Formal asymptotics of solutions to certain linear differential equations involving oscillation

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(Paris, 22 june 1998)

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Let $\mathcal{P} = \mathcal{P}^x$ denote the ring of analytic 2π-periodic functions in x on the real axis. Let $\mathcal{S} = \mathcal{S}^x$ denote the ring of formal Laurent series in $\mathcal{P}((e^{-x}))$, whose coefficients are defined on a *common* strip neighbourhood of the real axis. In this paper, we study the linear differential equation

$$L_r(x)h^{(r)}(x) + \cdots + L_0(x)h(x) = 0,$$

with coefficients $L_0, \ldots, L_r \neq 0$ in $\mathcal{S}$. We prove that, after a change of variables $x = p(\tilde{x} + \varphi(\tilde{x}))$ with $p \in \mathbb{N}^*$ and $\varphi \in \mathcal{P}^{\tilde{x}}$, this equation admits a basis of r formal solutions of the form

$$h = (\varphi_{r-1}(\tilde{x})\tilde{x}^{r-1} + \cdots + \varphi_0(\tilde{x}))\ \exp(\xi\tilde{x})\ \exp(\psi_d(\tilde{x})e^{d\tilde{x}} + \cdots + \psi_1(\tilde{x})e^{\tilde{x}}),$$

where $\varphi_0, \ldots, \varphi_{r-1} \in \mathcal{S}^{\tilde{x}}$, $\xi \in \mathbb{C}$ and $\psi_1, \ldots, \psi_d \in \mathcal{P}^{\tilde{x}}$. This generalizes a well known result when $\mathcal{P}$ is replaced by $\mathbb{C}$.

1. Introduction

Consider the linear differential equation

$$Lh = L_r h^{(r)} + \cdots + L_0 h = 0. \tag{1.1}$$

It is well known, e.g. (Ince, 1926), that if the coefficients $L_0, \ldots, L_r$ are power series in $\mathbb{C}[[z]]$, then there exists a basis of r formal solutions to (1.1) of the form

$$h = (h_{r-1}\log^{r-1} z + \cdots + h_1 \log z + h_0) z^\lambda e^{P(\sqrt[p]{z})},$$

where $h_0, \ldots, h_{r-1}$ are power series in $\mathbb{C}[[\sqrt[p]{x}]], p \in \mathbb{N}^*, \lambda \in \mathbb{C}$ and $P = P_d x^{-d/p} + \cdots + P_1 x^{-1/p}$ a polynomial in $\mathbb{C}[\sqrt[p]{z^{-1}}]$ without constant term. When replacing z by e^{-x}, it follows that, if the coefficients $L_0, \ldots, L_r$ are in $\mathbb{C}[[e^{-x}]]$, then the differential equation (1.1) admits a basis of r formal solutions of the form

$$h = (h_{r-1}x^{r-1} + \cdots + h_1 x + h_0)e^{\lambda x} e^{P(e^{x/p})},$$

where $h_0, \ldots, h_{r-1}x^{r-1} \in \mathbb{C}[[e^{-x/p}]], p \in \mathbb{N}^*, \lambda \in \mathbb{C}$ and $P \in \mathbb{C}[e^{x/p}]e^{x/p}$. This classical result was generalized in (van der Hoeven, 1997) to the case when the coefficients

† The difference between the dates in the title and on the cover are due to the fact that this paper was declared to be "uninteresting" after a long period of refereeing. Nevertheless, several people have asked me for the paper since then, which made me decide to publish this preprint a long time after its time of writing.

$L_0, \ldots, L_r$ are transseries. This allows for instance to find bases of formal solutions to equations like

$$e^{\Gamma(e^x)} f'' + \frac{e^x}{e^{e^x}+1} f' + \operatorname{erf} e^{1998x} f = 0.$$

A major actual drawback of the actual transseries theory (Écalle, 1992; van der Hoeven, 1997) is that it only modelizes "strongly monotonic" asymptotic behaviour, i.e. we do not allow oscillatory behaviour. In this paper, we make a first step towards the formal study of asymptotic linear differential equations which do involve oscillation.

In section 3, we start by studying the equation (1.1) when $L_r = 1, L_{r-1}, \ldots, L_0 \in \mathcal{P}$, where $\mathcal{P} = \mathcal{P}^x$ is the set of analytic 2π-periodic functions on the real axis in x. Notice that elements of $\mathcal{P}$ are actually defined on a small strip neighbourhood of the real axis. We show that there exists a basis of solutions to (1.1) of the form $h \in \mathcal{P}[x]e^{\lambda x}$. We next study the inhomogeneous equation

$$Lh = L_r h^{(r)} + \cdots + L_0 h = g, \tag{1.2}$$

with $g \in \mathcal{P}[x]$ and show that this equation always admits a solution (and even a very special, so called "distinguished solution") in $\mathcal{P}[x]$. This result persists in the case when $L_r \neq 1$, modulo a change of variables of the form $x = \tilde{x} + \varphi(\tilde{x})$, where $\varphi \in \mathcal{P}^{\tilde{x}}$ is an analytic 2π-periodic function in $\tilde{x}$ on the real axis.

In section 4, we consider the case when the coefficients L_i are in the set $\mathcal{S}$ of Laurent series in $\mathcal{P}((e^{-x}))$, whose coefficients are defined on a common strip neighbourhood of $\mathbb{R}$. We prove that, modulo a change of variables $x = p(\tilde{x} + \varphi)$, with $p \in \mathbb{N}^*$ and $\varphi \in \mathcal{P}^{\tilde{x}}$, there exists a basis of r solutions to (1.1) of the form

$$h = (\varphi_{r-1}(\tilde{x})\tilde{x}^{r-1} + \cdots + \varphi_0(\tilde{x})) \ \exp(\xi\tilde{x}) \ \exp(\psi_d(\tilde{x})e^{d\tilde{x}} + \cdots + \psi_1(\tilde{x})e^{\tilde{x}}),$$

where $\varphi_0, \ldots, \varphi_{r-1} \in \mathcal{S}^{\tilde{x}}$, $\xi \in \mathbb{C}$ and $\psi_1, \ldots, \psi_d \in \mathcal{P}^{\tilde{x}}$. We will follow a similar proof strategy as in (van der Hoeven, 1997), based on the Newton polygon method and distinguished solutions. Further generalizations of this result will be treated in a forthcoming paper.

2. Preliminaries

2.1. The coefficients

Let $\mathcal{P}$ be the space of analytic, 2π-periodic functions on the real axis. Such functions are actually analytic on a strip neighbourhood of the real axis (i.e. a set of the form $\{z \in \mathbb{C} | \varepsilon > |\Im z|\}$). Let $\mathcal{S}$ be the set of Laurent series $f \in \mathcal{P}((e^{-x}))$, such that the coefficients f_α are analytic on a common strip neighbourhood of the real axis. Clearly, $\mathcal{S}$ forms a ring. We will denote by v_f the valuation of $f \in \mathcal{S}$ in e^{-x}.

When solving algebraic or differential equations with coefficients in $\mathcal{P}$ or $\mathcal{S}$, we will encounter $2p\pi$-periodic functions with $p \in \mathbb{N}^*$, as well as singularities on the real axis, which need be circumvented by passing in the complex plane. For these reasons, we will consider changes of variables

$$x = p(\tilde{x} + \varphi(\tilde{x})) = p\gamma(\tilde{x}), \tag{2.1}$$

where $p \in \mathbb{N}^*$, $\varphi \in \mathcal{P}$ and the mapping $\gamma : \tilde{x} \mapsto \tilde{x} + \varphi(\tilde{x})$ is bijective in a strip neighbourhood of $\mathbb{R}$. Such a change of variables is called a *narrowing* and a composition of two narrowings is again a narrowing. Usually, x and $\tilde{x}$ are bound to certain strip neighbourhoods U resp. $\tilde{U}$ of $\mathbb{R}$ with $\gamma(\tilde{U}) \subseteq U$ and γ bijective on $\tilde{U}$. The number p is called the *multiplicator* of the narrowing.

