\circ h)\big)' &= \big(\ell_\gamma' \circ (g\circ h)\big)(g'\circ h)h' = \big((\ell_\gamma'\circ g)\circ h\big)\big)(g'\circ h)h' \\ &= \big(\big((\ell_\gamma'\circ g)g'\big)\circ h\big)h' = \big((\ell_\gamma\circ g)'\circ h\big)h' = \big((\ell_\gamma \circ g)\circ h\big)'\end{aligned}$$

so it remains to check that $\ell_\gamma\circ(g\circ h)$ and $(\ell_\gamma\circ g)\circ h$ have the same constant terms. This follows from Corollary 7.13. □

Characterizing composition recursively. The above shows that

$$(f,g)\mapsto f\circ g\ :\ \mathbb{L}_{<\alpha}\times\mathbb{L}_{<\alpha}^{>\mathbb{R}}\to\mathbb{L}_{<\alpha}$$

is a composition on $\mathbb{L}_{<\alpha}$ as defined in Section 4. We have also shown that

$$(f,g)\mapsto f\circ g\ :\ \mathbb{L}\times\mathbb{L}^{>\mathbb{R}}\to\mathbb{L}$$

is a composition on $\mathbb{L}$ as defined in the Introduction: it satisfies (CL1)–(CL5). This composition satisfies the following recursion:

Corollary 7.15. *Let $\gamma \geqslant \omega$ and $g\in\mathbb{L}^{>\mathbb{R}}$. Then*

$$\ell_\gamma\circ g \;=\; \int\big[\big(\ell_\gamma'\circ g\big)\,g'\big]-\lambda_{g;\gamma}.$$

Proof. By Corollary 7.10 the constant term of $\ell_\gamma\circ g$ equals $-\lambda_{g;\gamma}$. It remains to use the Chain Rule, Proposition 7.8. □

Note that $\ell_\gamma'\circ g = \big(\prod_{\beta<\gamma}\ell_\beta\circ g\big)^{-1}$ for $g\in\mathbb{L}^{>\mathbb{R}}$. In combination with the identities $\ell_{n+1}\circ g = \log(\ell_n\circ g)$ for such g, this gives us the right to speak of a recursion.

Corollary 7.16. *There is a unique composition $*$ on $\mathbb{L}$, namely $\circ$, that satisfies the above recursion: $\ell_\gamma * g = \int\big[\big(\ell_\gamma'\circ g\big)\,g'\big]-\lambda_{g;\gamma}$ for all $\gamma\geqslant\omega$ and $g\in\mathbb{L}^{>\mathbb{R}}$.*

Proof. Let $*$ be a composition on $\mathbb{L}$ that satisfies the above recursion. By (CL4) (strong linearity) it suffices that $\mathfrak{m}*g=\mathfrak{m}\circ g$ for $\mathfrak{m}\in\mathfrak{L}$ and $g\in\mathbb{L}^{>\mathbb{R}}$. By Lemma 4.8 this reduces to $\ell_\gamma * g = \ell_\gamma\circ g$ for such g. This is taken care of by the recursion and transfinite induction. □

8. Taylor Expansion and Compositional Inversion

As before, $\alpha=\omega^\lambda$, where λ is an infinite limit ordinal. In some proofs below we use the notation $\operatorname{supp} S := \bigcup_{f\in S}\operatorname{supp} f$ for $S\subseteq\mathbb{L}_{<\alpha}$.

Taylor expansion. In this subsection we fix $g,h\in\mathbb{L}_{<\alpha}$ with $g>\mathbb{R}$ and $h\prec g$. Our goal here is the following Taylor identity:

Proposition 8.1. *If $f\in\mathbb{L}_{<\alpha}$, then the sum $\sum_{n=0}^{\infty}\frac{f^{(n)}\circ g}{n!}h^n$ exists and*

$$f\circ(g+h) \;=\; \sum_{n=0}^{\infty}\frac{f^{(n)}\circ g}{n!}h^n.$$

Note that the assumption on h is weaker than $h\prec 1$; this weaker assumption works here because $\operatorname{supp}\partial_\alpha \preccurlyeq x^{-1}$. We need three lemmas:

Lemma 8.2. *If $f \in \mathbb{L}_{<\alpha}$, then $\sum_n \frac{f^{(n)}\circ g}{n!} h^n$ exists. The map $T : \mathbb{L}_{<\alpha} \to \mathbb{L}_{<\alpha}$ given by $T(f) := \sum_n \frac{f^{(n)}\circ g}{n!} h^n$ is an $\mathbb{L}_{<\alpha}$-composition with $g+h$.*

Proof. First note that $\mathfrak{S} := \operatorname{supp}\big((\operatorname{supp}\partial_\alpha)\circ g\big)$ is well-based: this is because $\operatorname{supp}\partial_\alpha = \{\ell_\beta^\dagger : \beta < \alpha\}$ is well-based, and the map $f \mapsto f\circ g : \mathbb{L}_{<\alpha} \to \mathbb{L}_{<\alpha}$ is strongly additive. Thus $\mathfrak{S}\cdot\operatorname{supp} h$ is well-based, and we claim that $\mathfrak{S}\cdot\operatorname{supp} h \prec 1$: this is because for $\mathfrak{m} \in \operatorname{supp}(\partial_\alpha)$ we have $\mathfrak{m} \preccurlyeq x^{-1}$, so $\mathfrak{m}\circ g \preccurlyeq x^{-1}\circ g = g^{-1}$, so for $\mathfrak{n} \in \mathfrak{S}$ we have $\mathfrak{n} \preccurlyeq g^{-1}$, and thus $\mathfrak{n}h \preccurlyeq g^{-1}h \prec 1$.

