D-algebraic power series

Joris van der Hoeven

(Paris, 27 january 1999)

†

In this paper, we study power series in several variables, which are completely determined as solutions of sets of differential equations with initial conditions. We introduce a large class of such power series, which we call "D-algebraic". We prove several closure properties for this class and give an effective zero test for D-algebraic series.

1. Introduction

A popular topic in computer algebra is finding closed form solutions to systems of differential equations. Unfortunately, such solutions rarely exist. Nevertheless, one often wishes to do some further computations with the solutions of such systems of equations, even if no closed form solutions exist. The next best thing to do then is to find a more implicit way to represent the solutions and to show how to do computations with the solutions via these representations.

Adopting this point of view, two main problems arise. First, the class of representable solutions should be made as large as possible and it should have as many closure properties as possible. Secondly, it should still be possible to perform all operations in the class effectively. In particular, we need an effective zero test.

Now in the case of analytic functions, it suffices to check whether an expression is locally zero in order to test whether it vanishes globally. In this paper, we are therefore interested in giving a zero test for expressions in power series, which are defined by algebraic differential equations and suitable initial conditions.

Some theoretically effective results in this direction were first obtained in (Denef and Lipshitz, 1989). Subsequently, more practical algorithms for zero testing were obtained in (Shackell, 1989; Shackell, 1993; Péladan-Germa, 1995) (see also (Péladan-Germa, 1997; van der Hoeven, 1997a). Unfortunately, the class of power series to which these latter algorithms apply does not admit many closure properties. Therefore, in this paper, we introduce a larger class of so called D-algebraic power series (which includes even divergent series), which does admit several closure properties. Yet, we are still able to generalize Shackell's first zero-equivalence algorithm from (Shackell, 1993) to this larger class. In a forthcoming paper we will describe an even larger class of D-implicit series in which an implicit function theorem holds.

† The difference between the dates in the title and on the cover is due to the fact that this paper was originally submitted to a journal. Although it was refused there, there has been some interest for this work afterwards. This made me decide to publish this preprint a long time after its time of writing.

2. Definition and examples of D-algebraic series

In all what follows, K is assumed to be an effective field of characteristic zero, i.e. all field operations including the zero test can be performed effectively. We also assume that all integer solutions to polynomial equations over K can be found. This holds for instance for the rationals.

DEFINITION 2.1. *An effective representation D-ring for power series in $z_1, \ldots, z_k$ over K is a polynomial ring $R = K[f_1, \ldots, f_n]$ together with effective derivations $d_1, \ldots, d_k : R \to R$ and a K-algebra homomorphism $\varepsilon : R \to K[[z_1, \ldots, z_k]]$, called the evaluation mapping, such that*

1. *For each $1 \leqslant i \leqslant k$, there exists a μ_i with $\varepsilon(d_i g) = z_i^{\mu_i} \frac{\partial \varepsilon(g)}{\partial z_i}$ for all $g \in R$.*
2. *There exists an algorithm, which given $g \in R$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{N}$ computes the coefficient of $z_1^{\alpha_1} \cdots z_k^{\alpha_k}$ in $\varepsilon(g)$.*

REMARK. In order to specify an effective representation D-ring R, it suffices to define the derivations $d_1, \ldots, d_k$ and the evaluation mapping ε for $f_1, \ldots, f_k$.

REMARK. Without loss of generality, one may always assume that R contains elements z_i^R with $\varepsilon(z_i^R) = z_i$ for $1 \leqslant i \leqslant k$. Indeed, assume for instance that no z_k^R with $\varepsilon(z_k^R)$ exists. Then we may extend R into $\hat{R} = K[f_1, \ldots, f_n, z_k^R]$, using the previous remark, by defining $d_1, \ldots, d_k$ and ε as in R on $f_1, \ldots, f_n$, and by setting $d_1 z_k^R = \cdots = d_{k-1} z_k^R = 0, d_k z_k^R = (z_k^R)^{\mu_k}$ and $\varepsilon(z_k^R) = z_k$.

DEFINITION 2.2. *An effective D-algebraic series in $z_1, \ldots, z_k$ is a series φ which can be represented by an element φ^R of an effective representation D-ring R; i.e. $\varepsilon(\varphi^R) = \varphi$.*

In what follows, *we will shortly write representation D-ring resp. D-algebraic power series for effective representation D-ring resp. effective D-algebraic power series*, since we will only be working in an effective context.

