Creative telescoping using reductions

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Creative telescoping is a popular method for proving combinatorial identities and the computation of parametric integrals that involve special functions. Traditional implementations of this method admit an exponential bit complexity and it is an open problem under which conditions creative telescoping can be achieved in polynomial time. More efficient *reduction-based* algorithms were recently introduced in order to get a better grip on such complexity issues. Initially, reduction-based algorithms only applied to special cases such as rational, algebraic, or hyperexponential functions. More recently, constructions of reductions appeared for larger classes of Fuchsian D-finite and general differentially-finite functions.

In this paper, we show how to construct reductions for mixed differential-difference systems, where the difference operators are either shift operators or q-difference operators. We recall how this yields an algorithm for creative telescoping and specify under which precise conditions on singularities this algorithm works. For creative telescoping of differential type, we next examine the complexity of our algorithms and prove a polynomial complexity bound. The algorithm for which this bound holds computes generators for a D-finite ideal of telescopers, but not necessarily a Gröbner basis for the ideal of all telescopers.

KEYWORDS: creative telescoping, holonomic function, Hermite reduction, residues

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1. INTRODUCTION

1.1. Creative telescoping

The technique of creative telescoping is a powerful tool for proving and finding combinatorial identities and the computation of parametric integrals that involve special functions. The name was coined by van der Poorten [77] in relation with Apéry's irrationality proof of $\zeta(3)$, but the first systematic algorithms were only developed about one decade later by Zeilberger and followers [90, 4, 91, 62, 85, 86]. Early precursors are [35, 89, 42].

Let us briefly recall the main idea behind creative telescoping on an extremely simple example: assume that we wish to prove the well known identity

$$F(n) = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Setting $f(n,k) = \binom{n}{k}$ and $g(n,k) = -\binom{n}{k-1}$, the defining relation of f in Pascal's triangle yields

$$f(n+1,k) - 2f(n,k) = g(n,k+1) - g(n,k).$$

Summing this relation from $k=0$ until $n+1$, the right hand side becomes a telescoping sum, and we obtain $F(n+1) - 2F(n) = g(n,n+1) - g(n,0) = 0$. Since $F(0) = 1$, the recurrence relation $F(n+1) = 2F(n)$ proves $F(n) = 2^n$.

More generally, for the evaluation of a sum of the form $F(n)=\sum_{k=a(n)}^{b(n)} f(n,k)$, the idea is to construct a relation of the form

$$c_s(n)\, f(n+s,k)+\cdots+c_0(n)\, f(n,k) \;=\; g(n,k+1)-g(n,k), \tag{1.1}$$

where the left-hand side does not depend on k and the right-hand telescopes when summing over k. If we manage to construct such a relation, then the creative telescoping process provides us with a non-trivial linear recurrence relation for $F(n)$. Writing $S_{1,n}$ for the operator that sends $\varphi(n,k)$ to $\varphi(n+1,k)$, the difference operator $c_s(n)\, S_{1,n}^s+\cdots+c_0(n)$ that annihilates f in (1.1) is called a *telescoper*, and the function g its corresponding *certificate*.

A similar idea works for integrals of the form $F(x)=\int_{a(y)}^{b(y)} f(x,y)\,\mathrm{d}y$ instead of discrete sums. In that case, we need a relation of the form

$$c_s(x)\,\frac{\partial^s f}{\partial x^n}(x,y)+\cdots+c_0(x)\, f(x,y) \;=\; \frac{\partial g}{\partial y}(x,y) \tag{1.2}$$

instead of (1.1). Again, $c_s(x)\,\partial^s/\partial x^s+\cdots+c_0(x)$ is called the telescoper and g the certificate. Several other variants can be considered as well, such as q-difference equations, multiple sums and integrals, etc.

After Zeilberger's initial impetus, the development of creative telescoping focussed on making implementations more efficient, while guaranteeing termination and widening the class of functions on which the theory can be applied. Whereas early implementations were particularly successful for first or low order recurrence relations (e.g. hypergeometric summation), efficient algorithms soon appeared for more general holonomic functions and D-finite ideals in Ore algebras [26, 29, 27, 63]. For a historical perspective on these advances and further references, we refer to [28].

One major application of creative telescoping is proving functional identities. In the purely differential setting, it should be noticed that alternative methods have been developed for larger classes of non-linear equations [32, 33, 83, 75, 46, 49].

1.2. From Hermite reduction to reduction-based telescoping

Despite the above successes, the efficiency of creative telescoping remained a major issue. Part of the problem was due to the fact that the construction of relations of the form (1.1) or (1.2) often relied on brute force computations rather than a better insight into their precise shape.

The next major step was the introduction of reduction-based algorithms as a remedy to these problems. The concept of a *reduction* goes back to the works of Ostrogradsky and Hermite on the integration of rational functions [74, 44]. Hermite reduction is nowadays a fundamental tool in symbolic integration and generalizations for algebraic, hyperexponential and hypergeometric functions were introduced with this kind of application in mind [88, 30, 2]. It is interesting to notice that the importance of Hermite reduction for integrals that depend on parameters was understood quite early [76]; see also [39].

Hermite reduction was first related to creative telescoping in [41]; the complexity perspective appeared in [13]. Initially, only quite restricted types of functions could be treated this way and it became a major challenge to develop reduction-based algorithms for similarly general classes of functions as for the other approaches. This triggered a lot of activity in recent years [24, 14, 67, 22, 34, 23, 54, 68], culminating with an algorithm for Fuchsian differential equations [21]. This last result made it plausible that a fully general algorithm in the differential case might be quite complex, since it might involve elaborate arguments in order to deal with irregular singularities. Fortunately this concern turned out to be unwarranted: last year we constructed suitable reductions for arbitrary differential equations in a purely algebraic way [50]. Our technique has recently been further improved in [16]. In the present paper, we continue its development and extend it to mixed differential-difference equations.

