Complex transseries solutions to algebraic differential equations

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Abstract

In our PhD. we have given an algorithm for the algebraic resolution of algebraic differential equations with real transseries coefficients. Unfortunately, not all equations do admit solutions in this strongly monotonic setting, even though we recently proved an intermediate value theorem.

In this paper we show that the algorithm from our PhD. generalizes to the setting of weakly oscillatory or complex transseries. Modulo a finite number of case separations, we show how to determine the solutions of an arbitrary algebraic differential equation over the complex transseries. We will show that such equations always admit complex transseries solutions. However, the field of complex transseries is not differentially algebraically closed.

1 Introduction

In [vdH97], we have studied the asymptotic behaviour of solutions to algebraic differential equations in the setting of strongly monotonic or real transseries. We have given a theoretical algorithm to find all such solutions, which is actually effective for suitable subclasses of transseries. More recently, we have proved the following "differential intermediate value theorem".

Theorem 1. [vdH00a] *Let $\mathbb{T}$ be the real field of grid-based transseries in x and let P be a differential polynomial with coefficients in $\mathbb{T}$. Then, given transseries $f < g \in \mathbb{T}$ with $P(f) < 0$ and $P(g) > 0$, there exists a $h \in \mathbb{T}$ with $f < h < g$ and $P(h) = 0$.*

This theorem implies in particular that any algebraic differential equation of odd degree, such as

$$f^7 + e^{e^x} f^3 f''' + \Gamma(\log x + 1) f'' + \log\left(e^x + \Gamma(\Gamma(x))\right) = 0,$$

has at least one real transseries solution. This theorem is striking in the sense that it suggests the existence of theories of ordered and/or valuated differential algebra.

However, a main drawback of the setting of real transseries, is that not every algebraic differential equation can be solved; actually, even an equation like $f^2 + 1 = 0$ has no solutions. In order to get a better understanding of the asymptotic behaviour of solutions to algebraic differential equations, it is therefore necessary to search for a complex analogue of the theory of real transseries. This paper is a first contribution in this direction.

The first problem is to actually define complex transseries. The difficulty is that it is not clear *a priori* whether an expression like e^{z^i} should be seen as an infinitely large or an infinitely small transmonomial. Several approaches can be followed. A first approach, based on pointwise algebras, was already described in chapter 6 of [vdH97]. However, this approach has the drawback that it is not easy to compute with complex transseries.

A second more computational approach is described in section 3. Roughly speaking, it is based on the observation that all computations with complex transseries can be done in a similar way as in the real setting, except for testing whether a monomial like e^{z^i} is infinitely large or small. Now whenever we have to make such a choice, we will actually consider both cases, by applying the automatic case separation strategy (see [vdH97]). We implicitly reject the case when e^{z^i} is bounded, which is "degenerate", but which deserves to be studied later.

The last approach, which is described in section 2, is more structural and really allows us to define a complex transseries in a not too difficult way. The underlying idea is analogue to the concept of a maximal ideal. Intuitively speaking, we assume the existence of some "god", who has decided *a priori* for us which monomials like e^{z^i} are infinitely large and which ones are infinitely small. It turns out that all possible choices lead to isomorphic fields of transseries. However, the geometric significance of these fields is hard to grasp.

In section 4, we introduce parameterized complex transseries, which are necessary to express generic solutions to differential equations. Indeed, such solutions may involve integration constants. As usual, our approach is based on the automatic case separation strategy.

The remaining sections deal with the resolution of asymptotic algebraic differential equations with complex transseries coefficients. Our approach is similar to the one followed in [vdH97], but we have made a few simplifications and we corrected an error (see section 9.4). Our main results are stated in sections 9.1 and 9.2. We show that there exists a theoretical algorithm to express the generic solution to an algebraic differential equation by means of parameterized complex transseries and we give a bound for the logarithmic depth of the generic solution. We also show that an algebraic differential equation of degree d admits at least d complex transseries solutions when counting with multiplicities. As a consequence, each linear differential equation admits a full system of solutions. However, our fields of complex transseries are not differentially algebraically closed and several interesting problems still need to be solved (see section 9.5).

The reader should be aware of a few changes in notations w.r.t. [vdH97], which are summarized in the following table:

old	≺≺	≼≼	≍	∼	≪≪	≫≫	≍≍	
new	≺	≼	≍	∼	≪	≫	≍	≈

2 Complex transseries

2.1 Real trigonometric fields

In all what follows, let R be a *real trigonometric field* and $C = R + i\,R$ its complexification. This means that R has the structure of a totally ordered field and functions $\exp, \sin\colon R \to R$, which are compatible with this ordering.

More precisely, we assume that $\exp$ admits an inverse function $\log$ with domain R_*^+ and that the function $\tan$ restricted to $]-\pi/2, \pi/2[$ admits a totally defined inverse. Here $\tan(x) = \sin(x)/\cos(x)$, where $\cos(x) = \sin(x + \pi/2)$ and $\pi = 4 \arctan 1$. Furthermore,

$$\begin{aligned}
\exp(x + y) &= \exp x \exp y; \\
\sin(x + y) &= \sin x \cos y + \sin y \cos x,
\end{aligned}$$

for all $x, y \in R$. Finally, for each $n \in \mathbb{N}$ resp. $n \in \mathbb{N}^*$ and $x \in R$, we require that

$$\exp x \;\geqslant\; 1 + x + \tfrac{1}{2} x^2 + \cdots + \frac{1}{(2\,n-1)!}\, x^{2n-1};$$
$$\cos x \;\geqslant\; 1 - \tfrac{1}{2} x^2 + \cdots + \frac{1}{(4\,n-4)!}\, x^{4n-4} - \frac{1}{(4\,n-2)!}\, x^{4n-2}.$$

Proposition 2. *The real numbers* $\mathbb{R}$ *form a real trigonometric field.*

Proof. The functional equations are classical. The inequality for $\exp x$ was first proved in [?]. As to the inequality for $\cos x$, we have

$$\cos x - \sum_{k=0}^{2n-1} \frac{(-1)^k}{(2\,k)!}\, x^{2k} \;=\; \sum_{k \geqslant n} \frac{x^{4k}}{(4\,k)!} \left(1 - \frac{x^2}{(4\,k+1)\,(4\,k+2)} \right) \;\geqslant\; 0,$$

if $|x| \leqslant 4\,n$. Otherwise,

$$\sum_{k=0}^{2n-1} \frac{(-1)^k}{(2\,k)!}\, x^{2k} = \left(1 - \frac{x^2}{2} \right) + \sum_{k=1}^{n-1} \frac{x^{4k}}{(4\,k)!} \left(1 - \frac{x^2}{(4\,k+1)\,(4\,k+2)} \right) \leqslant -1 \leqslant \cos x,$$

since $n \geqslant 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Remark 3. Analogue inequalities can be proved for $\sin x$ and for expansions at order $4\,n-3$ instead of $4\,n-1$.

Remark 4. Most of the most classical computations on complex numbers can be carried out in the context of real trigonometric fields. For instance, we the functions $\exp$ and $\sin$ may naturally be extended to C, numbers in C may naturally be written in polar form, etc.

Remark 5. We also have a natural partial analogue of real trigonometric fields; in this case, the functions $\exp$ and $\sin$ are no longer required to be total and the functional equations resp. inequalities are only required to hold, whenever they make sense. For instance, if $x, y \in \operatorname{dom} \exp$, then we require that $x + y \in \operatorname{dom} \exp$ and $\exp(x + y) = \exp x \exp y$.

2.2 Series with complex coefficients and monomials

Let $\mathfrak{M}$ be a totally ordered monomial group (or set) with R-powers. Then we recall that the field $R[\![\mathfrak{M}]\!]$ of grid-based power series is naturally totally ordered by $f > 0 \Leftrightarrow c_f > 0$, for all $f \neq 0$. This ordering is compatible with the multiplication: $f \geqslant 0 \wedge g \geqslant 0 \Rightarrow f g \geqslant 0$. More generally, if $\mathfrak{M}$ is only partially ordered, then we define an ordering on $R[\![\mathfrak{M}]\!]$ to be *compatible* with the asymptotic ordering on $\mathfrak{M}$, if

$$f \prec g \wedge g \geqslant 0 \;\Rightarrow\; f + g \geqslant 0 \tag{1}$$

for all $f, g \in R[\![\mathfrak{M}]\!]$.

In what follows, we are rather interested in the complexification $C[\![\mathfrak{M}]\!]$ of $R[\![\mathfrak{M}]\!]$. Obviously, this C-algebra can not be given an ordering which is compatible with the multiplication. Nevertheless, it is interesting to consider orderings on $C[\![\mathfrak{M}]\!]$ which are only compatible with the R-algebra structure of $C[\![\mathfrak{M}]\!]$. Such an ordering is again said to be compatible with the asymptotic ordering on $\mathfrak{M}$, if (1) holds for all $f, g \in R[\![\mathfrak{M}]\!]$.

Assuming that such orderings on $\mathfrak{M}$ and $C[\![\mathfrak{M}]\!]$ are total, the condition (1) implies that $\tau_f = \tau_g \Leftrightarrow \operatorname{sign} f = \operatorname{sign} g$ for all non zero $f, g \in C[\![\mathfrak{M}]\!]$. Consequently, the ordering on $C[\![\mathfrak{M}]\!]$ is totally determined by the sets

$$P_{\mathfrak{m}} = \{ c \in C \,|\, c\,\mathfrak{m} > 0 \},$$

where $\mathfrak{m}$ runs over $\mathfrak{M}$. Each $P_{\mathfrak{m}}$ is actually the set of strictly positive elements of a total ordering on C, which is compatible with the R-module structure of C. Therefore, each $P_{\mathfrak{m}}$ is characterized by an angle $\theta_{\mathfrak{m}} \in [-\pi, \pi)$ and a direction $\epsilon_{\mathfrak{m}} \in \{-1, 1\}$, via

$$P_{\mathfrak{m}} = \{c \in C \,|\, (\mathrm{Re}\,(c\,e^{-i\theta_{\mathfrak{m}}}) > 0) \vee (\mathrm{Re}\,(c\,e^{-i\theta_{\mathfrak{m}}}) = 0 \wedge \mathrm{Im}(\epsilon_{\mathfrak{m}}\,c\,e^{-i\theta_{\mathfrak{m}}}) > 0)\}.$$

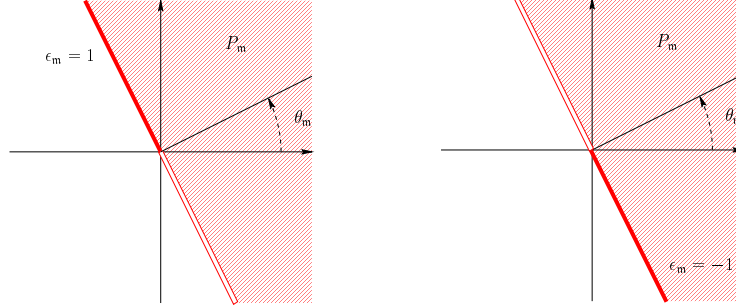This situation is illustrated in figure 1.

Figure 1. Shape of the region $P_{\mathfrak{m}}$ for $\epsilon_{\mathfrak{m}} = 1$ resp. $\epsilon_{\mathfrak{m}} = -1$.

In is also possible to consider complex powers of monomials: a *complex monomial group* is a monomial group $\mathfrak{M}$ with C-powers, with an asymptotic ordering $\succcurlyeq$ which is compatible with the expo-linear R-vector space structure of $\mathfrak{M}$. For instance the formal group $(\log z)^{\mathbb{C}} z^{\mathbb{C}}$ is a monomial group with $\mathbb{C}$-powers for the ordering $(\log z)^{\alpha} z^{\beta} \succ 1 \Longleftrightarrow (\mathrm{Re}\,\beta > 0) \vee (\mathrm{Re}\,\beta = 0 \wedge \mathrm{Re}\,\alpha > 0)$. This group is not totally ordered, since z^i and 1 are incomparable. We may make the ordering total by deciding that $z^{\mathbb{R}_*^+ i} \succ (\log z)^{\mathbb{R}_*^+ i} \succ 1$.

2.3 Pre-fields of complex transseries

Now consider a totally ordered grid-based algebra of the form $\mathbb{T} = C[[\mathfrak{M}]]$, where $\mathfrak{M}$ is a totally ordered complex monomial group, and where the ordering on $\mathbb{T}$ is assumed to be compatible with the asymptotic ordering on $\mathfrak{M}$. Assume that we also have a partial logarithmic function log on $\mathbb{T}$, such that

L1. log coincides with the usual logarithm on R_*^+;

L2. If $x, y \in \mathrm{dom}\,\log$, then $y/x \in \mathrm{dom}\,\log$ and $\log y = \log\,(y/x) + \log x$.

We say that $\mathbb{T}$ is a *pre-field of complex transseries* if the following conditions are satisfied:

T1. $\mathrm{dom}\,\log = \{f \in \mathbb{T}^* \,|\, c_f \in R_*^+\}$;

T2. $\log \mathfrak{m} \in \mathbb{T}^{\uparrow}$, for all $\mathfrak{m} \in \mathfrak{M}$;

T3. For all $\varepsilon \in \mathbb{T}^{\downarrow}$, we have $\log\,(1 + \varepsilon) = l \circ \varepsilon$, where $l = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \in C[[z]]$,

as well as the following conditions for the logarithm:

L3. $\log \mathfrak{m}^{\lambda} = \lambda \log \mathfrak{m}$ for all $\mathfrak{m} \in \mathfrak{M}$ and $\lambda \in C$;

L4. $\mathfrak{m} \prec \mathfrak{n} \Leftrightarrow \log \mathfrak{m} < \log \mathfrak{n}$ for all $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$;

L5. $\log \mathfrak{m} \lll \mathfrak{m}$ for all $\mathfrak{m} \in \mathfrak{M} \backslash \{1\}$.

In **L5**, we write $f \lll g$, if and only if $\log \mathfrak{d}_f \prec \log \mathfrak{d}_g$ for all $f, g \in \mathbb{T}^*$. In view of **L3**, this means that $f \lll g \Leftrightarrow f^{\lambda} \lll g^{\mu}$ for all $\lambda, \mu \in C^*$.

Remark 6. The conditions **L1** until **L5** replace the requirement that log should be a logarithmic function in the definition of real fields of transseries. It can indeed be checked that our conditions are *equivalent* to the usual conditions on log in this case.

Remark 7. The domain of the logarithm on $\mathbb{T}$ may be further extended, by setting $\log f = \log c_f + \log (f/c_f)$ for all $f \in \mathbb{T}^*$, where $\log c_f$ is defined using the arctan function on C. Of course, such an extension of the logarithm to $\mathbb{T}$ involves a choice of a principal determination. Furthermore, such an extension cannot satisfy both the properties **L1** and **L2**.

On the other hand, the partial inverse exp of log may be extended canonically in such a way that the equation $\exp f = g$ admits a solution for each $g \in \mathbb{T}^*$. Indeed, it suffices to extend exp via $\exp (f + c) := \exp(f) \exp(c)$ for all $f \in \operatorname{dom} \exp$ and $c \in C$. In what follows, we will always assume that the partial inverse exp of log has been extended in this way.

2.4 Logarithmic complex transseries in z

Consider the formal C-vector space $\log \mathfrak{L} = C \log z \oplus C \log \log z \oplus \cdots$ generated by the formal symbols $\log z$, $\log \log z$, ... Given angles $\theta_{\log z}$, $\theta_{\log \log z}$, $\ldots \in [-\pi, \pi)$ and directions $\theta_{\log z}$, $\theta_{\log \log z}$, $\ldots \in \{-1, 1\}$, we define a total ordering on $\log \mathfrak{L}$ as explained in section 2.2. Then the formal exponential $\mathfrak{L} = z^C (\log z)^C \cdots$ of $\log \mathfrak{L}$ is a complex monomial group for the asymptotic ordering $\succcurlyeq$ defined by $\mathfrak{m} \succcurlyeq \mathfrak{n} \Leftrightarrow \log \mathfrak{m} \geqslant \log \mathfrak{n}$, where $\log(z^{\alpha_0} (\log z)^{\alpha_1} \cdots) = \alpha_0 \log z + \alpha_1 \log \log z + \cdots$. In order to avoid confusion, we will sometimes write $\mathfrak{L}_{\boldsymbol{\theta}, \boldsymbol{\epsilon}}$ instead of $\mathfrak{L}$.

Assume from now on that $\boldsymbol{\theta}$ and $\boldsymbol{\epsilon}$ were chosen such that $R_*^+ \log z > R_*^+ \log \log z > \cdots > 0$. Given a non-zero grid-based series $f \in \mathbb{L} = C [\![\mathfrak{L}]\!]$ with $c_f \in R_*^+$, we define its logarithm by

$$\log f = \log c_f + \log \mathfrak{d}_f + l \circ (1 + \delta_f).$$

We may extend the total ordering on $\log \mathfrak{L} \subseteq \mathbb{L}$ to $\mathbb{L}$ in a similar way as in section 2.2, by extending the angle and direction families $\boldsymbol{\theta}$ resp. $\boldsymbol{\epsilon}$ into larger families $\hat{\boldsymbol{\theta}}$ resp. $\hat{\boldsymbol{\epsilon}}$. It is easily verified that the field $\mathbb{L}_{\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\epsilon}}} := \mathbb{L}$ with this ordering is a pre-field of complex transseries.

Actually, the structure of $\mathbb{L}_{\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\epsilon}}}$ does not really depend on the choices of $\boldsymbol{\theta}$, $\boldsymbol{\epsilon}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\epsilon}}$, modulo rotations and conjugations. Indeed, assume that $\boldsymbol{\xi}$ and $\boldsymbol{\varsigma}$ are a second family of angles and directions with indices in $\{\log z, \log \log z, \ldots\}$. Then we define an increasing isomorphism φ between $\log \mathfrak{L}_{\boldsymbol{\theta}, \boldsymbol{\epsilon}}$ and $\log \mathfrak{L}_{\boldsymbol{\xi}, \boldsymbol{\varsigma}}$ by

$$\varphi: \sum_{\mathfrak{m} \in \{\log z, \log \log z, \ldots\}} f_\mathfrak{m} \mathfrak{m} \longmapsto \sum_{\mathfrak{m} \in \{\log z, \log \log z, \ldots\}} e^{i \xi_\mathfrak{m}} \iota_{\epsilon_\mathfrak{m} \varsigma_\mathfrak{m}} \left(f_\mathfrak{m} e^{-i \theta_\mathfrak{m}} \right) \mathfrak{m},$$

where

$$\iota_1(z) = z;$$
$$\iota_{-1}(z) = \bar{z},$$

for all $z \in C$. We infer that $\acute{\varphi}: \mathfrak{L}_{\boldsymbol{\theta}, \boldsymbol{\epsilon}} \to \mathfrak{L}_{\boldsymbol{\xi}, \boldsymbol{\varsigma}}; \exp f \mapsto \exp \varphi(f)$ is an isomorphism of complex monomial groups. Now if $\hat{\boldsymbol{\xi}}$ and $\hat{\boldsymbol{\varsigma}}$ are families of angles resp. directions with indices in $\mathfrak{L}$, and which extend $\boldsymbol{\xi}$ and $\boldsymbol{\varsigma}$, then we define an increasing isomorphism $\hat{\varphi}$ between $\mathbb{L}_{\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\epsilon}}}$ and $\mathbb{L}_{\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\varsigma}}}$ by

$$\hat{\varphi}: \sum_{\mathfrak{m} \in \mathfrak{L}_{\boldsymbol{\theta}, \boldsymbol{\epsilon}}} f_\mathfrak{m} \mathfrak{m} \longmapsto \sum_{\mathfrak{m} \in \mathfrak{L}_{\boldsymbol{\theta}, \boldsymbol{\epsilon}}} e^{i \xi_\mathfrak{m}} \iota_{\epsilon_\mathfrak{m} \varsigma_\mathfrak{m}} \left(f_\mathfrak{m} e^{-i \theta_\mathfrak{m}} \right) \varphi(\mathfrak{m}). \tag{2}$$

We notice that $\hat{\varphi}$ extends φ if and only if $\acute{\varphi} = \operatorname{Id}$, which is again equivalent to the condition that for each $\mathfrak{m} \in \{\log z, \log \log z, \ldots\}$ we have

$$\xi_\mathfrak{m} = \theta_\mathfrak{m} \ \vee \ \varsigma_\mathfrak{m} \xi_\mathfrak{m} = \epsilon_\mathfrak{m} \theta_\mathfrak{m}.$$

In this case, we say that $(\boldsymbol{\xi}, \boldsymbol{\varsigma})$ and $(\boldsymbol{\theta}, \boldsymbol{\epsilon})$ are *strongly compatible*. We say that $(\boldsymbol{\xi}, \boldsymbol{\varsigma})$ and $(\boldsymbol{\theta}, \boldsymbol{\epsilon})$ are *compatible* if the relation holds for all $\mathfrak{m} = \log_l z$ with sufficiently large $l \in \mathbb{N}$.

2.5 Complex transseries in z

Assume now that we are given a complex field of transseries $\mathbb{T} = C[\![\mathfrak{M}]\!]$, which is not stable under exponentiation (modulo the extension of the exponentiation as described in remark 7). Let $\boldsymbol{\theta}$ and $\boldsymbol{\epsilon}$ be the associated families of angles and directions. Now consider the formal complex monomial group

$$\mathfrak{M}_{\exp} = \exp \mathbb{T}^{\uparrow},$$

whose asymptotic ordering is given by $\exp f \succcurlyeq \exp g \Leftrightarrow f \geqslant g$, for all $f, g \in \mathbb{T}^{\uparrow}$. Given extensions $\boldsymbol{\theta}_{\exp}$ and $\boldsymbol{\epsilon}_{\exp}$ of $\boldsymbol{\theta}$ and $\boldsymbol{\epsilon}$ to families indexed by monomials in $\mathfrak{M}_{\exp}$, we may totally order $\mathbb{T}_{\exp} = C[\![\mathfrak{M}_{\exp}]\!]$ as explained in section 2.2. It is easily verified that $\mathbb{T}_{\exp}$ is a pre-field of complex transseries, which we call the *exponential extension* of $\mathbb{T}$, relative to $\boldsymbol{\theta}$ and $\boldsymbol{\epsilon}$. In cases of confusion, we will write $\mathbb{T}_{\exp, \boldsymbol{\theta}_{\exp}, \boldsymbol{\epsilon}_{\exp}}$ instead of $\mathbb{T}$. Notice that the exponential of any series in $\mathbb{T}$ is defined in $\mathbb{T}_{\exp}$.

Again, the structure of $\mathbb{T}_{\exp, \boldsymbol{\theta}_{\exp}, \boldsymbol{\epsilon}_{\exp}}$ does not really depend on the choice of $(\boldsymbol{\theta}_{\exp}, \boldsymbol{\epsilon}_{\exp})$. Indeed, if $(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\epsilon}})$ and $(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\epsilon}})$ are two different such choices, then

$$\varphi_{\exp} \colon \sum_{\mathfrak{m} \in \mathfrak{M}_{\exp}} f_{\mathfrak{m}} \mathfrak{m} \longmapsto \sum_{\mathfrak{m} \in \mathfrak{M}_{\exp}} e^{i \xi_{\mathfrak{m}}} \iota_{\epsilon_{\mathfrak{m}} \varsigma_{\mathfrak{m}}} \left(f_{\mathfrak{m}} e^{-i \theta_{\mathfrak{m}}} \right) \mathfrak{m} \tag{3}$$

is an increasing isomorphism between $\mathbb{T}_{\exp, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\epsilon}}}$ and $\mathbb{T}_{\exp, \tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\epsilon}}}$.

Starting with $\mathbb{L}$ from the previous section, we may now consider the iterated exponential extensions $\mathbb{T}^0 = C[\![\mathfrak{M}^0]\!] = \mathbb{L}$, $\mathbb{T}^1 = C[\![\mathfrak{M}^1]\!] = \mathbb{L}_{\exp}$, $\mathbb{T}^2 = C[\![\mathfrak{M}^2]\!] = \mathbb{L}_{\exp, \exp}$ of $\mathbb{L}$. The union

$$\mathbb{T} = C[\![\![z]\!]\!] = \mathbb{T}^0 \cup \mathbb{T}^1 \cup \mathbb{T}^2 \cup \cdots = C[\![\mathfrak{M}^0 \cup \mathfrak{M}^1 \cup \mathfrak{M}^2 \cup \cdots]\!]$$

of these fields is called a *field of complex transseries in z*. Of course, the construction of $\mathbb{T}$ depends on the successive choices of angles $\boldsymbol{\theta}^i$ and directions $\boldsymbol{\epsilon}^i$ for $\mathbb{T}^i$, with indices in $\mathfrak{M}^i$. The angles $\boldsymbol{\theta}$ and directions $\boldsymbol{\epsilon}$ for $\mathbb{T}$ coincide with these choices on each $\mathfrak{M}^i$. We will write $\mathbb{T}_{\boldsymbol{\theta}, \boldsymbol{\epsilon}}$ instead of $\mathbb{T}$ whenever confusion may arise.