Next, let $f \in \mathbb{L}_{<\alpha}$ and $\mathfrak{m} \in \operatorname{supp}(f^{(n)}\circ g)$. Then $\mathfrak{m} \in \operatorname{supp}(\mathfrak{n}\circ g)$ with $\mathfrak{n} \in \operatorname{supp} f^{(n)}$, so $\mathfrak{n} = \mathfrak{n}_1\cdots\mathfrak{n}_n\mathfrak{v}$ with $\mathfrak{n}_1,\dots,\mathfrak{n}_n \in \operatorname{supp}(\partial_\alpha)$ and $\mathfrak{v} \in \operatorname{supp} f$, hence

$$\mathfrak{n}\circ g \;=\; (\mathfrak{n}_1\circ g)\cdots(\mathfrak{n}_n\circ g)\cdot(\mathfrak{v}\circ g),$$

which gives $\mathfrak{m} \in \mathfrak{S}^n\cdot\mathfrak{S}_f$, where $\mathfrak{S}_f := \operatorname{supp}\big((\operatorname{supp} f)\circ g\big)$. Thus we have shown:

$$\operatorname{supp}\big((f^{(n)}\circ g)h^n\big) \;\subseteq\; \big(\mathfrak{S}\cdot\operatorname{supp} h\big)^n\cdot\mathfrak{S}_f.$$

Now $\mathfrak{S}_f$ is well-based, so by Neumann's Lemma and what we proved about $\mathfrak{S}\cdot\operatorname{supp} h$ we may conclude that $\sum_n \frac{f^{(n)}\circ g}{n!} h^n$ does exist. The map T is clearly $\mathbb{R}$-linear with $T(1) = 1$ and $T(x) = g+h$. Let $(f_i)_{i\in I}$ in $\mathbb{L}_{<\alpha}$ be a summable family. Then $\bigcup_i \mathfrak{S}_{f_i}$ is well-based and the set $\{i \in I : \mathfrak{m} \in \mathfrak{S}_{f_i}\}$ is finite for every $\mathfrak{m} \in \mathfrak{L}_{<\alpha}$. It follows that $\sum_{n,i} \frac{f_i^{(n)}\circ g}{n!} h^n$ exists, and so $\sum_i T(f_i)$ exists as well, and both sums equal $T(\sum_i f_i)$. Thus T is strongly $\mathbb{R}$-linear. A routine computation using Lemma 2.4 also gives $T(f_1 f_2) = T(f_1)T(f_2)$ for $f_1, f_2 \in \mathbb{L}_{<\alpha}$. □

Lemma 8.3. *For $f \in \mathbb{L}_{<\alpha}^{>}$ we have $T(f) \sim f\circ g$ and $\log T(f) = T(\log f)$.*

Proof. For nonzero $f \in \mathbb{L}_{<\alpha}$ and $n \geqslant 1$ we have

$$\operatorname{supp}(f^{(n)}\circ g)h^n \;\prec\; \max\mathfrak{S}_f \;=\; \max\operatorname{supp}(f\circ g)$$

with notations from the proof of Lemma 8.2, and thus $T(f) \sim f\circ g$. In view of Proposition 7.4, the rest now follows as in the proof of Lemma 2.14, with $f \mapsto f\circ g$ in the role of Φ and h instead of ε. □

Lemma 8.4. *Let $\gamma \geqslant \omega$ and $\gamma < \alpha$. Then $T(\ell_\gamma) - \big(\ell_\gamma\circ(g+h)\big) \prec 1$.*

Proof. We have $\lambda_{g+h} = \lambda_g$, so by Lemma 7.9,

$$\ell_\gamma\circ g - \ell_\gamma\circ\ell_{\lambda_g} \prec 1, \qquad \ell_\gamma\circ(g+h) - \ell_\gamma\circ\ell_{\lambda_g} \prec 1,$$

and hence $\ell_\gamma\circ g - \ell_\gamma\circ(g+h) \prec 1$. Since $T(\ell_\gamma) = \ell_\gamma\circ g + \sum_{n=1}^\infty \frac{1}{n!}(\ell_\gamma^{(n)}\circ g)h^n$, it therefore suffices to show that $(\ell_\gamma^{(n)}\circ g)h^n \prec 1$ for all $n \geqslant 1$. Let $n \geqslant 1$ and let $\mathfrak{S}$ be as in the proof of Lemma 8.2. That proof for $f = \ell_\gamma'$ gives

$$\operatorname{supp}\big((\ell_\gamma^{(n)}\circ g)h^n\big) \;\subseteq\; \big(\mathfrak{S}\cdot\operatorname{supp} h\big)^{n-1}\cdot\operatorname{supp}\big((\operatorname{supp}\ell_\gamma')\circ g\big)\cdot(\operatorname{supp} h).$$

Now $\operatorname{supp}\ell_\gamma' = \{\ell_\gamma'\} \prec x^{-1}$, so $(\operatorname{supp}\ell_\gamma')\circ g \prec g^{-1}$. In view of $h \prec g$, this yields $\operatorname{supp}\big((\operatorname{supp}\ell_\gamma')\circ g\big)\cdot(\operatorname{supp} h) \prec 1$. Also $\mathfrak{S}\cdot\operatorname{supp} h \prec 1$, and thus $(\ell_\gamma^{(n)}\circ g)h^n \prec 1$. □

Proof of Proposition 8.1. Our job is to show that the above maps $f \mapsto f\circ(g+h)$ and T agree. By the strong linearity of these maps this reduces to $\mathfrak{m}\circ(g+h) = T(\mathfrak{m})$ for $\mathfrak{m} \in \mathfrak{L}_{<\alpha}$. Taking logarithms and using that these maps commute with taking logarithms (Lemma 8.3 and Proposition 7.4) this reduces further to $\ell_\gamma\circ(g+h) = T(\ell_\gamma)$ for $\gamma < \alpha$. We prove this by induction on γ. The case $\gamma = 0$ is obvious, and for the step from γ to $\gamma+1$ we use again that the two maps commute with taking logarithms. Let now $\gamma<\alpha$ be an infinite limit ordinal. A routine computation using Proposition 7.8 gives $T(\ell_\gamma)'=T(\ell_\gamma')(g+h)'$. Moreover, by Lemma 8.3,

$$\log T(\ell_\gamma') \;=\; T(\log \ell_\gamma') \;=\; T(-\sum_{\beta<\gamma}\ell_{\beta+1}) \;=\; -\sum_{\beta<\gamma} T(\ell_{\beta+1}),$$

and likewise $\log\big(\ell_\gamma'\circ(g+h)\big)=-\sum_{\beta<\gamma}\ell_{\beta+1}\circ(g+h)$, so $\log T(\ell_\gamma')=\log(\ell_\gamma'\circ(g+h))$ by the natural inductive assumption, hence $T(\ell_\gamma')=\ell_\gamma'\circ(g+h)$, and thus

$$T(\ell_\gamma)' \;=\; T(\ell_\gamma')(g+h)' \;=\; \big(\ell_\gamma'\circ(g+h)\big)\,(g+h)' \;=\; \big(\ell_\gamma\circ(g+h)\big)'$$

by the chain rule. It remains to check that $T(\ell_\gamma)$ and $\ell_\gamma\circ(g+h)$ have the same constant term. This follows from Lemma 8.4. □

This gives the existence part of Theorem 1.3: we just showed that our composition $\circ$ admits Taylor expansion as stated in that theorem, and the other three items are respectively contained in Corollary 5.9, Lemma 5.6, and Lemma 5.13.