Before we proceed, it is useful to recall how reductions show up in relation with creative telescoping. Assume that we wish to find a linear differential equation over $\mathbb{Q}(x)$ for the function $F(y)=\int_{a(y)}^{b(y)} f(x,y)\,\mathrm{d}y$, where $f\in\mathbb{Q}(x,y)$. Hermite reduction in y provides us with a unique decomposition of any $g\in\mathbb{Q}(x,y)$ as a sum

$$g = [g]+\frac{\partial h}{\partial y}, \tag{1.3}$$

where $[g]$ admits only simple poles in y and $h\in\mathbb{Q}(x,y)$. When restricting ourselves to functions g whose poles in y belong to a fixed finite set (i.e. $g\in\mathbb{Q}(x)[y,\psi(x,y)^{-1}]$ for some fixed square-free polynomial $\psi\in\mathbb{Q}[x,y]$), it follows that $[g]$ is "confined" to a finite dimensional $\mathbb{Q}(x)$-vector space $\mathbb{I}=[\mathbb{Q}(x)[y,\psi(x,y)^{-1}]]$. Indeed, for some algebraic extension $\mathbb{L}$ of $\mathbb{Q}(x)$ that contains all roots of ψ, the reduction $[g]$ is always an $\mathbb{L}$-linear combination of the fractions $1/(y-\alpha)$, where α runs over the set of roots of ψ in $\mathbb{L}$. In our case, we simply take ψ to be the square-free part of the denominator of f.

Now the crucial observation is that $[f]$, $[\partial f/\partial x]$, $[\partial^2 f/\partial x^2]$, etc. all belong to the above finite dimensional vector space, whence there exists a non-trivial relation

$$c_s(x)\left[\frac{\partial^s f}{\partial x^s}\right]+\cdots+c_0(x)\,[f] = 0 \tag{1.4}$$

with $c_0,\ldots,c_s\in\mathbb{Q}(x)$. By construction, there exists an $h_i\in\mathbb{Q}(x,y)$ with $\partial^i f/\partial x^i=[\partial^i f/\partial x^i]+\partial h_i/\partial y$ for $i=0,\ldots,s$. It is not hard to check that the reduction $[\,]$ commutes with partial derivation with respect to y, whence (1.4) implies

$$c_s(x)\frac{\partial^s f}{\partial x^s}+\cdots+c_0(x)\,f = \frac{\partial}{\partial y}(c_s(x)\,h_s+\cdots+c_0(x)\,h_0). \tag{1.5}$$

This new relation is of the desired form (1.2). Another advantage of this method with respect to previous ones is that the computation of certificates is possible, but only optional. Since certificates tend to be more voluminous than the telescopers themselves, this is very interesting from a complexity point of view.

In order to generalize the above reduction-based approach to functions f that satisfy higher order differential equations, one essentially needs to define a confined reduction on a so called *narrow* $\mathbb{K}(x)[y,\partial/\partial x,\partial/\partial y]$-submodule $\mathbb{D}$ of $\mathbb{K}(x,y)$ that contains f. In the differential case, such narrow modules are still of the same form $\mathbb{D}=\mathbb{Q}(x)[y,\psi(x,y)^{-1}]$. In the difference case, the set of allowed poles are translates of a finite number of points and their orders need to be bounded in a way that will be made precise.

1.3. On the choice of our setup

The theory of creative telescoping has become very wide, with many variants that cover specific cases more or less efficiently. For the exposition in this paper, we therefore had to make several choices concerning the setup. Let us briefly comment on these choices.

Non-commutative operator algebras. It was recognized by Zeilberger that the framework of holonomic functions is particularly suitable for the development of creative telescoping [90]. The reason is that there exists a systematic elimination theory for the equations satisfied by such functions that can be regarded as a non-commutative counterpart of the theory of Gröbner bases.

Elimination methods for differential equations actually predate Buchberger's famous algorithm [19]. One line of development is due to Riquier–Janet–Thomas [79, 56, 87] and another one to Ritt–Kolchin [80, 61]. The most “effective” variant of the latter theory [81, 17] essentially covers both commutative Gröbner basis theory, its skew differential counterpart for Weyl algebras, and non-linear generalizations that specialize to triangular system type solving in the algebraic case. Yet another line of development is due to Ore, who introduced a systematic theory for non-commutative polynomials [72, 73]. A more direct inspiration for Zeilberger's work were developments in the theory of D-modules in the seventies [9, 10, 12] and the development of non-commutative Gröbner basis theory in the computer algebra community in the eighties and later [40, 84, 60, 71, 64, 26, 70].

In this paper, we have chosen to work over differential-difference algebras generated by a finite number of coordinates $u_1, \ldots, u_n$, derivations $\partial / \partial u_i$, shift operators S_{η,u_i}, and q-difference operators Q_{q,u_i}. All derivations $\partial / \partial u_i$ and automorphisms S_{η,u_i}, Q_{q,u_i} are assumed to commute pairwise. Our setting is not the most general possible, but it has a natural geometric interpretation, as well as a non-linear counterpart [45]. The commutation of the derivations is not really essential, except for the main derivation that will be eliminated. It should also be quite straightforward to replace the algebra $\mathbb{A}$ in section 6.4 by more general algebras of solvable type [60, 64], Ore algebras [26], or G-algebras [70, 43]. On the other hand, when telescoping with respect to a difference operator it is essential that singularities are moved in a bijective way. It is therefore natural to restrict one's attention to homographies. Then, modulo a change of variables, homographies always operate like shift operators or q-difference operators: see section 5.1. Other interesting generalizations would concern nested extensions by solutions of linear differential-difference equations as in [78, 82].

Scalar equations versus first order systems. An eternal dilemma when dealing with linear differential or difference equations is whether one should privilege scalar higher order equations or first order systems. Thanks to the cyclic vector lemma [3], both approaches are essentially equivalent, so the choice is mainly a matter of taste. The best solution is probably to work with the representation that was used for the input, whenever possible. Our personal taste is slightly in favour of scalar equations, although we opted for first order systems in [50]. In [16], the authors preferred to work with scalar equations instead. In the present paper, we mainly consider scalar equations, but also outline how to adapt the theory to first order systems. At the end of the day, it seems that both approaches are more or less equally diligent.