Since we will sometimes work concurrently with several variables $x, \tilde{x}$, it will be convenient to write $\mathcal{P}^x$ instead of $\mathcal{P}$ if we want to emphasize that its elements are 2π-periodic in x (similarly, we will consider $\mathcal{P}^{\tilde{x}}$, $\mathcal{S}^x$, etc.)

PROPOSITION 2.1. *Consider a polynomial equation with coefficients in* $\mathcal{P}^x$*:*

$$P(f) = P_d f^d + \cdots + P_0 = 0 \quad (P_d \neq 0). \tag{2.2}$$

Then there exists a narrowing $x = p(\tilde{x} + \varphi(\tilde{x}))$*, such that (2.2) admits* d *solutions in* $\mathcal{P}^{\tilde{x}}$*, when counted with multiplicities.*

PROOF. Without loss of generality, we may assume that P is irreducible. Let $\gamma : t \mapsto t + \psi(t)$ be any immersion with $\psi \in \mathcal{P}^t$, such that the resultant of P and P' does not vanish on $\operatorname{Im} \gamma$. Then each solution y_i to

$$P_d(x) y^d + \cdots + P_0(x) = 0 \tag{2.3}$$

in a point $x_0 \in \operatorname{Im} \gamma$ determines a unique analytic solution f_i to (2.2) on $\operatorname{Im} \gamma$ such that $f_i(x_0) = y_i$. Since $P_0, \ldots, P_d$ are 2π-periodic, there exists a permutation σ of $\{1, \ldots, d\}$, such that $f_i(x_0 + 2\pi) = y_{\sigma(i)}$ for all $1 \leqslant i \leqslant d$. By the uniqueness of analytic continuation and induction over k, we infer that $f_i(x + 2\pi k) = f_{\sigma^k(i)}(x)$ for all x and $k \in \mathbb{N}$. Consequently, if $p \in \mathbb{N}^*$ is such that $\sigma^p = Id$, then $f_1, \ldots, f_d$ are all $2\pi p$-periodic and the narrowing $x = p\gamma(\tilde{x})$ satisfies our requirements. $\square$

For each ring R and $n \in \mathbb{N}$, let $R[x]_d$ be the set of polynomials of degrees at most d in x over R. In what follows we shall often consider polynomials in $\mathcal{S}[x]$ and $\mathcal{S}[x]_d$ and interpret such polynomials as Laurent series in $\mathcal{S}$ with coefficients in $\mathcal{P}[x]$ resp. $\mathcal{P}[x]_d$.

Let $\mathcal{E}_{d,0}$ denote the set of finite linear combinations $\varphi_1 e^{\lambda_1 x} + \cdots + \varphi_k e^{\lambda_k x}$, with $\varphi_1, \ldots, \varphi_k \in \mathcal{S}[x]_d$ and $\varphi_1, \ldots, \varphi_k \in \mathbb{C}$. For each polynomial without constant term $P = P_d e^{dx} + \cdots + P_1 e^x$ in $\mathcal{P}[e^x]$, we denote $\mathcal{E}_{d,P} = \mathcal{E}_{d,0} e^P$. We define

$$\mathcal{E}_d = \bigoplus_P \mathcal{E}_{d,P}.$$

We will search for solutions to (1.1) in $\mathcal{E}_{r-1}$, modulo a suitable narrowing.

2.2. LINEAR DIFFERENTIAL OPERATORS

Let $\partial_x = \frac{d}{dx}$ denote the differentiation operator with respect to x. Given a linear differential operator

$$L = L_r \partial_x^r + \cdots + L_0,$$

we define the *derivative* L' of L by

$$L' = r \partial_x^{r-1} + \cdots + L_1.$$

For any f and g, we have the product formula

$$L(fg) = (Lf)g + (L'f)g' + \cdots + \frac{1}{r!}(L^{(r)} f) g^{(r)}. \tag{2.4}$$

The operator L is said to be *monic*, if $L_r = 1$. In that case, $\frac{1}{r} L'$ is monic as well. If the L_i are in $\mathcal{S}$, then we will denote by $L_{i,\alpha}$ the coefficient of $e^{-\alpha x}$ in L_i for each i, α.

Given a linear differential operator L and a function h, there exists a unique linear differential operator $L_{\times h}$ such that

$$L_{\times h}(f) = L(hf)$$

for all f. We call $L_{\times h}$ a *multiplicative conjugate* of L. The coefficients of $L_{\times h}$ are given explicitly by

$$L_{\times h, i} = L^{(i)} h = \sum_{j=i}^{r} \binom{j}{i} L_j h^{(j-i)}.$$

We notice that if L has coefficients in $\mathcal{P}$, then $\mathcal{P} e^{\lambda x}$ and $\mathcal{P}[x] e^{\lambda x}$ are stable under L for each $\lambda \in \mathbb{C}$. Consequently, if $h \in \mathcal{P} e^{\lambda x}$, then $e^{-\lambda x} L_{\times h}$ has coefficients in $\mathcal{P}$.

Given a linear differential operator L and a function γ, we also define $L \circ \gamma$ to be the unique differential operator with

$$(L \circ \gamma)(f \circ \gamma) = (Lf) \circ \gamma$$

for all f. Such operators are encountered when performing a change of variables $x = \gamma(\tilde{x})$. Setting $\tilde{f} = f \circ \gamma$, $\widetilde{Lf} = (Lf) \circ \gamma$ and $\tilde{L} = L \circ \gamma$, we then have $f(x) = \tilde{f}(\tilde{x})$ and $\tilde{L} \tilde{f} = \widetilde{Lf}$. The coefficients of $\tilde{L}$ are obtained from the relations

$$\begin{array}{rcl} f(x) & = & \tilde{f}(\tilde{x}); \\ f'(x) & = & \gamma'(\tilde{x})^{-1} \tilde{f}'(\tilde{x}); \\ f''(x) & = & \gamma'(\tilde{x})^{-2} \tilde{f}''(\tilde{x}) - \gamma''(\tilde{x}) \gamma'(\tilde{x})^{-3} \tilde{f}'(\tilde{x}); \\ & \vdots & \end{array}$$

In particular, if $\gamma(\tilde{x}) = \tilde{x} + c$ for some constant c, then $f^{(j)}(x) = \tilde{f}^{(j)}(\tilde{x})$ and $\tilde{L}_j = L_j \circ \gamma$ for all j.

3. Linear differential equations with periodic coefficients

3.1. The monic homogeneous case

Consider the homogeneous linear differential equation (1.1), for coefficients $L_0, \ldots, L_r \in \mathcal{P}$ with $L_r = 1$. Let $\mathcal{H}$ be the space of analytic solutions to (1.1) on the real axis. Since $L_r = 1$, we have $\dim \mathcal{H} = r$. Let $\mathcal{C}$ be the space of analytic functions on the real axis and consider the mapping $\Phi : \mathcal{C} \to \mathcal{C}$ defined by

$$(\Phi f)(x) = f(x + 2\pi).$$

Since the coefficients of (1.1) are periodic, $\mathcal{H}$ is stable under Φ. From now on, we will only consider the restriction of Φ to $\mathcal{H}$, which is an isomorphism, since Φ is invertible and $\mathcal{H}$ finite dimensional. In particular, all eigenvalues of Φ are non zero; let $e^{2\pi\lambda}$ be such an eigenvalue. Modulo the change of function $h \to h / e^{\lambda x}$, we may assume without loss of generality that $\lambda = 0$.