Compositional Inversion. For $f,g\in\mathbb{L}_{<\alpha}^{>\mathbb{R}}$ we have $f\circ g\in\mathbb{L}_{<\alpha}^{>\mathbb{R}}$. Thus $\mathbb{L}_{<\alpha}^{>\mathbb{R}}$ is a monoid with respect to composition and with x as its identity element. Let us say that $g\in\mathbb{L}_{<\alpha}^{>\mathbb{R}}$ is (*compositionally*) *invertible* if $f\circ g=x$ for some $f\in\mathbb{L}_{<\alpha}$; note that such f is unique and satisfies $f\in\mathbb{L}_{<\alpha}^{>\mathbb{R}}$ and $g\circ f=x$, since $(g\circ f)\circ g=g\circ(f\circ g)=g$; we denote this unique f by g^{inv}. Note that if $f,g\in\mathbb{L}_{<\alpha}^{>\mathbb{R}}$ are invertible, then so are f^{inv} and $f\circ g$, with $(f\circ g)^{\text{inv}}=g^{\text{inv}}\circ f^{\text{inv}}$. Thus the invertible elements of $\mathbb{L}_{<\alpha}^{>\mathbb{R}}$ are exactly the elements of a group G_α with the group operation given by composition. Our goal here is to identify G_α as a subset of $\mathbb{L}_{<\alpha}^{>\mathbb{R}}$. By $\lambda_x=0$ and Lemma 7.11, $\lambda_f=0$ is necessary for $f\in\mathbb{L}_{<\alpha}^{>\mathbb{R}}$ to belong to G_α. It is also sufficient:

Proposition 8.5. *For $f\in\mathbb{L}_{<\alpha}^{>\mathbb{R}}$ we have: $f\in G_\alpha\Leftrightarrow\lambda_f=0$.*

We begin by considering the series *tangent to the identity*. These are the $x+h$ with $h\in\mathbb{L}_{<\alpha}^{\prec x}$. Fix such h and note that then

$$f\circ(x+h) \;=\; \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}h^n \qquad (f\in\mathbb{L}_{<\alpha})$$

and that the map $f\mapsto f\circ(x+h):\mathbb{L}_{<\alpha}\to\mathbb{L}_{<\alpha}$ equals $I+D$ where I is the identity map on $\mathbb{L}_{<\alpha}$ and the strongly $\mathbb{R}$-linear map $D:\mathbb{L}_{<\alpha}\to\mathbb{L}_{<\alpha}$ is given by $D(f)=\sum_{n=1}^{\infty}\frac{1}{n!}f^{(n)}h^n$. Now $\operatorname{supp}\partial_\alpha=\{\ell_\beta^\dagger:\ \beta<\alpha\}\preccurlyeq x^{-1}$, so $\operatorname{supp}\partial_\alpha$ is well-based. Moreover, $\operatorname{supp}h\prec x$, so D has well-based support

$$\operatorname{supp}D \;\subseteq\; \bigcup_{n=1}^{\infty}(\operatorname{supp}\partial_\alpha)^n\cdot(\operatorname{supp}h)^n \;\prec\; 1.$$

Thus by Lemma 2.12 the map $I+D$ on $\mathbb{L}_{<\alpha}$ is bijective with inverse $I+E$ where the strongly $\mathbb{R}$-linear map $E:\mathbb{L}_{<\alpha}\to\mathbb{L}_{<\alpha}$ is given by $E(f)=\sum_{n=1}^{\infty}(-1)^nD^n(f)$ and has well-based support $\operatorname{supp}E\subseteq\bigcup_{n=1}^{\infty}(\operatorname{supp}D)^n\prec 1$.

Lemma 8.6. *Let $h\in\mathbb{L}_{<\alpha}^{\prec x}$. Then the operator $f\mapsto f\circ(x+h):\mathbb{L}_{<\alpha}\to\mathbb{L}_{<\alpha}$ maps $x+\mathbb{L}_{<\alpha}^{\prec x}$ bijectively onto itself. In particular, $x+h$ is invertible.*

Proof. For $f=x+g$ with $g\in\mathbb{L}_{<\alpha}^{\prec x}$ we have

$$f\circ(x+h) \;=\; (x+g)\circ(x+h) \;=\; x+h+g^*$$

with $g^* = g \circ (x+h) \prec x \circ (x+h) = x+h \asymp x$, so the above operator does map $x + \mathbb{L}^{\prec x}_{<\alpha}$ injectively into itself. Conversely, with $g \in \mathbb{L}^{\prec x}_{<\alpha}$ we use the above inverse $I+E$ of $I+D$ to get $f := (I+E)(x+g)$ with $f \circ (x+h) = x+g$. It remains to note that $f = x+g+E(x+g)$ and $\operatorname{supp} E(x+g) \subseteq (\operatorname{supp} E)\operatorname{supp}(x+g) \prec x$, so $E(x+g) \prec x$. □

Thus if $f, g \in \mathbb{L}^{>\mathbb{R}}_{<\alpha}$ are tangent to the identity, then so are $f \circ g$ and g^{inv}:

$$G_{\alpha,1} \ := \ \{f \in \mathbb{L}^{>\mathbb{R}}_{<\alpha} : \ f \text{ is tangent to the identity}\} \text{ is a subgroup of } G_\alpha.$$

Below we improve this by showing that $G_{\alpha,1}$ is a *normal* subgroup of G_α.