Algebraic extensions versus rational counterparts. From the early days of symbolic computation, algorithms that require computations in algebraic extensions have acquired the reputation of being slow. Starting with Trager's work on the integration of rational functions [88], it has therefore become common practice to develop “rational” counterparts for reduction algorithms that avoid explicit computations in algebraic extensions. Nevertheless, from a complexity point of view, it is not so clear that working in algebraic extensions is necessarily that bad: computing in an extension of degree d is roughly d times more expensive, but typically deals with d conjugate roots in a single pass. Although the development of rational counterparts remains an interesting topic, we therefore do not regard such algorithms as intrinsically better. In this paper, we have chosen to systematically work in algebraic extensions whenever appropriate, which is somewhat simpler from a conceptual standpoint.

Characteristic zero. It will be convenient to assume that all fields considered in this paper are of characteristic zero, although most results can be generalized in a straightforward way to fields of (sufficiently large) positive characteristic.

1.4. Structure of the paper and outline of the main results

The main purpose of this paper is to present reduction-based algorithms for creative telescoping in the above setting of differential-shift-q-difference equations and to derive complexity bounds for several of these algorithms. In the purely differential case, our algorithms are completely systematic. In the presence of difference operators it is well known that indefinite summation and integration under the integral sign do not always preserve D-finiteness. Nevertheless, we provide a sufficient condition (explicit telescopability) for our algorithms to work, as well as a reduction-based algorithm that is able to check D-finiteness in a particular case. Let us briefly outline the structure of the paper and summarize our contributions.

Abstract reductions. For a gentle introduction to reduction-based creative telescoping, we refer to [34, Section 1.2]. In section 2, we start with a more abstract presentation of the basic theory that covers all cases that occurred in the literature so far. We also introduce the notion of "local reductions" and describe various related constructions on an abstract level.

The Lagrange identity. Let $\mathbb{K}$ be a field. One major new idea in [16] is the consideration of reductions with respect to general differential operators $K \in \mathbb{K}[x, \partial / \partial x]$ instead of just the derivation as in (1.3). More precisely, a reduction with respect to K is a $\mathbb{K}$-linear projection $[]: \mathbb{K}(x) \to \mathbb{K}(x)$ such that $f - [f] \in \operatorname{im} K$ for all $f \in \mathbb{K}(x)$. Now in the case of a rational function f, ordinary Hermite reduction provided us with a way to derive telescoping relations (1.5) from linear dependencies (1.4). In the case of a function f that satisfies a linear differential equation $L(f) = 0$, it is shown how to do something similar using reductions with respect to the adjoint operator L^* of L. The technical tool that makes this possible is the Lagrange identity [55, section 5.3]. We recall this formula in section 3 and generalize it to difference and matrix operators.

Differential reductions. In section 4, we show how to construct reductions for differential equations. This case was already covered in [50], but it is instructive to present it with the formalisms from sections 2 and 3 at hand. In sections 4.2 and 4.3, we start with a variant of the construction from [16], whereas section 4.5 explains the link with [50] and also clarifies how the Lagrange identity can be used to simplify the head/tail chopping process. We also show how new-style reductions with respect to the adjoint operator L^* give rise to old-style reductions in the sense of [50] and *vice versa*. In section 4.6, we finally consider mixtures of both reduction styles. This makes it possible to extend creative telescoping to the resolution of more general linear differential or difference equations that involve parameters: see Remark 6.7.

Difference reductions. The analogue construction of reductions for difference operators is the subject of section 5. Whereas derivations do not introduce new singularities, but only aggravate the order of poles, difference operators move singularities, but do not increase their severity. The idea behind the construction of differential reductions is to diminish the order of the poles of the function to be reduced. Likewise, difference reductions are based on the idea to shift singularities back until they are confined in a finite set. Another difference with section 4 is that the construction is somewhat less canonical due to the absence of a privileged section of orbits of singularities (see Remark 5.7), but this is of no consequence for the application to creative telescoping. Apart from that, the theory from section 4 naturally adapts to the difference setting.

Creative telescoping. After the formal introduction of the setting of DD-operator algebras in section 6.1, we next consider the application to creative telescoping. For suitable "telescopable" D-finite ideals I of a DD-operator algebra $\mathbb{B}$, the theory from sections 4 and 5 allows us to define a computable confined reduction on a narrow submodule $\mathbb{D}$ of $\mathbb{B}/I$. In section 6.4 we show how this can be used to construct telescopers. The algorithm is a straightforward generalization of the technique from section 1.2 and only relies on linear algebra. As an end-result, it produces a finite set of generators for a D-finite ideal of telescopers; in the favourable case when the reduction is "normal", these generators actually form a Gröbner basis of the ideal of all telescopers. In the context of Gröbner basis, the algorithm is reminiscent of the FGML algorithm [37] and a variant of the one used in [16].

D-finiteness tests for telescoping ideals. In the purely differential case, it is a well-known fact from the theory of D-modules (reproved in section 7) that all D-finite ideals are telescopable, so the algorithm from section 6.4 systematically works. In the presence of difference operators, the telescopability of a D-finite ideals depends on the nature of its singularities. In section 7, we examine such singularities in more detail and derive sufficient conditions for the telescopability of a D-finite ideal with a given basis for the quotient module (a particular was treated before in [16, section 4.4]). Sometimes these "explicit telescopability" conditions are only met after a change of basis. For differential reductions, we present an algorithm to compute such a base change if it exists. Even when the explicit telescopability conditions do not hold, it sometimes still happens that the telescoping ideal is D-finite; in the last subsection 7.4, we present an algorithm that decides whether this is the case and that computes a Gröbner basis of the telescoping ideal if it is D-finite.

Complexity of rational function arithmetic. The last two sections are devoted to analyzing the complexity of reduction-based algorithms for creative telescoping. For this, we first recall basic complexity results for computations with rational functions in several variables, including partial fraction decomposition. When using the common dense representation, these complexity bounds are quite pessimistic, yet always polynomial if the number of variables is fixed. When allowing for probabilistic algorithms of Las Vegas type, it is more efficient to represent rational functions by the straight line programs (SLPs) [20], and fast dense univariate polynomial arithmetic can be used at every evaluation point.