EXAMPLE. The series $\sin z_1 = 1 - \frac{1}{2} z_1^2 + \frac{1}{24} z_1^4 + \cdots$ is D-algebraic, since we may represent it by f_1 in the representation D-ring $R = K[f_1, f_2]$ with derivation $d_1 f_1 = f_2, d_1 f_2 = -f_1$ and evaluation $\varepsilon(f_1) = \sin z_1, \varepsilon(f_2) = \cos z_1$.

EXAMPLE. For each k, the series $\exp(z_1 + \cdots + z_k)$ is D-algebraic, since we may represent it by f_1 in the representation D-ring $R = K[f_1]$ with derivations $d_i f_1 = f_1$.

EXAMPLE. The divergent series $1 + z + 2z^2 + 6z^3 + 24z^4 + \cdots$ is D-algebraic. Indeed, we may represent it by f_1 in the representation D-ring $R = K[f_1, z_1^R]$ with derivation $d_1 f_1 = (1 - z_1^R) f_1 + 1$. In this case, $\varepsilon(d_1 \cdot) = z_1^2 \frac{\partial \varepsilon(\cdot)}{\partial z_1}$.

3. Closure properties for D-algebraic series

PROPOSITION 3.1. *The D-algebraic power series in $z_1, \ldots, z_k$ form a K-algebra.*

PROOF. Clearly, elements in K are D-algebraic. Assume that $R = K[f_1, \ldots, f_n]$ and $R' = K[f'_1, \ldots, f'_{n'}]$ are representation D-rings, with derivations $d_1, \ldots, d_k$ resp. $d'_1, \ldots, d'_k$ and evaluation mappings ε resp. ε'. We will prove that both D-rings can be embedded in common super representation D-ring $\hat{R}$. This will prove in particular that sums and products of D-algebraic power series are again D-algebraic.

Let the μ_i and μ'_i be such that

$$\begin{aligned}\varepsilon(d_i \cdot) &= z_i^{\mu_i} \frac{\partial \varepsilon(\cdot)}{\partial z_i}; \\ \varepsilon(d'_i \cdot) &= z_i^{\mu'_i} \frac{\partial \varepsilon'(\cdot)}{\partial z_i}.\end{aligned}$$

Modulo substitution of d_i by $(z_i^R)^{\max(\mu_i,\mu'_i)-\mu_i} d_i$ and d'_i by $(z_i^{R'})^{\max(\mu_i,\mu'_i)-\mu'_i} d'_i$, we may assume without loss of generality that $\mu_i = \mu'_i$ for each i. We now take $\hat{R} = K[f_1, \ldots, f_n, f'_1, \ldots, f'_{n'}]$ and we define the derivations and the evaluation mapping on the f_j as in R and on the f'_j as in R'. □

PROPOSITION 3.2. *Let φ be an invertible D-algebraic power series in $z_1, \ldots, z_k$. Then so is its inverse φ^{-1}.*

PROOF. Let φ be represented by φ^R in $R = K[f_1, \ldots, f_n]$. Then φ^{-1} is represented by g in the extension $\hat{R} = K[f_1, \ldots, f_n, g]$ of R, in which the derivatives of g are given by $\hat{d}_i g = (d_i \varphi^R) g^2$ and its evaluation by $\hat{\varepsilon}(g) = \varphi^{-1}$. □

PROPOSITION 3.3. *The class of D-algebraic power series in $z_1, \ldots, z_k$ is stable under $\frac{\partial}{\partial z_k}$.*

PROOF. Let $R = K[f_1, \ldots, f_n]$ be a representation D-ring such as in definition 2.1 and let us show how to extend R into a representation D-ring $\hat{R}$ in which $\frac{\partial \varepsilon(g)}{\partial z_k}$ can be represented for any $g \in R$. We take $\hat{R} = K[f_1, \ldots, f_n, \hat{f}_1, \ldots, \hat{f}_n]$, with evaluation mapping $\hat{\varepsilon}$, such that $\hat{\varepsilon}(f_j) = \varepsilon(f_j)$ and $\hat{\varepsilon}(\hat{f}_j) = \frac{\partial \varepsilon(f_j)}{\partial z_k}$ for all j. The derivations $\hat{d}_i$ are defined on the f_j as the d_i in R. In order to define them on the $\hat{f}_j$, we first notice that, given $1 \leqslant j \leqslant n$, we may write $d_i f_j = \hat{d}_i f_j = P(f_1, \ldots, f_n)$ for some polynomial P, so that