Lemma 8.7. *Let $f \in G_\alpha$ and $r \in \mathbb{R}^>$. Then $rf, f^r \in G_\alpha$.*

Proof. From $f \circ f^{\mathrm{inv}} = x$ we get $rf \circ f^{\mathrm{inv}} = rx$. In view of $rx \circ r^{-1}x = x$, this gives $rf \circ (f^{\mathrm{inv}} \circ r^{-1}x) = x$. Likewise $f^r \circ (f^{\mathrm{inv}} \circ x^{1/r}) = x$, using $x^r \circ x^{1/r} = x$. □

For $f, g, h \in \mathbb{L}$ we use the notation $f = g + o(h)$ to mean $f - g \prec h$. So far we defined λ_g only for $g \in \mathbb{L}^{>\mathbb{R}}$. We now extend this to all $g \in \mathbb{L}^\times$ in the obvious way:

$$\lambda_g \ := \min \sigma(\mathfrak{d} g) \ \text{ if } \ \mathfrak{d} g \neq 1, \qquad \lambda_g \ := \ \infty \ \text{ if } \ \mathfrak{d} g = 1, \text{ with } \infty > \alpha \text{ for all } \alpha.$$

Lemma 8.8. *Let $\mathfrak{m} = \ell_1^{r_1} \ell_2^{r_2} \cdots = \prod_{1 \leqslant \beta < \alpha} \ell_\beta^{r_\beta} \in \mathfrak{L}_{<\alpha}$, and let $g \in \mathbb{L}^{>}_{<\alpha}$, $\lambda_g > 0$. Then $\mathfrak{m} \circ (xg) = \mathfrak{m} + o(\mathfrak{m})$, and thus $x\mathfrak{m} \circ (xg) = x\mathfrak{m} g + o(x\mathfrak{m} g)$.*

Proof. We have $\mathfrak{m} \circ (xg) = \prod_{1 \leqslant \beta < \alpha} \big(\ell_\beta \circ (xg)\big)^{r_\beta}$, so it suffices to show:

$$\ell_\beta \circ (xg) \ = \ \ell_\beta + o(\ell_\beta) \ \text{ for } 1 \leqslant \beta < \alpha.$$

For $\omega \leqslant \beta < \alpha$ this holds by Corollary 7.10. It holds for $\beta = 1$ by observing $\ell_1 \circ (xg) = \log(xg) = \ell_1 + \log g$ and $\log g \preccurlyeq \ell_2 \prec \ell_1$. An easy induction then gives $\ell_n \circ (xg) = \ell_n + o(\ell_n)$ for all $n \geqslant 1$. □

Corollary 8.9. *Let $G^1_\alpha := \{x\mathfrak{m}(1+\varepsilon) : \ \mathfrak{m} \in \mathfrak{L}_{<\alpha}, \ \lambda_{\mathfrak{m}} > 0, \ \varepsilon \in \mathbb{L}^{\prec 1}_{<\alpha}\}$. Then G^1_α is a subgroup of G_α.*

Proof. Note that $G^1_\alpha \supseteq G_{\alpha,1}$. Let $f, g \in G^1_\alpha$, so we have $\mathfrak{m}, \mathfrak{n} \in \mathfrak{L}_{<\alpha}$ with $\lambda_{\mathfrak{m}}, \lambda_{\mathfrak{n}} > 0$ and $\varepsilon_1, \varepsilon_2 \in \mathbb{L}^{\prec 1}_{<\alpha}$, such that

$$f \ = \ x\mathfrak{m}(1+\varepsilon_1), \quad g \ = \ x\mathfrak{n}(1+\varepsilon_2).$$

Now $x\mathfrak{m} \circ g = x\mathfrak{m} \circ x\mathfrak{n}(1+\varepsilon_2) = x\mathfrak{m}\mathfrak{n}(1+\varepsilon_2) + o(x\mathfrak{m}\mathfrak{n}) = x\mathfrak{m}\mathfrak{n} + o(x\mathfrak{m}\mathfrak{n})$ by Lemma 8.8, so $f \circ g = x\mathfrak{m}\mathfrak{n} + o(x\mathfrak{m}\mathfrak{n}) \in G^1_\alpha$. Given $f \in G^1_\alpha$ as before, and taking $\mathfrak{n} := \mathfrak{m}^{-1}$, the above shows that $f \circ x\mathfrak{n} = x + o(x) \in G_{\alpha,1}$, so $f \circ x\mathfrak{n} \circ h = x$ where $h \in G_{\alpha,1}$. Thus $f \in G_\alpha$ and $f^{\mathrm{inv}} = x\mathfrak{n} \circ h = x\mathfrak{n} + o(x\mathfrak{n}) \in G^1_\alpha$. □

Here is a useful way to summarize the proof of Corollary 8.9: let $f, g \in \mathbb{L}_{<\alpha}$ and $f = x\mathfrak{m} + o(x\mathfrak{m})$, $g = x\mathfrak{n} + o(x\mathfrak{n})$ with $\mathfrak{m}, \mathfrak{n} \in \mathfrak{L}_{<\alpha}$, $\lambda_{\mathfrak{m}}, \lambda_{\mathfrak{n}} > 0$. Then

$$f \circ g \ = \ x\mathfrak{m}\mathfrak{n} + o(x\mathfrak{m}\mathfrak{n}), \qquad f^{\mathrm{inv}} \ = \ x/\mathfrak{m} + o(x/\mathfrak{m}).$$

Proof of Proposition 8.5. Let $f \in \mathbb{L}^{>\mathbb{R}}_{<\alpha}$. The direction $f \in G_\alpha \Rightarrow \lambda_f = 0$ was already explained. For the converse, assume $\lambda_f = 0$; our job is to derive $f \in G_\alpha$. By Lemma 8.7 we can arrange that f has leading coefficient 1 with $\mathfrak{d} f = x\mathfrak{m}$ and $\mathfrak{m} = \prod_{1 \leqslant \beta < \alpha} \ell_\beta^{r_\beta}$, so $f = x\mathfrak{m} + o(x\mathfrak{m}) \in G^1_\alpha \subseteq G_\alpha$. □

The proof shows that $G_\alpha = \{ax^b \circ g :\ a, b \in \mathbb{R}^>,\ g \in G_\alpha^1\}$. Note in this connection that $G_{\mathbb{R},x} := \{ax^b :\ a, b \in \mathbb{R}^>\}$ is a subgroup of G_α:

$$ax^b \circ sx^t \ =\ as^b x^{bt} \qquad (a, b, s, t \in \mathbb{R}^>).$$

It is easy to see that G_α^1 is *not* a normal subgroup of G_α. On the other hand:

Corollary 8.10. *$G_{\alpha,1}$ is a normal subgroup of G_α.*

Proof. Let $f \in G_{\alpha,1}$. By the above description of G_α it suffices to show that $g \circ f \circ g^{\mathrm{inv}} \in G_{\alpha,1}$, for all $g = sx^t$ with $s, t \in \mathbb{R}^>$, and for all $g \in G_\alpha^1$. For $s, t \in \mathbb{R}^>$ and $g = sx^t$ we have $g^{\mathrm{inv}} = ax^b$ with $a = s^{-1/t}$ and $b = 1/t$, so with $f = x + o(x)$ we get $f \circ g^{\mathrm{inv}} = g^{\mathrm{inv}} + o(g^{\mathrm{inv}}) = ax^b(1+\varepsilon)$ with $\varepsilon \prec 1$, and thus

$$g \circ f \circ g^{\mathrm{inv}} \ =\ s(ax^b)^t(1+\varepsilon)^t \ =\ x(1+\varepsilon)^t \ =\ x + o(x) \in G_{\alpha,1}.$$