Complexity of creative telescoping. In the last section we finally turn to the cost of creative telescoping itself. For this, we use a separate analysis for the complexity of the reduction process and the complexity of the FGLM-type algorithm for computing linear relations. The latter algorithm relies on standard linear algebra and its complexity analysis is straightforward in terms of the dimension of the vector space $\mathbb{I} = [\mathbb{D}]$ of reduced functions. We next turn to the reduction process and restrict our attention to differential reductions. In this case, we prove a polynomial complexity bound, as well as a polynomial bound for $\dim \mathbb{I}$ as a function of the input size. Altogether, this yields a polynomial time algorithm for creative telescoping, as promised in [50]. The complexity exponent is again extremely bad when using dense representations, but essentially cubic per evaluation point in the SLP model.

Remark 1.1. It is well known that it is easy to construct operators in $\mathbb{Q}[z, \partial/\partial z]$ of small bit size that admit polynomial solutions with a large number of terms. For instance, the operator $(\partial/\partial z + (z-1)^{-1})\,(z\,(\partial/\partial z + (z-1)^{-1}) - n)$ admits $(z^n - 1)/(z-1)$ as a solution. For the design of polynomial time algorithms for operators in $\mathbb{Q}[z, \partial/\partial z]$ it is therefore crucial to avoid the explicit computation of such solutions. For a similar reason, telescopers of minimal order (e.g. of the form $c_s \partial^s/\partial x^s + \cdots + c_0$ in (1.5) with minimal s) can admit very large coefficients. The telescopers computed by our polynomial time algorithm in section 9 are therefore not necessarily minimal. The above phenomenon also implies that the algorithms at the end of section 7 (for base changes to reach an explicitly telescopable form and creative telescoping for not explicitly telescopable ideals) do not run in polynomial time, in general.

1.5. Pending issues

Let us briefly list a few problems that have not been tackled in the present draft by lack of time. We intend to address them in a future version or in a follow-up paper.

Base changes for explicit telescopability. The base change algorithm to remove apparent singularities from the end of section 7.3 has been presented in the differential case only. We are still looking for existing work on this topic in the literature and a similar algorithm for the difference case.

Testing D-finiteness. The general algorithm for creative telescoping from section 7.4 (in absence of explicit telescopability) has also been presented in the differential case only. The difference analogue seems to require a bit more work, but it should be feasible to work out an algorithm along similar lines.

Fast arithmetic for rational functions. Several complexity bounds from section 8 are not very sharp or a bit sketchy. We intend to improve this section in the next version.

Complexity bounds for the difference case. The final complexity bounds in sections 9.2 and 9.3 have only been presented in the differential case (which still allows for difference operators among the θ_i). If p and q in Proposition 7.7 do not grow too fast, then it should be possible to prove polynomial complexity bounds in the difference case along similar lines (this in particular covers the case when $\theta_i = \partial_i$ for $i = 1, \ldots, n$). If p or q becomes large, then the defining equations of I tend to become large as well due to Proposition 7.2. The hope is that a polynomial complexity bound can somehow be derived from this.

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2. ABSTRACT REDUCTIONS

2.1. Definitions and basic properties

Let $\mathbb{K}$ be a field and consider a linear map $\theta: V \to V$ on some $\mathbb{K}$-vector space V. A *reduction* with respect to θ is a linear map $[\,]: V \to V$ that satisfies:

R1. $x - [x] \in \operatorname{im} \theta$ for all $x \in V$;

R2. $[[x]] = [x]$ for all $x \in V$.

Such a reduction is said to be *confined* if

R3. $\dim_{\mathbb{K}} \operatorname{im} [\,]$ is finite;

and *normal* if

R4. $[\operatorname{im} \theta] = 0$.

The condition **R2** stipulates that $[\,]$ is a projection. It will be convenient to also introduce the complementary projection $\{\}: V \to V$ involved in **R1**, called the *remainder*:

$$\{x\} \;=\; x - [x].$$

A normal reduction $[\,]: V \to V$ with respect to θ can be regarded as a projection of V onto a subvector space $C = \operatorname{im} [\,]$ of V that is isomorphic to the cokernel of θ.

PROPOSITION 2.1. *If $\xi: V \to V$ is a bijective $\mathbb{K}$-linear map, then a (normal, confined) reduction $[\,]$ with respect to θ is also a (normal, confined) reduction with respect to $\theta \circ \xi$ and vice versa.*

Proof. This follows from the observation that **R1** and **R4** only involve $\operatorname{im} \theta = \operatorname{im} \theta \circ \xi$. □

Remark 2.2. Sometimes, the linear map $\theta: W \to W$ is only defined on a $\mathbb{K}$-vector space W that contains V. Our definitions easily extend to this case module the replacement of $\operatorname{im} \theta$ by $\operatorname{im} \theta \cap V$ in the conditions **R1** and **R4**.

PROPOSITION 2.3. *Let $\mathbb{L}$ be an algebraic extension of $\mathbb{K}$ and consider a reduction $[\,]: \mathbb{L} \otimes V \to \mathbb{L} \otimes V$ with respect to the natural lift $\bar{\theta}: \mathbb{L} \otimes V \to \mathbb{L} \otimes V$ of θ. If $[\sigma(x)] = \sigma([x])$ for all automorphisms $\sigma: \mathbb{L} \to \mathbb{L}$ over $\mathbb{K}$ and $x \in \mathbb{L} \otimes V$, then $[x] \in V$ for all $x \in V$.*

Proof. For any $x \in V$, we have $\sigma([x]) = [\sigma(x)] = [x]$ for all automorphisms σ of $\mathbb{L}$, which implies $[x] \in V$. □

2.2. Local reductions

Let $\pi: V \to V$ be a projection and $\bar{\pi} = \mathrm{Id}_V - \pi$. Given a reduction $[]: V \to V$ with respect to a linear map $\theta: V \to V$, we say that the reduction is *local* for the projection $\pi: V \to V$ if

$$[x] \ = \ [\pi(x)] + \bar{\pi}(x) \tag{2.1}$$

for all $x \in V$.