We claim that $\mathbb{T}_{\boldsymbol{\theta}, \boldsymbol{\epsilon}}$ and $\mathbb{T}_{\boldsymbol{\xi}, \boldsymbol{\varsigma}}$ are isomorphic as soon as the restrictions of $(\boldsymbol{\xi}, \boldsymbol{\varsigma})$ and $(\boldsymbol{\theta}, \boldsymbol{\epsilon})$ to $\{\log z, \log \log z, ...\}$ are compatible. We have already shown (see formulas (2) and (3)) that there exist isomorphisms

$$\varphi^i \colon \mathbb{T}^i_{\boldsymbol{\theta}^i, \boldsymbol{\epsilon}^i} \to \mathbb{T}^i_{\boldsymbol{\xi}^i, \boldsymbol{\varsigma}^i}$$

for each i. Now let l_0 be such that $\xi_{\log_l z} = \theta_{\log_l z}$ for each $l \geqslant l_0$. Then we observe that $\varphi^j(\log_l z) = \varphi^i(\log_l z)$ for all $l \in \mathbb{N}$ and $j \geqslant i \geqslant l_0 - l$. By induction over $i \geqslant l_0$, it then follows that for $\varphi^j(\mathfrak{m}) = \varphi^i(\mathfrak{m})$ for all $\mathfrak{m} \in \mathfrak{M}^{i-l_0}$ and $j \geqslant i$. Given $\mathfrak{m} \in \mathfrak{M}$, this shows that the value of $\varphi^i(\mathfrak{m})$ does not depend on the choice of i, for sufficiently large i. In other words, the φ^i can be glued together into an isomorphism between $\mathbb{T}_{\boldsymbol{\theta}, \boldsymbol{\epsilon}}$ and $\mathbb{T}_{\boldsymbol{\xi}, \boldsymbol{\varsigma}}$.

Remark 8. It is possible to slightly generalize the construction of pre-fields of complex transseries in z, when starting with $\log \mathfrak{L} = C \log_{l+1} z \oplus C \log_{l+2} z \oplus \cdots$ instead of $\log \mathfrak{L} = C \log z \oplus \log_2 z \oplus \cdots$. Notice that z is not necessarily a monomial when adopting this generalization.

2.6 Fields of complex transseries

Actually, in our construction of pre-fields of complex transseries in z, it is reasonable to require that require that $\theta_{\log_l x} = 0$ for all sufficiently large $l \in \mathbb{N}$, thereby eliminating all ambiguity (up to isomorphism) in the construction of $\mathbb{T}$. More generally, a pre-field of complex transseries $\mathbb{T}$ is a *field of complex transseries*, if it satisfies the following axiom:

T4. For each $\mathfrak{m} \in \mathfrak{M}$, there exists an $i_0 \in \mathbb{N}$, such that for all $i \geqslant i_0$ we have

- $\mathfrak{d}(\log_{i+1} \mathfrak{m}) = \log \mathfrak{d}(\log_i \mathfrak{m})$.

- $\theta_{\mathfrak{d}(\log_i \mathfrak{m})} = 0$.

Then up to isomorphism, we have constructed *the* field of grid-based complex transseries in z. Actually, the same procedure of exponential extensions and direct limits can be used to close any field of complex transseries under exponentiation. Again, this closure is unique up to isomorphism.

Remark 9. In this paper we restrict our attention to grid-based complex transseries. Nevertheless, the results of this sections can easily be generalized to the case of Noetherian complex transseries. In this case, we recommend to replace the axiom **T4** by the following more complicated but better axiom:

T4. Let $(\mathfrak{m}_i)_{i \in \mathbb{N}}$ be a sequence of monomials in $\mathfrak{M}$, such that $\mathfrak{m}_{i+1} \in \operatorname{supp} \log \mathfrak{m}_i$. Then there exists an $i_0 \in \mathbb{N}$, such that for all $i \geqslant i_0$ we have

- $\mathfrak{n} \succcurlyeq \mathfrak{m}_{i+1}$ for all $\mathfrak{n} \in \log \operatorname{supp} \mathfrak{m}_i$.

- $\theta_{\mathfrak{m}_i} = 0$.

This axiom allows the resolution of certain functional equations like

$$f(z) = e^{\sqrt{z} + f(\log z)},$$

which admits natural solutions of the form

$$f = e^{\sqrt{z} + e^{\sqrt{\log z} + e^{\cdot^{\cdot^{\cdot}}}}},$$

which are called *nested transseries*.

2.7 Extra structure on the field of transseries in z

Consider a tuple $\mathfrak{B} = (\mathfrak{b}_1, ..., \mathfrak{b}_n)$ of non zero complex transseries in z with $1 \prec \mathfrak{b}_1 \prec \cdots \prec \mathfrak{b}_n$. We call $\mathfrak{B}$ a *complex transbasis* if the following conditions are satisfied:

TB1. $\mathfrak{b}_1 = \log_l z$ for some $l \in \mathbb{Z}$, which is called the *level* of $\mathfrak{B}$.

TB2. $\log \mathfrak{b}_i \in C[\![\mathfrak{b}_1; ...; \mathfrak{b}_{i-1}]\!]$ for each $i > 1$.

TB3. $\mathfrak{b}_1 \lll \cdots \lll \mathfrak{b}_n$ (i.e. $\mathfrak{d}_{\log \mathfrak{b}_2} \prec \cdots \prec \mathfrak{d}_{\log \mathfrak{b}_n}$).

Such a transbasis generates a *complex asymptotic scale* $\mathfrak{B}^C$. We say that $f \in C[\![z]\!]$ can be expanded w.r.t. $\mathfrak{B}$ if $f \in C[\![\mathfrak{B}^C]\!]$. If $l = 1$, then we say that $\mathfrak{B}$ (and any $f \in C[\![\mathfrak{B}^C]\!]$) is *purely exponential*. The following *incomplete transbasis theorem* is proved in a similar way as in the case of real transseries:

Theorem 10. *Let $\mathfrak{B}$ be a transbasis and $f \in C[\![z]\!]$ a complex transseries. Then f can be expanded w.r.t. a super-transbasis $\hat{\mathfrak{B}} \supseteq \mathfrak{B}$ of $\mathfrak{B}$.*

We define a strong derivation w.r.t. z on $\mathbb{T} = C \llbracket\!\llbracket z \rrbracket\!\rrbracket$ in the usual way: we take

$$(z^{\alpha_0} \cdots (\log_l z)^{\alpha_l})' = \left(\frac{\alpha_0}{z} + \cdots + \frac{\alpha_l}{z \log z \cdots \log_{l-1} z} \right) z^{\alpha_0} \cdots (\log_l z)^{\alpha_l}$$

for all monomials $z^{\alpha_0} \cdots (\log_l z)^{\alpha_l} \in \mathfrak{M}^0$. This yields a derivation on $\mathbb{T}^0$ through extension by strong linearity. Given a derivation on $\mathbb{T}^i$, we define

$$(\exp f)' = f' \exp f,$$

for all monomials $\exp f \in \exp(\mathbb{T}^i)^{\uparrow} = \mathfrak{M}^{i+1}$. This again yields a derivation on $\mathbb{T}^{i+1}$ through extension by strong linearity. By induction over i, we thus obtain a derivation on $\mathbb{T}$.

We recall that a derivation on $\mathbb{T}$ is said to be *strictly valuated* resp. *strictly positive* if the following conditions are satisfied:

VD $f \prec g \Rightarrow f' \prec g'$, for all $f, g \in \mathbb{T}$ with $g \not\asymp 1$;

PD $f \succ 1 \Rightarrow (f > 0 \Rightarrow f' > 0)$, for all $f \in \mathbb{T}$.

Contrary to the case of real transseries, our derivation on $\mathbb{T}$ cannot be strictly positive. Indeed, either $e^{iz} \succ 1$ or $e^{-iz} \succ 1$, say $e^{iz} \succ 1$. Then we have $(e^{iz})'' = - e^{iz}$, so either $\operatorname{sign}(e^{iz})' \neq \operatorname{sign} e^{iz}$ or $\operatorname{sign}(e^{iz})'' \neq \operatorname{sign}(e^{iz})'$. On the other hand, the following may be proved in the usual way:

Theorem 11. *The derivation on $C \llbracket\!\llbracket z \rrbracket\!\rrbracket$ is strictly valuated.*

Actually, the proof involves upward shiftings of transseries: given $f \in C \llbracket\!\llbracket z \rrbracket\!\rrbracket$, its upward (resp. downward) shifting is defined by $f{\uparrow} = f \circ \exp$ (resp. $f{\downarrow} = f \circ \log$). Contrary to the case of real transseries, this transseries does not necessarily live in the same field of transseries as f: if $f \in C \llbracket\!\llbracket z \rrbracket\!\rrbracket_{\theta, \epsilon}$, then we have $f{\uparrow} \in C \llbracket\!\llbracket z \rrbracket\!\rrbracket_{\theta\uparrow, \epsilon\uparrow}$, where $\theta\uparrow_{\mathfrak{m}\uparrow} = \theta_{\mathfrak{m}}$ and $\epsilon\uparrow_{\mathfrak{m}\uparrow} = \epsilon_{\mathfrak{m}}$ for all transmonomials $\mathfrak{m} \in \mathfrak{M}$. In the case of downward shiftings, one may have to consider the generalized fields of complex transseries in z from remark 8.

It is more difficult to extend functional composition from the real to the complex setting due to possible incompatibility between the angles and directions. For instance, if $\theta_z = 0$, then the transseries $e^{-z} + e^{-2z} + \cdots$ can not be composed on the right with $-z$. In general, right composition with a given transseries is only defined on a certain subfield of $C \llbracket\!\llbracket z \rrbracket\!\rrbracket$. Contrary to the case of real transseries, certain functional equations like

$$f(z) = e^{iz} + f(\alpha z),$$

with $\alpha \in C$ seem to fall outside the scope of the theory of complex transseries, unless someone comes up with some really new ideas to incorporate the solutions to such equations inside this theory.

2.8 Further generalizations

One of the main ideas behind the construction of fields of complex transseries is that we do not longer require the ordering on the constant field to be compatible with the multiplication. Indeed, we just need the compatibility with the addition (or multiplication with reals), in order to obtain ordered monomial groups via exponentiation.

The above idea may be used to generalize the results from this section to other circumstances. Consider for instance the set $C = \mathbb{C}_p$ of p-adic complex numbers, where $p > 2$. Then it is classical that there exists a partial logarithm on C, which is defined for all $z \in C$ with $\operatorname{val}_p z = 0$. By Zorn's lemma, there exists a total ordering on the $\mathbb{Q}$-vector space C. The theory of this section may now be adapted in order to construct the field $\mathbb{T}_p$ of complex p-adic transseries.

A first change concerns the condition **T1**, which should now become

$$\text{dom} \log = \{ f \in \mathbb{T}^* | \text{val}_p c_f = 0 \} = \{ f \in \mathbb{T}^* | c_f \in \text{dom} \log_C \}.$$

Furthermore, it is not as easy as before to characterize the total orderings on C, which are compatible with the $\mathbb{Q}$-vector space structure. Consequently, there is no natural analogue to the condition **T4** and we have to satisfy ourselves with the construction of pre-fields of complex transseries. Also, the exponentiation on $\mathbb{T}_p$ is not total.

Notice that it seems to be possible to take p itself for the indeterminate z in the construction of $\mathbb{T}_p$. This would yield a field of transseries which contains $\mathbb{C}_p$ and such that the logarithm is defined for all non zero elements.

3 Generic complex transseries

In practical computations with complex transseries the angles $\boldsymbol{\theta}$ and directions $\boldsymbol{\epsilon}$ are not known in advance and we have to choose them (or more precisely, to put constraints on them) as the computation progresses. This can be done by introducing a closed interval $\Theta_{\mathfrak{m}} \in R/(2\,\pi\,\mathbb{Z})$ for each transmonomial $\mathfrak{m}$, which corresponds to the constraint

$$\forall \alpha \in \Theta_{\mathfrak{m}}: \quad |\alpha - \theta_{\mathfrak{m}}| < \pi/2 \tag{4}$$

on $\theta_{\mathfrak{m}}$. Given such sets $\Theta_{\mathfrak{m}}$, we will work with *generic complex transseries* which are in the "intersection" of all $C [\![z]\!]_{\boldsymbol{\theta}, \boldsymbol{\epsilon}}$ such that $\boldsymbol{\theta}$ and $\boldsymbol{\epsilon}$ satisfy the above constraints. Actually, it is convenient to always work w.r.t. generic complex transbases, which we will introduce now.

3.1 Generic complex transbases

Let $\mathfrak{B} = (\mathfrak{b}_1, ..., \mathfrak{b}_n)$ be an n-tuple of symbols. Assume that each $\mathfrak{b}_i$ comes with closed interval $\Theta_i \subseteq \mathbb{R}/2\,\pi\,i$ modulo $2\,\pi$, such that $\Theta_i \cap (\Theta_i + \pi) = \varnothing$. Then we may order the monomial group $\mathfrak{B}^C = \mathfrak{b}_1^C \cdots \mathfrak{b}_n^C$ by

$$\mathfrak{b}_1^{\alpha_1} \cdots \mathfrak{b}_i^{\alpha_i} \succ 1 \iff \arg \alpha_i \in \Theta_i,$$

for each non zero monomial $\mathfrak{b}_1^{\alpha_1} \cdots \mathfrak{b}_i^{\alpha_i}$ with $\alpha_i \neq 0$. We call $\mathfrak{B}$ a *generic complex asymptotic basis* of the scale $\mathfrak{B}^C$. Such a basis is called a *generic complex transbasis*, if

TB1. $\mathfrak{b}_1 = \log_l z$ for some $l \in \mathbb{Z}$, which is called the *level* of $\mathfrak{B}$, and $0 \in \Theta_1$.

TB2. $\log \mathfrak{b}_i$ is a regular, infinitely large transseries in $C [\![\mathfrak{b}_1; ...; \mathfrak{b}_{i-1}]\!]$ for each $i > 1$.

TB3. $\mathfrak{d}_{\log \mathfrak{b}_2} \prec \cdots \prec \mathfrak{d}_{\log \mathfrak{b}_n}$.

An important question is whether the asymptotic constraints on the $\mathfrak{b}_i$ determine a non empty region of the complex transplane (see chapter 6 of [vdH97]). This question will be addressed in a forthcoming paper.

Example 12. The triple $(z, e^z, e^{-e^z/(1-z^{-1})})$ is a transbasis, for the constraints $1 \prec z \prec e^z \prec e^{-e^z/(1-z^{-1})}$. Computations with respect to this transbasis are valid in regions of $\mathbb{C}$, where $|e^{-e^z/(1-z^{-1})}| \succ |e^z| \succ |z| \succ 1$. This is for instance the case for $z = x + i\,y$, such that $x \to +\infty$ in a region where $k + \frac{1}{4} + \varepsilon \leqslant \frac{y}{2\,\pi} \leqslant k + \frac{3}{4} - \varepsilon$ for some small $\varepsilon \in (0, \frac{1}{4})$ and $k \in \mathbb{Z}$.

A *generic complex transseries* is an element of $C[\![\mathfrak{b}_1;...;\mathfrak{b}_n]\!]$ for some complex transbasis $(\mathfrak{b}_1,...,\mathfrak{b}_n)$. It can be shown that two transbases which have a non empty region of definition in common can be merged together. In the remainder of the paper we will follow an easier approach, which consists of working with respect to a *current transbasis*, which may be enlarged and on which we may impose additional asymptotic constraints during computations with complex transseries.

3.2 Case separations and the field operations

By construction, all ring operations can already be carried out in an algebra of the form $C[\![\mathfrak{b}_1;...;\mathfrak{b}_n]\!]$. In order to invert a complex transseries, we first have to be able to compute its dominant monomial. In principle, both 1 or e^{iz} might be "the" dominant monomial of a transseries like $1+e^{iz}$. Nevertheless, given a transseries $f \in C[\![\mathfrak{b}_1;...;\mathfrak{b}_n]\!]$ with dominant monomials $\mathfrak{d}_1,...,\mathfrak{d}_r$, then we may always separate r cases

$$\begin{cases} \mathfrak{d}_1 \succ \mathfrak{d}_2 \wedge \mathfrak{d}_1 \succ \mathfrak{d}_3 \wedge \cdots \wedge \mathfrak{d}_1 \succ \mathfrak{d}_r \\ \mathfrak{d}_2 \succ \mathfrak{d}_1 \wedge \mathfrak{d}_2 \succ \mathfrak{d}_3 \wedge \cdots \wedge \mathfrak{d}_2 \succ \mathfrak{d}_r \\ \vdots \\ \mathfrak{d}_r \succ \mathfrak{d}_1 \wedge \mathfrak{d}_r \succ \mathfrak{d}_2 \wedge \cdots \wedge \mathfrak{d}_r \succ \mathfrak{d}_1 \end{cases},$$

in each of which f has only one dominant monomial. This case separation technique is explained in detail in [vdH97]. In the present context, the imposition of a constraint

$$\mathfrak{b}_1^{\alpha_1} \cdots \mathfrak{b}_i^{\alpha_i} \succ 1$$

with $\alpha_i \neq 0$ reduces to the insertion of $\arg \alpha_i$ in the interval $\Theta_{\mathfrak{b}_i}$. If the length of the new interval exceeds π, then (4) can not be satisfied, so that the corresponding case does not need to be considered.

Remark 13. In order to be really complete, we should also consider the cases when several dominant monomials are asymptotic. For instance, in the case of the series $1+e^{iz}$, we should consider the cases $1 \prec e^{iz}$ and $1 \succ e^{iz}$, but also $1 \asymp e^{iz}$. However, in the present paper, we argue that the situation when $1 \asymp e^{iz}$ is "degenerate" in the sense that it corresponds to a single "direction" $\arg z \equiv 0 \ [\pi]$ among a continuous number if possibilities.

As a consequence, we notice that the process of "regularization" of a complex transseries is much easier than in the case of multivariate transseries studied in [vdH97]. Indeed, in the case when one has to consider the possibility that $1 \asymp e^{iz}$, one also has to consider the possibility of cancellation $1+e^{iz}=0$ or $1+e^{iz} \prec 1$. This would necessitate refinements of the coordinates and rewriting of the series in $C[\![\mathfrak{b}_1;...;\mathfrak{b}_n]\!]$.

Example 14. Modulo cases separations, we may thus carry out all field operations. For instance, the inverse of $1+e^{iz}$ is either given by

$$\frac{1}{1+e^{iz}} = 1 - e^{iz} + e^{2iz} + \cdots, \quad \text{if } e^{iz} \prec 1,$$

or

$$\frac{1}{1+e^{iz}} = e^{-iz} - e^{2iz} + e^{3iz} + \cdots, \quad \text{if } e^{iz} \succ 1,$$

3.3 Logarithms of complex transseries

Consider a non zero complex transseries $f \in C[\![\mathfrak{b}_1;...;\mathfrak{b}_n]\!]$. Modulo case separations, we may assume that f is regular, so that we can write

$$f = c\,\mathfrak{b}_1^{\alpha_1} \cdots \mathfrak{b}_n^{\alpha_n}\,(1+\varepsilon),$$

with $c, \alpha_1, ..., \alpha_n \in C$ and $\varepsilon \prec 1$. Consequently,

$$\log f = \alpha_n \log \mathfrak{b}_n + \cdots + \alpha_1 \log \mathfrak{b}_1 + c + \log(1 + \varepsilon).$$

If $\alpha_1 = 0$, then this series is already in $C[\![\mathfrak{b}_1; ...; \mathfrak{b}_n]\!]$. Otherwise, it still is, modulo the insertion of a new element $\mathfrak{b}_0 = \log \mathfrak{b}_1 = \log_{l+1} z$ in front of the transbasis, subject to the constraint $1 \prec \mathfrak{b}_0$. Since $\mathfrak{b}_0$ is a new symbol, this constraint is non contradictory with the existing expo-linear constraints on the $\mathfrak{b}_i$. The relation $\mathfrak{b}_0 \prec \mathfrak{d}_{\log \mathfrak{b}_2}$ is automatically verified, since $1 \prec \mathfrak{d}_{\log \mathfrak{b}_2} \in z_1^C$.

3.4 Exponentials of complex transseries

Consider a complex transseries $f \in C[\![\mathfrak{b}_1; ...; \mathfrak{b}_n]\!]$. Modulo case separations, we may assume that f is regular. In order to compute the exponential of f, we distinguish three cases:

Case 1: f is bounded. We may write $f = c + \varepsilon$, with $c \in C$ and $\varepsilon \prec 1$. Hence, $e^f = e^c e^\varepsilon$, with $c \in C$ and $e^\varepsilon \in C[\![\mathfrak{b}_1; ...; \mathfrak{b}_n]\!]$.

Case 2: $\log \mathfrak{b}_i \prec \mathfrak{d}_f \prec \log \mathfrak{b}_{i+1}$ for some $0 \leqslant i \leqslant n$ (where we understand that the left resp. right hand side relation is verified if $i = 0$ resp. $i = n$). We decompose $f = f^+ + f^-$, where $f^+ = [\mathfrak{b}_{i+1}^0 \cdots \mathfrak{b}_n^0] f \in C[\![\mathfrak{b}_1; ...; \mathfrak{b}_i]\!]$ and $f^- \prec 1$. Inserting e^{f^+} into $\mathfrak{B}$ by $\mathfrak{B} := (\mathfrak{b}_1, ..., \mathfrak{b}_i, e^{f^+}, \mathfrak{b}_{i+1}, ..., \mathfrak{b}_n)$, we then have $e^f = e^{f^+} e^{f^-} \in C[\![\mathfrak{b}_1; ...; \mathfrak{b}_i; e^{f^+}; \mathfrak{b}_{i+1}; ...; \mathfrak{b}_n]\!]$.

Case 3: $\mathfrak{d}_f \asymp \log \mathfrak{b}_i$ for some i. We may write $f = \lambda \log \mathfrak{b}_i + g$, with $\lambda \in C^*$ and $g \prec f$. Then $e^f = \mathfrak{b}_i^\lambda e^g$ and we compute e^g using the same algorithm. The computation of e^g cannot give rise to infinite loops, since the transbasis $\mathfrak{B}$ would remain invariant in such a loop, while the index i would strictly decrease.

3.5 A worked example

Consider the complex "exp-log function"

$$f = \log\left(e^{e^z + iz} + e^{ie^z}\right)$$

and let us show how to expand it generically with respect to a generic complex transbasis. We start with $\mathfrak{B} := (z)$ and recursively expand all subexpressions of f.

Expansion of e^z. In order to expand e^z, we fall into the second case of the exponentiation algorithm, since $\log z \prec z$ and $z \succ 1$. Consequently, we insert e^z into $\mathfrak{B}$ using $\mathfrak{B} := (z, e^z)$, so that e^z expands as e^z.

Expansions of $i\,z$ and $e^z + i\,z$. Since $C[\![\mathfrak{B}^C]\!]$ is a ring, we immediately have z, $e^z + i z \in C[\![z; e^z]\!]$. Since the expansions of sums and products do not present any problems, we will omit them in what follows.

Expansion of $e^{e^z + iz}$. In order to expand $e^{e^z + iz}$, we first have to determine the dominant of $e^z + i z$. Two cases need to be distinguished for this, namely $\Theta_{e^z} = \{0\}$, which corresponds to $e^z \succ 1$, and $\Theta_{e^z} = \{\pi\}$, which corresponds to $e^z \prec 1$. In the first case, $e^z \succ i\,z \asymp \log e^z$, so that $e^{e^z + iz}$ needs to be inserted into $\mathfrak{B}$. In the second case, $e^z \prec i\,z$, so we rewrite $e^{e^z + iz} = e^{iz} e^{e^z} = e^{iz} + e^{iz+z} + \frac{1}{2} e^{iz+2z} + \cdots \in C[\![z; e^z]\!]$.

Expansion of e^{ie^z}. In the case when $\Theta_{e^z} = \{0\}$, we have $i\,e^z \asymp \log e^{e^z + iz}$, so we rewrite $e^{ie^z} = (e^{e^z + iz})^i e^z \in C[\![z; e^z; e^{e^z + iz}]\!]$. In the other case, when $\Theta_{e^z} = \{\pi\}$, the argument $i\,e^z$ is bounded, so that $e^{ie^z} = 1 + i\,e^z - \frac{1}{2} e^{2z} + \cdots \in C[\![z; e^z]\!]$.