Next, let $g \in G_\alpha^1$, so $g = x\mathfrak{m} + o(x\mathfrak{m})$ with $\mathfrak{m} \in \mathfrak{L}_{<\alpha}$, $\lambda_{\mathfrak{m}} > 0$. Then $g^{\mathrm{inv}} = x/\mathfrak{m} + o(x/\mathfrak{m})$, so $f \circ g^{\mathrm{inv}} = (x + o(x)) \circ (x/\mathfrak{m} + o(x/\mathfrak{m})) = x/\mathfrak{m} + o(x/\mathfrak{m})$, and thus $g \circ f \circ g^{\mathrm{inv}} = (x\mathfrak{m} + o(x\mathfrak{m})) \circ (x/\mathfrak{m} + o(x/\mathfrak{m})) = x + o(x) \in G_{\alpha,1}$. □

9. Uniqueness, Embedding $\mathbb{L}$ into **No**, and Final Remarks

We continue to let $\circ$ denote the composition on $\mathbb{L}$ constructed in Sections 5 and 6. Corollary 7.16 characterizes this composition uniquely, but in the first subsection below we establish the more elegant characterization given by Theorem 1.3 from the introduction. Note that in Section 8 (end of first subsection) we already observed that $\circ$ witnesses the existence part of Theorem 1.3.

In the second subsection we indicate the natural embedding of $\mathbb{L}$ into **No**, and in the last subsection we finish with some remarks.

Uniqueness. Let $*$ denote any composition on $\mathbb{L}$ and let f, g, h range over $\mathbb{L}$.

Lemma 9.1. *Let $f \in \mathbb{L}_{<\omega}$ and $g > \mathbb{R}$. Then $f * g = f \circ g$.*

Proof. By induction on n and using (CL2), (CL3) we obtain

$$\ell_n * g \ = \log_n(g) \ =\ \ell_n \circ g.$$

Hence for $\mathfrak{m} = \prod_n \ell_n^{r_n} \in \mathfrak{L}_{<\omega}$ we have $\mathfrak{m} * g = \prod_n \log_n(g)^{r_n} = \mathfrak{m} \circ g$ by Lemma 4.8. The rest is an application of (CL4) (strong linearity). □

We say that $*$ *obeys the Chain Rule* if $(f * g)' = (f' * g) \cdot g'$ for all f, g with $g > \mathbb{R}$. We say that $*$ *admits Taylor expansion* if for all f, g, h with $g > \mathbb{R}$ and $h \prec g$ the sum $\sum_n \frac{f^{(n)} \circ g}{n!} h^n$ exists and equals $f * (g + h)$. Note that if $*$ admits Taylor expansion, then, with ε ranging over (sufficiently small) nonzero elements of $\mathbb{L}$,

$$f' \ =\ \lim_{\varepsilon \to 0} \frac{f * (x + \varepsilon) - f}{\varepsilon}.$$

Lemma 9.2. *If $*$ admits Taylor expansion, then it obeys the chain rule.*

Proof. Assume $*$ admits Taylor expansion, and let $g > \mathbb{R}$. Then

$$f' * g \ =\ \lim_{\varepsilon \to 0} \frac{f * (g + \varepsilon) - f * g}{\varepsilon}$$

as is easily verified. The usual argument shows that then $*$ obeys the chain rule: for all sufficiently small $\varepsilon \neq 0$ we have $g * (x + \varepsilon) \neq g$ and

$$\frac{(f * g) * (x + \varepsilon) - f * g}{\varepsilon} \ =\ \frac{f * (g * (x + \varepsilon)) - f * g}{g * (x + \varepsilon) - g} \cdot \frac{g * (x + \varepsilon) - g}{\varepsilon}.$$

Now $g*(x+\varepsilon)=g+\varepsilon_g$ with $\varepsilon_g\to 0$ as $\varepsilon\to 0$; thus letting ε go to 0 in the above displayed equality yields $(f*g)'=(f'*g)\cdot g'$. □

In the rest of this subsection we assume that $*$ admits Taylor expansion and has the following property: for all β,γ,

- $\ell_\gamma * \ell_{\omega^\beta} \;=\; \ell_{\omega^\beta+\gamma}$ *if* $\gamma<\omega^{\beta+1}$;
- $\ell_{\omega^{\beta+1}} * \ell_{\omega^\beta} \;=\; \ell_{\omega^{\beta+1}}-1$;
- $\ell_{\omega^\gamma} * \ell_{\omega^\beta}$ *has constant term* 0 *if* $\gamma>\beta$ *is a limit ordinal.*

We have to derive that then $f*g=f\circ g$, where $g>\mathbb{R}$. Here is the main lemma:

Lemma 9.3. *If $\rho>\omega^{\beta+1}$, then $\ell_\rho*\ell_{\omega^\beta}=\ell_\rho+\varepsilon_\rho$ with $\varepsilon_\rho\prec 1$.*

Proof. Set $\mu=\omega^{\beta+1}$. For $\rho=\mu+1$ we have

$$\begin{aligned}\ell_\rho*\ell_{\omega^\beta} &= \log(\ell_\mu*\ell_{\omega^\beta}) \;=\; \log(\ell_\mu-1)\\ &= \log\big(\ell_\mu(1-\ell_\mu^{-1})\big) \;=\; \log(\ell_\mu)+\log(1-\ell_\mu^{-1})\\ &= \ell_\rho+\varepsilon_\rho, \qquad \varepsilon_\rho\asymp\ell_\mu^{-1}\prec 1.\end{aligned}$$

Next, let $\rho>\mu+1$, and assume inductively that for every ordinal ν with $\mu<\nu<\rho$ we have $\ell_\nu*\ell_{\omega^\beta}=\ell_\nu+\varepsilon_\nu$ with $\varepsilon_\nu\prec 1$, so $\ell_\nu*\ell_{\omega^\beta}=\ell_\nu(1+h_\nu)$ with $h_\nu\prec\ell_\nu^{-1}$. Take $\gamma\geqslant\beta+1$ such that $\omega^\gamma\leqslant\rho<\omega^{\gamma+1}$. We distinguish three cases; only in the second case do we use the full inductive assumption.