By Jordan's theorem, the characteristic space associated to the eigenvalue $e^{2\pi\lambda} = 1$ can be written as a direct sum of invariant subspaces, each on which there exists a basis

$h_0, \dots, h_{\nu-1}$ with respect to which Φ is represented by the matrix

$$\begin{pmatrix} 1 & & & \mathrm{O} \\ 1 & 1 & & \\ & \ddots & \ddots & \\ \mathrm{O} & & 1 & 1 \end{pmatrix}.$$

On such a subspace, we have in particular $\Phi h_0 = h_0$, whence $h_0 \in \mathcal{P}$. Next, $\Phi h_1 = h_1 + h_0$ and setting $\varphi_1 = h_1 - h_0 \frac{x}{2\pi}$, we observe that $\Phi \varphi_1 = h_1 + h_0 - h_0 \frac{x+2\pi}{2\pi} = \varphi_1$. Therefore, $h_1 \in \mathcal{P}[x]_1$. Similarly, for each $1 < j < \nu_0$, one has $\Phi \varphi_j = \varphi_j$, where $\varphi_j = h_j - h_{j-1} \frac{x}{2\pi}$. By induction on j, it follows that $h_j \in \mathcal{P}[x]_j$.

For each $\lambda \in \mathbb{C}$, let ν_λ be the dimension of the characteristic space $\mathcal{H}_\lambda$ associated to the eigenvalue $e^{2\pi\lambda}$. We have just shown that

$$\mathcal{H}_\lambda \subseteq \mathcal{P}[x]_{\nu_\lambda - 1} e^{\lambda x}.$$

In other words,

THEOREM 3.1. *Assume that $L_r = 1$ and $L_{r-1}, \dots, L_0 \in \mathcal{P}$. Then the solution space $\mathcal{H}$ to (1.1) admits a basis of elements of the form*

$$h \in \mathcal{P}[x]_{\nu_\lambda - 1} e^{\lambda x} \quad (\lambda \in \mathbb{C}),$$

where $\nu_\lambda = \dim \mathcal{H} \cap \mathcal{P}[x] e^{\lambda x}$ for each $\lambda \in \mathbb{C}$. □

3.2. INTEGRATION

LEMMA 3.1. *Let $g = \psi e^{\lambda x}$, with $\psi = \psi_d x^d + \cdots + \psi_0 \in \mathcal{P}[x]_d$.*

(a) If $\lambda \notin \mathbb{Z}i$, then there exists a unique primitive $\int g$ of g in $\mathcal{P}[x]_d e^{\lambda x}$.
(b) If $\lambda \in \mathbb{Z}i$, there exists a unique primitive $\int g$ of g in $\mathcal{P}[x]_{d+1}$, such that $\langle (\int g)_0 | 1 \rangle = 0$.

PROOF. Setting $f = \varphi e^{\lambda x}$, solving $f' = g$ in $\mathcal{P}[x] e^{\lambda x}$ is equivalent to solving

$$\varphi' + \lambda \varphi = \psi$$

in $\mathcal{P}[x]$. We will search for a solution of the form

$$\varphi = \varphi_{d+1} x^{d+1} + \cdots + \varphi_0.$$

Then we have to solve the following system of equations:

$$\begin{aligned} \varphi_{d+1}' + \lambda \varphi_{d+1} &= 0; \\ \varphi_d' + \lambda \varphi_d &= \psi_d - (d+1) \varphi_{d+1}; \\ &\vdots \\ \varphi_0' + \lambda \varphi_0 &= \psi_0 - \varphi_1. \end{aligned}$$

In what follows, we will denote by a_j the coefficient of $e^{-\lambda x}$ in the Fourier series of ψ_j, for each j. If $\lambda \notin i\mathbb{Z}$, then $a_j = 0$.

We take $\varphi_{d+1} = \frac{1}{d+1} a_d e^{\lambda x}$, whence $\varphi_{d+1} = 0$, if $\lambda \notin \mathbb{Z}i$. The remaining φ_j are computed by induction over $j = d, \ldots, 0$. We make the induction hypothesis that $\varphi_{j+1} \in \mathcal{P}$ and that the coefficients of $e^{-\lambda x}$ in the Fourier series of $(j+1)\varphi_{j+1}$ and ψ_j coincide. Now let

$$\sum_{k \in \mathbb{Z}} c_k e^{ikx} = \psi_j - (j+1)\varphi_{j+1}$$

be the convergent Fourier series of $\psi_j - (j+1)\varphi_{j+1}$. Then we take

$$\varphi_j = \frac{a_{j-1}}{j} e^{-\lambda x} + \sum_{k \in \mathbb{Z}, ik+\lambda \neq 0} \frac{c_k}{ik+\lambda} e^{ikx},$$

which is convergent and periodic (in the case $j = 0$, we understand a_{j-1}/j to be zero). Since any solution to $\varphi_j' + \lambda \varphi_j = \psi_j - (j+1)\varphi_{j+1}$ is analytic, we have $\varphi_j \in \mathcal{P}$. The second induction hypothesis is again satisfied at the next stage, by definition of φ_j.

We have thus shown how to compute a primitive $f = \varphi e^{\lambda x}$ of g, with $\varphi \in \mathcal{P}[x]_{d+1}$. Moreover, if $\lambda \notin \mathbb{Z}i$, then $\varphi_{d+1} = 0$ and $f \in \mathcal{P}[x]_d e^{\lambda x}$. Finally, the primitive of g is unique up to a constant factor. If $\lambda \notin \mathbb{Z}i$, this implies that f is unique in $\mathcal{P}[x]_d e^{\lambda x}$ with $f' = g$. If $\lambda \in \mathbb{Z}i$, f is unique in $\mathcal{P}[x]_{d+1}$ with the property that the constant term $\langle \varphi_0 e^{\lambda x} | 1 \rangle$ of f vanishes. □

The primitive $\int g$ as constructed in the lemma is called the *distinguished primitive* of g. Notice that the mapping $g \mapsto \int g$ is injective and linear on $\mathcal{P}[x]e^{\lambda x}$, for each $\lambda \in \mathbb{C}$: this is clear if $\lambda \notin \mathbb{Z}i$; otherwise, it follows from the fact that $\langle \varphi + \psi | 1 \rangle = \langle \varphi | 1 \rangle + \langle \psi | 1 \rangle$ for all φ and ψ. Consequently, the mapping $\int$ may be extended uniquely to a linear, injective mapping from the subvector space of $\mathcal{C}$ generated by the the vector spaces of the form $\mathcal{P}[x]e^{\lambda x}$ into itself.

Let us denote by $\nu_L : \mathbb{C} \to \mathbb{N}$ the mapping which associates ν_λ to λ. Notice that ν_L factors through $\mathbb{C}/\mathbb{Z}i$, since $\nu_L(\lambda + i) = \nu_L(\lambda)$. We will now study the dependence of ν_L on L.