$$\begin{aligned}\frac{\partial}{\partial z_k} \hat{\varepsilon}(\hat{d}_i f_j) &= \frac{\partial}{\partial z_k} P(\hat{\varepsilon}(f_1), \ldots, \hat{\varepsilon}(f_k)) \\ &= \hat{\varepsilon}\left(\frac{\partial P}{\partial f_1}\right) \frac{\partial \hat{\varepsilon}(f_1)}{\partial z_k} + \cdots + \hat{\varepsilon}\left(\frac{\partial P}{\partial f_n}\right) \frac{\partial \hat{\varepsilon}(f_n)}{\partial z_k} \\ &= \hat{\varepsilon}\left(\frac{\partial P}{\partial f_1} \hat{f}_1 + \cdots + \frac{\partial P}{\partial f_n} \hat{f}_n\right).\end{aligned}$$

If $i < k$, we now take

$$\hat{d}_i \hat{f}_j = \frac{\partial P}{\partial f_1} \hat{f}_1 + \cdots + \frac{\partial P}{\partial f_n} \hat{f}_n,$$

therefore

$$\begin{aligned}\hat{\varepsilon}(\hat{d}_i \hat{f}_j) &= \hat{\varepsilon}\left(\frac{\partial P}{\partial f_1}\hat{f}_1 + \cdots + \frac{\partial P}{\partial f_n}\hat{f}_n\right) \\ &= \frac{\partial}{\partial z_k}\varepsilon(\hat{d}_i f_j) \\ &= \frac{\partial}{\partial z_k}\left[z_i^{\mu_i}\frac{\partial \hat{\varepsilon}(f_j)}{\partial z_i}\right] = z_i^{\mu_i}\frac{\partial}{\partial z_i}\left[\frac{\partial \hat{\varepsilon}(f_j)}{\partial z_k}\right] = z_i^{\mu_i}\frac{\partial \hat{\varepsilon}(\hat{f}_j)}{\partial z_i}.\end{aligned}$$

If $i = k$, we take

$$\hat{d}_k \hat{f}_j = \frac{\partial P}{\partial f_1}\hat{f}_1 + \cdots + \frac{\partial P}{\partial f_n}\hat{f}_n - \mu_k (z_k^R)^{\mu_k - 1}\hat{f}_j.$$

Again, we obtain

$$\begin{aligned}\hat{\varepsilon}(\hat{d}_k \hat{f}_j) &= \hat{\varepsilon}\left(\frac{\partial P}{\partial f_1}\hat{f}_1 + \cdots + \frac{\partial P}{\partial f_n}\hat{f}_n - \mu_k (z_k^R)^{\mu_k-1}\hat{f}_j\right) \\ &= \frac{\partial}{\partial z_k}\hat{\varepsilon}(\hat{d}_k f_j) - \mu_k z_k^{\mu_k - 1}\hat{\varepsilon}(\hat{f}_j) \\ &= z_k^{\mu_k}\frac{\partial}{\partial z_k}\left[\frac{\hat{\varepsilon}(\hat{d}_k f_j)}{z_k^{\mu_k}}\right] = z_k^{\mu_k}\frac{\partial \hat{\varepsilon}(\hat{f}_j)}{\partial z_k}.\end{aligned}$$

□

PROPOSITION 3.4. *The coefficients in z_k of a D-algebraic power series in $z_1, \ldots, z_k$ is a D-algebraic power series in $z_1, \ldots, z_{k-1}$.*

PROOF. Let $R = K[f_1, \ldots, f_n]$ be a representation D-ring for power series in $z_1, \ldots, z_k$ and let $\nu \in \mathbb{N}$. We will construct a representation D-ring $\hat{R}$ for power series in $z_1, \ldots, z_{k-1}$, such that for each g in R and $\alpha \leqslant \nu$, the coefficient $\varepsilon(g)_\alpha$ of z_k^α in $\varepsilon(g)$ can be represented in $\hat{R}$. This will clearly prove the proposition.