Case $\rho=\omega^\gamma$ and γ is a successor ordinal. Then $\gamma=\xi+1$, $\xi\geqslant\beta+1$ and from $\ell_\rho=(\ell_\rho*\ell_{\omega^\xi})+1$ and $\ell_{\omega^\xi}*\ell_{\omega^\beta}=\ell_{\omega^\xi}+\varepsilon$ with $\varepsilon\preccurlyeq 1$, we obtain

$$\begin{aligned}\ell_\rho*\ell_{\omega^\beta} &= \big((\ell_\rho*\ell_{\omega^\xi})+1\big)*\ell_{\omega^\beta} \;=\; \big(\ell_\rho*(\ell_{\omega^\xi}*\ell_{\omega^\beta})\big)+1\\ &= \big((\ell_\rho*(\ell_{\omega^\xi}+\varepsilon)\big)+1 \;=\; \left(\sum_{n=0}^{\infty}\frac{\ell_\rho^{(n)}*\ell_{\omega^\xi}}{n!}\varepsilon^n\right)+1\\ &= \ell_\rho+\varepsilon_\rho, \quad \varepsilon_\rho\prec 1,\end{aligned}$$

since $\ell_\rho^{(n)}\prec 1$ for $n\geqslant 1$.

Case $\rho=\omega^\gamma$ and γ is a limit ordinal. By Lemma 9.2 the composition $*$ obeys the Chain Rule, so by our assumption that the constant term of $\ell_\rho*\ell_{\omega^\beta}$ is 0:

$$\ell_\rho*\ell_{\omega^\beta} \;=\; \int[(\ell_\rho'*\ell_{\omega^\beta})\cdot\ell_{\omega^\beta}'].$$

Now $\ell_\rho'=\prod_{\nu<\rho}\ell_\nu^{-1}$ and likewise for ℓ_{ω^β}', so

$$\begin{aligned}(\ell_\rho'*\ell_{\omega^\beta})\cdot\ell_{\omega^\beta}' &= \big((\prod_{\nu<\rho}\ell_\nu^{-1})*\ell_{\omega^\beta}\big)\cdot\prod_{\nu<\omega^\beta}\ell_\nu^{-1}\\ &= \prod_{\nu<\mu}(\ell_\nu^{-1}*\ell_{\omega^\beta})\cdot\prod_{\mu\leqslant\nu<\rho}(\ell_\nu^{-1}*\ell_{\omega^\beta})\cdot\prod_{\nu<\omega^\beta}\ell_\nu^{-1}\\ &= \prod_{\nu<\mu}\ell_{\omega^\beta+\nu}^{-1}\cdot\prod_{\mu\leqslant\nu<\rho}(\ell_\nu^{-1}*\ell_{\omega^\beta})\cdot\prod_{\nu<\omega^\beta}\ell_\nu^{-1}\\ &= \prod_{\nu<\mu}\ell_\nu^{-1}\cdot\prod_{\mu\leqslant\nu<\rho}(\ell_\nu^{-1}*\ell_{\omega^\beta})\\ &= \big(\prod_{\nu<\mu}\ell_\nu^{-1}\big)\cdot(\ell_\mu^{-1}*\ell_{\omega^\beta})\cdot\prod_{\mu<\nu<\rho}(\ell_\nu^{-1}*\ell_{\omega^\beta}).\end{aligned}$$

Now $\ell_\mu * \ell_{\omega^\beta} = \ell_\mu - 1$ gives $\ell_\mu^{-1} * \ell_{\omega^\beta} = \ell_\mu^{-1}(1-\ell_\mu^{-1})^{-1}$. Together with the equality derived above and the inductive assumption this yields

$$(\ell'_\rho * \ell_{\omega^\beta}) \cdot \ell'_{\omega^\beta} \;=\; \ell'_\rho(1-\ell_\mu^{-1})^{-1} \prod_{\mu<\nu<\rho} (1+h_\nu)^{-1} \;=\; \ell'_\rho(1+h)$$

with $h \prec \ell_\rho^{-2}$. Hence $\ell_\rho * \ell_{\omega^\beta} = \ell_\rho + \int \ell'_\rho h$. Now $\int \ell'_\rho \ell_\rho^{-2} = -\ell_\rho^{-1}$, so $\int \ell'_\rho h = \varepsilon_\rho$ with $\varepsilon_\rho \prec \ell_\rho^{-1} \prec 1$.

Case $\rho > \omega^\gamma$. Then $\rho = \omega^\gamma + \nu$ where $0 < \nu < \omega^{\gamma+1}$, so $\ell_\rho = \ell_\nu * \ell_{\omega^\gamma}$. Now $\ell_{\omega^\gamma} * \ell_{\omega^\beta} = \ell_{\omega^\gamma} + \varepsilon$ with $\varepsilon = -1$ if $\gamma = \beta+1$ and $\varepsilon \prec 1$ if $\gamma > \beta + 1$. Thus

$$\begin{aligned} \ell_\rho * \ell_{\omega^\beta} \;&=\; \ell_\nu * (\ell_{\omega^\gamma} * \ell_{\omega^\beta}) \;=\; \ell_\nu * (\ell_{\omega^\gamma} + \varepsilon) \;=\; \sum_{n=0}^\infty \frac{\ell_\nu^{(n)} * \ell_{\omega^\gamma}}{n!}\varepsilon^n \\ &=\; (\ell_\nu * \ell_{\omega^\gamma}) + \varepsilon_\rho \;=\; \ell_\rho + \varepsilon_\rho \quad \text{where } \varepsilon_\rho \prec 1. \end{aligned}$$

□

Lemma 9.4. $f * \ell_{\omega^\beta} = f \circ \ell_{\omega^\beta}$.