LEMMA 3.2. *Let L be a monic linear differential operator in $\mathcal{P}[\partial_x]$. Then*

$$\nu_{L\partial_x} = \nu_L + \nu_{\partial_x}.$$

PROOF. Let $\mathcal{I}$ be the solution space to $(L\partial_x)h = 0$ and for each $\lambda \in \mathbb{C}$, let $\mathcal{I}_\lambda \subseteq \mathcal{P}[x]e^{\lambda x}$ be the characteristic space associated to $e^{2\pi\lambda}$, for Φ restricted to $\mathcal{I}$. Then the distinguished primitivation $\int$ maps $\mathcal{H}$ into $\mathcal{I}$ and $\mathcal{H}_\lambda$ into $\mathcal{I}_\lambda$ for each $\lambda \in \mathbb{C}$, while ∂_x maps $\mathcal{I}$ onto $\mathcal{H}$ and $\mathcal{I}_\lambda$ onto $\mathcal{H}_\lambda$ for each $\lambda \in \mathbb{C}$. For each $\lambda \notin \mathbb{Z}i$, we infer that

$$\nu_{L\partial_x}(\lambda) = \dim \mathcal{I}_\lambda = \dim \textstyle\int \mathcal{H}_\lambda = \dim \mathcal{H}_\lambda = \nu_L(\lambda).$$

For $\lambda \in \mathbb{Z}i$, we get

$$\nu_{L\partial_x}(\lambda) = \dim \mathcal{I}_\lambda = \dim(\textstyle\int \mathcal{H}_\lambda \oplus \mathbb{C}) = \dim \mathcal{H}_\lambda + 1 = \nu_L(\lambda) + 1.$$

This proves the lemma, since $\nu_{\partial_x}(\lambda) = 1$ if $\lambda \in \mathbb{Z}i$ and $\nu_{\partial_x}(\lambda) = 0$ otherwise. □

3.3. The monic inhomogeneous case

Lemma 3.2 may be generalized as follows:

Lemma 3.3. *Let L, K be two monic linear differential operators in $\mathcal{P}[\partial_x]$. Then*

$$\nu_{LK} = \nu_L + \nu_K.$$

Proof. Let us prove the lemma by induction over the order s of K. For $s = 0$, we have nothing to do. Assume that $s > 0$ and let h be a solution to $Kh = 0$ in $\mathcal{P}e^{\lambda x}$ for some λ (such a solutions exists always: see section 3.1). We will first assume that $h^{-1} \in \mathcal{P}e^{-\lambda x}$.

Since each solution of $h^{-1}L_{\times h}f = 0$ in $\mathcal{P}[x]e^{\mu x}$ determines a unique solution to $Lf = 0$ in $\mathcal{P}[x]e^{(\mu+\lambda)x}$ via multiplication by h, we have

$$\nu_{h^{-1}L_{\times h}}(\mu) = \nu_L(\mu + \lambda) \tag{3.1}$$

for all $\mu \in \mathbb{C}$. Given $\mu \in \mathbb{C}$, we have in a similar way

$$\nu_{h^{-1}K_{\times h}}(\mu) = \nu_K(\mu + \lambda), \tag{3.2}$$

and

$$\nu_{h^{-1}(LK)_{\times h}}(\mu) = \nu_{LK}(\mu + \lambda). \tag{3.3}$$

Since $h^{-1}Kh = h^{-1}K_{\times h}1 = 0$, we can factor $h^{-1}K_{\times h} = \Omega\partial_x$. By the induction hypothesis and (3.1), we get

$$\nu_{h^{-1}L_{\times h}\Omega}(\mu) = \nu_L(\mu + \lambda) + \nu_\Omega(\mu).$$

By lemma 3.2, we therefore have

$$\nu_{(h^{-1}L_{\times h})(h^{-1}K_{\times h})}(\mu) = \nu_L(\mu + \lambda) + \nu_\Omega(\mu) + \nu_{\partial_x}(\mu).$$

Applying the lemma again, we also have

$$\nu_{h^{-1}K_{\times h}}(\mu) = \nu_\Omega(\mu) + \nu_{\partial_x}(\mu).$$

Combining these two equations with (3.2), we obtain

$$\nu_{(h^{-1}L_{\times h})(h^{-1}K_{\times h})}(\mu) = \nu_L(\mu + \lambda) + \nu_K(\mu + \lambda).$$

But

$$(h^{-1}L_{\times h})(h^{-1}K_{\times h}) = h^{-1}(LK)_{\times h},$$

whence the lemma follows from (3.3) in the case when $h^{-1} \in \mathcal{P}e^{-\lambda x}$.

In general, when $e^{-\lambda x}h$ is not invertible in $\mathcal{P}$, we consider a change of variables $x = \tilde{x} + i\varepsilon$, with $\varepsilon \in \mathbb{R}^*$ sufficiently small, such that h does not vanish on $i\varepsilon + \mathbb{R}$. Applying the previous argument to the operators $\tilde{L} = L \circ \gamma$, $\tilde{K} = K \circ \gamma$ and $\widetilde{LK} = (LK) \circ \gamma = \tilde{L}\tilde{K}$, we then find $\nu_{\widetilde{LK}} = \nu_{\tilde{L}} + \nu_{\tilde{K}}$. Moreover, $\nu_{\tilde{L}} = \nu_L$, since any solution $f \in \mathcal{P}[x]e^{\mu x}$ to $Lf = 0$ determines a unique solution $\tilde{f} = f \circ \gamma \in \mathcal{P}^{\tilde{x}}[\tilde{x}]e^{\mu\tilde{x}}$ to $\tilde{L}\tilde{f} = 0$. Similarly, $\nu_{\tilde{K}} = \nu_K$ and $\nu_{\widetilde{LK}} = \nu_{LK}$, whence the lemma. □

Theorem 3.2. *Assume that $L_r = 1, L_{r-1}, \ldots, L_0 \in \mathcal{P}$ and $g \in \mathcal{P}$. Then (1.2) admits at least one solution in $\mathcal{P}[x]_{\nu_L(0)}$.*

Proof. Assume first that g is invertible in $\mathcal{P}$. Then $\partial_x(g^{-1}L_{\times g})$ is a monic operator with coefficients in $\mathcal{P}$ and $(\partial_x(g^{-1}L_{\times g}))(f/g) = (\partial_x(g^{-1}L))(f) = 0$, for any solution f to (1.2). Inversely, there exists a solution h to

$$\partial_x(g^{-1}L)(h) = 0, \tag{3.4}$$

such that $g^{-1}Lh = 1$: otherwise, Lh would vanish for all solutions to (3.4) and the dimension of $\mathcal{H}$ would be at least $r+1$.

Let us now write $h = h_0 e^{\lambda_0 x} + \cdots + h_k e^{\lambda_k x}$, with $h_0, \ldots, h_k \in \mathcal{P}[x]$, $\lambda_0 = 0$ and pairwise distinct λ_j modulo i. For each $j > 0$, we observe that $g^{-1}L(h_j\lambda_j) \in \mathcal{P}[x]e^{\lambda_j x}$, whence $g^{-1}L(h_j\lambda_j) = 0$. Hence $f = h_0 \in \mathcal{P}[x]$ is again a solution to (3.4) and $g^{-1}Lf = 1$. Now lemma 3.3 implies that

$$\nu_{\partial_x(g^{-1}L_{\times g})}(0) = \nu_{g^{-1}L_{\times g}}(0) + 1 = \nu_L(0) + 1,$$

whence $f \in \mathcal{P}[x]_{\nu_L(0)}$. This completes the proof in the case when g is invertible in $\mathcal{P}$.