PROPOSITION 2.4. *If $[]: V \to V$ is a local reduction with respect to θ for π, then $\pi \circ []: \pi(V) \to \pi(V)$ is a reduction with respect to $\pi \circ \theta$.*

Proof. Given $x \in \pi(V)$, there exists a $y \in V$ with $x - [x] = \theta(y)$, whence $x - \pi([x]) = \pi(x - [x]) = \pi(\theta(y))$. Since $\pi([x]) = [x] - \bar{\pi}([x])$, we also have $[\pi([x])] = [x] - [\bar{\pi}([x])] = [x] - \bar{\pi}([x]) = [x] - ([x] - \pi([x])) = \pi([x])$. We conclude that $\pi([\pi([x])]) = \pi(\pi([x])) = \pi([x])$. □

PROPOSITION 2.5. *If $[]: \pi(V) \to \pi(V)$ is a reduction with respect to $\pi \circ \theta$ and $\xi: \pi(\mathrm{im}\, \theta) \to \mathrm{im}\, \theta$ a linear mapping with $\pi \circ \xi = \mathrm{Id}_{\pi(\mathrm{V im}\theta)}$, then we define a local reduction $[]^*: V \to V$ with respect to θ for π by*

$$[x]^* \ = \ x - \xi(\{\pi(x)\}).$$

Proof. Given $x \in V$, there exists a $y \in V$ with $\xi(\{\pi(x)\}) = y$ and $x - [x]^* = y$. Furthermore, $\pi([x]^*) = \pi(x) - \pi(\xi(\{\pi(x)\})) = \pi(x) - \{\pi(x)\} = [\pi(x)]$, whence $\{\pi([x]^*)\} = \pi([x]^*) - [\pi([x]^*)] = [\pi(x)] - [[\pi(x)]] = 0$ and $[[x]^*]^* = [x]^* - \xi(0) = [x]^*$. We finally have $[\pi(x)]^* + \bar{\pi}(x) = \pi(x) - \xi(\{\pi(\pi(x))\}) + \bar{\pi}(x) = x - \xi(\{\pi(x)\}) = [x]^*$. □

2.3. Gluing local reductions together

Two projections $\pi_1, \pi_2: V \to V$ are said to be *orthogonal* if $\pi_1 \circ \pi_2 = \pi_2 \circ \pi_1 = 0$. A family $(\pi_i)_{i \in I}$ of projections $\pi_i: V \to V$ is said to be *orthogonal* if its members are pairwise orthogonal. Such a family is said to be *summable* if $\{i \in I: \pi_i(x) \neq 0\}$ is finite for all $x \in V$. In that case, the sum $\pi = \sum_{i \in I} \pi_i$ with $\pi(x) = \sum_{i \in I} \pi_i(x)$ is well defined for all $x \in V$. Moreover, $\bar{\pi} = \mathrm{Id}_V - \pi$ is a projection that is orthogonal to each π_i and any $x \in V$ can be decomposed canonically as

$$x \ = \ \bar{\pi}(x) + \sum_{i \in I} \pi_i(x).$$

Two local reductions $[]_1, []_2$ for π_1, π_2 are said to be *independent* if π_1 and π_2 are orthogonal and for all $x \in V$, we have

$$\pi_1(\{x\}_2) = \pi_2(\{x\}_1) = 0. \tag{2.2}$$

Such reductions necessarily commute: we have $[[x]_1]_2 = [x - \{x\}_1]_2 = [x]_2 - [\{x\}_1]_2 = [x_2] - \{x\}_1 = x - \{x\}_1 - \{x\}_2$, and $[[x]_2]_1 = x - \{x\}_1 - \{x\}_2$ in a similar way. A family $([]_i)_{i \in I}$ of reductions for projections $(\pi_i)_{i \in I}$ is said to be *independent* if the $[]_i$ are pairwise independent and if $(\pi_i)_{i \in I}$ is summable.

PROPOSITION 2.6. *Let $([]_i)_{i \in I}$ be a family of independent local reductions $[]_i: V \to V$ with respect to θ and for the projections $\pi_i: V \to V$. Let $\pi = \sum_{i \in I} \pi_i$ and $\bar{\pi} = 1 - \pi$. Then we may define a reduction $[]: V \to V$ with respect to θ by*

$$[x] \ = \ \bar{\pi}(x) + \sum_{i \in I} [\pi_i(x)]_i. \tag{2.3}$$

This reduction is local for π. For all $x \in V$, we have

$$[x] \ = \ x - \sum_{i \in I} \{x\}_i \tag{2.4}$$

$$[x] \ = \ [\cdots[[x]_{i_1}]_{i_2} \cdots]_{i_k}, \tag{2.5}$$

where $i_1, \ldots, i_k \in I$ are the indices i with $\pi_i(x) \neq 0$.

Proof. For all $i \in I$ and $x \in V$, we get from (2.1) that $[x - \pi_i(x)]_i = x - \pi_i(x)$, whence $\{\pi_i(x)\}_i = \{x\}_i - \{x - \pi_i(x)\}_i = \{x\}_i$. It follows that $[x] = \bar{\pi}(x) + \sum_{i \in I} (\pi_i(x) - \{\pi_i(x)\}_i) = x - \sum_{i \in I} \{x\}_i$, which proves (2.4). We also have $[\{x\}_i]_i = [x]_i - [x - \{x\}_i]_i = [x]_i - [[x]_i]_i = 0$, whence $[\pi_i(\{x\}_i)]_i = \pi_i(\{x\}_i) - \{x\}_i$, and $[\{x\}_i] = \bar{\pi}(\{x\}_i) + [\pi_i(\{x\}_i)]_i + \sum_{j \neq i} [\pi_j(\{x\}_i)]_j = \bar{\pi}(\{x\}_i) + [\pi_i(\{x\}_i)]_i = \bar{\pi}(\{x\}_i) + \pi_i(\{x\}_i) - \{x\}_i = \{x\}_i - \{x\}_i = 0$. It follows that $[[x]] = [x - \sum_{i \in I} \{x\}_i] = [x] - \sum_{i \in I} [\{x\}_i] = [x]$. Since $\{x\} = \sum_{i \in I} \{\pi_i(x)\}_i \in \operatorname{im} \theta$, this shows that $[]$ is a reduction with respect to θ. We also have $[x] = [\pi(x)] + [\bar{\pi}(x)] = [\pi(x)] + \bar{\pi}(x) = [\pi(x)] + x - \pi(x)$, so $[]$ is local for π.