Proof. By the usual reductions it suffices to verify the identity for hyperlogarithms $f = \ell_\rho$. For $\rho \leqslant \omega^{\beta+1}$ our assumptions on $*$ take care of this. Let $\rho > \omega^{\beta+1}$ and assume inductively that $\ell_\nu * \ell_{\omega^\beta} = \ell_\nu \circ \ell_{\omega^\beta}$ for all $\nu < \rho$. Then by the chain rule, $(\ell_\rho * \ell_{\omega^\beta})' = (\ell_\rho \circ \ell_{\omega^\beta})'$. By Lemma 9.3, $\ell_\rho * \ell_{\omega^\beta}$ and $\ell_\rho \circ \ell_{\omega^\beta}$ both have the constant term 0, so they are equal. □

We now finish the proof that $f * g = f \circ g$ for all f, g with $g > \mathbb{R}$. First, for nonzero γ we have $\gamma = \omega^{\beta_1} + \cdots + \omega^{\beta_k}$ with $\beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant \beta_k$, $k \geqslant 1$. For $k=1$ we have $f * \ell_\gamma = f \circ \ell_\gamma$ by the last lemma. For $k > 1$ we have $\gamma = \omega^{\beta_1} + \nu$ with $\nu = \omega^{\beta_2} + \cdots + \omega^{\beta_k} < \omega^{\beta_1+1}$, and thus

$$f * \ell_\gamma \;=\; f * (\ell_\nu * \ell_{\omega^{\beta_1}}) \;=\; (f * \ell_\nu) * \ell_{\omega^{\beta_1}} \;=\; (f \circ \ell_\nu) \circ \ell_{\omega^{\beta_1}} \;=\; f \circ \ell_\gamma,$$

where for the third equality we use an obvious induction assumption on k. We have now shown that $f * \ell_\gamma = f \circ \ell_\gamma$ for all f and γ.

Next, let $f \in \mathbb{L}_{\geqslant\omega}$ and $g > \mathbb{R}$. In Section 6 we defined $f^{\uparrow 3} \in \mathbb{L}_{\geqslant\omega}$ and observed that $\log_3(g) = \ell_\gamma + \varepsilon$ with $\gamma = \lambda_g + 3$ and $\varepsilon \prec 1$. Then $f = f^{\uparrow 3} \circ \ell_3 = f^{\uparrow 3} * \ell_3$. Using also $\ell_3 \circ g = \log_3(g) = \ell_3 * g$ we obtain

$$\begin{aligned} f \circ g \;&=\; (f^{\uparrow 3} \circ \ell_3) \circ g \;=\; f^{\uparrow 3} \circ (\ell_3 \circ g) \;=\; f^{\uparrow 3} \circ (\ell_\gamma + \varepsilon), \\ f * g \;&=\; (f^{\uparrow 3} * \ell_3) * g \;=\; f^{\uparrow 3} * (\ell_3 * g) \;=\; f^{\uparrow 3} * (\ell_\gamma + \varepsilon), \end{aligned}$$

which by Taylor expansion yields $f \circ g = f * g$.

Finally, for arbitrary f, g with $g > \mathbb{R}$ we have $f = \sum_{\mathfrak{m} \in \mathfrak{L}_{<\omega}} f_{[\mathfrak{m}]} \mathfrak{m}$ where all $f_{[\mathfrak{m}]}$ lie in $\mathbb{L}_{\geqslant\omega}$. In view of Lemma 9.1 this gives

$$f \circ g \;=\; \sum_{\mathfrak{m} \in \mathfrak{L}_{<\omega}} (f_{[\mathfrak{m}]} \circ g)(\mathfrak{m} \circ g) \;=\; \sum_{\mathfrak{m} \in \mathfrak{L}_{<\omega}} (f_{[\mathfrak{m}]} * g)(\mathfrak{m} * g) \;=\; f * g.$$

This concludes the proof of Theorem 1.3. Note that for the above proof of $* = \circ$ we only needed the Taylor identity for $f * (g+h)$ with $g > \mathbb{R}$ and $h \preccurlyeq 1$.

Embedding $\mathbb{L}$ into No. We use here [3] and its notations, viewing **No** as a logarithmic-exponential field extension of $\mathbb{R}$ in the usual way. In that paper we defined $\log_\alpha \omega := \omega^{\omega^{-\alpha}} \in \mathbf{No}$, thinking of it as the "α times iterated function log evaluated at ω" in view of $\log(\log_\alpha \omega) = \log_{\alpha+1} \omega$. Berarducci and Mantova [5] constructed a derivation on **No** which in [3] and here we denote by ∂_{BM}.

Proposition 9.5. *There is a unique ordered field embedding $\iota : \mathbb{L} \to \mathbf{No}$ such that:*

(i) *ι is the identity on $\mathbb{R}$ and $\iota(\ell_\alpha) = \log_\alpha \omega$ for all α;*
(ii) *for every summable family $(f_i)_{i\in I}$ in $\mathbb{L}$ the family $\iota(f_i)$ is summable in* **No** *and $\iota(\sum_i f_i) = \sum_i \iota(f_i)$;*
(iii) *$\iota(\log f) = \log \iota(f)$ for all $f \in \mathbb{L}^>$.*

This embedding also preserves the derivation: $\iota(f') = \partial_{\mathrm{BM}}(\iota(f))$ for all $f \in \mathbb{L}$.

Proof. In [3, Section 2] we defined for any summable family (a_i) in **No** the product $\prod_i \omega^{a_i} := \omega^{\sum_i a_i}$. In [3, remarks preceding lemma 3.3] we defined $a^r := \exp(r \log a)$ for $a \in \mathbf{No}^>$ and $r \in \mathbb{R}$, and recorded the fact that $(\log_\alpha \omega)^r = \omega^{r\omega^{-\alpha}}$ for $r \in \mathbb{R}$. Thus for any logarithmic hypermonomial $\prod_{\beta<\alpha} \ell_\beta^{r_\beta}$ of $\mathbb{L}$ we have a product

$$\prod_{\beta<\alpha} (\log_\beta \omega)^{r_\beta} \;=\; \prod_{\beta<\alpha} \omega^{r_\beta \omega^{-\beta}} \;=\; \omega^{\sum_{\beta<\alpha} r_\beta \omega^{-\beta}}.$$