In general, let $c \in \mathbb{R}$ be such that $c > |\sup_{x \in \mathbb{R}} g(x)|$ and decompose $g = c + \tilde{g}$. Then c and $\tilde{g}$ are both invertible and by what precedes, there exist solutions to $Lf_1 = c$ and $Lf_2 = \tilde{g}$ in $\mathcal{P}[x]_{\nu_L(0)}$. Consequently, $f = f_1 + f_2$ is a solution to (1.2) in $\mathcal{P}[x]_{\nu_L(0)}$. $\square$

Corollary 3.3. *Assume that $L_r = 1, L_{r-1}, \ldots, L_0 \in \mathcal{P}$ and $g \in \mathcal{P}[x]_d$. Then (1.2) has at least one solution in $\mathcal{P}[x]_{d+\nu_L(0)}$.*

Proof. We prove the corollary by induction over d. In the case $d = -1$ we have nothing to do. Assume therefore that $d \geqslant 0$. By theorem 3.2, there exists a $\varphi \in \mathcal{P}[x]_{\nu_L(0)}$, with $L\varphi = g_d$. Then

$$L(\varphi x^d) = g_d x^d + d(L'\varphi)x^{d-1} + \cdots + L^{(d)}\varphi.$$

Consequently, $g - L(\varphi x^d) \in \mathcal{P}[x]_{d-1}$. By the induction hypothesis, there exists a $\psi \in \mathcal{P}[x]_{d+\nu_L(0)-1}$, such that $L\psi = g - L(\varphi x^d)$. We conclude that $f = \varphi x^d + \psi$ is an element in $\mathcal{P}[x]_{d+\nu_L(0)}$ with $Lf = g$. $\square$

Let us now show how to privilege a particular solution to (1.2) among the solutions in $\mathcal{P}[x]_{d+\nu_L(0)}$. This solution will be called the "distinguished primitive" to $Lf = g$ and coincides with the distinguished integral if $L = \partial_x$. We first recall that $\mathcal{P}$ is a Hilbert space for the Hermitian form defined by

$$\langle f | g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(x)\overline{g(x)}dx.$$

For each $j \geqslant 0$, let H_j be the vector space of $h_j \in \mathcal{P}$, such that there exists a solution $h \in \mathcal{P}[x]$ to $Lh = 0$ of the form $h = h_j x^j + \cdots + h_0$. For each $f = f_k x^k + \cdots + f_0$, we define $\pi_{L,x^j}(f)$ to be the orthogonal projection of f on H_j. Notice that the operator π_{L,x^j} is linear.

Theorem 3.4. *Assume that $L_r = 1, L_{r-1}, \ldots, L_0$ are in $\mathcal{P}$ and $g \in \mathcal{P}[x]_d$. Then there exists a unique solution f in $\mathcal{P}[x]_{d+\nu_L(0)}$ to (1.2), such that $\pi_{L,x^j}(f) = 0$ for all j. This solution, which is denoted by $L^{-1}g$, is called the* distinguished solution *to $Lf = g$. The mapping $g \mapsto L^{-1}g$ is linear.*

PROOF. Let f be a solution to $Lf = g$ in $\mathcal{P}[x]_{d+\nu_L(0)}$. Let j be maximal such that $\pi_{L,x^j}(f) \neq 0$, if such a j exists, and let $h = h_j x^j + \cdots + h_0$ be a solution to $Lh = 0$ with $\pi_{L,x^j}(f) = h_j$. Then $\tilde{f} = f - h$ is again a solution to (1.2) in $\mathcal{P}[x]_{d+\nu_L(0)}$, but the minimal index $\tilde{\jmath}$ with $\pi_{L,x^{\tilde{\jmath}}}(\tilde{f}) \neq 0$ is strictly smaller than j, if such a $\tilde{\jmath}$ exists. Repeating the procedure, we therefore obtain a solution to (1.2) with $\pi_{L,x^j}(f) = 0$ for all j.

Assume that $\tilde{f}$ is a second solution to (1.2) with $\pi_{L,x^j}(\tilde{f}) = 0$ for all j. If $\tilde{f} \neq f$, then we would be able to write $h = \tilde{f} - f = h_j x^j + \cdots + h_0$, with $h_j \neq 0$ and $0 = \pi_{L,x^j}(\tilde{f} - f) = \pi_{L,x^j}(h_j x^j) = h_j$, which is impossible. Therefore, $\tilde{f} = f$.

Now consider $g_1, g_2 \in \mathcal{P}[x]$ and let $f_1 = L^{-1}g_1, f_2 = L^{-1}g_2$. We have $L(f_1 + f_2) = g_1 + g_2$ and $\pi_{L,x^j}(f_1 + f_2) = \pi_{L,x^j}(f_1) + \pi_{L,x^j}(f_2) = 0$ for all j. Consequently, $L^{-1}(g_1 + g_2) = f_1 + f_2$, i.e. L^{-1} is linear. □

4. Asymptotic linear differential equations

4.1. The Newton polygon method

Consider the linear differential equation (1.1), with coefficients $L_0, \ldots, L_r \in \mathcal{S}$. Each iterated derivative of h may be expressed as h times a differential polynomial $h^{(j)} = R_j(f)h$ in the logarithmic derivative $f = h'/h$ of h. For instance, $R_0(f) = 1, R_1(f) = f, R_2(f) = f^2 + f', R_3 = f^3 + 3f'f + f''$. Hence, solving (1.1) is equivalent to solving the Ricatti equation

$$L_r R_r(f) + \cdots + L_0 R_0(f) = 0,$$

modulo one integration and one exponentiation: $h = e^{\int f}$. We will use the Newton polygon method in order to solve this equation.

For this purpose, we will actually show how to solve the slightly more general, *asymptotic Ricatti equation*

$$R(f) = L_r R_r(f) + \cdots + L_0 R_0(f) = 0 \quad (v_f > \omega), \tag{4.1}$$

with coefficients $L_0, \ldots, L_r \in \mathcal{S}$ and integer $\omega < 0$ or $\omega = -\infty$. We recall that $v_f \in \mathbb{Q} \cup \{\infty\}$ denotes the valuation of f in e^{-x}. Two main types of solutions can be distinguished: those for which $v_f \geqslant 0$ and those for which $v_f < 0$. Actually, the Newton polygon method will be used in order to reduce the resolution of (4.1) to the case when we only need to find the solutions with $v_f \geqslant 0$. In section 4.2, we will show how to solve this special case using the results from section 3.

If $v_f < 0$, then $R_j(f)$ and f^j coincide up to lower order terms for all j, i.e. $v_{R_j(f)-f^j} > v_{R_j(f)}$. Hence, the first term $ce^{-\mu x}$ of a solution to (4.1) with $v_f < 0$ must also be the first term of a solution to the *asymptotic algebraic equation*

$$L_r f^r + \cdots + L_0 = 0 \quad (0 > v_f > \omega). \tag{4.2}$$

The exponent $\mu \in \mathbb{Q}$ of such a first term can be read of from the Newton polygon and the coefficient c is a root of a Newton polynomial (see section 4.3), which is an algebraic equation over $\mathcal{P}$. Furthermore, proposition 2.1 ensures that we may assume without loss of generality that these "potential dominant terms" $ce^{-\mu x}$ of f are in $\mathcal{S}$, modulo a narrowing of x.