Expansion of f. We first have to determine the dominant monomial of $e^{e^z+iz} + e^{ie^z}$. If $\Theta_{e^z} = \{0\}$, then we separate the cases $\Theta_{e^{e^z+iz}} = \{-\frac{\pi}{4}\}$ in which $e^{e^z+iz} \succ e^{ie^z}$, and $\Theta_{e^{e^z+iz}} = \{\frac{3\pi}{4}\}$ in which $e^{e^z+iz} \prec e^{ie^z}$. In the first case, we obtain

$$
\begin{aligned}
f &= e^z + i\,z + \log\left(1 + e^{(i-1)e^z - iz}\right) \\
&= e^z + i\,z + e^{(i-1)e^z - iz} + \frac{1}{2}\,e^{2(i-1)e^z - 2iz} + \cdots \in C[\![z; e^z; e^{e^z+iz}]\!].
\end{aligned}
$$

In the second case, we get

$$
\begin{aligned}
f &= i\,e^z + \log\left(1 + e^{(1-i)e^z + iz}\right) \\
&= i\,e^z + e^{(1-i)e^z + iz} + \frac{1}{2}\,e^{2(1-i)e^z + 2iz} + \cdots \in C[\![z; e^z; e^{e^z+iz}]\!].
\end{aligned}
$$

If $\Theta_{e^z} = \{\pi\}$, then $e^{e^z+iz} + e^{ie^z} = e^{iz} + 1 + e^{(i+1)z} + i\,e^z + \cdots$, so we separate the cases $\Theta_{e^z} = [\frac{\pi}{2}, \pi]$ in which $e^{iz} \succ 1$, and $\Theta_{e^z} = [\pi, \frac{3\pi}{2}]$ in which $e^{iz} \prec 1$. If $\Theta_{e^z} = [\frac{\pi}{2}, \pi]$, then

$$
\begin{aligned}
f &= i\,z + \log\left(1 + (e^{e^z} - 1) + e^{-iz}\,e^{ie^z}\right) \\
&= i\,z + e^z + e^{-iz} + (i-1)\,e^{(1-i)z} - \frac{1}{2}\,e^{-2iz} + \cdots \in C[\![z; e^z]\!].
\end{aligned}
$$

Otherwise, we obtain

$$
\begin{aligned}
f &= \log\left(1 + (e^{ie^z} - 1) + e^{iz}\,e^{e^z}\right) \\
&= i\,e^z - e^{iz} + (1+i)\,e^{(1+i)z} - \frac{1}{2}\,e^{2iz} + \cdots \in C[\![z; e^z]\!].
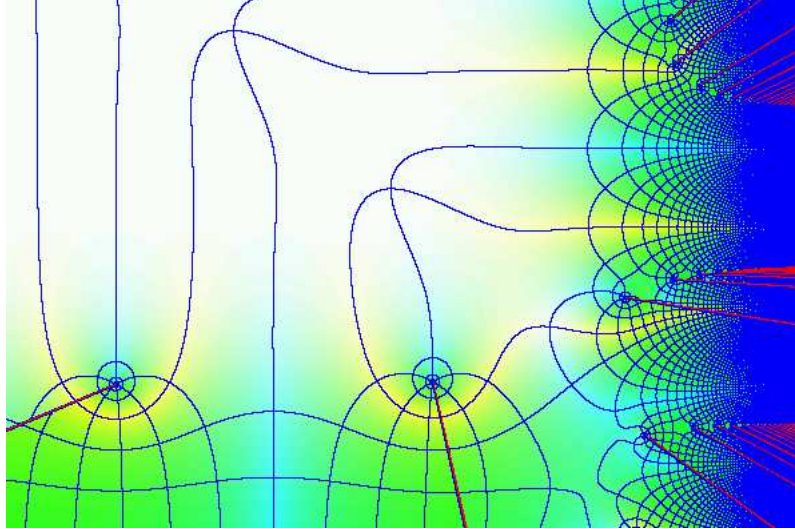\end{aligned}
$$

Figure 2. A plot of the function $f = \log\left(e^{e^z+iz} + e^{ie^z}\right)$, which illustrates the four possible asymptotic behaviours of f on non degenerate regions. The "rows" of singularities correspond to the borders between regions of different types.

4   Parameterized complex transseries

In order to deal with integration constants when solving differential equations, we need to consider parameterized transseries. As in the case of generic transseries, if will often be necessary to distinguish several cases as a function of the values of the parameters. Again, this can be done by putting constraints on the parameters.

4.1 Definition of parameterized complex transseries

Let $\lambda = (\lambda_1, ..., \lambda_\ell)$ be a ℓ-tuple of complex parameters. We call a subset Λ of C^ℓ a *region*, if Λ is the set of solutions of a system of polynomial equations or inequations

$$
\begin{aligned}
c(\lambda) &= 0; \\
c(\lambda) &\neq 0,
\end{aligned}
$$

where $c \in C[\lambda_1, ..., \lambda_\ell]$, and "rational function inequalities on the real parts"

$$
\mathrm{Re}\, \frac{c_1(\lambda)}{c_2(\lambda)} > 0,
$$

where $c_1, c_2 \in C(\lambda_1, ..., \lambda_\ell)$ and c_2 does not vanish on Λ. Notice that Λ may be seen as a special kind of semi-algebraic set, under the isomorphism $C^\ell \cong R^{2\ell}$. The polynomial algebra $P = C[\lambda_1, ..., \lambda_\ell]$ will also be called the *coefficient* or *parameter algebra*.

Given a non empty region $\Lambda \subseteq C^\ell$, let $\mathfrak{B} = (\mathfrak{b}_1, ..., \mathfrak{b}_n)$ be an n-tuple of symbols. Assume that each $\mathfrak{b}_i$ comes with a finite set $\Delta_i = \Delta_{\mathfrak{b}_i} = \{\delta_{1,1}, ..., \delta_{i,d_i}\} \subseteq P$ of *directions*, such that $\delta_{i,j}$ does not vanish on Λ for all $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant d_i$, and such that

$$
\left\{ \lambda \in C^p \,\middle|\, \mathrm{Re}\, \frac{\delta_{i,j}(\lambda)}{\delta_{i,j'}(\lambda)} > 0 \right\} \subseteq \Lambda, \tag{5}
$$

for all $1 \leqslant i \leqslant r$ and $1 \leqslant j, j' \leqslant d_i$ with $j' \neq j$. In the case when $\ell = 0$, the directions $\delta_{i,j}$ correspond to the extremal angles in the intervals Θ_i from the previous section.

For each $1 \leqslant i \leqslant r$, there exists a natural partial ordering $\leqslant_i$ on the R-vector space P, which is generated by the relations $\delta_{i,j} >_i 0$ for all j. Indeed, the constraints (5) in an arbitrary point $\lambda \in \Lambda$ guarantee the absence of relations

$$
\alpha_1 \delta_{i,1} + \cdots + \alpha_{d_i} \delta_{d_i} = 0,
$$

with $(\alpha_1, ..., \alpha_{d_i}) \in (R^+)^{d_i} \setminus (0, ..., 0)$. Consequently, we may define a natural neglection relation $\succ$ on the asymptotic scale $\mathfrak{B}^P = \mathfrak{b}_1^P \cdots \mathfrak{b}_p^P$ by

$$
\mathfrak{b}_1^{\alpha_1} \cdots \mathfrak{b}_i^{\alpha_i} \succ 1 \iff \alpha_i >_i 0,
$$

for each non zero monomial $\mathfrak{b}_1^{\alpha_1} \cdots \mathfrak{b}_i^{\alpha_i}$ with $\alpha_i \neq 0$. We say that $\mathfrak{B}$ is a *parameterized transbasis*, if

TB1. $\mathfrak{b}_1 = \log_l z$ for some $l \in \mathbb{Z}$, which is called the *level* of $\mathfrak{B}$, and $1 \in \Delta_1$.

TB2. $\log \mathfrak{b}_i$ is a regular, infinitely large transseries in $P[\![\mathfrak{b}_1; ...; \mathfrak{b}_{i-1}]\!]$ for each $i > 1$.

TB3. $\mathfrak{d}_{\log \mathfrak{b}_2} \prec \cdots \prec \mathfrak{d}_{\log \mathfrak{b}_n}$.

A parameterized transseries is an element of $P[\![\mathfrak{b}_1; ...; \mathfrak{b}_n]\!]$ for some transbasis $(\mathfrak{b}_1, ..., \mathfrak{b}_n)$.

4.2 Uniform regularization

A regular parameterized transseries $f \in P[\![\mathfrak{B}]\!]$ is said to be *uniformly regular*, if either $f = 0$, or $f_{\mathfrak{d}_f}(\lambda) \neq 0$ for all $\lambda \in \Lambda$. In this section we prove that any parameterized transseries $f \in P[\![\mathfrak{B}]\!]$ can be uniformly regularized modulo case separations. We notice that a uniformly regular parameterized transseries on a region Λ remains uniformly regular on any subregion of Λ.

Lemma 15. *Let* $\mathfrak{m} \in \mathfrak{b}_1^P \cdots \mathfrak{b}_n^P$ *be a monomial. Then, modulo case separations, we may assume that either* $\mathfrak{m} \prec 1$, $\mathfrak{m} = 1$ *or* $\mathfrak{m} \succ 1$.

Proof. Write $\mathfrak{m} = \mathfrak{b}_1^{\alpha_1} \cdots \mathfrak{b}_n^{\alpha_n}$, with $\alpha_1, ..., \alpha_n \in P$ and separate the following $2n + 1$ cases :

cases A   For some $1 \leqslant i \leqslant n$, we have $\alpha_i <_i 0, \alpha_{i+1} = 0, ..., \alpha_n = 0$;

cases B   For some $1 \leqslant i \leqslant n$, we have $\alpha_i >_i 0, \alpha_{i+1} = 0, ..., \alpha_n = 0$;

case C. $\alpha_1 = \cdots = \alpha_n = 0$.

In the n cases **A**, we have $\mathfrak{m} \prec 1$. In the n cases **B** we have $\mathfrak{m} \succ 1$. In case **C**, we have $\mathfrak{m} = 1$.

Notice that the imposition of the constraints of the form $\alpha_i = 0$ or $\alpha_i >_i 0$ may involve a reduction of the region Λ and/or the insertion of new directions into Δ_i. Indeed, $\alpha_i = 0$ is an additional algebraic constraint on Λ. In order to impose $\alpha_i >_i 0$, we first impose the constraints $\alpha_i \neq 0$ and $\operatorname{Re} \frac{\alpha_i}{\delta_{i,j}} > 0$ on Λ, for all $1 \leqslant j \leqslant d_i$. Next we insert α_i into Δ_i. $\square$

Lemma 16. *Let* $\mathfrak{m}_1, ..., \mathfrak{m}_{k+1}$ *be infinitesimal monomials in an arbitrary monomial group* $\mathfrak{M}$ *with* $\mathbb{Q}$-*powers, such that* $\mathfrak{m}_{k+1} = \mathfrak{m}_1^{\alpha_1} \cdots \mathfrak{m}_k^{\alpha_k}$, *for certain* $\alpha_1, ..., \alpha_k \in \mathbb{Z}$. *Then there exist infinitesimal monomials* $\mathfrak{n}_1, ..., \mathfrak{n}_k \in \mathfrak{M}$, *such that* $\mathfrak{m}_i \in \{\mathfrak{n}_1, ..., \mathfrak{n}_k\}^\star$ *for all* $1 \leqslant i \leqslant k + 1$.

Proof. Since $\mathfrak{m}_{k+1} \prec 1$, we may assume without loss of generality that $\alpha_k > 0$, modulo a permutation of indices. We will prove the lemma by induction over k. For $k = 1$ the lemma is trivial. So assume that $k > 1$ and let $\mathfrak{w} = \mathfrak{m}_1^{\alpha_1} \cdots \mathfrak{m}_{k-1}^{\alpha_{k-1}}$. Then we have either $\mathfrak{w} \prec 1$, $\mathfrak{w} = 1$ or $\mathfrak{w} \succ 1$.

If $\mathfrak{w} \prec 1$, then there exist $\mathfrak{v}_1, ..., \mathfrak{v}_{k-1} \in \mathfrak{M}$, such that $\mathfrak{m}_1, ..., \mathfrak{m}_{k-1}, \mathfrak{w} \in \{\mathfrak{v}_1, ..., \mathfrak{v}_{k-1}\}^\star$, by the induction hypothesis. Consequently, $\mathfrak{m}_1, ..., \mathfrak{m}_{k+1} \in \{\mathfrak{v}_1, ..., \mathfrak{v}_{k-1}, \mathfrak{m}_k\}^\star$. If $\mathfrak{w} = 1$, then $\mathfrak{m}_{k+1} = \mathfrak{m}_k^{\alpha_k}$, whence *a fortiori* $\mathfrak{m}_1, ..., \mathfrak{m}_{k+1} \in \{\mathfrak{m}_1, ..., \mathfrak{m}_k\}^\star$. If $\mathfrak{w} \succ 1$, then there exist $\mathfrak{v}_1, ..., \mathfrak{v}_{k-1} \in \mathfrak{M}$, such that $\mathfrak{m}_1^{1/\alpha_k}, ..., \mathfrak{m}_{k-1}^{1/\alpha_k}, \mathfrak{w}^{-1/\alpha_k} \in \{\mathfrak{v}_1, ..., \mathfrak{v}_{k-1}\}^\star$, by the induction hypothesis. Hence $\mathfrak{m}_1, ..., \mathfrak{m}_{k+1} \in \{\mathfrak{v}_1, ..., \mathfrak{v}_{k-1}, \mathfrak{m}_k \mathfrak{w}^{1/\alpha_k}\}^\star$. The lemma follows by induction. $\square$

Theorem 17. *Any* $f \in P[\![\mathfrak{B}]\!]$ *can be uniformly regularized modulo case separations.*

Proof. Let $\mathfrak{m}_1 \prec 1, ..., \mathfrak{m}_p \prec 1, \mathfrak{n}_1, ..., \mathfrak{n}_q \in \mathfrak{b}_1^P \cdots \mathfrak{b}_n^P$ be such that $\operatorname{supp} f \subseteq \{\mathfrak{m}_1, ..., \mathfrak{m}_p\}^\star \{\mathfrak{n}_1, ..., \mathfrak{n}_q\}$. By lemma 15, we may assume without loss of generality that either $\mathfrak{n}_i \prec 1$, $\mathfrak{n}_i = 1$ or $\mathfrak{n}_i \succ 1$ for each i, modulo some case separations. Without loss of generality, we may therefore assume that f admits a Cartesian representation in $\mathfrak{m}_1, ..., \mathfrak{m}_p$, i.e. $\operatorname{supp} f \subseteq \{\mathfrak{m}_1, ..., \mathfrak{m}_p\}^\star \mathfrak{m}_1^{\alpha_1} \cdots \mathfrak{m}_p^{\alpha_p}$ for certain $\alpha_1, ..., \alpha_p \in \mathbb{Z}$. Choosing p minimal, we will prove the theorem by induction over p. If $p = 0$, then $f = 0$, and we have nothing to prove. So assume that $p > 0$.

We will first show how to regularize f modulo case separations. So let $\mathfrak{d}_1, ..., \mathfrak{d}_d$ be the set of dominant monomials of f. By repeated application of lemma 15, and modulo reordering, we may assume that $\mathfrak{d}_1 \preccurlyeq \cdots \preccurlyeq \mathfrak{d}_d$. If all these inequalities are strict, then we are done, since $\mathfrak{d}_d$ will be the only dominant monomial. Otherwise, we have $\mathfrak{d}_i = \mathfrak{d}_j$ for certain $i < j$, which yields a non trivial relation $\mathfrak{m}_1^{\alpha_1} \cdots \mathfrak{m}_q^{\alpha_q} = 1$ for certain $\alpha_1, ..., \alpha_q \in \mathbb{Z}$. Then lemma 16 implies that we may find a Cartesian representation for f in $p - 1$ variables only, and we are done again, by the induction hypothesis.

In order to make f uniformly regular modulo case separations, we use the following algorithm:

Step 1   Regularize f modulo case separations and let $\mathfrak{d}$ be its dominant monomial (if $f \neq 0$).

Step 2 If $f = 0$, or $f_{\mathfrak{d}}(\lambda) \neq 0$ for all $\lambda \in \Lambda$, then we are done.

Step 3 Separate the cases when $f_{\mathfrak{d}} = 0$ and $f_{\mathfrak{d}} \neq 0$ and go back to step 1.

We have to show that this algorithm terminates. Assume the contrary and let $\mathfrak{d}_1, \mathfrak{d}_2, \ldots$ be the successive dominant monomials of f in step 1 on smaller and smaller subregions $\Lambda_1 \supsetneq \Lambda_2 \supsetneq \cdots$ of Λ. Ultimately, for each i, there exists a $\lambda_i \in \Lambda_i$ with $f_{\mathfrak{d}_i}(\lambda_i) = 0$ in step 2, and the next region is given by $\Lambda_{i+1} = \{\lambda \in \Lambda_i \,|\, f_{\mathfrak{d}_i}(\lambda) = 0\}$ in step 3. Now the numerators of all coefficients $f_{\mathfrak{d}_1}, f_{\mathfrak{d}_2}, \ldots$ belong to the Noetherian polynomial ring $P = C[\lambda_1, \ldots, \lambda_\ell]$. Consequently, the increasing chain of ideals $(f_{\mathfrak{d}_1}) \subseteq (f_{\mathfrak{d}_1}, f_{\mathfrak{d}_2}) \subseteq \cdots$ is stationary and so is the decreasing chain $\Lambda_1 \supsetneq \Lambda_2 \supsetneq \cdots$ of subregions of Λ: contradiction. $\qquad\square$

4.3 Computations with parameterized complex transseries

Using the tool of uniform regularization, we may compute with parameterized complex transseries in a similar way as explained in sections 3.2, 3.3 and 3.4. Of course, it may happen that we need to exponentiate or to take logarithms of parameterized constants in P. Nevertheless, this can only happen a finite number of times, so that we may see these exponentials resp. logarithms as new parameters. Furthermore, we will show that it is never necessary to exponentiate or take logarithms of parameterized constants during the resolution of algebraic differential equations.

Example 18. Consider the expansion of the function

$$f = \log\left(e^{e^z + \lambda z} + e^{\mu e^z}\right).$$

Case $e^z \prec 1$. We insert -1 into Δ_{e^z} and get

$$e^{e^z + \lambda z} + e^{\mu e^z} = e^{\lambda z} + e^{(\lambda+1)z} + \cdots + 1 + \mu\, e^z + \cdots.$$

We thus have to determine whether $e^{\lambda z} \prec 1$ and $e^{\lambda z} \succ 1$, which leads to the following cases and expansions for f:

$$
\begin{aligned}
f &= \log 2 + \left(\tfrac{1}{2} + \tfrac{\mu}{2}\right) e^z + \left(-\tfrac{1}{8} + \tfrac{\mu}{4} - \tfrac{\mu^2}{8}\right) e^{2z} + \cdots && (\lambda = 0, \Delta_{e^z} = \{-1\});\\
f &= e^{\lambda z} + \cdots + \mu\, e^z + \cdots && (\lambda \neq 0, \Delta_{e^z} = \{-1, -\lambda\});\\
f &= \lambda\, z + e^z + \cdots + e^{-\lambda z} + \cdots && (\lambda \neq 0, \Delta_{e^z} = \{-1, \lambda\}).
\end{aligned}
$$

Case $e^z \succ 1$. We insert 1 into Δ_{e^z} and next need to determine whether $e^{e^z + \lambda z} \prec e^{\mu e^z + \lambda \mu z}$ or $e^{e^z + \lambda z} \succ e^{\mu e^z + \lambda \mu z}$. This leads to the following cases and expansions for f:

$$
\begin{aligned}
f &= e^z + \lambda\, z + e^{-\lambda \mu z} e^{(\mu-1)(e^z + \lambda z)} + \cdots && (\mu \neq 1, \Delta_{e^z} = \{1\}, \Delta_{e^z + \lambda z} = \{1 - \mu\});\\
f &= \mu\, e^z + e^{\lambda \mu z} e^{(1-\mu)(e^z + \lambda z)} + \cdots && (\lambda \neq 0, \Delta_{e^z} = \{1\}, \Delta_{e^z + \lambda z} = \{\mu - 1\}).
\end{aligned}
$$

In the last exceptional case when $\mu = 1$, we get

$$f = \log\left(e^{e^z + \lambda z}\left(1 + e^{-\lambda z}\right)\right),$$

so that we need to determine whether $1 \prec e^{-\lambda z}$ or $1 \succ e^{-\lambda z}$. This leads to the following final cases and expansions for f:

$$
\begin{aligned}
f &= e^z + \lambda\, z + \log 2 && (\lambda = 0, \mu = 1, \Delta_{e^z} = \{1\});\\
f &= e^z + e^{\lambda z} - \tfrac{1}{2} e^{2\lambda z} + \cdots && (\lambda \neq 0, \mu = 1, \Delta_{e^z} = \{1, -\lambda\});\\
f &= e^z + \lambda\, z + e^{-\lambda z} + \cdots && (\lambda \neq 0, \mu = 1, \Delta_{e^z} = \{1, \lambda\}).
\end{aligned}
$$

5 The differential Newton polygon method

In the remainder of this paper, we will be concerned with the resolution of asymptotic algebraic differential equations like

$$P(f) = 0 \qquad (f \prec \mathfrak{v}), \tag{6}$$

where $P \in \mathbb{T}[f, f', ..., f^{(r)}]$ is a differential polynomial with transseries coefficients and $\mathfrak{v} \in \mathfrak{M}$ a transmonomial.

In this section, we describe the differential Newton polygon method, which enables us to compute the successive terms of solutions one by one. In the next sections, we will be concerned with the transformation of this transfinite process into a finite algorithm. In sections 5, 6 and 7 the transseries in $\mathbb{T}$ are assumed to be as in section 2. In section 8, we will consider parameterized transseries solutions.

5.1 Notations

5.1.1 Asymptotic relations

Except for the usual asymptotic relations $\prec, \preccurlyeq, \sim, \asymp, \lll, \underline{\lll}, \widetilde{\approx}$ and $\underline{\asymp}$, we will also need the flattened relations $\prec_h, \preccurlyeq_h, \asymp_h$ and there variants $\prec_h^*, \preccurlyeq_h^*, \asymp_h^*$, where h is an infinitely large or small transseries. These relations are defined by

$$\begin{aligned}
f \prec_h g &\iff \forall \varphi \lll h \colon\ f\varphi \prec g; \\
f \preccurlyeq_h g &\iff \exists \varphi \lll h \colon\ f\varphi \preccurlyeq g; \\
f \asymp_h g &\iff f \preccurlyeq_h g \wedge g \preccurlyeq_h f; \\
f \prec_h^* g &\iff \forall \varphi \underline{\lll} \log h \colon\ f\varphi \prec g; \\
f \preccurlyeq_h^* g &\iff \exists \varphi \underline{\lll} \log h \colon\ f\varphi \preccurlyeq g; \\
f \asymp_h^* g &\iff f \preccurlyeq_h^* g \wedge g \preccurlyeq_h^* f.
\end{aligned}$$

Notice that $f \prec_h^* g \Rightarrow f \prec_h g$, $f \preccurlyeq_h^* g \Rightarrow f \preccurlyeq_h g$ and $f \asymp_h^* g \Rightarrow f \asymp_h g$.

5.1.2 Natural decomposition of P

The differential polynomial P is most naturally decomposed as

$$P(f) = \sum_{\boldsymbol{i}} P_{\boldsymbol{i}} f^{\boldsymbol{i}} \tag{7}$$

Here we use vector notation for tuples $\boldsymbol{i} = (i_0, ..., i_r)$ and $\boldsymbol{j} = (j_0, ..., j_r)$ of integers:

$$\begin{aligned}
|\boldsymbol{i}| &= r; \\
\|\boldsymbol{i}\| &= i_0 + \cdots + i_r; \\
\boldsymbol{i} \leqslant \boldsymbol{j} &\Leftrightarrow i_0 \leqslant j_0 \wedge \cdots \wedge i_r \leqslant j_r; \\
f^{\boldsymbol{i}} &= f^{i_0}(f')^{i_1} \cdots (f^{(r)})^{i_r}; \\
\binom{\boldsymbol{j}}{\boldsymbol{i}} &= \binom{j_0}{i_0} \cdots \binom{j_r}{i_r}.
\end{aligned}$$

The i-th *homogeneous part* of P is defined by

$$P_i = \sum_{\|\boldsymbol{i}\|=i} P_{\boldsymbol{i}} f^{\boldsymbol{i}},$$

so that

$$P = \sum_{i=0}^{\deg P} P_i.$$

5.1.3 Decomposition of P along orders

Another very useful decomposition of P is its *decomposition along orders*:

$$P(f) = \sum_{\boldsymbol{\omega}} P_{[\boldsymbol{\omega}]} f^{[\boldsymbol{\omega}]} \tag{8}$$

In this notation, $\boldsymbol{\omega}$ runs through tuples $\boldsymbol{\omega} = (\omega_1, ..., \omega_l)$ of integers in $\{0, ..., r\}$ of length $l \leqslant \deg P$, and $P_{[\boldsymbol{\omega}]} = P_{[\omega_{\sigma(1)}, ..., \omega_{\sigma(l)}]}$ for all permutations of integers. We again use vector notation for such tuples

$$
\begin{aligned}
|\boldsymbol{\omega}| &= l; \\
\|\boldsymbol{\omega}\| &= \omega_1 + \cdots + \omega_{|\boldsymbol{\omega}|}; \\
\boldsymbol{\omega} \leqslant \boldsymbol{\tau} &\Leftrightarrow |\boldsymbol{\omega}| = |\boldsymbol{\tau}| \wedge \omega_1 \leqslant \tau_1 \wedge \cdots \wedge \omega_{|\boldsymbol{\omega}|} \leqslant \tau_{|\boldsymbol{\tau}|}; \\
f^{[\boldsymbol{\omega}]} &= f^{(\omega_1)} \cdots f^{(\omega_{|\boldsymbol{\omega}|})}; \\
\binom{\boldsymbol{\tau}}{\boldsymbol{\omega}} &= \binom{\tau_1}{\omega_1} \cdots \binom{\tau_{|\boldsymbol{\tau}|}}{\omega_{|\boldsymbol{\omega}|}}.
\end{aligned}
$$

We call $\|\boldsymbol{\omega}\|$ the *weight* of $\boldsymbol{\omega}$ and

$$\|P\| = \max_{\boldsymbol{\omega} \mid P_{[\boldsymbol{\omega}]} \neq 0} \|\boldsymbol{\omega}\|$$

the *weight* of P.