It is routine to check that this yields a unique $\mathbb{R}$-linear map $\iota : \mathbb{L} \to \mathbf{No}$ such that for every logarithmic hypermonomial $\prod_{\beta<\alpha} \ell_\beta^{r_\beta}$ we have

$$\iota\Big(\prod_{\beta<\alpha} \ell_\beta^{r_\beta}\Big) \;=\; \prod_{\beta<\alpha} (\log_\beta \omega)^{r_\beta}$$

and for every summable family $(f_i)_{i\in I}$ in $\mathbb{L}$ the family $\iota(f_i)$ is summable in **No** and $\iota(\sum_i f_i) = \sum_i \iota(f_i)$. It is easy to verify that this map ι is an ordered field embedding satisfying (i) and (ii). It also satisfies (iii) in view of [5, Lemma 2.3]. As to uniqueness, let i also be an ordered field embedding satisfying (i), (ii), (iii) with i instead of ι. Then for $\mathfrak{m} = \prod_{\beta<\alpha} \ell_\beta^{r_\beta}$ we have $\log \mathfrak{m} = \sum_{\beta<\alpha} r_\beta \ell_{\beta+1}$. Therefore, using again [5, Lemma 2.3],

$$\log i(\mathfrak{m}) \;=\; i(\log \mathfrak{m}) \;=\; \sum_{\beta<\alpha} r_\beta \log_{\beta+1} \omega \;=\; \log \prod_{\beta<\alpha} (\log_\beta \omega)^{r_\beta},$$

so $i(\mathfrak{m}) = \prod_{\beta<\alpha} (\log_\beta \omega)^{r_\beta} = \iota(\mathfrak{m})$. Thus $i = \iota$. That ι is also an embedding of differential fields with the derivation ∂_{BM} on **No** uses the fact that

$$\partial_{\mathrm{BM}}(\log_\alpha \omega) \;=\; 1/\prod_{\beta<\alpha} \log_\beta \omega,$$

for which we refer to [3, two lines before Lemma 2.10]. □

In [3] we also defined a canonical embedding of the differential field $\mathbb{T}$ into **No**, and we observe here that on $\mathbb{T}_{\log} = \mathbb{L}^{\cup}_{<\omega}$ this embedding agrees with the embedding of Proposition 9.5.

Final Remarks. One issue we didn't touch is the monotonicity of composition on the right: for $f, g, h \in \mathbb{L}^{>\mathbb{R}}$ with $g < h$, do we have $f \circ g < f \circ h$? We believe this to be true, and it would reflect how composition behaves for germs of functions in Hardy fields. But we only have proofs for special cases. It might be better to deal with this in the wider setting of the (conjectural) field $\mathbb{H}$ of all hyperseries where every positive infinite element should have a compositional inverse.

It would be interesting to represent right composition with various $g \in \mathbb{L}^{>\mathbb{R}}$ on certain subfields of $\mathbb{L}$ in the form $\mathrm{e}^{\phi\partial}$ for suitable $\phi \in \mathbb{L}$, as we did for $g = \ell_{\omega^\beta}$ in the remark following the proof of Lemma 5.1.

The identity $\ell_{\omega^{\beta+1}} \circ \ell_{\omega^\beta} = \ell_{\omega^{\beta+1}} - 1$ reflects a choice of integration constant -1. It is surely the most natural choice, but for any family (c_β) of real numbers there is a composition $*$ on $\mathbb{L}$ such that instead for all β,

$$\ell_{\omega^{\beta+1}} * \ell_{\omega^\beta} \;=\; \ell_{\omega^{\beta+1}} + c_\beta.$$

Such a composition $*$ is obtained by replacing (5.1) with

$$f * \ell_{\omega^\beta} \;:=\; \sum_{n=0}^{\infty} \frac{c_\beta^n}{n!} \delta^n f \qquad (f \in \mathbb{L}_{[\mu,\alpha)})$$

and following otherwise the definitions in Sections 5 and 6. Theorem 1.3 goes through for $*$ in the role of $\circ$, except that the above identity involving the constants c_β replaces "$\ell_{\omega^{\beta+1}} \circ \ell_{\omega^\beta} \;=\; \ell_{\omega^{\beta+1}} - 1$". The proofs for $\circ$ are easily adapted to $*$. Note that any $c_\beta \geqslant 0$ would give a failure of monotonicity of $*$ on the right.

Another topic is the connection to Hardy fields. Kneser [13] yields a real analytic function $\ell_{\mathrm{K}} : \mathbb{R} \to \mathbb{R}^{>}$ with $\ell_{\mathrm{K}}(\log t) = \ell_{\mathrm{K}}(t) - 1$ for $t > 0$; its germ at $+\infty$, also denoted by ℓ_{K} below, generates a Hardy field extension of $\mathbb{R}(x, \log x, \log_2 x, \dots)$ such that $\mathbb{R} < \ell_{\mathrm{K}} < \log_n(x)$ for all n, with x here the germ of the identity function on $\mathbb{R}$. Clearly, ℓ_{K} has ℓ_ω as a kind of formal counter part. In the appendix to [14], Schmeling constructs likewise for all $n > 1$ a real analytic function with ℓ_{ω^n} as a formal counter part. Much remains to be done to strengthen this connection. There is ongoing work along these lines with partial results announced in [4] .

References

[1] M. Aschenbrenner, L. van den Dries, J. van der Hoeven, *Differentially algebraic gaps*, Selecta Mathematica, 11: 247–280, 2005.

[2] ——— *Asymptotic Differential Algebra and Model Theory of Transseries*, Ann. of Math. Stud. 195, Princeton U. Press, arXiv:1509.02588.

[3] ——— *The surreal numbers as a universal H-field*, J. Eur. Math. Soc. (JEMS), to appear.

[4] ——— *On Numbers, Germs, and Transseries*, Proc. Int. Cong. of Math.–2018, Rio de Janeiro, Vol. 2 (19–43).

[5] A. Berarducci, V. Mantova, *Surreal numbers, derivations, and transseries*, J. Eur. Math. Soc. (JEMS), 20: 339–390, 2018.

[6] ——— *Transseries as germs of surreal functions*, Trans. Amer. Math. Soc., to appear.

[7] L. Comtet, *Advanced Combinatorics*, D. Reidel Publishing Co., Dordrecht, 1974.

[8] B. Dahn, P. Göring, *Notes on exponential-logarithmic terms*, Fund. Math. **127** (1987), 45–50.

[9] J. Écalle, *Introduction aux Fonctions Analysables et Preuve Constructive de la Conjecture de Dulac*, Actualités Mathématiques, Hermann, Paris, 1992.

[10] J. van der Hoeven, *Asymptotique Automatique*, Thèse, École Polytechnique, Paris, 1997.

[11] ——— *Operators on generalized power series*, Illinois Journal of Mathematics, 45(4):1161–1190, 2001.

[12] ——— *Transseries and Real Differential Algebra*, Lecture Notes in Math., vol. 1888, Springer-Verlag, New York, 2006.

[13] H. Kneser, *Reelle analytische Lösungen der Gleichung $\phi(\phi(x)) = e^x$ und verwandter Funktionalgleichungen*, J. Reine Angew. Math. **187** (1950), 56–67.

[14] M. C. Schmeling, *Corps de Transséries*, Ph. D. thesis, Université Paris-VII, 2001.

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