We take $\hat{R} = K[f_{1,0}, \ldots, f_{1,\nu}, \ldots, f_{n,0}, \ldots, f_{n,\nu}]$ with evaluation mapping $\hat{\varepsilon} : \hat{R} \to K[[z_1, \ldots, z_{k-1}]]; f_{i,\alpha} \mapsto \varepsilon(f_i)_\alpha$. Then we have natural "extraction of coefficient mappings" $\rho_0, \ldots, \rho_\nu : R \to \hat{R}$, with $\rho_\alpha(f_j) = f_{j,\alpha}$, such that $\rho : g \mapsto \rho_0(g) + \cdots + \rho_\nu(g) z_k^\nu$ is a K-algebra homomorphism from R into $\hat{R}[z_k]/(z_k^{\nu+1})$.

We now define $\hat{d}_i$ by $\hat{d}_i(f_{j,\alpha}) = \rho_\alpha(d_i f_j)$. We indeed have $\hat{\varepsilon}(\hat{d}_i(f_{j,\alpha})) = \hat{\varepsilon}(\rho_\alpha(d_i f_j)) = \varepsilon(d_i f_j)_\alpha = (z_i^{\mu_i}\frac{\partial \varepsilon(f_j)}{\partial z_i})_\alpha = z_i^{\mu_i}\frac{\partial \varepsilon(f_j)_\alpha}{\partial z_i} = z_i^{\mu_i}\frac{\partial \hat{\varepsilon}(f_{j,\alpha})}{\partial z_i}$. □

The proof of the last proposition is based on an effective stabilization property, which will be proved in section 4.1.

PROPOSITION 3.5. *Let φ be a D-algebraic power series in $z_1, \ldots, z_k$, whose valuation in z_k is at least one. Then φz_k^{-1} is D-algebraic.*

PROOF. Let $R = K[f_1, \ldots, f_n]$ be a representation D-ring for φ as in definition 2.1. By proposition 4.3 from section 4.1, with $k' = k - 1$, we can compute a set of generators $\varphi_1^R, \ldots, \varphi_r^R$ for the smallest ideal of R, which contains φ and which is stable under $d_1, \ldots, d_{k-1}$. We notice that $\varphi_j = \varepsilon(\varphi_i^R)$ has valuation at least one in z_k, for each j, whence $z_k^{-1}\varphi_j$ is still a power series.

Now consider the representation D-ring $R = K[f_1, \ldots, f_n, g_1, \ldots, g_r]$ whose evaluation mapping $\hat{\varepsilon}$ is given by $\hat{\varepsilon}(f_j) = \varepsilon(f_j)$ for $1 \leqslant j \leqslant n$ and $\hat{\varepsilon}(g_j) = z_k^{-1}\varphi_j$ for $1 \leqslant j \leqslant r$. In particular, φz_k^{-1} will be represented in this ring.

For $i < k$, we take the derivation $\hat{d}_i$ on $\hat{R}$ to coincide with d_i on the f_j. As to $\hat{d}_i g_j$, we first recall that $d_i \varphi_j^R$ can be written as a linear combination

$$d_i \varphi_j^R = a_1 \varphi_1^R + \cdots + a_r \varphi_r^R,$$

with coefficients $a_1, \ldots, a_r \in R$. Then we take $\hat{d}_i g_j = a_1 g_1 + \cdots + a_r g_r$, so that

$$\begin{aligned} \hat{\varepsilon}(\hat{d}_i g_j) &= \hat{\varepsilon}(a_1 g_1 + \cdots + a_r g_r) \\ &= (\varepsilon(a_1)\varphi_1 + \cdots + \varepsilon(a_r)\varphi_r) z_k^{-1} \\ &= \varepsilon(d_i \varphi_j^R) z_k^{-1} = z_i^{\mu_i} z_k^{-1} \frac{\partial \varphi_j}{\partial z_i} = z_i^{\mu_i} \frac{\partial \hat{\varepsilon}(g_j)}{\partial z_i}. \end{aligned}$$

For $i = k$, we let $\hat{d}_k$ coincide with $z_k^R d_k$ on the f_j, so that $\hat{\varepsilon}(\hat{d}_k f_j) = z_k^{\mu_k+1} \frac{\partial \hat{\varepsilon}(f_j)}{\partial z_k}$. Next, we take $\hat{d}_k g_j = d_k \varphi_j^R - g_j (z_k^R)^{\mu_k}$, and again we obtain

$$\begin{aligned} \hat{\varepsilon}(\hat{d}_k g_j) &= \varepsilon(d_k \varphi_j^R) - \hat{\varepsilon}(g_j) z_k^{\mu_k} \\ &= z_k^{\mu_k} \frac{\partial \varphi_j}{\partial z_k} - \varphi_j z_k^{\mu_k - 1} \\ &= z_k^{\mu_k+1} \frac{\partial}{\partial z_k} \left(\frac{\varphi_j}{z_k} \right) = z_k^{\mu_k+1} \frac{\partial \hat{\varepsilon}(g_j)}{\partial z_k}. \end{aligned}$$

□

Example. From the proposition, it follows that the series $(\exp z_1 - 1)/z_1$ is D-algebraic in z_1. In previous settings, no zero tests were available for expressions involving such series.