5.1.4 Logarithmic decomposition of P

It is convenient to denote the successive logarithmic derivatives of f by

$$
\begin{aligned}
f^{\dagger} &= f'/f; \\
f^{\langle i \rangle} &= f^{\dagger \cdots \dagger} \qquad (i \text{ times}).
\end{aligned}
$$

Then each $f^{(i)}$ can be rewritten as a polynomial in $f, f^{\dagger}, ..., f^{\langle i \rangle}$:

$$
\begin{aligned}
f &= f; \\
f' &= f^{\dagger} f; \\
f'' &= ((f^{\dagger})^2 + f^{\dagger\dagger} f^{\dagger}) f; \\
f''' &= ((f^{\dagger})^3 + 3 f^{\dagger\dagger} (f^{\dagger})^2 + (f^{\dagger\dagger})^2 f^{\dagger} + f^{\dagger\dagger\dagger} f^{\dagger\dagger} f^{\dagger}) f; \\
&\vdots
\end{aligned}
$$

We define the *logarithmic decomposition* of P by

$$P(f) = \sum_{\boldsymbol{i} = (i_0, ..., i_r)} P_{\langle \boldsymbol{i} \rangle} f^{\langle \boldsymbol{i} \rangle}, \tag{9}$$

where

$$f^{\langle \boldsymbol{i} \rangle} = f^{i_0} (f^{\dagger})^{i_1} \cdots (f^{\langle r \rangle})^{i_r}.$$

Now consider the lexicographical ordering $\leqslant^{\mathrm{lex}}$ on $\mathbb{N}^{r+1}$, defined by

$$
\begin{aligned}
\boldsymbol{i} <^{\mathrm{lex}} \boldsymbol{j} \iff \ &(i_0 < j_0) \vee \\
&(i_0 = j_0 \wedge i_1 < j_0) \vee \\
&\qquad \vdots \\
&(i_0 = j_0 \wedge \cdots \wedge i_{r-1} = j_{r-1} \wedge i_r < j_r).
\end{aligned}
$$

This ordering is total, so there exists a maximal $\boldsymbol{i}$ for $\leqslant^{\mathrm{lex}}$ with $P_{\langle \boldsymbol{i} \rangle} \neq 0$, assuming that $P \neq 0$. For this $\boldsymbol{i}$, we have

$$P(f) \sim P_{\langle \boldsymbol{i} \rangle} f^{\langle \boldsymbol{i} \rangle} \tag{10}$$

for all f, whose dominant monomial is sufficiently large.

5.1.5 Additive and multiplicative conjugations and upward shifting.

Given a differential polynomial P and a transseries h it is useful to define the *additive* and *multiplicative conjugates* P_{+h} and $P_{\times h}$ of P w.r.t. h and the *upward shifting* $P{\uparrow}$ of P as being the unique differential polynomials, such that for all f, we have

$$
\begin{aligned}
P_{+h}(f) &= P(h+f); \\
P_{\times h}(f) &= P(h\,f); \\
P{\uparrow}(f{\uparrow}) &= P(f){\uparrow}.
\end{aligned}
$$

The coefficients of P_{+h} are explicitly given by

$$
P_{+h,\boldsymbol{i}} = \sum_{\boldsymbol{j}\geqslant\boldsymbol{i}} \binom{\boldsymbol{j}}{\boldsymbol{i}} h^{\boldsymbol{j}-\boldsymbol{i}} P_{\boldsymbol{j}}. \tag{11}
$$

The coefficients of $P_{\times h}$ are more easily expressed using decompositions along orders:

$$
P_{\times h,[\boldsymbol{\omega}]} = \sum_{\boldsymbol{\tau}\geqslant\boldsymbol{\omega}} \binom{\boldsymbol{\tau}}{\boldsymbol{\omega}} h^{[\boldsymbol{\tau}-\boldsymbol{\omega}]} P_{[\boldsymbol{\tau}]}. \tag{12}
$$

The coefficients of the upward shifting (or compositional conjugation by e^z) are given by

$$
(P{\uparrow})_{[\boldsymbol{\omega}]} = \sum_{\boldsymbol{\tau}\geqslant\boldsymbol{\omega}} s_{\boldsymbol{\tau},\boldsymbol{\omega}}\, e^{-\|\boldsymbol{\tau}\|z}\, (P_{[\boldsymbol{\tau}]}{\uparrow}), \tag{13}
$$

where the $s_{\boldsymbol{\tau},\boldsymbol{\omega}}$ are generalized Stirling numbers of the first kind:

$$
\begin{aligned}
s_{\boldsymbol{\tau},\boldsymbol{\omega}} &= s_{\tau_1,\omega_1}\cdots s_{\tau_{|\tau|},\omega_{|\omega|}}; \\
(f(\log z))^{(j)} &= \sum_{i=0}^{j} s_{j,i}\, x^{-j} f^{(i)}(\log z).
\end{aligned}
$$

5.2 Differential Newton polynomials

Given a differential polynomial P with transseries coefficients, its *dominant monomial* $\mathfrak{d}_P$ is defined by

$$
\mathfrak{d}_P = \max_{\boldsymbol{i},\preccurlyeq} \mathfrak{d}_{P_{\boldsymbol{i}}}. \tag{14}
$$

and its *dominant part* (or coefficient) $D_P \in C[c,c',\ldots,c^{(r)}]$ by

$$
D_P = \sum_{\boldsymbol{i}} P_{\boldsymbol{i},\mathfrak{d}_P} c^{\boldsymbol{i}}. \tag{15}
$$

The following theorem shows how D_P looks like after sufficiently many upward shiftings:

Proposition 19. *Let P be a differential polynomial with purely exponential coefficients. Then there exists a polynomial $Q \in C[c]$ and an integer ν, such that for all $i \geqslant \|P\|$, we have $D_{P{\uparrow}^i} = Q\,(c')^{\nu}$.*

Proof. Let ν be minimal, such that there exists an $\boldsymbol{\omega}$ with $\|\boldsymbol{\omega}\| = \nu$ and $(D_P{\uparrow})_{[\boldsymbol{\omega}]} \neq 0$. Then we have $\mathfrak{d}(D_P{\uparrow}) = e^{-\nu z}$ and

$$
D_{P{\uparrow}}(c) = \sum_{\|\boldsymbol{\omega}\|=\mu} \left(\sum_{\boldsymbol{\tau}\geqslant\boldsymbol{\omega}} s_{\boldsymbol{\tau},\boldsymbol{\omega}} D_{P,[\boldsymbol{\tau}]} \right) c^{[\boldsymbol{\omega}]}, \tag{16}
$$

by formula (13). Since $D_{P\uparrow} \neq 0$, we must have $\nu \leqslant \|D_P\|$. Consequently, $\|D_P\| \geqslant \nu = \|D_{P\uparrow}\| \geqslant \|D_{P\uparrow\uparrow}\| \geqslant \cdots$. Hence, for some $i \leqslant \|P\|$, we have $\|D_{P\uparrow^{i+1}}\| = \|D_{P\uparrow^i}\|$. But then (16) applied on $P\uparrow_i$ instead of P yields $D_{P\uparrow^{i+1}} = D_{P\uparrow^i}$. This shows that $D_{P\uparrow_i}$ is independent of i, for $i \geqslant \|P\|$.

In order to prove the proposition, it now suffices to show that $D_{P\uparrow} = D_P$ implies $D_{P\uparrow} = Q\,(c')^\nu$ for some polynomial $Q \in C[c]$. For all differential polynomials R of homogeneous weight ν, let

$$R^* = \sum_j \left([c^j\,(c')^\nu]\,R\right) c^j\,(c')^\nu. \tag{17}$$

Since $D^*_{P\uparrow} = D^*_P$, it suffices to show that $P = 0$ if and only if $D^*_P = 0$. Now $D^*_P = 0$ implies that $D_P(z) = 0$. Furthermore, (13) yields

$$D_P\uparrow = e^{-\nu z}\,D_P. \tag{18}$$

Consequently, we also have $D_P(e^z) = e^{\nu z}\,(D_P\uparrow)(e^z) = e^{\nu z}\,(D_P(z))\uparrow = 0$. By induction, it follows that $D_P(\exp_i z) = 0$ for any iterated exponential of z. We conclude that $D_P = P = 0$, by (10). $\qquad\square$

Given an arbitrary differential polynomial P, the above proposition implies that there exists a polynomial $Q \in C[c]$ and an integer ν, such that $D_{P\uparrow_i} = Q\,(c')^\nu$ for all sufficiently large i. We call

$$N_P = Q\,(c')^\nu$$

the *differential Newton polynomial* of P. More generally, given a monomial $\mathfrak{m}$, we call $N_{P_{\times\mathfrak{m}}}$ the *differential Newton polynomial* of P *associated to* $\mathfrak{m}$.

5.3 Potential dominant monomials and terms

Returning to the asymptotic differential equation (6), we call $\mathfrak{m} \prec \mathfrak{v}$ a *potential dominant monomial*, if $N_{P_{\times\mathfrak{m}}}$ admits a non trivial root $c \in (C^{\mathrm{alg}})^*$, where C^{alg} stands for the algebraic closure of C. If $c \in C^*$, then the corresponding term $c\,\mathfrak{m}$ is called a *potential dominant term*. The *multiplicity* of c (and of $c\,\mathfrak{m}$) is the differential valuation of $N_{P_{\mathfrak{m},+c}}$, i.e. the least i such that $N_{P_{\mathfrak{m},+c},i} \neq 0$. The *Newton degree* of (6) is the largest possible degree of $N_{P_{\times\mathfrak{m}}}$ for monomials $\mathfrak{m} \prec \mathfrak{v}$.

Proposition 20. *Assume that f is a regular, non-zero transseries solution to* (6). *Then τ_f is a potential dominant term.* $\qquad\square$

A potential dominant monomial $\mathfrak{m}$ is said to be *algebraic* if $N_{P_{\times\mathfrak{m}}}$ is non homogeneous, and *differential* if $N_{P_{\times\mathfrak{m}}} \notin C[c]$. A potential dominant monomial, which is both algebraic and differential, is said to be *mixed*. Notice that (12) implies

$$\mathfrak{d}(P_{\times\mathfrak{m}}) \ \asymp^*_{\mathfrak{m}} \ \mathfrak{m}\,\mathfrak{d}(P),$$

if the coefficients of P and $\mathfrak{m}$ are purely exponential.

5.3.1 Algebraic potential dominant monomials

The algebraic potential dominant monomials correspond to the slopes of the Newton polygon in a non differential setting. However, they can not be determined directly as a function of the dominant monomials of the $P_{\boldsymbol{i}}$, because there may be some cancellation of terms in the different homogeneous parts during multiplicative conjugations. Instead, the algebraic potential dominant monomials are determined by successive approximation:

Proposition 21. *Let $i < j$ be such that $P_i \neq 0$ and $P_j \neq 0$.*

   a) If P is purely exponential, then there exists a unique purely exponential monomial $\mathfrak{m}$, such that $\mathfrak{d}(P_{i,\times\mathfrak{m}}) = \mathfrak{d}(P_{j,\times\mathfrak{m}})$.

   b) Denoting by $\mathfrak{m}_{P,i,j}$ the monomial $\mathfrak{m}$ in (a), there exists an integer $k \leqslant \|P\|$, such that for all $l \geqslant k$ we have $\mathfrak{m}_{P\uparrow l, i, j} = \mathfrak{m}_{P\uparrow k, i, j}\uparrow_{l-k}$.

   c) There exists a unique monomial $\mathfrak{m}$, such that $N_{(P_i + P_j)\times\mathfrak{m}}$ is non homogeneous.

Proof. In (a), let $\mathfrak{B} = (\mathfrak{b}_1, ..., \mathfrak{b}_n)$ be a purely exponential transbasis for the coefficients of P. We prove the existence of $\mathfrak{m}$ by induction over the least possible k, such that we may write $\mathfrak{d}(P_i)/\mathfrak{d}(P_j) = \mathfrak{b}_1^{\alpha_1} \cdots \mathfrak{b}_k^{\alpha_k}$. If $k = 0$, then we have $\mathfrak{m} = 1$. Otherwise, let $Q = P_{\times\mathfrak{n}}$ with $\mathfrak{n} = \mathfrak{b}_k^{\alpha_k/(j-i)}$. Then

$$\mathfrak{d}(Q_i) \asymp_{\mathfrak{b}_k} \mathfrak{d}(P_i)\,\mathfrak{n}^i \asymp_{\mathfrak{b}_k} \mathfrak{d}(P_j)\,\mathfrak{n}^j \asymp_{\mathfrak{b}_k} \mathfrak{d}(Q_j),$$

so that $\mathfrak{d}(Q_i)/\mathfrak{d}(Q_j) = \mathfrak{b}_1^{\beta_1} \cdots \mathfrak{b}_l^{\beta_l}$ for some $l < k$ and $\beta_1, ..., \beta_l$. By the induction hypothesis, there exists a purely exponential monomial $\mathfrak{w}$, such that $\mathfrak{d}(Q_{i,\times\mathfrak{w}}) = \mathfrak{d}(Q_{j,\mathfrak{w}})$. Hence we may take $\mathfrak{m} = \mathfrak{n}\mathfrak{w}$. As to the uniqueness of $\mathfrak{m}$, assume that $\mathfrak{n} = \mathfrak{m}\,\mathfrak{b}_1^{\alpha_1} \cdots \mathfrak{b}_k^{\alpha_k}$ with $\alpha_k \neq 0$. Then

$$\mathfrak{d}(P_{i,\times\mathfrak{n}}) \asymp_{\mathfrak{b}_k} \mathfrak{d}(P_{i,\times\mathfrak{m}})\,\mathfrak{b}_k^{i\alpha_k} \not\asymp_{\mathfrak{b}_k} \mathfrak{d}(P_{j,\mathfrak{m}})\,\mathfrak{b}_k^{j\alpha_k} \asymp_{\mathfrak{b}_k} \mathfrak{d}(P_{j,\times\mathfrak{n}}).$$

This proves (a).

   With the notations from proposition 19, we have already shown that $\|D_{P_i\uparrow}\| \leqslant \|D_{P_i}\|$ and that equality occurs if and only if $D_{P_i} = c^{i - \|D_{P_i}\|}\,(c')^{\|D_{P_i}\|}$. Because of (12), we also notice that $\|D_{P_i,\times e^{\alpha z}}\| = \|D_{P_i}\|$ for all α. It follows that

$$\|D_{P_i,\times\mathfrak{m}_{P,i,j}}\| \geqslant \|D_{P_i\uparrow,\times\mathfrak{m}_{P\uparrow,i,j}}\| \geqslant \cdots$$

and similarly for P_j instead of P_i, since we necessarily have $\mathfrak{m}_{P_i\uparrow,i,j} = \mathfrak{m}_{P,i,j}\uparrow e^{\alpha z}$ for some α. We finally notice that $\|D_{P_i,\times\mathfrak{m}_{P,i,j}}\| = \|D_{P_i\uparrow,\times\mathfrak{m}_{P\uparrow,i,j}}\|$ and $\|D_{P_j,\times\mathfrak{m}_{P,i,j}}\| = \|D_{P_j\uparrow,\times\mathfrak{m}_{P\uparrow,i,j}}\|$ imply that $\mathfrak{m}_{P\uparrow,i,j} = \mathfrak{m}_{P,i,j}\uparrow$, since $\|D_{(c^\alpha(c')^\beta)\times e^{\gamma x}}\| = 0 \neq \beta = \|D_{c^\alpha(c')^\beta}\|$ whenever $\beta \neq 0$ and $\gamma \neq 0$. Consequently, $\|D_{P_i\uparrow l,\times\mathfrak{m}_{P\uparrow l,i,j}}\|$ and $\|D_{P_j\uparrow l,\times\mathfrak{m}_{P\uparrow l,i,j}}\|$ stabilize for $l \geqslant k$ with $k \leqslant \|P\|$. For this k, we have (b).

   With the notations from (b), $\mathfrak{m}_{P\uparrow k,i,j}\downarrow_k$ is actually the unique monomial $\mathfrak{m}$ such that

$$D_{(P_i+P_j)\times\mathfrak{m}\uparrow l} = D_{P_i,\times\mathfrak{m}\uparrow k} + D_{P_j,\times\mathfrak{m}\uparrow k}$$

is non homogeneous for all sufficiently large l. Now $N_{(P_i+P_j)\times\mathfrak{m}} = D_{(P_i+P_j)\times\mathfrak{m}\uparrow l}$ for sufficiently large l. This proves (c) for purely exponential differential polynomials P, and also for general differential polynomials, after sufficiently many upward shiftings. $\qquad\square$

   The unique monomial $\mathfrak{m}$ from part (c) of the above proposition is called an *equalizer* or the (i, j)-equalizer for P. An algebraic potential dominant monomial is necessarily an equalizer. Consequently, there are only a finite number of algebraic potential dominant monomials and they can be found as described in the proof of proposition 21. Notice that, given a transbasis $\mathfrak{B} = (\mathfrak{b}_1, ..., \mathfrak{b}_n)$ for the coefficients of P, all equalizers for P belong to $(\log_{\|P\|}^C \mathfrak{b}_1) \cdots (\log^C \mathfrak{b}_1)\,\mathfrak{B}^C$.

5.3.2 Differential potential dominant monomials

In order to find the differential potential dominant monomials, it suffices to consider the homogeneous parts P_i of P, since $N_{P\times\mathfrak{m},i} = N_{P_i,\times\mathfrak{m}}$, if $c'|N_{P\times\mathfrak{m}}$ and $N_{P\times\mathfrak{m},i} \neq 0$. Now we may rewrite P_i as f^i times a differential polynomial $R_{P,i}$ of order $r \leqslant 1$ in $f^\dagger$. We call $R_{P,i}$ the i-th Ricatti equation associated to P. Since solving $P_i(f) = 0$ is equivalent to solving $R_{P,i}(f^\dagger) = 0$, we are entitled to expect that finding the potential dominant monomials of f w.r.t. $P(f) = 0$ is equivalent to solving $R_{P,i}(f^\dagger) = 0$ "up to a certain extent".

Proposition 22. *The monomial $\mathfrak{m} \prec \mathfrak{v}$ is a potential dominant monomial of f w.r.t.*

$$P_i(f) = 0 \tag{19}$$

if and only if the equation

$$R_{P,i,+\mathfrak{m}^\dagger}(f^\dagger) = 0 \qquad \left(f^\dagger \prec \frac{1}{z \log z \log \log z \,\cdots} \right) \tag{20}$$

has strictly positive Newton degree.

Proof. We first notice that $R_{P\uparrow,i} = (R_{P,i}\uparrow)_{\times e^{-z}}$ for all P and i. We claim that the equivalence of the proposition holds for P and $\mathfrak{m}$ if and only if it holds for $P\uparrow$ and $\mathfrak{m}\uparrow$. Indeed, $\mathfrak{m}$ is potential dominant monomial w.r.t. (19), if and only if $\mathfrak{m}$ is a potential dominant monomial w.r.t.

$$P_i\uparrow(f\uparrow) = 0 \tag{21}$$

and (20) has strictly positive Newton degree if and only if

$$R_{P,i,+\mathfrak{m}^\dagger}\uparrow(f^\dagger\uparrow) = 0 \qquad \left(f^\dagger\uparrow \prec \frac{1}{e^z z \log z \,\cdots} \right) \tag{22}$$

has strictly positive Newton degree. Now the latter is the case if and only if

$$(R_{P,i,+\mathfrak{m}^\dagger}\uparrow)_{\times e^{-z}}(f\uparrow^\dagger) = 0 \qquad \left(f\uparrow^\dagger \prec \frac{1}{z \log z \, \log \log z \,\cdots} \right)$$

has strictly positive Newton degree. But

$$(R_{P,i,+\mathfrak{m}^\dagger}\uparrow)_{\times e^{-z}} = (R_{P,i}\uparrow)_{+\mathfrak{m}^\dagger\uparrow,\times e^{-z}} = (R_{P,i}\uparrow)_{\times e^{-z},+\mathfrak{m}\uparrow^\dagger} = R_{P\uparrow,i,+\mathfrak{m}\uparrow^\dagger}.$$

This proves our claim.

Now assume that $\mathfrak{m}$ is a potential dominant monomial w.r.t. (19). In view of our claim, we may assume without loss of generality that P and $\mathfrak{m}$ are purely exponential and that $N_{P_i,\times\mathfrak{m}} = D_{P_i,\times\mathfrak{m}}$. Since P_i is homogeneous, we have $D_{P_i,\times\mathfrak{m}} = \alpha \, (c')^i$ for some $\alpha \in C^*$ and

$$D_{R_{P,i,+\mathfrak{m}^\dagger}} = \alpha \, c^i.$$

Since $R_{P,i,+\mathfrak{m}^\dagger}$ is purely exponential, it follows that $N_{R_{P,i,+\mathfrak{m}^\dagger,\times z^{-2}}}$ has degree i, so that the Newton degree of (20) is at least i. Similarly, if $\mathfrak{m}$ is not a potential dominant monomial w.r.t. (19), then $D_{P_i,\times\mathfrak{m}} = \alpha \, c^i$ and

$$D_{R_{P,i,+\mathfrak{m}^\dagger}} = \alpha$$

for some $\alpha \in C^*$. Consequently, $N_{R_{P,i,+\mathfrak{m}^\dagger,\times\mathfrak{n}}} = \alpha$ for any infinitesimal monomial $\mathfrak{n}$, and the Newton degree of (20) vanishes. $\qquad\square$

5.4 Refinements

Now we know how to determine potential dominant terms of solutions to (6), let us show how to obtain more terms. A *refinement* is a change of variables together with an asymptotic constraint

$$f = \varphi + \tilde{f} \qquad (\tilde{f} \prec \tilde{\mathfrak{v}}), \tag{23}$$

where $\tilde{\mathfrak{v}} \prec \varphi$. Such a refinement transforms (6) into

$$P_{+\varphi}(\tilde{f}) = 0 \qquad (\tilde{f} \prec \tilde{\mathfrak{v}}). \tag{24}$$

We call the refinement *admissible*, if (24) has strictly positive Newton degree.

Proposition 23. *Let* $c\,\mathfrak{m}$ *be the dominant term of* φ *and assume that* $\tilde{\mathfrak{v}} = \mathfrak{m}$. *Then the Newton degree of* (24) *is equal to the multiplicity* $\tilde{d}$ *of* c *as a root of* $N_{P_{\times\mathfrak{m}}}$.

Proof. Let us first show that $\deg N_{P_{+\varphi,\times\mathfrak{n}}} \leqslant \tilde{d}$ for any monomial $\mathfrak{n} \prec \mathfrak{m}$. Modulo replacing P by $P_{\times\mathfrak{m}}$ we may assume without loss of generality that $\mathfrak{m} = 1$. Modulo a sufficient number of upward shiftings, we may also assume that $N_P = D_P$, that $N_{P_{+\varphi,\times\mathfrak{n}}} = D_{P_{+\varphi,\times\mathfrak{n}}}$, and that P, $\mathfrak{n}$ and φ are purely exponential. The differential valuation of $N_{P,+c} = D_{P,+\varphi}$ being $\tilde{d}$, we have in particular $\mathfrak{d}(P_{+\varphi,\tilde{d}}) = \mathfrak{d}(P_{+\varphi})$. Hence,

$$\mathfrak{d}(P_{+\varphi,\times\mathfrak{n},i}) \asymp_{\mathfrak{n}} \mathfrak{d}(P_{+\varphi,\times\mathfrak{n},i})\,\mathfrak{n}^i \prec_{\mathfrak{n}} \mathfrak{d}(P_{+\varphi,\times\mathfrak{n}})\,\mathfrak{n}^{\tilde{d}} \asymp_{\mathfrak{n}} \mathfrak{d}(P_{+\varphi,\times\mathfrak{n},\tilde{d}})$$

for all $i > \tilde{d}$. We infer that $\deg N_{P_{+\varphi,\times\mathfrak{n}}} \leqslant \tilde{d}$.