Assume that we have determined such a potential dominant term $ce^{-\mu x} \in \mathcal{S}$ of a solution f to 4.1. We then consider the *refinement*

$$f = ce^{-\mu x} + \tilde{f} \quad (v_{\tilde{f}} > \mu), \tag{4.3}$$

i.e. a simultaneous change of functions and the imposition of an asymptotic constraint. Then (4.1) transforms into a new asymptotic Ricatti equation

$$\tilde{L}_r R_r(\tilde{f}) + \cdots + \tilde{L}_0 R_0(\tilde{f}) = 0 \quad (v_{\tilde{f}} > \mu), \tag{4.4}$$

which has again coefficients in $\mathcal{S}$. In section 4.5, we shall see that the recursive application of this method enables us to find r linearly independent solutions to (1.1) in $\mathcal{E}_{r-1}$.

4.2. Distinguished solutions and applications

Assume that $L_r = 1, L_{r-1}, \ldots, L_0 \in \mathcal{S}$ and $g \in \mathcal{S}[x]_d$. Let v_L be the minimum of the valuations of the L_i in e^{-x}. We define the dominant part L^{dom} of L to be the linear differential operator with $L_i^{\mathrm{dom}} = L_{i,v_L}$, where L_{i,v_L} denotes the coefficient of $e^{-v_L x}$ in L_i. We notice that $L^{\mathrm{dom}}_{\times e^{\alpha x}} = (L_{\times e^{\alpha x}})^{\mathrm{dom}}$ and $L^{\mathrm{dom}} = (e^{\alpha x} L)^{\mathrm{dom}}$ for all $\alpha \in \mathbb{Z}$. Given $f \in \mathcal{S}$, $j \in \mathbb{N}$ and $\alpha \in \mathbb{Z}$, we denote $\pi_{L, x^j e^{\alpha x}}(f) = \pi_{(e^{-\alpha x} L_{\times e^{\alpha x}})^{\mathrm{dom}}, x^j}(f_\alpha)$, where $\pi_{(e^{-\alpha x} L_{\times e^{\alpha x}})^{\mathrm{dom}}, x^j}$ is as in section 3.3. We also denote $\nu_L(\alpha) = \nu_{L^{\mathrm{dom}}}(\alpha) = \nu_{e^{-\alpha x} L^{\mathrm{dom}}_{\times e^{\alpha x}}}(0)$ for all α and $\nu_L^+(\alpha) = \sum_{\beta \in \mathbb{N}} \nu_L(\alpha - \beta)$.

Theorem 4.1. *Let $L_0, \ldots, L_r \in \mathcal{S}, g \in \mathcal{S}[x]_d$ and assume that L^{dom} is monic. Then there exists a unique solution f to (1.2) in $\mathcal{S}_{d+\nu_L^+(v_L - v_g)}$, such that $\pi_{L, x^j e^{\alpha x}}(f) = 0$ for all $\alpha \in \mathbb{Z}$ and $j \in \mathbb{N}$. We call f the* distinguished solution *to (1.2) and denote it by $L^{-1} g$. The operator $g \mapsto L^{-1} g$ is linear.*

Proof. Without loss of generality, we may assume that $v_L = 0$, modulo a multiplication of (1.2) by $e^{v_L x}$. We first observe that $v_f \geqslant v_g$. Indeed, otherwise $L^{\mathrm{dom}}_{\times e^{-v_f x}} f_{v_f} = 0$, since $Lf = (L^{\mathrm{dom}}_{\times e^{-v_f x}} f_{v_f} + o(1)) e^{-v_f x}$. Consequently, if $f_{v_f, j} x^j$ is the leading term of f_{v_f}, we would have $0 = \pi_{L, x^j e^{v_f x}}(f) = f_{v_f, j} \neq 0$.

Let us now show how to compute the coefficients $f_{v_g}, f_{v_g+1}, \ldots$ of f by induction. Assume that $f_{v_g}, \ldots, f_{\alpha-1}$ have been constructed and that $\tilde{g} = g - L(f_{v_g} e^{-v_g x} + \cdots + f_{\alpha-1} e^{-(\alpha-1)x})$ is in $\mathcal{P}[x]_{d+\nu_{-v_g}+\cdots+\nu_{1-\alpha}}((e^{-x}))$, with valuation $v_{\tilde{g}} \geqslant \alpha$. By theorem 3.4,

$$f_\alpha = (L^{\mathrm{dom}}_{\times e^{-\alpha x}})^{-1} \tilde{g}_\alpha \tag{4.5}$$

is the only solution to the equation $L^{\mathrm{dom}}_{\times e^{-\alpha x}} f_\alpha = \tilde{g}_\alpha$ in $\mathcal{P}[x]_{d+\nu_{-v_g}+\cdots+\nu_\alpha}$ with $\pi_{L, x^j e^{-\alpha x}}(f_\alpha e^{-\alpha x}) = \pi_{L^{\mathrm{dom}}_{\times e^{-\alpha x}}, x^j}(f_\alpha) = 0$ for all j. By construction, the valuation of

$$g - L(f_{v_g} e^{-v_g x} + \cdots + f_\alpha e^{-\alpha x}) = (\tilde{g} - e^{-\alpha x} L^{\mathrm{dom}}_{\times e^{-\alpha x}} f_\alpha) - (L_{\times e^{-\alpha x}} - e^{-\alpha x} L^{\mathrm{dom}}_{\times e^{-\alpha x}} f_\alpha)$$

is at least $\alpha + 1$.

We conclude that $f_{v_g}, f_{v_g+1}, \ldots \in \mathcal{P}[x]_{d+\nu_L^+(-v_g)}$ are uniquely determined by the conditions that $\pi_{L, x^j e^{-\alpha x}}(f_\alpha e^{-\alpha x}) = 0$ for all j, α and $g - L(f_{v_g} e^{-v_g x} + \cdots + f_\alpha e^{-\alpha x})$ has valuation $> \alpha$ for all α. It follows that $f = f_{v_g} e^{-v_g x} + f_{v_g+1} e^{-(v_g+1)x} + \cdots$ is the unique solution in $\mathcal{P}[x]_{d+\nu_L^+(-v_g)}((e^{-x}))$ to (1.2), such that $\pi_{L, x^j e^{\alpha x}}(f) = 0$ for all $\alpha \in \mathbb{Z}$ and $j \in \mathbb{N}$. Since L^{dom} is monic, the operator $L^{\mathrm{dom}}_{\times e^{-\alpha x}}$ is monic for each α. Consequently, the f_α, which are given by (4.5), are defined on the same common strip neighbourhood of $\mathbb{R}$

as the coefficients of the L_i and g. The operator L^{-1} is linear for the same reason as in the proof of theorem 3.4. □

Corollary 4.2. *Let $L_0, \ldots, L_r \in \mathcal{S}$ be such that L^{dom} is monic and let d_0 be the order of L^{dom}. Then the solutions to (1.1) in $\mathcal{E}_{d_0-1,0}$ form a vector space of dimension d_0.*

Proof. By theorem 3.1, the vector space of solutions to $L^{\mathrm{dom}}\varphi = 0$ in $\mathcal{E}_{d_0-1,0}$ admits a basis $\varphi_1, \ldots, \varphi_{d_0}$ of solutions of the form $\varphi_i \in \mathcal{P}[x]_{\nu_L(-\lambda_i)-1} e^{-\lambda_i x}$. Each φ_i determines a solution

$$h_i = e^{-\lambda_i x}(e^{\lambda_i x} L_{\times e^{-\lambda_i x}})^{-1}(e^{\lambda_i x} L_{\times e^{-\lambda_i x}})(e^{\lambda_i x}\varphi_i)$$

to (1.1) in $\mathcal{P}[x]_{\nu_L(-\lambda_i)-1+\nu_L^+(-\lambda_i-1)} e^{\lambda_i x}\mathcal{S} \subseteq \mathcal{E}_{d_0-1,0}$ with dominant term φ_i.