4. Zero tests for D-algebraic series

In this section, we fix once and for all a representation D-ring $R = K[f_1, \ldots, f_n]$ as in definition 2.1 and we assume that $z_1^R, \ldots, z_k^R$ are elements which represent $z_1, \ldots, z_k$.

In section 4.2, we are interested in testing whether a D-algebraic series φ, which can be represented by an element φ^R in R vanishes. In section 4.3, we will describe a simultaneous zero test for D-algebraic power series $\xi_1, \ldots, \xi_s$, which can be represented by elements $\xi_1^R, \ldots, \xi_s^R$ in R. Throughout this section, we make the induction hypothesis that we can answer both questions for D-algebraic power series in less than k variables; we notice that both problems are trivial for $k = 0$.

4.1. Differential stabilization

One of the main ingredients for our zero test algorithms is the observation (due to Shackell (Shackell, 1993) for the one dimensional case) that the differential ideal generated by φ^R is finitely generated, since R is Noetherian. An easy consequence of this, which leads to our zero test algorithm from section 4.2, is the following:

Proposition 4.1. *Given $\varphi^R \in R$, we can compute a linear differential relation*

$$d_k^r \varphi^R = a_{r-1} d_k^{r-1} \varphi^R + \cdots + a_0 \varphi^R, \tag{4.1}$$

with $a_0, \ldots, a_{r-1} \in R$.

Proof. The chain of ideals $(\varphi^R), (\varphi^R, d_k \varphi^R), (\varphi^R, d_k \varphi^R, d_k^2 \varphi^R), \ldots$ stabilizes and we can check whether $d_k^r \varphi^R \in (\varphi^R, \ldots, d_k^{r-1} \varphi^R)$ by computing a Groebner basis of $(\varphi^R, \ldots, d_k^{r-1} \varphi^R)$. □

Our collective zero test algorithm from section 4.3 will be based on the following, more general result:

Proposition 4.2. *Given $\xi_1^R, \ldots, \xi_s^R \in R$, we can compute generators $\varphi_1^R, \ldots, \varphi_r^R \in R$ for the differential ideal $(d_k^i \varphi_j)_{i \in \mathbb{N}, 1 \leqslant j \leqslant s}$. There exists a $r \times r$ matrix M^R with coefficients in R, such that*

$$d_k \begin{pmatrix} \varphi_1^R \\ \vdots \\ \varphi_r^R \end{pmatrix} = M^R \begin{pmatrix} \varphi_1^R \\ \vdots \\ \varphi_r^R \end{pmatrix}. \tag{4.2}$$

Proof. We use the following algorithm: we start by computing a Groebner basis G for $(\xi_1^R, \ldots, \xi_s^R)$. Next, as long as there exists an element $g \in G$, such that $d_k g$ does not reduce to zero modulo G, we compute the set G' of such elements $d_k g$ and replace G by a Groebner basis for $G \cup G'$. At the end of this loop, the derivative $d_k g$ of each element g in G can be expressed as a linear combination of elements in G, which leads to the expression (4.2). □

Instead of working with respect with the derivation d_k only, one can also work with respect to a finite set of derivations:

Proposition 4.3. *Given $\xi_1^R, \ldots, \xi_s^R \in R$ and $k' \leqslant k$, we can compute a set of generators $\varphi_1^R, \ldots, \varphi_r^R \in R$ for the ideal generated by all $d_{i_1} \cdots d_{i_l} \varphi_j$, with $i_1, \ldots, i_l \leqslant k'$ and $1 \leqslant j \leqslant s$. There exists $r \times r$ matrices $M_1^R, \ldots, M_{k'}^R$ with coefficients in R, such that for each $1 \leqslant i \leqslant k'$*

$$d_i \begin{pmatrix} \varphi_1^R \\ \vdots \\ \varphi_r^R \end{pmatrix} = M_i^R \begin{pmatrix} \varphi_1^R \\ \vdots \\ \varphi_r^R \end{pmatrix}. \tag{4.3}$$