At a second stage, we have to show that $\deg N_{P_{+\varphi,\times\mathfrak{n}}} \geqslant \tilde{d}$. Without loss of generality, we may again assume that $\mathfrak{m} = 1$, that $N_P = D_P$, and that P and φ are purely exponential. The differential valuation of $N_{P,+c} = D_{P,+\varphi}$ being $\tilde{d}$, we have $\mathfrak{d}(P_{+\varphi,i}) \prec \mathfrak{d}(P_{+\varphi})$ for all $i < \tilde{d}$. Taking $\mathfrak{n} = z^{-1}$, we thus get

$$\mathfrak{d}(P_{+\varphi,\times\mathfrak{n},i}) \asymp_{e^z} \mathfrak{d}(P_{+\varphi,i}) \prec_{e^z} \mathfrak{d}(P_{+\varphi}) = \mathfrak{d}(P_{+\varphi,\tilde{d}}) \asymp_{e^z} \mathfrak{d}(P_{+\varphi,\times\mathfrak{n},\tilde{d}})$$

for all $i < \tilde{d}$. We conclude that $\deg N_{P_{+\varphi,\times\mathfrak{n}}} \geqslant \tilde{d}$. $\square$

5.5 A worked example

Consider the algebraic differential equation

$$P(f) = f + f f'' - (f')^2 = 0. \tag{25}$$

Let us start by computing the potential dominant monomials of f. We first have to find the $(1,2)$-equalizer relative to (25). Since $D_{P_2} \in c^{\mathbb{N}}(c')^{\mathbb{N}}$, we cannot have $N_{P_2} = P_2$, so we have to compute

$$P{\uparrow} = f + e^{-2z}\left(-f f' + f f'' - (f')^2\right).$$

In order to "equalize" $P{\uparrow}_1$ and $P{\uparrow}_2$, we have to conjugate P multiplicatively with e^{2z}:

$$P{\uparrow}_{\times e^{2z}} = e^{2z}\left(f - 2 f^2 - f f' + f f'' - (f')^2\right).$$

At this point, we observe that $D_{P{\uparrow}_{\times e^{2z}}{\uparrow}} = c - 2 c^2 \in C[c]$, so we have found the $(1,2)$-equalizer, which is $\mathfrak{e} = e^{2z}{\downarrow} = z^2$. Since $N_{P_{\times\mathfrak{e}}} = c - 2 c^2$, the corresponding algebraic potential dominant term of f is $\tau^{\mathrm{alg}} = \frac{1}{2} z^2$. As to the differential potential dominant monomials, we have

$$\begin{aligned} R_{P,1} &= 1; \\ R_{P,2} &= f^{\dagger\prime}. \end{aligned}$$

Clearly, $R_{P,1}$ has no roots and $R_{P,2}(f^{\dagger}) = 0$ has all constants $\lambda \in C$ as its solutions modulo $1/(z \log z \log\log z \cdots)$. Consequently, $e^{\lambda z}$ is a potential dominant monomial of f for all $\lambda \in C$, such that $e^{\lambda z} \succ 1$. The corresponding differential potential dominant terms are of the form $\tau_{\lambda,\mu} = \mu\, e^{\lambda z}$, with $\mu \neq 0$ and $e^{\lambda z} \succ 1$.

In order to find more terms of the solution to (25), we have to refine the equation. First of all, consider the refinement

$$f = \tau^{\mathrm{alg}} + \tilde{f} \qquad (\tilde{f} \prec \tau^{\mathrm{alg}}),$$

which transforms (25) into

$$2\,\tilde f - 2\,z\,\tilde f' + \tfrac{1}{2}\,z^2\,\tilde f'' + \tilde f\,\tilde f'' - (\tilde f')^2 = 0 \qquad (\tilde f \prec z^2). \tag{26}$$

Since $P_0 = 0$, we first observe that $f = \tfrac{1}{2}\,z^2$ is actually a solution to (25). On the other hand, since τ^{alg} is a potential dominant term of multiplicity 1 of f, the Newton degree of (26) is one. The only potential dominant monomials of $\tilde f$ therefore necessarily correspond to solutions modulo $1/(z \log z \log\log z \cdots)$ of the Ricatti equation

$$2 - 2\,z\,\tilde f^\dagger + \tfrac{1}{2}\,z^2\,((\tilde f^\dagger)^2 + f^{\dagger\prime}) = 0.$$

These solutions are of the form $\tilde f^\dagger = \tfrac{1}{z} + \cdots$ and $\tilde f^\dagger = \tfrac{4}{z} + \cdots$, which leads to the potential dominant monomials z and z^4, from which we remove z^4, since $z^4 \not\prec z^2$. Expanding one term further, we see that the generic solution to (26) is

$$\tilde f = \lambda\,z + \tfrac{\lambda^2}{2},$$

with $\lambda \in C$ and where the case $\lambda = 0$ recovers the previous solution. In other words,

$$f = \tfrac{1}{2}\,z^2 + \lambda\,z + \tfrac{\lambda^2}{2}$$

is the first type of generic solution to (25).

As to the second case, we consider the refinement

$$f = \tau_{\lambda,\mu} + \tilde f \qquad (\tilde f \prec \tau_{\lambda,\mu}),$$

which transforms (25) into

$$\mu\,e^{\lambda z} + (\lambda^2 f - 2\,\lambda\,f' + f'')\,\mu\,e^{\lambda z} + f + \tilde f\,\tilde f'' - (\tilde f')^2 = 0 \qquad (\tilde f \prec e^{\lambda z}). \tag{27}$$

Again, this equation has Newton degree one. On the one hand, we observe that the linear part of this equation only admits solutions with dominant monomial $e^{\lambda z}$ or $z\,e^{\lambda z}$. Consequently, (27) admits at most one solution. On the other hand, we will show in the next section that quasi-linear equations (i.e. of Newton degree one) always admit at least one solution. In our case, this leads to the following second type of generic solution to (25):

$$f = \mu\,e^{\lambda z} - \tfrac{1}{\lambda^2} + \tfrac{1}{4\,\mu\,\lambda^4}\,e^{-\lambda z}.$$

For the present example, we actually even found exact solutions. Of course, the expansions are infinite in general.

6 Distinguished solutions

6.1 Distinguished left inverses of linear differential operators

Let $\mathfrak{B} = (\mathfrak{b}_1, ..., \mathfrak{b}_n)$ be a purely exponential transbasis. A linear operator on

$$\mathbb{S} = C[z]\,[\![\mathfrak{b}_1; ...; \mathfrak{b}_n]\!] \subseteq C\,[\![z; \mathfrak{b}_1; ...; \mathfrak{b}_n]\!]$$

is said to be *grid-based* if its *operator support*

$$\operatorname{supp}_* L = \bigcup_{\mathfrak{m}} \frac{\operatorname{supp} L\mathfrak{m}}{\mathfrak{m}}$$

is grid-based. For all transseries $f \in \mathbb{S}$ we have

$$\operatorname{supp} L f \subseteq (\operatorname{supp}_* L)(\operatorname{supp} f).$$

In particular, the differentiation ∂ on $\mathbb{S}$ is grid-based with

$$\operatorname{supp}_* \partial \subseteq \{z^{-1}\} \cup \operatorname{supp} \mathfrak{b}_1^{\dagger} \cup \cdots \cup \operatorname{supp} \mathfrak{b}_n^{\dagger}.$$

Consequently, any linear differential operator $L = L_0 + L_1 \partial + \cdots + L_r \partial^r$ with coefficients in $C[\![\mathfrak{b}_1; ...; \mathfrak{b}_n]\!]$ is also grid-based, since

$$\operatorname{supp}_* L \subseteq \operatorname{supp} L_0 \cup (\operatorname{supp} L_1)(\operatorname{supp}_* \partial) \cup \cdots \cup (\operatorname{supp} L_r)(\operatorname{supp}_* \partial)^r.$$

We will now show that L also admits a so called *distinguished left inverse* L^{-1}, which is linear and grid-based. Here a *distinguished solution* to the equation

$$L f = g$$

is a solution f, such that for all other solutions $\hat{f}$, we have $f_{\mathfrak{d}_{\hat{f}-f}} = 0$. Distinguished solutions are clearly unique. We say that L^{-1} is a distinguished left inverse of L, if $L^{-1} g$ is a distinguished solution to $L f = g$ for each g.

In what follows, we will often consider linear differential operators L as linear differential polynomials. In this case, you should keep in mind that L_i denotes the coefficient of $f^{(i)}$ in L and not the i-th homogeneous part. We will also denote $\operatorname{supp} L = \operatorname{supp} L_0 \cup \cdots \cup \operatorname{supp} L_r$ for any linear differential operator L as above.

Theorem 24. *Let $L = L_0 + L_1 \partial + \cdots + L_r \partial^r$ be a linear differential operator with coefficients in $C[\![\mathfrak{b}_1; ...; \mathfrak{b}_n]\!]$ and $L_r \neq 0$. Then L admits a distinguished linear left inverse L^{-1} on $C[z][\![\mathfrak{b}_1; ...; \mathfrak{b}_n]\!]$. This left inverse is grid-based and*

$$\operatorname{supp}_* L^{-1} \subseteq \mathfrak{V} \mathfrak{W}^*,$$

where

$$\mathfrak{V} = z^{r-\mathbb{N}} \left\{ \frac{\mathfrak{m}}{\mathfrak{d}(L_{\times \mathfrak{m}})} \,\middle|\, \mathfrak{m} \in \mathfrak{B}^C \right\};$$

$$\mathfrak{W} = z^{-\mathbb{N}} \cup z^{r-\mathbb{N}} \left(\bigcup_{\mathfrak{m} \in \mathfrak{B}^C} \frac{\operatorname{supp} L_{\times \mathfrak{m}}}{\mathfrak{d}(L_{\times \mathfrak{m}})} \,\middle\backslash\, \{1\} \right).$$

Proof. Let $\mathfrak{M} = z^{\mathbb{N}} \mathfrak{B}^C$ and $\mathfrak{H} = \{\mathfrak{d}_h | h \in \mathbb{S}, L h = 0\}$. There exists a unique strongly linear operator $\Delta: C[[\mathfrak{M} \backslash \mathfrak{H}]] \to C[[\mathfrak{M}]]$, such that

$$\Delta \mathfrak{m} = \tau_{L \mathfrak{m}}$$

for all $\mathfrak{m} \in \mathfrak{M} \backslash \mathfrak{H}$. The operator Δ admits a natural left inverse $\Delta^{-1}: C[[\mathfrak{M}]] \to C[[\mathfrak{M} \backslash \mathfrak{H}]]$, which is constructed as follows. Let $z^i \mathfrak{n} \in \mathfrak{M}$, where $\mathfrak{n}$ is purely exponential. By proposition $21(a)$, there exists a purely exponential monomial $\mathfrak{m}$ with $\mathfrak{d}(L_{\times \mathfrak{m}}) = \mathfrak{n}$. Let $\mathfrak{d}$ and D respectively denote the dominant monomial and dominant part of $L_{\times \mathfrak{m}}$. Let

$$\tau = \frac{c_f i!}{D_{L_{\times \mathfrak{m}}, j}(i+j)!} z^{i+j},$$

where j is minimal with $D_j \neq 0$. Then we observe that

$$L_{\times \mathfrak{m}} \tau \sim (D_{L_{\times \mathfrak{m}}} \tau) \mathfrak{d}_{L_{\times \mathfrak{m}}} \sim z^i \mathfrak{n},$$

so that $\Delta \tau = z^i \mathfrak{n}$. Consequently, we may take $\Delta^{-1}(z^i \mathfrak{n}) = \tau$ and extend Δ^{-1} to the whole of $C[[\mathfrak{M}]]$ by strong linearity.

Let $R = L - \Delta$. By construction, the operator $R\,\Delta^{-1}$ is strictly extensive, and the operator $(\mathrm{Id} + R\,\Delta^{-1})\,\Delta$ coincides with L on $C[[\mathfrak{M}\backslash\mathfrak{H}]]$. Now consider the functional

$$\Phi(f, g) = g - R\,\Delta^{-1}\,f.$$

By the implicit function theorem from [vdH00b], there exists a linear operator

$$\Psi = (\mathrm{Id} + R\,\Delta^{-1})^{-1} = \mathrm{Id} - R\,\Delta^{-1} + (R\,\Delta^{-1})^2 + \cdots,$$

such that $\Phi(\Psi(g), g) = \Psi(g)$ for all $g \in C[[\mathfrak{M}]]$. Consequently,

$$L^{-1} = \Delta^{-1}\,(\mathrm{Id} + R\,\Delta^{-1})^{-1} : C[[\mathfrak{M}]] \to C[[\mathfrak{M}\backslash\mathfrak{H}]]$$

is a strongly linear left inverse for L.

In order to prove that L^{-1} is actually a grid-based operator, we first notice that, by construction,

$$\begin{aligned}
\mathrm{supp}_*\,\Delta^{-1} &\subseteq \mathfrak{V}; \\
\mathrm{supp}_*\,(R\,\Delta^{-1}) &\subseteq \mathfrak{W}.
\end{aligned}$$

Since

$$\mathrm{supp}_*\,\Delta^{-1}\,(\mathrm{Id} + R\,\Delta^{-1})^{-1} \subseteq (\mathrm{supp}_*\,\Delta^{-1})\,(\mathrm{supp}_*\,(R\,\Delta^{-1}))^*,$$

it therefore suffices to prove that $\mathfrak{V}$ and $\mathfrak{W}$ are grid-based. But this follows from theorem 17 when considering $\mathfrak{m}^{-1}\,L_{\times\mathfrak{m}}$ as a generic transseries in $\lambda_1, \ldots, \lambda_n$, for $\mathfrak{m} = \mathfrak{b}_1^{\lambda_1} \cdots \mathfrak{b}_n^{\lambda_n}$. Indeed, there exist a finite number of regions, each on which $\mathfrak{m}^{-1}\,L_{\times\mathfrak{m}}$ is uniformly regular. Consequently, $\mathfrak{m}/\mathfrak{d}(L_{\times\mathfrak{m}})$ can only take a finite number of values and

$$\bigcup_{\mathfrak{m}\in\mathfrak{B}^C} \frac{\mathrm{supp}\,L_{\times\mathfrak{m}}}{\mathfrak{d}_{L_{\times\mathfrak{m}}}}$$

is contained in the union of the supports of the generic transseries $\mathfrak{m}^{-1}\,L_{\times\mathfrak{m}}$ on each of the regions. $\qquad\square$

Remark 25. Each $\mathfrak{h} \in \mathfrak{H}$ induces a *canonical solution* $h = \mathfrak{h} + L^{-1}\,L\,\mathfrak{h} \in C[[\mathfrak{M}]]$ to $Lh = 0$. This canonical solution satisfies $h_{\mathfrak{h}} = 1$ and $h_{\mathfrak{i}} = 0$ for all $\mathfrak{i} \in \mathfrak{H}\backslash\{\mathfrak{h}\}$. Actually, the canonical solutions h are polynomials of degree $< r$ in z with coefficients in $C[[\mathfrak{b}_1; \ldots; \mathfrak{b}_n]]$. In order to see this, let

$$\begin{aligned}
\mathfrak{D} &= \{z^i\,\mathfrak{m} \in z^{\mathbb{N}}\mathfrak{B}^C \,|\, i \leqslant \mathrm{Card}\,\{\mathfrak{i} \in \mathfrak{H}|\mathfrak{h} \succ \mathfrak{i} \succcurlyeq_{e^z} \mathfrak{m}\} \wedge z^i\,\mathfrak{m} \prec \mathfrak{h}\}; \\
\mathfrak{I} &= (\mathfrak{d} \circ L)(\mathfrak{D}).
\end{aligned}$$

Then we observe that L maps $C[[\mathfrak{D}]]$ into $C[[\mathfrak{I}]]$ and that L^{-1} maps $C[[\mathfrak{I}]]$ into $C[[\mathfrak{D}]]$.

6.2 Distinguished solutions of quasi-linear equations

Let $\mathfrak{M}$ be a subset of a monomial group. The notion of operator support can be extended to strongly k-linear operators $M : C[[\mathfrak{M}]]^k \to C[[\mathfrak{M}]]$ by

$$\mathrm{supp}_*\,M = \bigcup_{(\mathfrak{m}_1, \ldots \mathfrak{m}_k) \in \mathfrak{M}^k} \frac{\mathrm{supp}\,M(\mathfrak{m}_1, \ldots, \mathfrak{m}_k)}{\mathfrak{m}_1 \cdots \mathfrak{m}_k}.$$

More generally, if $\Phi : C[[\mathfrak{M}]] \to C[[\mathfrak{M}]]$ is a Noetherian operator, then we define its *operator support* by

$$\mathrm{supp}_*\,\Phi = \bigcup_{i \in \mathbb{N}} \mathrm{supp}_*\,\Phi_i,$$

where Φ_i stands for the i-th homogeneous part of Φ. We have

$$\operatorname{supp}\Phi(f) \subseteq (\operatorname{supp}_* \Phi)(\operatorname{supp} f)^*$$

for all $f \in C[[\mathfrak{M}]]$. We say that Φ is *grid-based*, if $\operatorname{supp}_* \Phi$ is grid-based.

Let $\mathfrak{B} = (\mathfrak{b}_1, ..., \mathfrak{b}_n)$ be a purely exponential transbasis and P a differential polynomial with coefficients in $C[\![\mathfrak{b}_1; ...; \mathfrak{b}_n]\!]$. Notice that we may naturally consider P as a grid-based operator on $C[\![\mathfrak{b}_1; ...; \mathfrak{b}_n]\!]$. The equation (6) is said to be *quasi-linear* if its Newton degree is one. A solution f to such an equation is again said to be *distinguished* if we have $f_{\mathfrak{d}(\tilde{f} - f)} = 0$ for all other solutions $\tilde{f}$ to (6).

Theorem 26. *Assume that the equation* (6) *is quasi-linear. Then it admits a distinguished transseries solution.*

Proof. Without loss of generality, we may assume that $\mathfrak{d}_P = 1$ and $\mathfrak{v} = 1$. We prove the proposition by induction over n. If $n = 0$, then we must have $P_0 = 0$, so that 0 is the distinguished solution to (6). So assume that $n \neq 0$ and let

$$D = \sum_{\mathbf{i}, \mathfrak{m} \asymp_{\mathfrak{b}_n} 1} P_{\mathbf{i}, \mathfrak{m}} \mathfrak{m} f^{\mathbf{i}}$$

be the dominant part of P w.r.t. $\mathfrak{b}_n$. By the induction hypothesis, there exists a distinguished solution to the quasi-linear equation

$$D(\varphi) = 0 \qquad (\varphi \prec 1). \tag{28}$$

We first proceed with the refinement

$$f = \varphi + \tilde{f} \qquad (\tilde{f} \prec \varphi),$$

so that $D_{+\varphi, 0} = 0$, and a sufficient number of upward shiftings, so that $P_{+\varphi}$ is purely exponential. We next decompose $P_{+\varphi}$ as

$$P_{+\varphi} = \Delta + R - \tilde{g},$$

where

$$\begin{aligned} \Delta &= D_{+\varphi, 1}; \\ \tilde{g} &= -P_{+\varphi, 0}; \\ R &= P_{+\varphi} - P_{+\varphi, 0} - D_{+\varphi, 1}. \end{aligned}$$

Let $\mathfrak{I} = \{\mathfrak{m} \in z^{\mathbb{N}} \mathfrak{B}^C | \mathfrak{m} \prec_{\mathfrak{b}_n} 1\}$. Since $C[z][[\mathfrak{b}_1; ...; \mathfrak{b}_{n-1}]] \mathfrak{b}_n^\alpha$ is stable under Δ and Δ^{-1} for each $\alpha \in C$, the operator $R \Delta^{-1}$ is strictly extensive on $C[[\mathfrak{I}]]$. Consequently, the implicit function theorem from [vdH00b] implies that the operator $\operatorname{Id} + R \Delta^{-1}$ can be inverted, like in the proof of theorem 24:

$$(\operatorname{Id} + R \Delta^{-1})^{-1} = \operatorname{Id} - R \Delta^{-1} + (R \Delta^{-1})^2 + \cdots.$$

In particular,

$$\tilde{f} = \Delta^{-1} (\operatorname{Id} + R \Delta^{-1})^{-1} \tilde{g}$$

is a solution to $P_{+\varphi}(\tilde{f}) = 0$. Furthermore, we have

$$\operatorname{supp}_* (\operatorname{Id} + R \Delta^{-1})^{-1} \subseteq (\operatorname{supp}_* R \Delta^{-1})^*,$$

so that

$$\operatorname{supp} \tilde{f} \subseteq (\operatorname{supp}_* R \Delta^{-1})^* (\operatorname{supp} g)^*.$$

We claim that $f = \varphi + \tilde{f}$ is the distinguished solution. Indeed, let $\hat{f} \neq f$ be another solution and let $\mathfrak{d} = \mathfrak{d}_{\hat{f} - f}$. If $\mathfrak{d} \asymp_{\mathfrak{b}_n} 1$, then

$$\hat{\varphi} = \sum_{\mathfrak{m} \succ_{\mathfrak{b}_n} 1} \hat{f}_{\mathfrak{m}} \mathfrak{m}$$

is a solution to (28), so that $f_{\mathfrak{d}} = \varphi_{\mathfrak{d}} = 0$. If $\mathfrak{d} \prec_{\mathfrak{b}_n} 1$, then let

$$\delta = \sum_{\mathfrak{m} \succ_{\mathfrak{b}_n} \mathfrak{d}} (\hat{f} - f)_{\mathfrak{m}} \mathfrak{m}.$$

Since $P(\hat{f}) - P(f) = 0$, we have $\Delta \delta = 0$, so that $\mathfrak{d} = \mathfrak{d}_\delta$ is the dominant monomial of a solution to the homogeneous equation $\Delta h = 0$. Consequently, $f_{\mathfrak{d}} = 0$, since $\tilde{f} \in \operatorname{Im} \Delta^{-1}$. $\square$

Remark 27. By induction over n, it also follows that we need at most n upward shiftings in order to express the distinguished solutions. In other words, if P has coefficients in $C [\![\mathfrak{b}_1; \ldots; \mathfrak{b}_n]\!]$, where $\mathfrak{B}$ is purely exponential, then $f \in C [\![\log_{n-1} z; \ldots; z; \mathfrak{b}_1; \ldots; \mathfrak{b}_n]\!]$. Actually, if r is the order of P, then the number of upward shiftings we need is also bounded by r.

Indeed, denoting $\mathfrak{J} = \{ \mathfrak{m} \in \mathfrak{B}^C | \mathfrak{m} \prec_{\mathfrak{b}_n} 1 \}$ and using a similar argument as in remark 25, we first observe that Δ is bijective on $C [\![\mathfrak{J}]\!]$ if $\Delta h = 0$ admits no solutions in $C [\![\mathfrak{J}]\!]$. Moreover, if $\Delta h = 0$ admits such a solution, then $P_{+f,1}$ has a root with the same dominant part w.r.t. $\mathfrak{b}_n$. The same observations recursively hold for all Δ involved in the resolution of (28). Now if f is the distinguished solution of (6), then the linear equation $P_{+f,1} h = 0$ admits at most r solutions. Hence, there are at most r transbasis elements $\mathfrak{b}_i$ for which we need to make an upward shifting.

6.3 A worked example

Consider the linear differential equation

$$L f = f'' - e^{\lambda z} f' + e^{(\lambda+\mu) z} f = 1, \tag{29}$$

under the assumptions $z \succ 1$ and $e^{\lambda z} \succ e^{\mu z} \succ 1$. Then L has coefficients in $C [\![e^{\mathbb{C} z}]\!]$ and

$$L_{\times e^{\alpha z}} = e^{\alpha z} \left(f'' + 2\alpha f' + \alpha^2 f - e^{\lambda z} f' - \alpha e^{\lambda z} f + e^{(\lambda+\mu) z} f \right)$$

for each $\alpha \in \mathbb{C}$, so that

$$\begin{aligned}
\mathfrak{d}(L_{\times e^{\alpha z}}) &= e^{(\alpha+\lambda+\mu) z}; \\
\operatorname{supp} L_{\times e^{\alpha z}} &\subseteq e^{(\alpha+\lambda+\mu) z} \{ 1, e^{-\mu z}, e^{-(\lambda+\mu) z} \}.
\end{aligned}$$

Hence, theorem 24 implies that (29) has a distinguished solution in $C[z] [\![e^{\mathbb{C} z}]\!]$ with $\operatorname{supp} f_{\mathrm{dis}} \subseteq z^{2-\mathbb{N}} e^{-(\lambda+\mu) z} \{ e^{-\mu z}, e^{-(\lambda+\mu) z} \}^*$. Actually, it is easily seen that

$$\operatorname{supp} f_{\mathrm{dis}} \subseteq e^{-(\lambda+\mu) z} \{ e^{-\mu z}, e^{-(\lambda+\mu) z} \}^*$$

and the first terms of f are given by

$$f_{\mathrm{dis}} = e^{-(\lambda+\mu) z} + (\lambda+\mu) e^{-(\lambda+2\mu) z} + (\lambda+\mu)(\lambda+2\mu) e^{-(\lambda+3\mu) z} - (\lambda+\mu)^2 e^{-(2\lambda+2\mu) z} + \cdots.$$

In order to find all solutions to (29), we have to solve the Ricatti equation associated to the linear part of (29):

$$g^2 + g' - e^{\lambda z} g + e^{(\lambda+\mu) z} = 0. \tag{30}$$

This equation has two potential dominant terms $e^{\lambda z}$ and $e^{\mu z}$ of multiplicities one. Consequently, we get quasi-linear equations when refining $g = e^{\lambda z} + \tilde{g}$ ($\tilde{g} \prec e^{\lambda z}$) or $g = e^{\mu z} + \tilde{g}$ ($\tilde{g} \prec e^{\mu z}$). When setting $\tilde{g} = e^{\mu z} h_1$ resp. $\tilde{g} = e^{(2\mu - \lambda)z} h_2$, these equations are conveniently rewritten as

$$e^{-(\lambda - \mu)z} h_1^2 + h_1 - e^{-\lambda z} h_1' - \mu\, e^{-\lambda z} h_1 + 1 + \lambda\, e^{-\mu z} = 0 \qquad (h_1 \prec e^{(\lambda - \mu)z}) \tag{31}$$

and

$$e^{-2(\lambda - \mu)z} h_2^2 - h_2 + 2\, e^{-(\lambda - \mu)z} h_2 + e^{-\lambda z} h_2' + (2\,\mu - \lambda)\, e^{\lambda z} h_2 + 1 + \mu\, e^{-\mu z} = 0 \quad (h_2 \prec e^{(\lambda - \mu)z}). \tag{32}$$

By theorem 26, these equations admit distinguished solutions

$$
\begin{aligned}
h_1 &= -1 - e^{-(\lambda - \mu)z} - \lambda\, e^{-\mu z} + \cdots; \\
h_2 &= 1 + 2\, e^{-(\lambda - \mu)z} + \mu\, e^{-\mu z} + \cdots.
\end{aligned}
$$

More precisely, in the proof of theorem 26, and for (31), we would have $\Delta = h_1$, $g = -1$ and $R = e^{-(\lambda - \mu)z} h_1^2 - e^{-\lambda z} h_1' - \mu\, e^{-\lambda z} h_1 + \lambda\, e^{-\mu z}$. It follows that $\operatorname{supp}_* R\,\Delta^{-1} \subseteq \{e^{-\mu z}, e^{-(\lambda - \mu)z}\}^*$, so that $h_1 \in \mathbb{C}[[e^{-\mu z}, e^{-(\lambda - \mu)z}]]$. Similarly, $h_2 \in \mathbb{C}[[e^{-\mu z}, e^{-(\lambda - \mu)z}]]$. Returning to (30), we obtain the following solutions:

$$
\begin{aligned}
g_1 &= e^{\lambda z} - e^{\mu z} - e^{(2\mu - \lambda)z} - \lambda\, e^{-2\mu z} + \cdots; \\
g_2 &= e^{\mu z} + e^{(2\mu - \lambda)z} + 2\, e^{(3\mu - 2\lambda)z} + \mu\, e^{(\mu - \lambda)z} + \cdots,
\end{aligned}
$$

which yield a basis

$$
\begin{aligned}
\varphi_1 &= e^{\frac{1}{\lambda} e^{\lambda z} - \frac{1}{\mu} e^{\mu z} - \frac{1}{2\mu - \lambda} e^{(2\mu - \lambda)z} + \frac{\lambda}{2\mu} e^{-2\mu z} + \cdots}; \\
\varphi_2 &= e^{\frac{1}{\mu} e^{\mu z} + \frac{1}{2\mu - \lambda} e^{(2\mu - \lambda)z} + \frac{2}{3\mu - 2\lambda} e^{(3\mu - 2\lambda)z} + \frac{\mu}{\mu - \lambda} e^{(\mu - \lambda)z} + \cdots}.
\end{aligned}
$$

of solutions to $L\varphi = 0$.