We claim that the h_i are linearly independent. Assume for contradiction that $h = c_1 h_1 + \cdots + c_{d_0} h_{d_0} = 0$ for certain constants $c_1, \ldots, c_{d_0}$, not all zero. We may reorder the $h_1, \ldots, h_{d_0-1}$, such that $c_1, \ldots, c_k$ are the non zero constants, for which $\Re\lambda_1 = \cdots = \Re\lambda_k$ are minimal. Then the dominant term of h (as a series in e^{-x} whose coefficients are linear combinations of elements in $\mathcal{P}$ times exponentials $e^{-\lambda x}$ with $\Re\lambda = 0$) is $c_1\varphi_1 + \cdots + c_k\varphi_k$, which is non zero; contradiction.

On the other hand, the dominant term φ of a solution to (1.1) in $\mathcal{E}_{d_0-1,0}$ necessarily satisfies $L^{\mathrm{dom}}\varphi = 0$. Consequently, we may rewrite φ as a linear combination of the h_i plus an asymptotically smaller solution to (1.1). Repeating this procedure, we conclude that $h_1, \ldots, h_{d_0}$ forms a basis for the solutions to (1.1) in $\mathcal{E}_{d_0-1,0}$. □

Corollary 4.3. *Let $L_0, \ldots, L_r \in \mathcal{S}$ and let d_0 be the order of L^{dom}. Then there exists a narrowing $\tilde{x}$ of x, such that the solutions to (1.1) in $\mathcal{E}^{\tilde{x}}_{d_0-1,0}$ form a vector space of dimension d_0.*

Proof. Apply the previous corollary to $L/L^{\mathrm{dom}}_{d_0}$, for any narrowing $x = \tilde{x} + \varphi \quad (\varphi \in \mathcal{P}^{\tilde{x}})$, such that $L^{\mathrm{dom}}_{d_0}$ does not vanish for $\tilde{x} \in \mathbb{R}$. □

4.3. Finding the potential dominant terms

In this section, we are interested in finding potential dominant terms $ce^{-\mu x}$ of solutions to (4.1) with $v_f < 0$. We already noticed that such terms coincide with the potential dominant terms of the solutions to (4.2).

We say that μ with $\omega < \mu < 0$ is a *potential dominant exponent* of f, if there exist indices $j < k$ with $v_{L_j} + j\mu = v_{L_k} + k\mu$ and $v_{L_l} + l\mu \geqslant v_{L_j} + j\mu$ for all other indices l. There are only a finite number of such μ, which can be read of graphically from the Newton polygon associated to (4.2); for instance, see (van der Hoeven, 1997).

Given any $\mu < 0$, let j be an index such that $v_{L_l} + l\mu \geqslant v_{L_j} + j\mu = \alpha$ for all other l. Then we call

$$\Lambda(\lambda) = L_{r,\alpha-r\mu}\lambda^r + \cdots + L_{0,\alpha} \tag{4.6}$$

the *Newton polynomial* associated to μ, where $L_{j,\beta}$ denotes the coefficient of $e^{-\beta x}$ in L_j. We call $ce^{-\mu x}$ a *potential dominant term* of f, if c is a non zero root (in the algebraic closure of $\mathcal{P}$) of the Newton polynomial Λ associated to λ. The *multiplicity* of $ce^{-\mu x}$ is the multiplicity of c as a root of Λ.

Clearly, if $ce^{-\mu x}$ is a potential dominant term, then μ must be a potential dominant exponent, since Λ should contain at least two terms in order to admit a non zero root. It is also readily checked that the dominant term $ce^{-\mu x}$ of a solution f to (4.1) with $v_f < 0$ must necessarily be a potential dominant term: otherwise, $R(f)$ would be equal to $\Lambda(c)e^{-\alpha x}$ plus lower order terms.

The *Newton degree* d of (4.1) is the largest index d, such that $v_{L_d} + d\omega \leqslant v_{L_j} + j\omega = \alpha$ for all other indices j. It can be shown that this degree either coincides with the largest possible degree of a Newton polynomial associated to a potential dominant exponent $\mu < 0$, or with the order d_0 of the dominant part of L, if there are no potential dominant exponents.

LEMMA 4.1. *Let d_0 be the order of L^{dom}. Then there are precisely $d - d_0$ potential dominant terms $ce^{-\mu x}$ of f with $\mu < 0$, when counted with multiplicities.*

PROOF. Let $0 > \mu_1 > \cdots > \mu_m$ be the potential dominant exponents of f. Each potential dominant exponent μ_i is determined by two indices $j_i < k_i$, which are the first projections of the extremities of the corresponding edge of the Newton polygon; j_i and k_i are respectively the valuation and the degree of the Newton polynomial associated to μ_i, therefore this polynomial has $k_i - j_i$ non zero roots. But $d_0 = j_1 < k_1 = j_2 < \cdots < k_{m-1} = j_m < k_m = d$, whence, counting with multiplicities, there are $(k_1 - j_1) + \cdots + (k_m - j_m) = d - d_0$ potential dominant terms of f. $\square$

4.4. NARROWINGS AND REFINEMENTS

Assuming that we know the potential dominant terms of solutions to (4.1), we now want to perform a narrowing followed by a refinement in order to find the next terms of the solutions.

LEMMA 4.2.

(a) There exists a narrowing (2.1), such that all potential dominant terms of solutions to (4.1) are in $\mathcal{S}^{\tilde{x}}$.
(b) If there exists a potential dominant term whose multiplicity is equal to the Newton degree d of (4.1), then this narrowing may be chosen with multiplicator $p = 1$.
(c) The Newton degree of the asymptotic Ricatti equation (4.1) rewritten with respect to the new coordinate $\tilde{x}$ is again d.

PROOF. Part (a) results from an iterative application of proposition 2.1 to all Newton polynomials associated to a potential dominant exponent of a solution to (1.1).

Now assume that the Newton polynomial Λ associated to some potential dominant exponent μ has a root of multiplicity d. Then $\Lambda(\lambda)$ is a constant multiple of $(\lambda - c)^d$. In particular, $\Lambda_0 = L_{0,\alpha}$ and $\Lambda_1 = L_{1,\alpha-\mu}$ both do not vanish, so that $\mu \in \mathbb{Z}$. Furthermore, c is actually the root of the polynomial $P^{(d-1)}$ of degree one with coefficients in $\mathcal{P}$. Consequently, c is 2π-periodic and meromorphic on $\mathbb{R}$. Therefore, any narrowing (2.1) with $p = 1$, such that $\Im\gamma$ contains no poles of c, meets our requirements in (b).