Proof. Analogue proof as the previous proposition: now G' consists of those elements $d_i g$ which do not reduce to zero modulo G, with $1 \leqslant i \leqslant k'$ and $g \in G$. □

4.2. A first zero test

Assume that we want to test whether a series φ, represented by $\varphi^R \in R$, vanishes. Application of proposition 4.1 and evaluation of (4.1) yields a relation

$$z_k^{(\mu_k - 1) r} \delta^r \varphi = \psi_{r-1} \delta^{r-1} \varphi + \cdots + \psi_0 \varphi,$$

where

$$\delta = z_k \frac{\partial}{\partial z_k}$$

and where $\psi_0, \ldots, \psi_{r-1}$ are D-algebraic power series, which can be represented by elements $\psi_0^R, \ldots, \psi_{r-1}^R$ in R. This relation can be rewritten as

$$\psi_r \delta^r \varphi + \cdots + \psi_0 \varphi = 0, \tag{4.4}$$

when setting $\psi_r = -z_k^{(\mu_k-1)r}$.

By proposition 3.4 and the induction hypothesis, we can test whether each coefficient of the form $\psi_{i,j}$ with $1 \leqslant i \leqslant r-1$ and $0 \leqslant j \leqslant (\mu_k - 1)r$ vanishes. Denoting by $\nu_0, \ldots, \nu_r$ the valuations of $\psi_0, \ldots, \psi_{r-1}$ in z_k, we can therefore compute

$$\nu = \min(\nu_0, \ldots, \nu_r) \leqslant (\mu_k - 1)r.$$

Now recall the rule $(\delta\xi)_\alpha = \alpha\xi_\alpha$ for the extraction of the coefficient of z_k^α in ξ. Then extraction of the coefficient of $z_k^{\alpha+\nu}$ in (4.4) yields

$$(\psi_{r,\nu}\alpha^r + \cdots + \psi_{0,\nu})\varphi_\alpha = L, \tag{4.5}$$

where L is a $K[[z_1, \ldots, z_{k-1}]]$-linear combination of previous coefficients $\psi_{i,j}$ with $j < \alpha + \nu$. Hence, in the case when the polynomial

$$\Psi(\alpha) = \psi_{r,\nu}\alpha^r + \cdots + \psi_{0,\nu}$$

does not vanish for a given α, the relation (4.5) yields a recurrence relation for φ_α.

LEMMA 4.1. *There exists an algorithm to compute the integer roots of Ψ.*

PROOF. Since $\Psi \neq 0$, let $\beta_1, \ldots, \beta_{k-1}$ be such that the coefficient of $z_1^{\beta_1} \cdots z_{k-1}^{\beta_{k-1}}$ in Ψ, which is a polynomial P in $K[\alpha]$ does not vanish. Then each root of Ψ is in particular a root of P. Now we assumed that our effective field of constants K is such that the finite number of integer roots of P can be determined. For each such a root α, we can check whether $\Psi(\alpha) = 0$ by proposition 3.4 and the induction hypothesis. □

THEOREM 4.1. *There exists an effective zero test for D-algebraic power series.*

PROOF. Let β be the largest positive integer root of Ψ, as computed in the previous proposition. Then for each $\alpha > \beta$, the coefficient φ_α is given as a linear combination of previous coefficients $\varphi_0, \ldots, \varphi_{\alpha-1}$. Hence, $\varphi = 0$ if and only if $\varphi_0 = \cdots = \varphi_\beta = 0$. But this can be tested using the induction hypothesis and proposition 3.4. □

4.3. An alternative zero test

If several D-algebraic series $\xi_1, \ldots, \xi_s$ have all to be tested for zero (i.e. such as in the recursive step of the proof of theorem 4.1), then it might be interesting to have a simultaneous zero test, rather than applying the previous zero test sequentially on $\xi_1, \ldots, \xi_s$. In this section, we therefore describe a variant of the algorithm from the previous section, by starting from the relation (4.2) in proposition 4.2 (or from (4.3) in proposition 4.3, when taking $k' = k$), rather than (4.1).