Let us prove (2.5) by induction over k. To simplify notations, assume that $i_j = j$ for all j, and write $x_i = \pi_i(x)$ for $i = 1, \ldots, k$. For $k = 0$, we have nothing to prove. For $k = 1$, the result follows from (2.1). If $k \geqslant 2$, then let $\bar{x} = \bar{\pi}(x)$ and $[]' = []_{k-1} \circ \cdots \circ []_1$. Notice that $[x_i]_k = [\pi_k(x_i)]_k + x_i - \pi_k(x_i) = x_i$ for $i = 1, \ldots, k-1$. The induction hypothesis yields $[\bar{x} + x_1 + \cdots + x_{k-1}]' = \bar{x} + [x_1]_1 + \cdots + [x_{k-1}]_{k-1}$, whence

$$
\begin{aligned}
[[x]']_k &= [\bar{x} + [x_1]_1 + \cdots + [x_{k-1}]_{k-1} + x_k]_k \\
&= \bar{x} + [x_k]_k + [[x_1]_1]_k + \cdots + [[x_{k-1}]_{k-1}]_k \\
&= \bar{x} + [x_k]_k + [[x_1]_k]_1 + \cdots + [[x_{k-1}]_k]_{k-1} \\
&= \bar{x} + [x_k]_k + [x_1]_1 + \cdots + [x_{k-1}]_{k-1} \\
&= [x].
\end{aligned}
$$

We conclude by induction. □

2.4. Composition of reductions

PROPOSITION 2.7. *The composition $[] = []_1 \circ \cdots \circ []_k$ of pairwise commuting reductions $[]_1, \ldots, []_k$: $V \to V$ with respect to θ is again a reduction with respect to θ.*

Proof. Let us first assume that $k = 2$. For each $x \in V$, we have $\{x\} = x - [[x]_1]_2 = x - [x]_2 + x_2 - [[x]_1]_2 = \{x\}_2 - [\{x\}_1]_2 = \{x\}_2 - \{x\}_1 + \{\{x\}_1\}_2 \in \operatorname{im} \theta$. We also have $[[x]] = [[[[x]_1]_2]_1]_2 = [[[[x]_1]_1]_2]_2 = [[x]_1]_2 = [x]$. The general case follows using an easy induction on k. □

Let $[]_1, []_2: V \to V$ be two local reductions with respect to $\theta: V \to V$ for two orthogonal projections $\pi_1, \pi_2: V \to V$. We have seen that Proposition 2.6 applies whenever (2.2) holds. If we only have $\pi_1(\{x\}_2) = 0$ for all $x \in V$, then the following still holds:

PROPOSITION 2.8. *If $\pi_1(\{x\}_2) = 0$ for all $x \in V$, then $[] = []_2 \circ []_1$ is a reduction with respect to θ.*

Proof. For each $x \in V$, we again have $\{x\} = x - [[x]_1]_2 = \{x\}_2 - [\{x\}_1]_2 = \{x\}_2 - \{x\}_1 + \{\{x\}_1\}_2 \in \operatorname{im} \theta$. Furthermore, setting $y = \{[x]_1\}_2$ with $\pi_1(y) = 0$, we have

$$[[[x]_1]_2]_1 = [[x]_1 - y]_1 = [x]_1 - [y]_1 = [x]_1 - [\pi_1(y)]_1 - y + \pi_1(y) = [x]_1 - y = [[x]_1]_2.$$

It follows that $[[x]] = [[[[x]_1]_2]_1]_2 = [[[x]_1]_2]_2 = [[x]_1]_2 = [x]$. □

2.5. Normalization of confined reductions

Consider a confined reduction $[]: V \to V$ with respect to θ. Then $I = \operatorname{im} []$ is a finite dimensional vector space and $E = [\operatorname{im} \theta]$ a finite dimensional subvector space of I that we call the space of *exceptional functions* for $[]$. Notice that $E \subseteq \operatorname{im} \theta$. Given any projection $\pi: I \to I$ with $\pi(I) = E$ and $\bar{\pi} = \operatorname{Id}_I - \pi$, let $[\![]\!]: V \to V$ be defined by

$$[\![x]\!] = \bar{\pi}([x]). \tag{2.6}$$

PROPOSITION 2.9. *The relation (2.6) defines a normal confined reduction with respect to θ.*

Proof. For any $x \in V$, we have

$$x - [\![x]\!] = \{x\} + \pi([x]) \in \operatorname{im}\theta.$$

Secondly, for any $y \in I$, there exists a $z \in V$ with $y = [z]$, whence $[y] = [[z]] = [z] = y$. For any $x \in V$, we thus obtain

$$[\![[\![x]\!]]\!] = \bar{\pi}([[\![x]\!]]) = \bar{\pi}([\![x]\!]) = \bar{\pi}(\bar{\pi}([x])) = \bar{\pi}([x]) = [\![x]\!].$$

Of course, we have $\dim_{\mathbb{K}} \operatorname{im} [\![]\!] = \dim_{\mathbb{K}} E \leqslant \dim_{\mathbb{K}} I < +\infty$. Finally, for any $x \in \operatorname{im}\theta$, the construction ensures that $[x] \in E$ whence $\bar{\pi}([x]) = 0$. □

This proposition also admits a local analogue. Assume that we are given a local reduction $[]: V \to V$ with respect to $\theta: V \to V$ for the projection $\pi: V \to V$. We say that $[]$ is *locally confined* if $\pi([V])$ is finite dimensional, and *locally normal* if $\pi([\operatorname{im}\theta]) = 0$. Assume that $[]$ is locally confined and decompose $I = \pi([V]) = E \oplus \bar{E}$ with $E = \pi([\operatorname{im}\theta])$. Let $\xi: I \to [V]$ a linear mapping such that $\pi \circ \xi$ is a projection of I on E (such a map is easily constructed from a basis for E). Then we define

$$[\![x]\!] = [x] - \xi(\pi([x])). \tag{2.7}$$

PROPOSITION 2.10. *The relation* (2.7) *defines a local reduction with respect to θ that is locally normal and confined for π.*