It is interesting to study the solutions $f = f_{\mathrm{dis}} + \alpha_1\, \varphi_1 + \alpha_2\, \varphi_2$ to (29) from an analytical point of view. Indeed, the asymptotic conditions $\varphi_1 \prec 1$ or $\varphi_1 \succ 1$ and $\varphi_2 \prec 1$ or $\varphi_2 \succ 1$ divide complex space into four non degenerate regions. However, each of these regions has infinitely many "bounded connected components". When moving from one connected component to another one, a "generalized Stokes phenomenon" occurs. Consequently, a specific formal solution to (29) only makes sense on a bounded connected component from the analytical point of view.

Nevertheless, it is possible to give an asymptotic meaning to the generic formal solution to (29) on each region, by associating a "generalized Stokes matrix" to each connected component of the region. This issue will be detailed in a forthcoming paper. An interesting remaining question is the asymptotic behaviour of the Stokes matrices. Actually, the generalized Stokes phenomenon might be qualified as *multi-Stokes phenomenon*, since the Stokes phenomena occur with respect to several generalized sectors of different types. Equation (29) is one of the simplest examples which exhibits this multi-Stokes phenomenon.

7   Unravellings

7.1   Total unravellings

Theorem 26 together with propositions 21, 22 and 23 suggest that the solutions to an arbitrary asymptotic algebraic equation (6) can be expressed using the field operations, exponentiation, logarithm and distinguished solutions of quasi-linear equations. This is indeed so, if the Newton degree decreases at each refinement in proposition 23.

The remaining case, when the Newton degree repeatedly does not decrease in proposition 23, occurs when there are "almost multiple solutions". In order to "unravel" these solutions, we have to find their greatest common part. More precisely, consider an asymptotic algebraic differential equation (6) of Newton degree d. Then an *unravelling* (or *total unravelling*) is a refinement

$$f = \varphi + \tilde{f} \qquad (\tilde{f} \prec \tilde{\mathfrak{v}}),$$

such that

U1. The Newton degree of $P_{+\varphi}(\tilde{f}) = 0$ $(\tilde{f} \prec \tilde{\mathfrak{v}})$ equals d.

U2. For any $\tilde{\varphi} \prec \tilde{\mathfrak{v}}$, the Newton degree of $P_{\varphi+\tilde{\varphi}}(\tilde{\tilde{f}}) = 0$ $(\tilde{\tilde{f}} \prec \mathfrak{d}(\tilde{\varphi}))$ is $< d$.

Clearly, the series φ, which is also called an *unraveller*, may be replaced by any other series of the form $\varphi + \psi$ with $\psi \prec \tilde{\mathfrak{v}}$.

From a theoretical point of view it is possible to prove a certain number of facts about unravellings. First of all, any unraveller admits a truncation $\varphi^<$, which is a *canonical unraveller*, in the sense that

$$f = \varphi^< + \tilde{f} \qquad (\tilde{f} \prec \tilde{\mathfrak{v}}^<)$$

is an unravelling for all $\tilde{\mathfrak{v}}^<$ with $\operatorname{supp} \varphi^< \succ \tilde{\mathfrak{v}}^< \succ \tilde{\mathfrak{v}}$, and that a similar property does not hold for any proper truncation of $\varphi^<$.

Secondly, it is possible to construct the so called *canonical algebraic unraveller* φ by transfinite induction: having constructed the first α terms of φ, say of sum ψ, one looks at the equation $P_{+\psi}(\tilde{f}) = 0$ $(\tilde{f} \prec \mathfrak{v})$ of Newton degree d. If this equation has an algebraic potential dominant term τ of multiplicity d, then this term is unique, and we take it to be the next term of φ.

However, in what follows, we are interested in more constructive ways to obtain unravellings. For this purpose, we recall that in the more classical context of algebraic equations, multiple roots are usually found by solving the derivative (or a higher derivative) of the equation with respect to the indeterminate. In the next sections, we will describe a similar strategy in order to find the almost multiple solutions to asymptotic algebraic differential equations. The price to be paid is that we will need a sequence of so called partial unravellings (and adjusted partial unravellings) in order to construct a total unravelling.

7.2 Partial unravellings

Consider an asymptotic algebraic differential equation (6) of Newton degree d. Given a monomial $\mathfrak{m} \prec \mathfrak{v}$ such that $N_{P_{\times\mathfrak{m}}}$ admits a root of multiplicity d, we define Q by

- If $N_{P_{d,\times\mathfrak{m}}} = \alpha (c - \beta)^{d-k} (c')^k$ with $k < d$, then $Q = (\partial^{d-1} P_{\times\mathfrak{m}}/\partial f^{d-1-k} \partial (f')^k)_{\times\mathfrak{m}^{-1}}$;

- If $N_{P_{d,\times\mathfrak{m}}} = \alpha (c')^d$, then $Q = (\partial^{d-1} P_{\times\mathfrak{m}}/\partial (f')^{d-1})_{\times\mathfrak{m}^{-1}}$.

Now let $\tilde{\mathfrak{v}} \preccurlyeq \varphi \preccurlyeq \mathfrak{m}$, $\tilde{P} = P_{+\varphi}$ and $\tilde{Q} = Q_{+\varphi}$ be such that

PU1. $Q(\varphi) = 0$.

PU2. The Newton degree of $\tilde{P}(\tilde{f}) = 0$ $(\tilde{f} \prec \tilde{\mathfrak{v}})$ is d.

PU3. For any $\tilde{\varphi} \prec \tilde{\mathfrak{v}}$ with $\tilde{Q}(\tilde{\varphi}) = 0$, the Newton degree of $\tilde{P}_{+\tilde{\varphi}}(\tilde{\tilde{f}}) = 0$ $(\tilde{\tilde{f}} \prec \tilde{\varphi})$ is $< d$.

Then the refinement

$$f = \varphi + \tilde{f} \qquad (\tilde{f} \prec \tilde{\mathfrak{v}}), \tag{33}$$

is said to be a *partial unravelling* with $\mathfrak{m}$ as its *associated monomial*. Notice that the equations $Q(\varphi) = 0$ ($\varphi \preccurlyeq \mathfrak{m}$) and $\tilde{Q}(\tilde{\varphi}) = 0$ ($\tilde{\varphi} \prec \tilde{\mathfrak{v}}$) are quasi-linear. Partial unravellings are constructed as follows.

Proposition 28. *Let $\mathfrak{m}$ and Q be as above. Then there exists a $\varphi \preccurlyeq \mathfrak{m}$ which satisfies the conditions* **PU1**, **PU2** *and* **PU3**.

Proof. We construct sequences $\varphi^{[1]}$, $\varphi^{[2]}$, ... and $\tilde{\mathfrak{v}}^{[1]} \succ \tilde{\mathfrak{v}}^{[2]} \succ \cdots$ of approximations of φ and $\tilde{\mathfrak{v}}$, such that all $\varphi^{[i]}$ and $\tilde{\mathfrak{v}}^{[i]}$ satisfy the conditions **PU1** and **PU2**. We let $\varphi^{[1]}$ be the distinguished solution to the equation $Q(\varphi^{[1]}) = 0$ ($\varphi^{[1]} \preccurlyeq \mathfrak{m}$) and $\tilde{\mathfrak{v}}^{[1]} = \mathfrak{d}(\varphi^{[1]})$. As long as $\varphi^{[i]}$ and $\tilde{\mathfrak{v}}^{[i]}$ do not satisfy the condition **PU3**, there exists a $\psi^{[i]} \prec \mathfrak{v}^{[i]}$ with $Q_{+\varphi^{[i]}}(\psi^{[i]}) = 0$, such that the Newton degree of $P_{\varphi^{[i]}+\psi^{[i]}}(\tilde{f})$ ($\tilde{f} \prec \psi^{[i]}$) is d. Hence we may take $\varphi^{[i+1]} = \varphi^{[i]} + \psi^{[i]}$ and $\tilde{\mathfrak{v}}^{[i+1]} = \mathfrak{d}(\psi^{[i]})$.

We claim that the sequences $\varphi^{[1]}, \varphi^{[2]}, ...$ and $\tilde{\mathfrak{v}}^{[1]}, \tilde{\mathfrak{v}}^{[2]}, ...$ are of length at most $r + 1$, so that we may take their last elements for φ and $\tilde{\mathfrak{v}}$. Indeed, for each i, the series $\varphi^{[j]} - \varphi^{[i]}$ with $j < i$ are solutions to the quasi-linear equation $Q_{+\varphi^{[i]}}(\hbar) = 0$ ($\hbar \preccurlyeq \mathfrak{m}$). Consequently, the dominant monomials of these series, which are pairwise distinct, are all dominant monomials of solutions to the homogeneous linear differential equation $Q_{+\varphi^{[i]},1}(h) = 0$. But there are at most r linearly independent solutions to this equation. $\qquad\square$

Proposition 29. *Consider a partial unravelling* (33) *as above, followed by a refinement*

$$\tilde{f} = \tilde{\varphi} + \tilde{\tilde{f}} \qquad (\tilde{\tilde{f}} \prec \tilde{\tilde{\mathfrak{v}}}),$$

such that the Newton degree of

$$\tilde{\tilde{P}}(\tilde{\tilde{f}}) = \tilde{P}_{+\tilde{\varphi}}(\tilde{\tilde{f}}) = 0 \qquad (\tilde{\tilde{f}} \prec \tilde{\tilde{\mathfrak{v}}}) \tag{34}$$

is equal to d. Then, for $\tilde{\mathfrak{m}} = \mathfrak{d}(\tilde{\varphi})$, we have

$$\frac{\tilde{\mathfrak{m}}}{\tilde{\tilde{\mathfrak{v}}}} \preccurlyeq \log \frac{\mathfrak{m}}{\tilde{\mathfrak{m}}}.$$

Proof. Without loss of generality, we may assume that $\tilde{P}$, $\tilde{Q}$, $\tilde{\varphi}$, $\tilde{\mathfrak{v}}$ and $\tilde{\tilde{\mathfrak{v}}}$ are purely exponential, that $\mathfrak{m} = 1$ and that $\mathfrak{d}(\tilde{P}) = \mathfrak{d}(\tilde{Q}) = 1$. From **PU3** it follows that $\tilde{\mathfrak{m}}$ is neither a potential dominant monomial for $\tilde{Q}(\psi) = 0$ ($\psi \prec \tilde{\mathfrak{v}}$), nor for $\tilde{Q}_1(\psi) = 0$ ($\psi \prec \tilde{\mathfrak{v}}$). Proposition 22, applied to $\tilde{Q}_{\times \tilde{\mathfrak{m}},1}$ and the "non potential dominant monomial" 1, therefore yields

$$\mathfrak{d}(R_{\tilde{Q}_{\times \tilde{\mathfrak{m}},1,0}}) = \mathfrak{d}(R_{\tilde{Q}_{\times \tilde{\mathfrak{m}},1}}).$$

Consequently,

$$\tilde{Q}(\tilde{\varphi}) \asymp R_{\tilde{Q}_{\times \tilde{\mathfrak{m}},1}}\left(\frac{\tilde{\varphi}}{\tilde{\mathfrak{m}}}\right) \asymp \mathfrak{d}(R_{\tilde{Q}_{\times \tilde{\mathfrak{m}},1}}) \asymp \mathfrak{d}(\tilde{Q}_{\times \tilde{\mathfrak{m}}}).$$

On the other hand, we have

$$\frac{\mathfrak{d}(\tilde{Q}_{\times \tilde{\mathfrak{m}}})}{\mathfrak{d}(\tilde{Q})\,\tilde{\mathfrak{m}}} \preccurlyeq \tilde{\mathfrak{m}}^\dagger \succcurlyeq \log \tilde{\mathfrak{m}},$$

so that

$$\frac{\tilde{Q}(\tilde{\varphi})}{\tilde{\mathfrak{m}}} \preccurlyeq \log \tilde{\mathfrak{m}}.$$

Now recall that $\tilde{Q}(\tilde{\varphi})$ is the coefficient of $f^{d-1-k}(f')^k$ in $\tilde{\tilde{P}}(f)$ for some k. It follows that

$$\tilde{\tilde{P}}_{d-1} \succcurlyeq_{\tilde{\mathfrak{m}}}^* \tilde{\mathfrak{m}},$$

Now assume that $\mathfrak{n}$ is a monomial with $\mathfrak{n} \prec_{\tilde{\mathfrak{m}}}^* \tilde{\mathfrak{m}}$ (so that $\mathfrak{n} \prec_{\mathfrak{n}}^* \tilde{\mathfrak{m}}$). Then we have

$$\mathfrak{d}(\tilde{\tilde{P}}_{\times \mathfrak{n}, d-1}) \asymp_{\mathfrak{n}}^* \mathfrak{d}(\tilde{\tilde{P}}_{d-1}) \mathfrak{n}^{d-1} \succ_{\mathfrak{n}}^* \mathfrak{n}^d \asymp_{\mathfrak{n}}^* \mathfrak{d}(\tilde{\tilde{P}}_{\times \mathfrak{n}, d}).$$

We conclude that the degree of $N_{\tilde{\tilde{P}}_{\times \mathfrak{n}}}$ can not exceed $d-1$. If $\tilde{\tilde{\mathfrak{v}}}$ is chosen such that (34) has Newton degree d, it thus follows that

$$\tilde{\tilde{\mathfrak{v}}} \succcurlyeq_{\tilde{\mathfrak{m}}}^* \tilde{\mathfrak{m}},$$

which completes the proof. $\qquad\square$

7.3 Adjusted partial unravellings

Proposition 29 shows that by taking sequences of partial unravellings, we rapidly approach a total unraveling. The only problem which still remains to be solved is the appearance of highly iterated logarithms. We will first solve this problem in the particular case when the Newton degree of P coincides with its normal degree. In the next subsection, we will show that the general case can be reduced to this case.

In the sequel, we assume that (6) is an asymptotic differential equation of degree and Newton degree d, such that the following additional conditions are satisfied for a certain purely exponential transbasis $\mathfrak{B} = (\mathfrak{b}_1, ..., \mathfrak{b}_n)$:

E1. P_d has coefficients in $C[\![\mathfrak{b}_1; ...; \mathfrak{b}_n]\!]$.

E2. $P_0, ..., P_{d-1}$ have coefficients in $C[z][\![\mathfrak{b}_1; ...; \mathfrak{b}_n]\!]$.

E3. $P_d(f) = 0$ $(f \prec \mathfrak{v})$ admits only potential dominant monomials in $\mathfrak{B}^C$.

The two first conditions can clearly be met after a sufficient number of upward shiftings. In section 9, we will show that this is also the case for the last condition.

Proposition 30. *Let τ be a potential dominant term of multiplicity d for (6). Then*

   a) Modulo the insertion of new elements into $\mathfrak{B}$, we have $\tau = c \, z^\mu \, \mathfrak{m} \in C \, z^{\mathbb{N}} \, \mathfrak{B}^C$.

   b) There exists a unique $\varphi \in z \, C[z] \, \mathfrak{m}$, such that either

     i. $f = \varphi + \tilde{f}$ $(\tilde{f} \prec \tilde{\mathfrak{v}})$ is a total unravelling and $\tilde{\mathfrak{v}}$ is $\preccurlyeq$-minimal in $\operatorname{supp} \varphi$.

     ii. The Newton degree of $f = \varphi + \tilde{f}$ $(\tilde{f} \prec z \, \mathfrak{m})$ is d.

Proof. Let us first prove (a). If τ is differential, then **E3** implies that τ is purely exponential, so $\tau \in C \, \mathfrak{B}^C$ after a suitable extension of $\mathfrak{B}$. If τ is algebraic, then $\mathfrak{d}(\tau)$ is the (i, d)-equalizer for each $i < d$, since τ has multiplicity d. Proposition 21(a) implies that there exists a unique purely exponential monomial $\mathfrak{n} = e^{\mu z}(\mathfrak{m}\!\uparrow) \in e^{Cz}(\mathfrak{B}\!\uparrow)^C$ with $\tau\!\uparrow \asymp_{e^z} \mathfrak{n}$, such that $\mathfrak{d}(P_i\!\uparrow_{\times \mathfrak{n}}) = \mathfrak{d}(P_d\!\uparrow_{\times \mathfrak{n}})$ for all $i < d$. More precisely, in the algorithm in proposition 21(a), $\mathfrak{m}$ is chosen such that $\mathfrak{d}(P_d\!\uparrow_{\times \mathfrak{m}\!\uparrow}) \asymp_{\mathfrak{b}_1\!\uparrow} \mathfrak{d}(P_i\!\uparrow_{\times \mathfrak{m}\!\uparrow})$, whence $\mathfrak{d}(P_d\!\uparrow_{\times \mathfrak{m}\!\uparrow}) = \mathfrak{d}(P_i\!\uparrow_{\times \mathfrak{m}\!\uparrow}) \, e^{\nu_i z}$ for some $\nu_i \in \mathbb{N}$, and μ satisfies $(d-i) \, \mu = \nu_i$. In particular, for $i = d-1$, this yields $\mu \in \mathbb{N}$. We claim that $\tau \asymp \mathfrak{n}\!\downarrow$.

If $\mu = 0$, then we have $\mathfrak{d}(P_{d, \times \mathfrak{n}\downarrow}) = \mathfrak{d}(P_{i, \times \mathfrak{n}\downarrow})$ and $\mathfrak{d}(P_d\!\uparrow_{\times \mathfrak{n}}) = \mathfrak{d}(P_i\!\uparrow_{\times \mathfrak{n}})$ for all $i < d$. Since $P_0\!\uparrow_{\times \mathfrak{n}} = P_{0, \times \mathfrak{n}\downarrow}\!\uparrow$, this can only happen if $N_{P\uparrow_{\times \mathfrak{n}}} = D_{P\uparrow_{\times \mathfrak{n}}}$. Hence $\mathfrak{n}\!\downarrow$ is the (i, d)-equalizer w.r.t. P for all $i < d$ and $\tau \asymp \mathfrak{n}\!\downarrow$. If $\mu > 0$, then **E3** implies that $\mathfrak{n}$ is not a potential dominant monomial for $P_d\!\uparrow$. Consequently, the coefficients of c^0 and c^d in $D_{P\uparrow_{\times \mathfrak{n}}}$ both do not vanish. It follows that $\mathfrak{n}\!\downarrow$ is the $(0, d)$-equalizer w.r.t. P and again $\tau \asymp \mathfrak{n}\!\downarrow$.

We prove the existence of φ in (b) by induction over μ; the uniqueness of φ follows from **E3**. If $\mu = 0$, then $\varphi = 0$ clearly satisfies assumption ii. If $\mu > 0$, then we refine

$$f = \tau + \tilde{f} \qquad (\tilde{f} \prec \tau),$$

and remark that $P_{+\tau}$ satisfies the hypothesis **E1**, **E2** and **E3**, due to part (a). Now consider the equation

$$P_{+\tau}(\tilde{f}) = 0 \qquad (\tilde{f} \prec \tau) \tag{35}$$

of Newton degree d. If this equation admits a potential dominant term $\tilde{\tau}$ of multiplicity d with $\mathfrak{m} \prec \tilde{\tau} \prec \tau$, then the induction hypothesis implies that there exists a $\tilde{\varphi} \in z\, C[z]\, \mathfrak{m}$, which satisfies the assumption i or ii, and we may take $\varphi = \tau + \tilde{\varphi}$. If there does not exist such a potential dominant term $\tilde{\tau}$, then there either do not exist potential dominant terms of multiplicity d at all for (35), so that i holds for $\varphi = \tau$, or such potential dominant terms do exist, and we have ii for $\varphi = \tau$. $\qquad\square$

Given a potential dominant term $\tau = c\, z^\mu\, \mathfrak{m}$ of multiplicity d, let φ be as in proposition 30(b). In case i, we say by convention that $f = \varphi + \tilde{f}$ $(\tilde{f} \prec \tilde{\mathfrak{v}})$ is an *adjusted partial unravelling*. In case ii, let

$$\tilde{f} = \tilde{\varphi} + \tilde{\tilde{f}} \qquad (\tilde{\tilde{f}} \prec \tilde{\tilde{\mathfrak{v}}}) \tag{36}$$

be a partial unraveling relative to the equation

$$P_{+\varphi}(\tilde{f}) = 0 \qquad (\tilde{f} \prec \tilde{\mathfrak{v}}),$$

and with $\mathfrak{m}$ as its associated monomial. Then we say that

$$f = \varphi + \tilde{\varphi} + \tilde{\tilde{f}} \qquad (\tilde{\tilde{f}} \prec \tilde{\tilde{\mathfrak{v}}})$$

is an *adjusted partial unravelling*. Notice that a partial unravelling like (36) always exists, by propositions 28 and 30(a).

Notice also that we necessarily have $\tilde{\varphi} \in C[z]\,[\![\mathfrak{b}_1; ...; \mathfrak{b}_n]\!]$. Indeed, consider the differential polynomial Q with $Q(\tilde{\varphi}) = 0$ in **PU1**. Since $\deg P = d$, this differential polynomial is actually linear. Furthermore, since $\mathfrak{m} \in \mathfrak{B}^C$, the coefficients of Q_0 are in $C[z]\,[\![\mathfrak{b}_1; ...; \mathfrak{b}_n]\!]$ and the coefficients of Q_1 in $C\,[\![\mathfrak{b}_1; ...; \mathfrak{b}_n]\!]$. We conclude that all solutions to $Q(\psi) = 0$, and in particular $\psi = \tilde{\varphi}$, are in $C[z]\,[\![\mathfrak{b}_1; ...; \mathfrak{b}_n]\!]$. A consequence of our observation is that $P_{\varphi + \tilde{\varphi}}$ again satisfies the hypotheses **E1**, **E2** and **E3**, so that we may consider sequences of adjusted partial unravellings.

Proposition 31. *Any sequence of adjusted partial unravellings*

$$\begin{aligned} f = f^{[0]} &= \varphi^{[1]} + f^{[1]} & (f^{[1]} \prec \mathfrak{v}^{[0]}); \\ f^{[1]} &= \varphi^{[2]} + f^{[2]} & (f^{[2]} \prec \mathfrak{v}^{[1]}); \\ &\vdots \end{aligned}$$

is finite, say of length l, and its composition

$$f = \varphi^{[1]} + \cdots + \varphi^{[l]} + f^{[l]} \qquad (f^{[l]} \prec \mathfrak{v}^{[l-1]})$$

is a total unravelling.