As to (c), let $f(x) = \tilde{f}(\tilde{x})$ and $\tilde{L} = L \circ \gamma$. Then (4.1) transforms into an asymptotic Ricatti equation

$$\tilde{L}_r R_r(\tilde{f}) + \cdots + \tilde{L}_0 R_0(\tilde{f}) = 0 \quad (\tilde{v}_{\tilde{f}} > p\omega), \tag{4.7}$$

with coefficients $\tilde{L}_0, \ldots, \tilde{L}_r$ in $\mathcal{S}^{\tilde{x}}$ and where $\tilde{v}_{\tilde{f}}$ denotes the valuation of $\tilde{f}$ in $e^{-\tilde{x}}$. Since each $f^{(j)}(x)$ is a $\mathcal{P}^{\tilde{x}}$-linear combination of $\tilde{f}'(\tilde{x}), \ldots, \tilde{f}^{(j)}(\tilde{x})$, each $\tilde{L}_j(\tilde{x})$ is a $\mathcal{P}^{\tilde{x}}$-linear combination of $L_j(\gamma(\tilde{x})), \ldots, L_r(\gamma(\tilde{x}))$. Notice that $\tilde{v}_{L_j(\gamma(\tilde{x}))} = pv_{L_j(x)}$ for each j. By the definition of the Newton degree, $\alpha = v_{L_d} + d\omega$ is such that $v_{L_j} + j\omega \geqslant \alpha$ for $j \geqslant d$ and $v_{L_j} + j\omega > \alpha$ for $j < d$. Consequently, $\tilde{v}_{\tilde{L}_j} \geqslant p\min(\alpha - j\omega, \ldots, \alpha - r\omega) = p\alpha - p\omega j$ for $j \geqslant d$ and similarly $\tilde{v}_{\tilde{L}_j} > p\alpha - p\omega j$ for $j < d$. Furthermore, $\tilde{v}_{\tilde{L}_d(\tilde{x}) - L_d(\gamma(\tilde{x}))} \geqslant p\alpha - p\omega(d+1)$, whence $\tilde{v}_{\tilde{L}_d} = pv_{L_d} = p\alpha - p\omega d$. Therefore, the Newton degree of (4.7) is d. □

Lemma (4.2) ensures us that modulo a narrowing, and without altering the Newton degree of (4.1), we may assume without loss of generality that all potential dominant terms of solutions to (4.1) are in $\mathcal{S}$.

Given such a potential dominant monomial $ce^{-\mu x}$, the change of variables $f = ce^{-\mu x} + \tilde{f}$ in the refinement (4.3) corresponds to the change of variables $h = e^{\int ce^{-\mu x}} \tilde{h}$ in the linear differential equation (1.1). Consequently, $\tilde{h}$ also satisfies a linear differential equation, whence (4.4) is again an asymptotic Ricatti equation; actually, $\tilde{L} = e^{-\int ce^{-\mu x}} L_{\times \int ce^{-\mu x}}$.

Furthermore, each solution $\tilde{h}$ to $\tilde{L}\tilde{h} = 0$ in $\mathcal{E}_{r-1}$, whose logarithmic derivative $\tilde{f} = \tilde{h}'/\tilde{h}$ satisfies (4.4), induces a solution $h = e^{\int ce^{-\mu x}} \tilde{h}$ to (1.1) in $\mathcal{E}_{r-1}$, whose logarithmic derivative $f = h'/h$ satisfies (4.1). Indeed, we may take $\int ce^{-\mu x} = e^{-\mu x}(e^{\mu x}\partial_{\times e^{-\mu x}})^{-1} c \in \mathcal{P}e^{-\mu x}$.

Lemma 4.3. *Assume that $ce^{-\mu x}$ is a potential dominant term to a solution of (4.1) in $\mathcal{S}$. Then the Newton degree of (4.4) is equal to the multiplicity of c as a root of the Newton polynomial Λ associated to μ.*

Proof. Let d denote the Newton degree of (4.1) and let $\alpha = v_{L_d} + d\mu$. We notice that $v_{L_j} + j\mu \leqslant \alpha$ for all indices j, by the definition of the Newton degree d. Now using the fact that $R_k(f)$ and f^k coincide up to lower order terms for all j, we may express $\tilde{L}_j$ in terms of the L_k by

$$\begin{aligned} \tilde{L}_j &= \sum_{k=j}^{n} \binom{k}{j} (L_{k,\alpha - k\mu} + o(1)) e^{-(\alpha - k\mu)x} (ce^{-\mu x})^{k-j} \\ &= (\Lambda^{(j)}(c) + o(1)) e^{-(\alpha + j\mu)x}. \end{aligned}$$

Denoting by $\tilde{d}$ the multiplicity of c as a root of Λ, we have in particular $v_{\tilde{L}_{\tilde{d}}} = \alpha + \tilde{d}\mu$, $v_{\tilde{L}_j} \geqslant \alpha + j\mu$ for $j \geqslant \tilde{d}$ and $v_{\tilde{L}_j} > \alpha + j\mu$ for $j < \tilde{d}$. In other words, the Newton degree of (4.4) equals $\tilde{d}$. □

4.5. Solving the homogeneous equation

Theorem 4.4. *Assume that we are given an asymptotic Ricatti equation (4.1) of Newton degree d. Then there exists a narrowing $\tilde{x}$ of x, such that (1.1) admits d linearly independent solutions in $\mathcal{E}^{\tilde{x}}_{d-1}$, whose logarithmic derivatives are solutions to (4.1).*

Proof. We prove the theorem by a double induction over r and $-\omega$. Clearly, the theorem holds for $r = 0$ and for $\omega = 0$. Assume therefore that $r > 0$, $-\omega > 0$ and that we have proved the theorem for all smaller r and all smaller $-\omega$ with the same r. By lemma 4.2(a), there exists a narrowing x_0 of x, such that the potential dominant terms $c_1 e^{-\mu_1 x_0}, \ldots, c_m e^{-\mu_m x_0}$ of solutions to (4.1) are in $\mathcal{S}^{x_0}$. Moreover, by lemma 4.2(b) if there exists only one such potential dominant term of multiplicity d, then we may assume the multiplicator of this narrowing to be 1.

Now consider the refinement $f = c_1 e^{-\mu_1 x} + \tilde{f}_1 \quad (v_{\tilde{f}_1} > p\mu_1)$. The Newton degree of the resulting asymptotic Ricatti equation in $\tilde{f}_1$ is d_1, by lemmas 4.2(c) and 4.3. We have either $d_1 < d$, or $p = 1$ and $-p\mu_1 < -\omega$. In both cases, the induction hypothesis implies that there exists a narrowing x_1 of x_0, such that there exist d_1 linearly independent solutions $h_{1,1}, \ldots, h_{1,d_1}$ to (1.1) in $\mathcal{E}_{d_1-1}^{x_1}$, which correspond to solutions to the asymptotic Ricatti equation in $\tilde{f}_1$.

Similarly, for i running from 2 to m, assume that we are given a narrowing x_{i-1} of x_1 and consider the refinement $f = c_i e^{-\mu_i x} + \tilde{f}_i \quad (v_{\tilde{f}_i} > p\mu_i)$. The Newton degree of the resulting asymptotic Ricatti equation in $\tilde{f}_i$ is $d_i < d$. Hence, by the induction hypothesis, there exists a narrowing x_i of x_{i-1}, relative to which there exist d_i linearly independent solutions $h_{i,1}, \ldots, h_{i,d_i}$ to (1.1) in $\mathcal{E}_{d_i-1}^{x_i}$, which correspond to solutions to the asymptotic Ricatti equation in $\tilde{f}_i$.

Finally, by the second corollary of theorem 4.1, there exists a narrowing $\tilde{x}$ of x_m, such that (1.1) admits d_0 linearly independent solutions $h_{0,1}, \ldots, h_{0,d_0}$ in $\mathcal{E}_{d_0-1,0}^{\tilde{x}}$, whose logarithmic derivatives are solutions to (4.1). By construction, solutions $h_{i,j}$ and $h_{i',j'}$ necessarily belong to different direct summands of $\mathcal{E}_{d-1}^{\tilde{x}}$ for $i' \neq i$. Hence the $h_{i,j}$ are linearly independent. By lemma 4.1, we have $d = d_0 + \cdots + d_m$, which concludes the proof of the theorem. □

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