Proof. Given $x \in V$, let $\varepsilon = \xi(\pi([x])) \in \operatorname{im}\theta$. Then we have $x - [\![x]\!] = \{x\} - \varepsilon \in \operatorname{im}\theta$. Letting $y \in \operatorname{im}\theta$ be such that $\pi(\varepsilon) = \pi([y])$, we also have $[\varepsilon] = [\pi(\varepsilon)] + \bar{\pi}(\varepsilon) = [\pi([y])] + \bar{\pi}(\varepsilon) = \pi([y]) + \bar{\pi}(\varepsilon) = \pi(\varepsilon) + \bar{\pi}(\varepsilon) = \varepsilon$, whence $[\![\varepsilon]\!] = \varepsilon - \xi(\pi(\varepsilon)) = (\xi \circ \pi)(\pi([x])) - (\xi \circ \pi)^2(\pi([x])) = 0$, since $\xi \circ \pi$ and $(\xi \circ \pi)^2$ coincide on I. From $[\![[x]]\!] = [[x]] - \xi(\pi([[x]])) = [x] - \xi(\pi([x])) = [\![x]\!]$, we conclude that $[\![[\![x]\!]]\!] = [\![[x]]\!] - [\![\varepsilon]\!] = [\![x]\!]$, whence $[\![]\!]$ is a reduction with respect to θ.

The reduction is local for π, since $[\![\pi(x)]\!] + \bar{\pi}(x) = [\pi(x)] - \xi(\pi([\pi(x)])) + \bar{\pi}(x) = [x] - \xi(\pi([x])) = [\![x]\!]$. By construction, $\dim_{\mathbb{K}} \pi([\![V]\!]) = \dim_{\mathbb{K}} E \leqslant \dim_{\mathbb{K}} I = \dim_{\mathbb{K}} \pi(V)$. For $x \in \operatorname{im}\theta$ and $y = \pi([x]) \in E$, we finally have $\pi([\![x]\!]) = \pi([x]) - \pi(\xi(\pi([x]))) = y - \pi(\xi(y)) = 0$. □

3. LAGRANGE'S IDENTITY AND GENERALIZATIONS

3.1. The original differential case

Let $\mathbb{K}$ be a differential field for the derivation ∂ and let $\mathbb{K}[\partial]$ denote the skew ring of linear differential operators over $\mathbb{K}$. For instance, one may take $\mathbb{K} = \Bbbk(z)$ and $\partial = \partial_z$. Recall that the *adjoint* of a differential operator $L = L_r \partial^r + \cdots + L_1 \partial + L_0 \in \mathbb{K}[\partial]$ is defined by

$$L^* = (-\partial)^r L_r + \cdots + (-\partial) L_1 + L_0.$$

Given two indeterminates u and f, we write $(\mathbb{K}[\partial] \otimes \mathbb{K}[\partial])(u \otimes f)$ for the set of $\mathbb{K}$-linear combinations of products $(\partial^i u)(\partial^j f)$ with $i, j \in \mathbb{N}$. The following identity is due to Lagrange [66]:

PROPOSITION 3.1. *For any $L \in \mathbb{K}[\partial]$ of order r, there exists a*

$$P_L(u, f) \in (\mathbb{K}[\partial] \otimes \mathbb{K}[\partial])(u \otimes f),$$

called the "bilinear concomitant", such that

$$u L(f) - L^*(u) f = \partial(P_L(u, f)). \tag{3.1}$$

More specifically, we have

$$P_L(u, f) = \sum_{0 \leqslant i < r} \sum_{0 \leqslant j < r-i} (-1)^i L_{i+j+1} (\partial^i u)(\partial^j f). \tag{3.2}$$

Proof. Let us first prove the existence of P_L by induction on r. If $r=0$, then $L\in\mathbb{K}$, and we may take $P_L(u,f)=0$. If $r>0$, then we write $L=K\partial+L_0$ with K of order $r-1$ and $L_0\in\mathbb{K}$. By the induction hypothesis, there exists a $P_K(u,f)$ with

$$u\,K(f)-K^*(u)\,f \;=\; \partial\,(P_K(u,f)).$$

It follows that

$$\begin{aligned} u\,L(f)-L^*(u)\,f &= u\,(K\partial)(f)+(\partial K^*)(u)\,f\\ &= \partial(P_K(u,\partial f))+K^*(u)\,\partial f+\partial(K^*(u))\,f\\ &= \partial(P_K(u,\partial f)+K^*(u)\,f).\end{aligned}$$

We conclude by taking

$$P_L(u,f)=K^*(u)\,f+P_K(u,\partial f).$$

From this relation, it follows by induction over r that

$$P_L(u,f) \;=\; \sum_{0\leqslant j<r}(L_{\gg(j+1)})^*\,(\partial^j f), \qquad L_{\gg(j+1)}=L_r\partial^{r-j-1}+\cdots+L_{j+1},$$

which can be rewritten as (3.2). □

3.2. The difference case

Assume now that $\mathbb{K}$ is a difference field for the automorphism $\sigma:\mathbb{K}\to\mathbb{K}$ and let $\mathbb{K}[\sigma]$ denote the corresponding ring of skew difference operators. For instance, one may take $\mathbb{K}=\Bbbk(z)$ and $\sigma=S_\eta: z\mapsto z+\eta$ or $\sigma=Q_q: z\mapsto q\,z$ with $\eta,q\in\Bbbk^{\neq}:=\{c\in\Bbbk : c\neq 0\}$. We recall that the *adjoint* of a difference operator $L=L_r\sigma^r+\cdots+L_1\sigma+L_0\in\mathbb{K}[\sigma]$ is defined by

$$L^* \;=\; \sigma^{-r}L_r+\cdots+\sigma^{-1}L_1+L_0.$$

We also define the *finite difference operator* Δ associated to σ by $\Delta=\sigma-1$.