Proof. Let $l \in \mathbb{N} \cup \{\infty\}$ denote the length of the sequence of adjusted partial unravellings. For each $1 \leqslant i \leqslant l$, let $\mathfrak{m}^{[i]} = z^{k^{[i]}} \mathfrak{n}^{[i]} = \mathfrak{d}(\varphi^{[i]}) \in z^{\mathbb{N}} \mathfrak{B}^C$. For each $1 \leqslant i \leqslant l-1$, let $\chi^{[i]}$ denote the exponentiality of $\mathfrak{m}^{[i]}/\mathfrak{m}^{[i+1]}$. Given $2 \leqslant i \leqslant l-1$, proposition 29 implies that

$$\frac{\mathfrak{m}^{[i]}}{\mathfrak{v}^{[i+1]}} \preccurlyeq \log \frac{\mathfrak{n}^{[i-1]}}{\mathfrak{m}^{[i]}},$$

Since $\mathfrak{n}^{[i-1]} \preccurlyeq \mathfrak{m}^{[i-1]}$ and $\mathfrak{v}^{[i+1]} \preccurlyeq \mathfrak{m}^{[i+1]}$, this yields

$$\frac{\mathfrak{m}^{[i]}}{\mathfrak{m}^{[i+1]}} \;\preccurlyeq\; \log \frac{\mathfrak{m}^{[i-1]}}{\mathfrak{m}^{[i]}},$$

By induction, it follows that $\chi_1 > \cdots > \chi_{l-1} \geqslant 0$. We conclude that $l \leqslant \chi_1 + 1$. The composition of the sequence of adjusted partial unravellings is clearly a total unraveling. $\square$

7.4 Construction of total unravellings

Let us now return to the case of a general asymptotic differential equation (6) of Newton degree d. Assume that (33) is a partial unravelling with $\mathfrak{m} = \mathfrak{d}_\varphi$ and that $\tilde{\tau}$ is a potential dominant monomial of multiplicity d for

$$\tilde{P}(\tilde{f}) = P_{+\varphi}(\tilde{f}) = 0 \qquad (\tilde{f} \prec \tilde{\mathfrak{v}}). \tag{37}$$

Modulo a sufficient number of upward shiftings, we may assume that φ, $\tilde{\mathfrak{v}}$, $\tilde{\tau}$ and the coefficients of P can be expanded w.r.t. a purely exponential transbasis $\mathfrak{B} = (\mathfrak{b}_1, ..., \mathfrak{b}_n)$. Let $\mathfrak{b}_k$ be the transbasis element such that $\varphi/\tilde{\tau} \asymp \mathfrak{b}_k$, and consider the dominant part Π of $\tilde{P}_{\times \tilde{\tau}}$ with respect to $\mathfrak{b}_k$:

$$\Pi = \sum_{\mathfrak{n} \asymp_{\mathfrak{b}_k} \mathfrak{d}(\tilde{P}_{\times \tilde{\tau}})} \tilde{P}_{\times \tilde{\tau}, \mathfrak{n}} \, \mathfrak{n}$$

On the one hand, since $\deg N_{\tilde{P}_{\times \mathfrak{d}(\varphi)}} = \deg N_{P_{\times \mathfrak{d}(\varphi)}} = d$, we have

$$\mathfrak{d}(\tilde{P}_{+\varphi, \times \tilde{\tau}, i}) \;\asymp_{\mathfrak{b}_k}\; \mathfrak{d}(\tilde{P}_{\times \mathfrak{d}(\varphi), i}) \left(\frac{\tilde{\tau}}{\mathfrak{d}(\varphi)}\right)^i \;\prec_{\mathfrak{b}_k}\; \mathfrak{d}(\tilde{P}_{\times \mathfrak{d}(\varphi), d}) \left(\frac{\tilde{\tau}}{\mathfrak{d}(\varphi)}\right)^d \;\asymp_{\mathfrak{b}_k}\; \mathfrak{d}(\tilde{P}_{\times \tilde{\tau}, d}),$$

for all $i > d$, so that $\deg \Pi = d$. Consequently, Π satisfies the conditions **E1** and **E2** from the previous section; we will see in section 9 that it also satisfies **E3**, modulo some additional upward shiftings. On the other hand, the following proposition reduces the problem of determining the unravellings for (6) to a similar problem for Π. In view of the previous section, this completes the effective construction of unravellings.

Proposition 32. *With the above notations, a refinement*

$$\tilde{f} = \tilde{\varphi} + \tilde{\tilde{f}} \qquad (\tilde{\tilde{f}} \prec \tilde{\tilde{\mathfrak{v}}}) \tag{38}$$

with $\tilde{\varphi} \sim \tilde{\tau}$ is a total unravelling w.r.t. (37) if and only if

$$g = \frac{\tilde{\varphi}}{\tilde{\tau}} + \tilde{g} \qquad \left(\tilde{g} \prec \frac{\tilde{\tilde{\mathfrak{v}}}}{\tilde{\tau}}\right) \tag{39}$$

is a total unravelling w.r.t. the equation

$$\Pi(g) = 0 \qquad \left(g \prec \frac{\tilde{\tilde{\mathfrak{v}}}}{\tilde{\tau}}\right). \tag{40}$$

Proof. Modulo a multiplicative conjugation, we may assume without loss of generality that $\tilde{\tau} = 1$. Now if (38) is an unravelling, then proposition 29 implies that

$$\frac{1}{\tilde{\tilde{\mathfrak{v}}}} \;\preccurlyeq\; \log \mathfrak{m} \asymp \log \mathfrak{b}_k,$$

so that $\tilde{\tilde{\mathfrak{v}}} \asymp_{\mathfrak{b}_k} 1$. Actually, in the proof of proposition 29 we showed that

$$\tilde{P}_{+\tilde{\varphi},d-1} \succcurlyeq^*_{\mathfrak{b}_k} 1,$$

so that $\Pi_{+\tilde{\varphi},d-1} \neq 0$. We infer that $\deg N_{\Pi_{+\tilde{\varphi},\times\mathfrak{n}}} \leqslant d-1$ for all $\mathfrak{n} \prec 1$ with $\mathfrak{n} \succcurlyeq \mathfrak{b}_k$. In other words, for (39) to be an unravelling, it is again necessary that $\tilde{\tilde{\mathfrak{v}}} \asymp_{\mathfrak{b}_k} 1$.

The above argument shows that it suffices to prove the equivalence under the assumption that $\tilde{\tilde{\mathfrak{v}}} \asymp_{\mathfrak{b}_k} 1$. Now we notice that for each transseries $\psi \preccurlyeq_{\mathfrak{b}_k} 1$ and each monomial $\mathfrak{n} \asymp_{\mathfrak{b}_k} 1$, the dominant parts of $\tilde{P}_{+\psi,\times\mathfrak{n}}$ and $\Pi_{+\psi,\times\mathfrak{n}}$ w.r.t. $\mathfrak{b}_k$ coincide. Consequently,

$$N_{\tilde{P}_{+\psi,\times\mathfrak{n}}} = N_{\Pi_{+\psi,\times\mathfrak{n}}}$$

for such $\mathfrak{n}$ and ψ. In particular, we have

$$N_{\tilde{P}_{+\tilde{\varphi},\times\mathfrak{n}}} = N_{\Pi_{+\tilde{\varphi},\times\mathfrak{n}}}$$

for all $\mathfrak{n} \prec \tilde{\tilde{\mathfrak{v}}}$ sufficiently close to $\tilde{\tilde{\mathfrak{v}}}$, so that the Newton degrees of

$$\tilde{P}_{+\tilde{\varphi}}(\tilde{\tilde{f}}) = 0 \qquad (\tilde{\tilde{f}} \prec \tilde{\tilde{\mathfrak{v}}})$$

and

$$\Pi_{+\tilde{\varphi}}(\tilde{g}) = 0 \qquad (\tilde{g} \prec \tilde{\tilde{\mathfrak{v}}})$$

coincide. Hence **U1** holds for (38) if and only if it holds for (39). Similarly, for all $\psi \prec \tilde{\tilde{\mathfrak{v}}}$, such that $\psi \asymp_{\mathfrak{b}_k} 1$, the Newton degrees of

$$\tilde{P}_{+\psi}(\tilde{\tilde{f}}) = 0 \qquad (\tilde{\tilde{f}} \prec \mathfrak{d}_\psi) \tag{41}$$

and

$$\Pi_{+\psi}(\tilde{g}) = 0 \qquad (\tilde{g} \prec \mathfrak{d}_\psi) \tag{42}$$

coincide. Furthermore, for a similar reason as above, the Newton degrees of (41) and (42) are both bounded by $d-1$ if $\psi \prec_{\mathfrak{b}_k} 1$. In other words, **U2** holds for (38) if and only if it holds for (39). We conclude that (38) is a total unravelling w.r.t. (37) if and only if (39) is a total unravelling w.r.t. (37). $\qquad\square$

7.5   Worked examples

One of the easiest examples which illustrates the importance of unravellings is

$$P(f) = f^2 - \frac{2}{1-z^{-1}} f + \left(\frac{1}{1-z^{-1}}\right)^2 = e^{-z}, \tag{43}$$

where $1 \prec z \prec e^z$. This equation admits 1 as it's unique potential dominant term of multiplicity 2. However, the refinement

$$f = 1 + \tilde{f} \qquad (\tilde{f} \prec 1)$$

leads to the equation

$$\tilde{f}^2 - \frac{2\,z^{-1}}{1-z^{-1}} \tilde{f} + \left(\frac{z^{-1}}{1-z^{-1}}\right)^2 = e^{-z} \qquad (\tilde{f} \prec 1),$$

which again admits a unique potential dominant term z^{-1} of multiplicity 2. Continuing like this leads to an infinite sequence $f = 1 + \tilde{f} \ (\tilde{f} \prec 1)$, $\tilde{f} = z^{-1} + \tilde{\tilde{f}} \ (\tilde{\tilde{f}} \prec z^{-1})$, ... of refinements, and we do not succeed in separating the two solutions.

Therefore, we rather construct a total unravelling. In order to do so, we first compute the distinguished solution $\varphi = (1 - z^{-1})^{-1}$ to the quasi-linear differential equation

$$Q(\varphi) = \frac{\partial P}{\partial f}(\varphi) = 2\,\varphi - \frac{2}{1 - z^{-1}}\,\varphi = 0$$

and then refine

$$f = \varphi + \tilde{f} \qquad (\tilde{f} \prec 1).$$

This refinement (and partial unravelling with associated monomial 1) is actually already the total unravelling we were looking for and the equation (43) transforms into

$$\tilde{f}^2 = e^{-z} \qquad (\tilde{f} \prec 1).$$

This time, the new equation admits two potential dominant terms $e^{-z/2}$ and $-e^{-z/2}$ of multiplicities < 2, which allows us to compute the solutions to (43) by recursion.

In general, total unravellings can only be achieved via successions of partial unravellings. An important example which illustrates this phenomenon is the following:

$$f^2 + 2\,f' + \frac{1}{z^2} + \frac{1}{z^2 \log^2 z} + \cdots + \frac{1}{z^2 \log^2 z \cdots \log_{l-1}^2 z} = 0 \tag{44}$$

This equation becomes purely exponential after l upward shiftings:

$$f^2 + \frac{2}{e^z\,e^{e^z} \cdots \exp_l z}\,f' + \frac{1}{\exp_l^2 z} + \frac{1}{\exp_{l-1}^2 z \exp_l^2 z} + \cdots + \frac{1}{e^{2z}\,e^{2e^z} \cdots \exp_l^2 z} = 0. \tag{45}$$

This new equation admits a unique potential dominant monomial $\exp_l^{-1} z$ of multiplicity 2. Indeed, this is easily seen when substituting $(\exp_l^{-1} z)\,g$ for f:

$$\frac{1}{\exp_l^2 z}\left(g^2 + \frac{2}{e^z \cdots \exp_{l-1} z}\,g' - 2\,g + 1 + \frac{1}{\exp_{l-1}^2 z} + \cdots + \frac{1}{e^{2z} \cdots \exp_{l-1}^2 z} \right) = 0. \tag{46}$$

Now $g = 1 + \tilde{g}$ ($\tilde{g} \prec 1$) is a partial unravelling w.r.t. (46) and $f = \exp_l^{-1} z + \tilde{f}$ ($\tilde{f} \prec \exp_l^{-1} z$) a partial unravelling w.r.t. (45). However, the partial unravelling transforms (46) into an equation of the same form as (45), but with l decreased by 1. Consequently, we need a succession of l unravellings

$$\begin{aligned}
f &= \frac{1}{\exp_l z} + f_1 \qquad \left(f_1 \prec \frac{1}{\exp_l z}\right); \\
f_1 &= \frac{1}{\exp_{l-1} z \exp_l z} + f_2 \qquad \left(f_2 \prec \frac{1}{\exp_{l-1} z \exp_l z}\right); \\
&\ \vdots \\
f_{l-1} &= \frac{1}{e^z \cdots \exp_l z} + f_l \qquad \left(f_l \prec \frac{1}{e^z \cdots \exp_l z}\right).
\end{aligned}$$

in order to attain the total unravelling

$$f = \frac{1}{\exp_l z} + \frac{1}{\exp_{l-1} z \exp_l z} + \cdots + \frac{1}{e^z \cdots \exp_l z} + \tilde{f} \qquad \left(f_l \prec \frac{1}{e^z \cdots \exp_l z}\right).$$

Notice that the ratios $\exp_l z, \ldots, e^z$ of the successive potential dominant terms indeed satisfy proposition 29.

An open question is whether there exist examples which essentially need the technique of adjusted partial unravellings in order to limit the appearance of iterated logarithms. If not, this would allow some major simplifications in the algorithm `solve` in the next section. Actually, the whole computation process of total unravellings needs to be better understood in order to generalize it to other fields of transseries (see section 9.5).

8 Computation of parameterized solutions

8.1 About the algorithms in this section

In this section, we give algorithms to solve an asymptotic differential equation (6), where the coefficients of P and $\mathfrak{v}$ can be expanded with respect to a parameterized transbasis $\mathfrak{B} = (\mathfrak{b}_1, ..., \mathfrak{b}_n)$. Actually, we show how to solve such an equation modulo a transmonomial $\mathfrak{w}$. Here a *solution modulo* $\mathfrak{w}$ to (6) is a transseries $\varphi \prec \mathfrak{v}$, such that the Newton degree of $P_{+\varphi}(\tilde{f}) = 0$ ($\tilde{f} \prec \mathfrak{w}$) is strictly positive. In our algorithms, we will use the following conventions without further mention.

Automatic case separations. Our algorithms are non deterministic in the sense that we allow automatic case separations, in a similar way as sections 3 and 4. The algorithms are constructed in such a way that each solution modulo $\mathfrak{w}$ to (6) occurs in precisely one case. In other words, for each specialization of the initial parameters and, for each specialization of the directions (which corresponds to a choice of a field of transseries $\mathbb{T}$ as in section 2, which satisfies the constraints on the $\theta_{\mathfrak{m}}$) and for each solution φ in $\mathbb{T}$, there exists precisely one case and precisely one specialization of the newly introduced parameters, which yields φ as a solution.

The termination of our non deterministic algorithm follows from the termination of each of its branches by a similar Noetherianity argument as in the proof of theorem 17. For more details about the automatic case separation strategy, see chapter 8 in [vdH97].

Automatic upward shiftings. We assume throughout our algorithms that all computations are done w.r.t. a purely exponential parameterized transbasis $\mathfrak{B} = (\mathfrak{b}_1, ..., \mathfrak{b}_n)$. In order to do this, we are allowed to insert new elements into $\mathfrak{B}$ and to "shift the whole computation upwards" whenever necessary. Upward shifting can for instance be implemented via an exception, which starts over the whole non deterministic computation, after shifting the input upwards. A better strategy, which consists of associating an "exponential level" to each transseries, is explained in more detail on page 276 of [vdH97].

Non standard monomials. It is useful to allow a few non standard transmonomials in our algorithms, like the formal infinitely large and small monomials $\infty_{\mathfrak{M}}$ resp. $\infty_{\mathfrak{M}}^{-1}$, as well as monomials of the form $(\exp_l z \exp_{l-1} z \cdots z \log z \cdots)^{-1}$.

Effective computations with transseries. Of course, our algorithms are not really effective, as long as we do not provide algorithms to compute the expansions of parameterized transseries and to test them for being zero. In this paper, we assume that we have oracles for deciding such questions. Similarly, in our algorithms for computing distinguished solutions in the next section, we will merely give infinite formulas for the results. Nevertheless, we notice that for certain classes of transseries, the oracles may be replaced by real algorithms, in a similar way as described in chapter 12 of [vdH97].

8.2 Distinguished solutions

In order to get a parameterized version of theorem 24, we have to make sure that the operator $L^{-1} = \Delta^{-1} (\mathrm{Id} + R \, \Delta^{-1})^{-1}$ is well defined. This can be done as follows:

Algorithm $\mathtt{linear}(L, g)$
Input A linear differential operator L and transseries g
Output The distinguished solution f to $L f = g$

Step 1 [Introduce generic monomial]
    Let $\mathfrak{B} = (\mathfrak{b}_1, ..., \mathfrak{b}_n)$ denote the current purely exponential transbasis

Let $\lambda_1, \ldots, \lambda_n$ be n temporary new parameters in C and set $\mathfrak{m} := \mathfrak{b}_1^{\lambda_1} \cdots \mathfrak{b}_n^{\lambda_n}$

Step 2 [Compute the distinguished solution]

Uniformly regularize $\mathfrak{m}^{-1} L_{\times \mathfrak{m}}$

Let $f := \Delta^{-1} (\mathrm{Id} + R \, \Delta^{-1})^{-1} \, g$, with the notations from the proof of theorem 24

Step 3 [Clean up]

Destroy the parameters $\lambda_1, \ldots, \lambda_n$ by projection of the regions

Return f

In the algorithm, the uniform regularization of a differential operator or polynomial means that we uniformly regularize all its coefficients and that we make the corresponding dominant monomials pairwise comparable for $\preccurlyeq$ using theorem 17 and lemma 15.

In the parameterized context, the equation (6) has a Newton degree d, if each specialization of the directions and parameters leads to an equation of Newton degree d. We will show below how to compute the Newton degree modulo case separations. The distinguished solution to a quasi-linear equation (i.e. of Newton degree 1) is computed inductively as in the proof of theorem 26. The dominant part of P w.r.t. $\mathfrak{b}_n, \mathfrak{b}_{n-1}, \ldots$ is computed by the usual formula after the uniform regularization of P.

Algorithm `quasi_linear`$(P, \mathfrak{v}, k := n)$

Input

An integer $k \in \{0, \ldots, n\}$ (with n as default value), a differential polynomial P with coefficients in $C[\![\mathfrak{b}_1; \ldots; \mathfrak{b}_k]\!]$ and a monomial $\mathfrak{v} \in \mathfrak{b}_1^C \cdots \mathfrak{b}_k^C$, such that (6) is quasi-linear

Output The distinguished solution to (6)

Step 1 [Normalize]

If $\mathfrak{v} \neq 1$, then return $\mathfrak{v}$ times `quasi_linear`$(P_{\times \mathfrak{v}}, 1, k)$

Uniformly regularize P

If $\mathfrak{d}(P) \neq 1$ then return `quasi_linear`$(\mathfrak{d}(P)^{-1} P, 1, k)$

Step 2 [Recurse]

Compute the dominant part D of P w.r.t. $\mathfrak{b}_k$

Let $\varphi := $ `quasi_linear`$(D, 1, k-1)$

Step 3 [Return]

Return $\Delta^{-1} (\mathrm{Id} + R \, \Delta^{-1})^{-1} \, \tilde{g}$, with the notations from the proof of theorem 26

8.3 Determining the Newton polygon

The (i, j)-equalizer of a differential polynomial P is computed similarly as in the proof of proposition 21, by uniformly regularizing P_i and P_j at each recursion.

Algorithm `equalizer`(P, i, j)

Input A differential polynomial P and integers $0 \leqslant i < j \leqslant \deg P$

Output The (i, j)-equalizer for P

Step 1 [regularize]

Uniformly regularize P_i and P_j

Step 2 [equalize]

Let $\mathfrak{m} := \sqrt[j-i]{\mathfrak{d}(P_i)/\mathfrak{d}(P_j)}$

If $\mathfrak{m} \neq 1$ then return $\mathfrak{m}$ `equalizer`$(P_{\times \mathfrak{m}}, i, j)$

Step 3 [shift upwards]

If $\mathfrak{d}(P_i) = \mathfrak{d}(P_j))$ and $D_{P_i + P_j} \in C[c] (c')^{\mathbb{N}}$, then return 1

Shift upwards and return to step 2

The analogue of the Newton polygon associated to an algebraic differential equation in the differential case is the determination of (i, j)-equalizers, which occur as potential dominant monomials, and which are extremal in the sense that $j - i$ is maximal. These equalizers correspond to the slopes of the Newton polygon and the i and j to the first coordinates.

Algorithm `Newton_polygon`(P)
Input A differential polynomial P
Output

Indices $\mathrm{val}\, P = i_0 < \cdots < i_k = \deg P$ and potential dominant monomials $\mathfrak{m}_1 \prec \cdots \prec \mathfrak{m}_k$ for $P(f) = 0$, such that $\mathfrak{m}_j$ is the (i_{j-1}, i_j)-equalizer for P for each j

Step 1 [Initialize]
$j := \mathrm{val}\, P$
$k := 0$
Step 2 [Insert vertex]
$i_k := j$
If $j = \deg P$ then return $i_0, ..., i_k$ and $\mathfrak{m}_1, ..., \mathfrak{m}_{k-1}$
$k := k + 1$
Step 3 [Search edge]

Compute $\mathfrak{e}_{j,j'} = \sqrt[j'-j]{\texttt{equalizer}(P, j, j')}$ for all $j' > j$ with $P_{j'} \neq 0$
Let $\mathfrak{m}_k := \min_{\preccurlyeq} \{\mathfrak{e}_{j,j'}\}$ and let j' be minimal with $\mathfrak{m}_k = \mathfrak{e}_{j,j'}$.
Set $j := j'$ and go to step 2

The Newton degree of (6) can easily be read from the Newton polygon:

Algorithm `Newton_degree`$(P, \mathfrak{v})$
Input A differential polynomial P and a monomial $\mathfrak{v}$
Output The Newton degree of $P(f) = 0$ $(f \prec \mathfrak{v})$

Compute $i_0, ..., i_k$ and $\mathfrak{m}_1, ..., \mathfrak{m}_k$ by `Newton_polynomial`(P)
Let $j \in \{0, ..., k\}$ be maximal, such that $\mathfrak{m}_j \prec \mathfrak{v}$
Return i_j

8.4 Computing potential dominant terms

Given the Newton polygon associated to P, let us now show how to determine the potential dominant terms of solutions to (6) and their multiplicities. We separate a case for each edge and each vertex of the Newton polygon, and determine the corresponding algebraic or mixed resp. differential potential dominant monomials and terms. In order to determine the differential potential dominant monomials, we recursively have to solve Ricatti equations modulo $(x \log x \cdots)^{-1}$. The algorithm `solve` which does this will be specified in the next section.

Algorithm `pdt`$(P, \mathfrak{v})$ [non deterministic]
Input A differential polynomial P and a monomial $\mathfrak{v}$
Output A potential dominant term τ for (6)

Step 1 [Determine Newton degree]

Compute $i_0, ..., i_k$ and $\mathfrak{m}_1, ..., \mathfrak{m}_k$ by `Newton_polygon`(P)
Choose a $j \in \{0, ..., k\}$, such that $j = 0$ or $\mathfrak{m}_j \prec \mathfrak{v}$, and set $d := i_j$

If $j = 0$ then go to step 3
Separate two cases and go to step 2 resp. 3

Step 2 [Algebraic and mixed potential dominant terms]
Let $\mathfrak{m} := \mathfrak{m}_j$
Let c be a new parameter in C
Impose the constraint $D_{P_{\times \mathfrak{m}}}(c) = 0$ (as an algebraic constraint, since $c' = 0$)
Return $c\,\mathfrak{m}$

Step 3 [Differential but non mixed potential dominant terms]
Let $g := \texttt{solve}(R_{P,d}, \infty_{\mathfrak{M}}, (x \log x \log \log x \cdots)^{-1})$
Let $\mathfrak{m} := \exp \int g$, where the integral is computed using $\texttt{linear}$
If $j < k$ then impose the constraint $\mathfrak{m} \prec \mathfrak{m}_{j+1}$
Otherwise impose the constraint $\mathfrak{m} \prec \mathfrak{v}$
If $j > 0$ then impose the constraint $\mathfrak{m} \succ \mathfrak{m}_j$
Let c be a new parameter in C and impose the constraint $c \neq 0$
Return $c\,\mathfrak{m}$

Algorithm $\texttt{multiplicity}(P, \tau)$
Input A differential polynomial P and a term τ
Output The multiplicity of $c(\tau)$ as a root of $N_{P_{\times \mathfrak{d}(\tau)}}$

Repeat
    Uniformly regularize $P_{\times \mathfrak{d}(\tau)}$
    If $D_{P_{\times \mathfrak{d}(\tau)}} \notin C[c](c')^{\mathbb{N}}$ then shift upwards
Until $D_{P_{\times \mathfrak{d}(\tau)}} \in C[c](c')^{\mathbb{N}}$
Return the multiplicity of c_τ as a root of $D_{P_{\times \mathfrak{d}(\tau)}}$

8.5 Solving the differential equation

We can now state the main resolution algorithm for solving asymptotic algebraic differential equations (6) modulo monomials $\mathfrak{w}$.