After evaluation, the relation (4.2) becomes

$$z_k^{\mu_k-1}\begin{pmatrix}\delta_k\varphi_1\\ \vdots\\ \delta_k\varphi_r\end{pmatrix}=M\begin{pmatrix}\varphi_1\\ \vdots\\ \varphi_r\end{pmatrix}, \tag{4.6}$$

where M is obtained by evaluation of the entries of M^R. Now extraction of coefficients in (4.6) does not immediately lead to a suitable recurrence relation like (4.5). Instead, we will first perform some row operations, so that we get a relation of the form

$$Az_k^{\mu_k-1}\begin{pmatrix}\delta_k\varphi_1\\ \vdots\\ \delta_k\varphi_r\end{pmatrix}=AM\begin{pmatrix}\varphi_1\\ \vdots\\ \varphi_r\end{pmatrix}, \tag{4.7}$$

for some suitable invertible matrix A. Extraction of coefficients in (4.7) will yield the desired analogue of (4.5).

We obtain the invertible matrix A by starting with $A = Id$ and doing row operations while maintaining the following properties:

1. The coefficients of A are of the form $cz_k^{-\alpha}$, with $c \in K$ and $0 \leqslant \alpha \leqslant \mu_k - 1$.
2. The coefficients of AM are power series in z_k.
3. For a certain $l \geqslant 0$, we have $d_1 \leqslant \cdots \leqslant d_l < d_{l+1} = \cdots = d_r$, where d_i denotes the maximal degree in z_k^{-1} of the i-th row.
4. The first l rows of $A_{1-\mu_k}\alpha - (AM)_0$ are linearly independent.

Condition 2 implies that we obtain the following relation, when extracting the coefficient of z_k^α in (4.7):

$$[A_{1-\mu_k}\alpha-(AM)_0]\begin{pmatrix}\varphi_{1,\alpha}\\ \vdots\\ \varphi_{r,\alpha}\end{pmatrix}+[A_{2-\mu_k}\alpha-(AM)_1]\begin{pmatrix}\varphi_{1,\alpha+1}\\ \vdots\\ \varphi_{r,\alpha+1}\end{pmatrix}+\cdots=0. \tag{4.8}$$

We will now show that after a finite number of row operations, we are able to obtain a matrix A, such that $l = r$ in conditions 3 and 4. This means that, except for a finite number of α (namely the integer eigenvalues of $A_{1-\mu_k}\alpha - (AM)_0$), the relation (4.8) is actually a recurrence relation for $\varphi_{1,\alpha}, \ldots, \varphi_{r,\alpha}$.

Lemma 4.2. *We can compute a matrix A, which satisfies the above four conditions, and such that $l = r$.*

Proof. Starting with $A = Id$ and $l = 0$, we repeat the following process

- We compute $C = A_{1-\mu_k}\alpha - (AM)_0$ and determine the rank l' of its last $p - l$ rows.
- We permute rows, such that the $l + l'$ first rows of C become linearly independent, and we apply the same permutation of rows on A.
- Next, by taking K-linear row combinations of the l' middle rows (from row $l + 1$ until row $l + l'$), we make the last $p - (l + l')$ rows of C vanish, and we perform the same row operations on A.
- If $l + l' < r$, we finally multiply the last $r - (l + l')$ rows of A by z_k^{-1} and return to the first step of the loop with $l := l + l'$.

Clearly, throughout this process, A remains an invertible matrix which satisfies conditions 1, 3 and 4. Condition 2 is also preserved, since we made sure that the last $r-(l+l')$ rows of $Az_k^{\mu_k-1} - AM$ have non zero valuation in z_k (equivalently, the last $r-(l+l')$ rows of C vanish).

The loop terminates, since the rank l' is necessarily maximal if $d_{l+1} = \cdots = d_r = \mu_k - 1$. Indeed, the last $r-l$ rows of $\lim_{\alpha\to\infty} \alpha^{-1} C$ coincide with the last $r-l$ rows of $A_{1-\mu_k}$ in this case, which must have rank $r-l$, since A is invertible. □

THEOREM 4.2. *There exists an effective simultaneous zero test for D-algebraic series.*

PROOF. We first claim that the problem of deciding whether $\xi_1 = \cdots = \xi_s = 0$ is equivalent to the deciding whether $\varphi_1 = \cdots = \varphi_r = 0$. Indeed, the ξ_i are $C[[z_1, \ldots, z_k]]$-linear combinations of the φ_j, while the ξ_j are $C[[z_1, \ldots, z_k]]$-linear combinations of the ξ_i and their partial derivatives.