Algorithm $\texttt{solve}(P, \mathfrak{v}, \mathfrak{w})$
Input A differential polynomial P and monomials $\mathfrak{v}$ and $\mathfrak{w}$
Output A solution to $P(f)$ $(f \prec \mathfrak{v})$ modulo $\mathfrak{w}$

Step M1 [Initialize]
$\varphi := 0$
$mode := normal$

Step M2 [Are we done?]
If $\texttt{Newton\_degree}(P_{+\varphi}, \mathfrak{w}) > 0$, then separate two cases and respectively
    1. Return φ
    2. Proceed with step M3

Step M3 [Compute potential dominant term]
Let $d := \texttt{Newton\_degree}(P_{+\varphi}, \mathfrak{v})$
Let $\tau := \texttt{pdt}(P_{+\varphi}, \mathfrak{v})$
If $\tau \prec \mathfrak{w}$ then kill this process

Step M4 [Does τ have multiplicity $< d$?]
If $\texttt{multiplicity}(P, \tau) < d$ then
    $\varphi := \varphi + \tau$
    $\mathfrak{v} := \mathfrak{d}(\tau)$

$mode := normal$
Go to step M2

Step M5 [Dispatch on *mode*]
If $mode = normal$ then go to step H1
If $mode = unravel$ then go to step H3
If $mode = adjust$ then go to step U4

Step H1 [Prepare first step unravelling loop]
$\Pi := P$
$\mathfrak{m} := \mathfrak{d}(\tau)$
$mode := unravel$
$serial := head$

Step H2 [Prepare partial unravelling loop]
While $D_{\Pi_{+\varphi,\times\mathfrak{m}}} \notin C[c]\,(c')^{\mathbb{N}}$ shift upwards
If $D_{\Pi_{+\varphi,\times\mathfrak{m}}} \in C[c]\,(c')^k$ with $k < d$ then $Q := (\partial^{d-1}\Pi_{+\varphi,\times\mathfrak{m}}/\partial f^{d-1-k}\partial(f')^k)_{\times\mathfrak{m}^{-1},-\varphi}$
Otherwise $Q := (\partial^{d-1}\Pi_{+\varphi,\times\mathfrak{m}}/\partial(f')^{d-1})_{\times\mathfrak{m}^{-1},-\varphi}$

Step H3 [Partial unravelling]
If $\texttt{multiplicity}(Q_{+\varphi}, \tau) = 1$ then
$\psi := \texttt{quasi\_linear}(Q_{+\varphi+\tau}, \tau)$
$\varphi := \varphi + \tau + \psi$
$\mathfrak{v} := \mathfrak{d}(\tau)$
Go to step M2

Step H4 [Dispatch on *serial*]
If $serial = head$ then go to step T1
If $serial = tail$ then go to step T2

Step T1 [Prepare other steps unravelling loop]
Let k be such that $\mathfrak{b}_k \succcurlyeq \mathfrak{m}/\mathfrak{d}(\tau)$
Uniformly regularize $P_{+\varphi,\times\mathfrak{d}(\tau)}$
Compute the dominant part Π of $P_{+\varphi,\times\mathfrak{d}(\tau)}$ w.r.t. $\mathfrak{b}_k$ and set $\Pi := \Pi_{/\mathfrak{d}(\tau),-\varphi}$
$\zeta := z$
$serial := tail$

Step T2 [Prepare next step unravelling loop]
If there is no purely exponential monomial $\mathfrak{m}$ in ζ with $\mathfrak{d}(\tau) \in \zeta^{\{0,\dots r\}}\mathfrak{m}$, then set $\zeta := z$
Let $\mathfrak{m}$ be a purely exponential monomial in ζ, such that $\mathfrak{d}(\tau) \in \zeta^{\{0,\dots,r\}}\mathfrak{m}$
$mode := adjust$

Step T3 [Adjust partial unravelling]
If $\tau \in \zeta^{\{1,\dots r\}}\mathfrak{m}$ then
$\varphi := \varphi + \tau$
$\mathfrak{v} := \mathfrak{d}(\tau)$
Go to step M2
If $\tau \succ \mathfrak{m}$ then $\mathfrak{m} := \mathfrak{d}(\tau)$
$mode := unravel$
Go to step H3

The algorithm $\texttt{solve}$ gradually constructs a solution φ modulo $\mathfrak{w}$ to (6) via a succession of refinements. Each time we get back to the main entry **M2** of the loop, we actually have to solve the equation $P_{+\varphi}(f) = 0$ $(f \prec \mathfrak{v})$. Given a potential dominant term τ for this equation, the next refinement (and value of φ) depends on the *mode* variable.

The core of the algorithm consists of steps **M1-M5** in which case $mode = normal$. As long as we do not hit a potential dominant term τ of maximal multiplicity d, the algorithm only executes steps **M1-M5**. Given a potential dominant term τ of multiplicity $< d$, we can simply take $\varphi = \tau + \tilde{\varphi}$ $(\tilde{\varphi} \prec \tau)$ for our refinement, which corresponds to the assignments $\varphi := \varphi + \tau$ and $\mathfrak{v} := \mathfrak{d}(\tau)$ in step M4.

The steps **H1-H4** correspond to the first partial unravelling when we hit a potential dominant term τ of maximal multiplicity d. As long as we do not enter **T1-T3**, we will have $\Pi = P$, $mode = unravel$ and $serial = head$. We start by computing the differential polynomial Q from section 7.2 (modulo an additive conjugation by φ). We then keep refining $\varphi := \tau + \psi + \tilde{\varphi}$ $(\tilde{\varphi} \prec \tau)$ as far as possible in step H3, where ψ is the distinguished solution to the quasi-linear equation $Q_{\varphi+\tau}(\psi)$ $(\psi \prec \tau)$.

If the steps **H1-H4** do not lead to a complete unravelling, we apply the theory from sections 7.3 and 7.4 in steps **T1-T3**. We start by computing once and for all the differential polynomial Π in **T1**, with this change that we apply a multiplicative and additive conjugation to it in order to make it "compatible" with P. The transseries ζ, which is initialized with z, may become an iterated exponential $\exp_l z$ as a result of upward shiftings. As long as the current potential dominant term τ has not yet the required form in order to start a partial unravelling, we have $mode = adjust$, and we keep on adjusting in step T3.

The termination of `solve` is guaranteed by propositions 23 and 31, modulo the hypothesis that the resolution process requires only a finite number U of upward shiftings. An upper bound for U will be given in the next section.

9 Main theorems and final remarks

9.1 Complex transseries solutions to algebraic differential equations

The following main theorem describes the general form of solutions to asymptotic algebraic differential equations (6) with parameterized complex transseries coefficients.

Theorem 33. *Consider an asymptotic algebraic differential equation* (6) *with transseries coefficients. Then, modulo case separations, there exist a finite number of parameterized transseries solutions* $f_1, ..., f_s$ *to* (6), *with the following properties:*

   a) *The logarithmic depths of* $f_1, ..., f_s$ *do not exceed the logarithmic depths of the coefficients of* P *by more than a fixed constant* $U_{d,r,w} \leqslant d\,(4\,w)^r$, *which only depends on the Newton degree* d, *the order* r *and the weight* $w = \|P\|$ *of* (6).

   b) *For each specialization of the parameters occurring in the coefficients of* P, *for each specialization of the directions, and for each solution* f *to* (6) *after these specializations, there exists exactly one* f_i *and exactly one specialization of the remaining parameters on which depends* f_i, *for which* f_i *specializes to* f.

Proof. In view of the algorithm from the previous section, we only have to prove (a). We prove the bounds for $U_{d,r,w}$ by a double induction over r and d. For $r = 0$, we necessarily have $w = 0$, and clearly $U_{d,0,0} = 0$. So assume that $r > 0$. If $d = 0$, then (6) has no solutions, so that $U_{0,r,w} = 0$. Assume therefore that $d > 0$.

We first observe that the number of upward shiftings needed to compute a potential dominant term is bounded by

$$T_{d,r,w} \leqslant \max\,\{w, U_{w,r-1,w}\}$$

by propositions 21(b), 22 and the induction hypothesis. We have to estimate the maximal number of upward shiftings which may occur in the main loop of `solve`, before we reach a lower Newton degree. Now the first partial unravelling requires at most $T_{d,r,w} + r$ upward shiftings, in view of remark 27. By the induction hypothesis and proposition 30, the second loop of adjusted partial unravellings requires at most $T_{d,r,w} + U_{w,r-1,w} + 1$ upward shiftings. Finally, the decisive refinement, which decreases the Newton degree, again needs at most $T_{d,r,w}$ upward shiftings. Altogether, we obtain

$$U_{d,r,w} \leqslant U_{d-1,r,w} + 3\,T_{d,r,w} + U_{w,r-1,w} + r + 1.$$

Consequently,

$$U_{d,r,w} \leqslant d\,(3\,T_{d,r,w} + U_{w,r-1,w} + r + 1)$$

In particular, for $r = 1$, we obtain

$$U_{d,1,w} \leqslant U_{d,1,d} \leqslant d\,(3\,d + 2).$$

For $r > 1$, we obtain

$$U_{d,r,w} \leqslant d\,(4\,U_{w,r-1,w} + r + 1).$$

By induction, we finally notice that

$$U_{w,r,w} \leqslant (4\,w)^r\,(w+1),$$

which implies our bound. $\square$

Remark 34. Of course, it is possible to improve the bound for $U_{d,r,w}$ for particular values of d, r and w. First of all, in the case when $r = 1$, it is easily checked that 1 upward shifting is sufficient in proposition 21(b), so that

$$T_{d,1,w} \leqslant 1.$$

This observation implies the sharper bounds

$$\begin{aligned} U_{d,1,w} &\leqslant 5\,d; \\ U_{d,r,w} &\leqslant 8\,d\,(4\,w)^{r-1} \end{aligned}$$

for $U_{d,r,w}$. A careful analysis of the differential Newton polygon method will probably lead to even sharper bounds for small values of r. Similarly, it is possible to improve the bounds for small values of d, by using the fact that the weight of P_i is bounded by i^r for $i \leqslant d$.

Although the above theorem describes the general form of solutions to (6), it does not claim the actual existence of such solutions. We say that φ is a solution of multiplicity ν of (6), if the differential valuation of $P_{+\varphi}$ equals ν. The following theorem stipulates the existence of solutions to (6) of a very special form.

Theorem 35. *Consider an asymptotic algebraic differential equation* (6) *of Newton degree d and whose coefficients can be expanded w.r.t. a transbasis* $(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$. *Then there exist at least d solutions to* (6) *when counting with multiplicities. Moreover, these solutions can be expanded w.r.t.* $(\log_l \mathfrak{b}_1, \ldots, \log \mathfrak{b}_1, \mathfrak{b}_1, \ldots, \mathfrak{b}_n)$ *for some l.*

Proof. Without loss of generality, we may assume that $\mathfrak{b}_1 = e^z$. Let us prove the theorem by induction over d. For $d = 0$ we have nothing to prove. For $d = 1$, the equation is quasi-linear and the distinguished solution can be expanded w.r.t. $(\log_r z, \ldots, z, \mathfrak{b}_1, \ldots, \mathfrak{b}_n)$. Assume therefore that $d > 1$.

If there exists only one algebraic potential dominant term with multiplicity d, then consider the unravelling $f = \varphi + \tilde{f}$ ($\tilde{f} \prec \tilde{\mathfrak{v}}$) we obtain by executing `solve`, but where we always choose the unique algebraic potential dominant term in `pdt`. Since this branch only involves the computation of equalizers and solutions of quasi-linear equations, φ can be expanded w.r.t. a transbasis of the form $(\log_l z, ..., z, \mathfrak{b}_1, ..., \mathfrak{b}_n)$. Modulo replacing P by $P_{+\varphi}$, we may thus assume without loss of generality that (6) admits no algebraic potential dominant terms of multiplicity d.

If there exists a mixed potential dominant monomial $\mathfrak{m}$, then $c\mathfrak{m}$ is a potential dominant term of multiplicity $< d$ for each $c \neq 0$, and the coefficients of $P_{+c\mathfrak{m}}$ can be expanded w.r.t. $(\log_{\|P\|} z, ..., z, \mathfrak{b}_1, ..., \mathfrak{b}_n)$. By the induction hypothesis each equation $P_{+c\mathfrak{m}}(\tilde{f})$ ($\tilde{f} \prec \mathfrak{m}$) admits at least one solution which can be expanded w.r.t. $(\log_l z, ..., z, \mathfrak{b}_1, ..., \mathfrak{b}_n)$ for some l. Hence, there exists an infinity of solutions with the required properties. In what follows, we therefore assume that all potential dominant monomials are algebraic, but not mixed.

Now let $\mathrm{val}\, P \leqslant i_0 < \cdots < i_s \leqslant d$ and $\mathfrak{m}_1 \prec \cdots \prec \mathfrak{m}_s$ be such that $\mathfrak{m}_j$ is the (i_{j-1}, i_j)-equalizer for each $j \in \{1, ..., s\}$. For each $j \in \{1, ..., s\}$ the Newton polynomial $N_{P_{\times \mathfrak{m}_j}}$ is a polynomial with valuation i_{j-1} and degree i_j, which has $i_j - i_{j-1}$ roots (when counting with multiplicities). These root induce at least $d - i_0$ potential dominant terms, which can be expanded w.r.t. $(\log_{\|P\|} z, ..., z, \mathfrak{b}_1, ..., \mathfrak{b}_n)$, and whose multiplicities are $< d$. By proposition 23 and the induction hypothesis, this leads to at least $d - i_0$ solutions of the required form, when counting with multiplicities. The theorem now follows from the fact that 0 is a solution of multiplicity i_0. $\qquad \square$

9.2 Linear differential equations

We recall that a differential field F is said to be *differentially algebraically closed*, if for any pair (P, Q) of differential polynomials over F, such that the order of P is strictly larger than the order of Q, there exists a root of P in F, which is not a root of Q.

Let $\mathbb{T}$ be a field of complex transseries as in section 2. Unfortunately, theorem 35 is not sufficient for $\mathbb{T}$ to be differentially algebraically closed. Indeed, the only transseries solutions for the elliptic equation

$$f^3 + (f')^2 + f = 0$$

are $f = 0$, $f = i$ and $f = -i$. Consequently, there are no transseries solutions to this equation, which are not solutions of the equation of lower order

$$f^3 + f = 0.$$

Nevertheless, theorem 35 is sufficient for the following application.

Theorem 36. *Let L be a linear differential operator of order r with coefficients in $\mathbb{T}$. Then*

    a) L can be completely factored over $\mathbb{T}$.

    b) There exist r linearly independent solutions to $Lh = 0$ in $\mathbb{T}$.

Proof. By theorem 35 the Ricatti equation associated to L has at least one solution $\varphi \in \mathbb{T}$. Consequently, we may factor

$$L = \tilde{L}\,(\partial - \varphi),$$

for some linear differential operator $\tilde{L}$ of lower order $r - 1$ and coefficients in $\mathbb{T}$. Part (a) now follows by induction over r.

Now consider a factorization

$$L = (\partial - \varphi_r) \cdots (\partial - \varphi_1), \qquad (47)$$

with $\varphi_1, ..., \varphi_r \in \mathbb{T}$ and let

$$
\begin{aligned}
h_1 &= e^{\int \varphi_1}; \\
h_2 &= (\partial - \varphi_1)^{-1} e^{\int \varphi_2}; \\
&\vdots \\
h_r &= [(\partial - \varphi_{r-1}) \cdots (\partial - \varphi_1)]^{-1} e^{\int \varphi_r},
\end{aligned}
$$

where $\int$ stands for distinguished integration. Then $L h_1 = \cdots = L h_r = 0$. Moreover, by the distinguished properties of the left inverses $(\partial - \varphi_1)^{-1}, ..., [(\partial - \varphi_{r-1}) \cdots (\partial - \varphi_1)]^{-1}$, we have

$$
h_{i, \mathfrak{d}(h_j)} = 0
$$

for all $i > j$. This guarantees the linear independence of $h_1, ..., h_r$. Indeed assume that there exists a relation

$$
\lambda_i h_i + \cdots + \lambda_r h_r = 0
$$

with $\lambda_i \neq 0$. Then $0 = (\lambda_i h_i + \lambda_{i+1} h_{i+1} + \cdots + \lambda_r h_r)_{\mathfrak{d}(h_i)} = (\lambda_i h_i)_{\mathfrak{d}(h_i)} = \lambda_i \neq 0$. This contradiction completes the proof of (b). $\qquad\square$

Remark 37. When choosing the factorization 47 in such a way that $h_1 \prec \cdots \prec h_r$, we even obtain the canonical basis of solutions of $L h = 0$ in the proof of theorem 36.

Remark 38. In the case of real transseries, it can be shown that each linear differential operator may be factored as the product of a transseries and operators of the form

$$
\frac{\partial}{\partial z} + a
$$

or

$$
\frac{\partial^2}{\partial z^2} + \left(2a - \frac{b'}{b} \right) \frac{\partial}{\partial z} + \left(a^2 + b^2 + a' - \frac{a b'}{b} \right) = \left(\frac{\partial}{\partial z} + a - b i + \frac{b'}{b} \right) \left(\frac{\partial}{\partial z} + a + b i \right).
$$

9.3 Bounding the number of integration constants

Although the algorithm `solve` provides us with the generic solution to (6), it is not clear *a priori* that the number of new parameters on which the solution depends does not exceed r. In this section we sketch a proof of the fact that the number of such integration constants is indeed bounded by r.

We first notice that the only place where we introduce (continuous) integration constants is in step 3 of `pdt`. Each integration constant c can therefore be "attached" to a solution of a Ricatti equation of the form $c \, e^{\int \varphi}$. Given an arbitrary moment during the algorithm `solve`, we actually search solutions of the form

$$
f = \lambda_1 e^{\int \varphi_1 + \lambda_2 e^{\cdot^{\cdot^{\int \varphi_l + \tilde{f}}}}},
$$

where $\lambda_1, ..., \lambda_s$ are the "active integration constants". The idea is now to set

$$
F_1 = \lambda_1 e^{\int \varphi_1 + \lambda_2 e^{\cdot^{\cdot^{\int \varphi_l + \tilde{f}}}}}, F_2 = \lambda_2 e^{\int \varphi_2 + \lambda_3 e^{\cdot^{\cdot^{\int \varphi_l + \tilde{f}}}}}, ..., F_l = \lambda_l e^{\int \varphi_l + \tilde{f}}
$$

and to consider P as a differential polynomial of order $r - l$ in $\tilde{f}$, with coefficients in $C[\![\mathfrak{b}_1; ...; \mathfrak{b}_n, F_1, ..., F_l]\!]$. In other words, we consider $F_1, ..., F_l$ as new monomials and we give $\mathfrak{b}_1^C \cdots \mathfrak{b}_n^C F_1^C \cdots F_l^C$ the natural "pointwise" quasi-ordering $\preccurlyeq$ (see chapter 6 of [vdH97]).

The only obstruction for the computation with coefficients in $C[\![\mathfrak{b}_1;...;\mathfrak{b}_n, F_1, ..., F_l]\!]$ instead of coefficients in $C[\![\mathfrak{b}_1;...;\mathfrak{b}_n]\!]$ is when the uniform regularization of a transseries in $C[\![\mathfrak{b}_1;...;\mathfrak{b}_n, F_1, ..., F_l]\!]$ is not possible. Now this obstruction corresponds to the imposition of an algebraic constraint on an active integration constant λ_i, when performing the same computation in $C[\![\mathfrak{b}_1;...;\mathfrak{b}_n]\!]$. In order to solve this problem, an "error handler" is installed each time that we introduce a new continuous integration constant λ_i. Whenever we impose an algebraic constraint on λ_i, we go back to the error handler and reperform the same computations while assuming that λ_i either did or did not (non determinism) satisfy the algebraic constraint right from the start.

In all branches of the new resolution process, the order of the asymptotic differential equation, when rewritten as an equation in $\tilde{f}$, does not exceed $r - l$. Consequently, $l \leqslant r$ at the end of each branch of the process.

9.4 Comparison with previous work and errata

The reader may have noticed a certain number of changes with respect to the treatment of algebraic different equations in [vdH97]. Although the results of this paper were stated in the context of grid-based transseries, they may easily be adapted to the well-ordered context from [vdH97], except for the results about parameterized transseries, which become more complicated. The algorithm `solve` may still be applied in the well-ordered context, except that the introduction of new parameters should then be interpreted as a new source of (continuous) case separations.

During a careful reexamination of our previous work, we noticed that proposition $5.7(c)$ in [vdH97] does not hold for all $\boldsymbol{j}$. Consequently, our previous treatment of almost double solutions in section 5.5.1 does not work. The present, more complicated, treatment using unravellings corrects this error. When calling a refinement occurring in our construction of a total unravelling a *privileged refinement*, the proof of theorem 5.2 in [vdH00a] remains correct (except for the bound for the maximal length l of a chain of privileged refinements, which may have to be replaced by a larger bound).

Some other changes with respect our previous work are the following:

- In view of theorem 3.3 in [vdH00a] it is no longer necessary to develop the theory from section 5 in the purely exponential setting first (as we did in [vdH97]).

- We simplified and improved the construction of distinguished solutions to linear and quasi-linear equations, through a new application of the generalized implicit function theorem from [vdH00b].

- In comparison with the effective asymptotic resolution of algebraic differential equations in chapter 12 from [vdH97], we noticed that we actually never need to impose exponential constraints on the parameters. After correcting the error related to privileged refinements, we therefore no longer need to assume the existence of an oracle to determine the consistency of first order systems of exp-log constraints in theorem 12.4.

- In the corrected version, we also consider the case when $N_{P_{d, \times \mathfrak{m}}} = \alpha(c - \beta)^{d-k}(c')^k$ with $0 < k < d$. We forgot that case in the original version.

9.5 Conclusion and perspectives

In this paper, we have generalized the transseries technique for solving algebraic differential equation as far as reasonably possible. Three main problems remain to be solved.

Analytic counterpart.   First of all, one has to show that the complex transseries solutions to algebraic equations have a genuine analytic meaning. This problem, which will be treated in a forthcoming paper, can actually be subdivided in two parts:

- We have to show that a consistent system of asymptotic constraints on the directions corresponds to a non empty asymptotic region of the complex plane. In general, this region does not need to be connected.

- We have to give an analytic meaning to our transseries solutions on regions as above. This analytic meaning should be compatible with the asymptotic relations, which have in particular to be formalized on disconnected regions.

Differentially algebraic closure.   We have already remarked that our fields of complex transseries are not differentially algebraically closed. In other words, we still miss most of the solutions to algebraic differential equations in our formalism. In order to get a full understanding of the asymptotic behaviour of solutions to algebraic differential equations, two approaches may be followed:

- In order to solve an equation like

$$f^3 + (f')^2 + f = 0$$

  one may start with studying the solutions f in the neighbourhoods of singularities other than ∞. This can be done by performing a change of variable $z = c + \tilde{z}^{-1}$, which transforms the equation into an equation which does admit a solution space of dimension 1.

  More generally, for a general asymptotic algebraic differential equation (6), the above trick leads to transseries solutions in $C \llbracket z_1 \rrbracket \cdots \llbracket z_k \rrbracket$, where $z = z_1$ is the original variable, and

$$z_1 \;\; = \;\; \varphi_1 + \tau_1 \, z_2^{-1};$$
$$\vdots$$
$$z_{k-1} \;\; = \;\; \varphi_{k-1} + \tau_{k-1} \, z_k^{-1}.$$

  It is not yet clear to us how to alternate usual refinements with substitutions of the form $z = c + \tilde{z}^{-1}$.

- Assuming for simplicity that P has constant coefficients, one may also start with studying the singularities of the dynamical system associated to the algebraic differential equation. For instance, one may use the theory from chapter 10 in [vdH97] to desingularize P as a polynomial in $f, f', ..., f^{(r)}$. This leads to a better understanding of the behaviour of the dynamical system for different subregions of "the $(f, f', ..., f^{(r)})$-space". We next apply the asymptotic tools from this paper to obtain full solutions on these regions. Finally, one has to study how the solutions globally glue together.

In any case, a purely local treatment seems not to be possible in order to describe all solutions to an algebraic differential equation. A good combination of a more global theory with our local results might lead to the resolution of interesting questions, such as

> Is it possible for an analytic solution to an algebraic differential equation with coefficients in $\mathbb{C}$ to admit a natural boundary somewhere on its Riemann surface?

For Liouvillian functions and, in view of the theorem 36, for functions which are obtained via the repeated resolution of linear differential equations, the answer seems to be negative.

Unravellings. In relation to a joint project with Aschenbrenner and Van den Dries, which aims to describe the model theory of "real differential fields" and "valuated differential fields", it seems important to better understand our technique of unravellings in cases where the transseries in $\mathbb{T}$ do not necessarily admit finite logarithmic depths. A typical example of an equation which is "hard to unravel" is

$$f^2 + 2\,f' + \frac{1}{x^2} + \frac{1}{x^2 \log^2 x} + \frac{1}{x^2 \log^2 x \log^2 \log x} + \cdots = 0.$$

A good question is whether there are essentially different examples of equations which are hard to unravel. Another question is whether we may avoid the adjusted partial unravellings from section 7.3.

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