Now let A be as in lemma 4.2 and let β be the largest positive integer root of the characteristic polynomial $\Psi \in K[[z_1, \ldots, z_{k-1}]][\alpha]$ of $A_{1-\mu}\alpha - (AM)_0$. We compute β as in lemma 4.1. Because of the recurrence relation (4.8), testing whether $\varphi_1 = \cdots = \varphi_r$ is now equivalent to testing whether $\varphi_{j,\alpha} = 0$ for all j and $\alpha \leqslant \beta$. Again, we can test this using the induction hypothesis and proposition 3.4. □

5. Conclusion

5.1. Complexity issues

We have introduced a large class of formal D-algebraic power series, which enjoys many closure properties and we gave an effective zero test for such series. However, from the complexity point of view, our algorithm has two drawbacks. First, it may require a long time to compute the relations (4.1), (4.2) or (4.3), since these computations both require ideal stabilization and Groebner basis computations. Secondly, we have no nice *a priori* bound for the largest integer root β computed in lemma 4.1.

Nevertheless, from a practical point of view we think that the first drawback is the most serious one. Indeed, we think that Groebner basis computations are generically expensive, while we expect the computed largest integer root β to be reasonable *in general*. It would be interesting to know whether the first drawback can be removed by using ideas from (Péladan-Germa, 1997). We also notice that the hard cases in zero testing are usually when the power series are actually zero. If a power series is non zero, then a non zero coefficient can usually be found quite efficiently using techniques from (Brent and Kung, 1978; van der Hoeven, 1997b).

5.2. Power series involving parameters

Let us finally remark that the algorithms presented in this paper are compatible with the "automatic case separation" strategy†. This means that we can apply the algorithms in the case when the field K has the form $K = C[\lambda_1, \ldots, \lambda_l]$, where C is an effective field and $\lambda_1, \ldots, \lambda_l$ are a finite numbers of formal parameters in C.

† In computer algebra some authors also use the less suggestive terminology of "dynamic evaluation" instead of "automatic case separation". For a discussion, see (van der Hoeven, 1997a).

The field K is then an effective field in the sense that at each zero test $P=0$ in K, we separate the cases when $P=0$ and $P\neq 0$, by interpreting the conditions $P=0$ resp. $P\neq 0$ as constraints on the parameters $\lambda_1,\ldots,\lambda_l$. We automatically eliminate branches which contain contradictory constraints (this can be tested for instance using Groebner basis techniques). For more details, we refer to (van der Hoeven, 1997a).

The compatibility of the zero test algorithms in this paper with the automatic case separation strategy stems from the fact that the only potentially infinite branches in the computation tree correspond to the imposition of an infinite number of polynomial equalities (and no inequalities). But in such a chain of constraints, all constraints can be deduced from a finite number, since $C[\lambda_1,\ldots,\lambda_l]$ is a Noetherian ring.

References

Brent, R., Kung, H. (1978). Fast algorithms for manipulating formal power series. *Journal of the ACM*, **25**:581–595.

Denef, J., Lipshitz, L. (1989). Decision problems for differential equations. *The Journ. of Symb. Logic*, **55**(3):941–950.

Péladan-Germa, A. (1995). Testing identities of series defined by algebraic partial differential equations. In Cohen, G., Giusti, M., Mora, T., editors, *Proc. of AAECC-11*, volume 948 of *Lect. Notes in Comp. Science*, pages 393–407, Paris. Springer.

Péladan-Germa, A. (1997). *Tests effectifs de nullité dans des extensions d'anneaux différentiels.* PhD thesis, Gage, École Polytechnique, Palaiseau, France.

Shackell, J. (1989). A differential-equations approach to functional equivalence. In *Proc. ISSAC '89*, pages 7–10, Portland, Oregon, A.C.M., New York. ACM Press.

Shackell, J. (1993). Zero equivalence in function fields defined by differential equations. *Proc. of the A.M.S.*, **336**(1):151–172.

van der Hoeven, J. (1997a). *Automatic asymptotics*. PhD thesis, École polytechnique, France.

van der Hoeven, J. (1997b). Lazy multiplication of formal power series. In Küchlin, W. W., editor, *Proc. ISSAC '97*, pages 17–20, Maui, Hawaii.