PROPOSITION 3.2. *For any $L\in\mathbb{K}[\sigma]$ of order r, there exists a*

$$P_L(u,f)\in(\mathbb{K}[\sigma^{-1}]\otimes\mathbb{K}[\sigma])(u\otimes f)$$

such that

$$u\,L(f)-L^*(u)\,f \;=\; \Delta(P_L(u,f)). \tag{3.3}$$

More specifically, we have

$$P_L(u,f) \;=\; \sum_{0<i\leqslant r}\;\sum_{0\leqslant j\leqslant r-i}\sigma^{-i}(L_{i+j})\,(\sigma^{-i}u)\,(\sigma^j f). \tag{3.4}$$

Proof. We again prove the existence of P_L by induction on r. If $r=0$, then $L\in\mathbb{K}$, and we may take $P_L(u,f)=0$. If $r>0$, then we write $L=K\sigma+L_0$ with K of order $r-1$ and $L_0\in\mathbb{K}$. By the induction hypothesis, there exists a $P_K(u,f)$ with

$$u\,K(f)-K^*(u)\,f \;=\; \Delta\,(P_K(u,f)).$$

It follows that

$$\begin{aligned} u\,L(f)-L^*(u)\,f &= u\,(K\sigma)(f)-(\sigma^{-1}K^*)(u)\,f\\ &= \Delta(P_K(u,\sigma f))+K^*(u)\,\sigma f-\sigma^{-1}(K^*(u))\,f\\ &= \Delta(P_K(u,\sigma f)+\sigma^{-1}(K^*(u))\,f)\\ &= \Delta(P_K(u,\sigma f)+(L-L_0)^*(u)\,f).\end{aligned}$$

We conclude by taking

$$P_L(u,f) = (L-L_0)^*(u)\, f + P_K(u,\sigma f).$$

From this relation, it follows by induction over r that

$$P_L(u,f) \;=\; \sum_{0\leqslant j<r} (L_{\gg j}-L_j)^*(u)\,(\sigma^j f), \qquad L_{\gg j} = L_r \sigma^{r-j} + \cdots + L_j,$$

which can be rewritten as (3.4). □

3.3. Matrix versions

Let $\mathbb{K}$ again be a differential field for ∂ and consider a not necessarily commutative differential algebra $\mathbb{A}$ over $\mathbb{K}$. One important example of such an algebra is the algebra of $n\times n$ matrices $\mathbb{K}^{n\times n}$. Proposition 3.1 naturally generalizes to the case when $L\in\mathbb{A}[\partial]$ admits coefficients in $\mathbb{A}$, modulo a few precautions: in this setting, it is important to distinguish between usual operators $L\in\mathbb{A}[\partial]$ that operate on the left

$$L(f) \;=\; (L_r \partial^r + \cdots + L_0)(f) \;=\; L_r(\partial^r f) + \cdots + L_0 f,$$

and their adjoints $L^*\in\mathbb{K}^{r\times r}[\partial]^*$ that operate on the right:

$$L^*(u) \;=\; ((-\partial)^r L_r + \cdots + L_0)(u) \;=\; ((-\partial)^r u)\, L_r + \cdots + u\, L_0,$$

Similarly, a bilinear operator $Q\in(\mathbb{A}[\partial]^*\otimes\mathbb{A}[\partial])(u\otimes f)$ acts *via*

$$Q(u,f) \;=\; \sum_{i,j} (\partial^i u)\, Q_{i,j}\,(\partial^j f),$$

where $Q_{i,j}\in\mathbb{A}$ are the coefficients of Q. Similar precautions apply to the difference setting, and we have:

PROPOSITION 3.3. *Let $\mathbb{A}$ be a differential algebra over a differential field $\mathbb{K}$ for ∂. For any $L\in\mathbb{A}[\partial]$ of order r, there exists a*

$$P_L(u,f)\in(\mathbb{A}[\partial]^*\otimes\mathbb{A}[\partial])(u\otimes f)$$

such that

$$u\,L(f) - L^*(u)\, f \;=\; \partial(P_L(u,f)).$$

More specifically, we have

$$P_L(u,f) \;=\; \sum_{0\leqslant i<r}\ \sum_{0\leqslant j<r-i} (-1)^i\,(\partial^i u)\, L_{i+j+1}\,(\partial^j f).$$ □

PROPOSITION 3.4. *Let $\mathbb{A}$ be a difference algebra over a difference field $\mathbb{K}$ for σ. For any $L\in\mathbb{A}[\sigma]$ of order r, there exists a*

$$P_L(u,f)\in(\mathbb{A}[\sigma]^*\otimes\mathbb{A}[\sigma])(u\otimes f)$$

such that

$$u\,L(f) - L^*(u)\, f \;=\; \Delta(P_L(u,f)).$$

More specifically, we have

$$P_L(u,f) \;=\; \sum_{0<i\leqslant r}\ \sum_{0\leqslant j\leqslant r-i} (\sigma^{-i} u)\, \sigma^{-i}(L_{i+j})\,(\sigma^j f).$$ □

3.4. Twisting

Let $\mathbb{K}$ be a differential field for ∂ and assume that $\varphi \in \mathbb{K}$ is the logarithmic derivative $\varphi = E^\dagger = \partial(E)/E$ of another element $E \in \mathbb{K}^{\neq}$. In fact, we may allow E to live in some abstract extension field $\hat{\mathbb{K}}$ of $\mathbb{K}$. Then we have $E^{-1}\partial(Ef) = (\partial + \varphi) f$ for all $f \in \mathbb{K}$ and we call $\partial_{\ltimes\varphi} = E^{-1}\partial E = \partial + \varphi \in \mathbb{K}[\partial]$ the *twist* of ∂ by φ. More generally, for any $L \in \mathbb{K}[\partial]$, we define the twist $L_{\ltimes\varphi} = E^{-1} L E \in \mathbb{K}[\partial]$. One may check that $L_{\ltimes\varphi} = L_r(\partial + \varphi)^r + \cdots + L_0$ and $(KL)_{\ltimes\varphi} = K_{\ltimes\varphi} L_{\ltimes\varphi}$ for all $K, L \in \mathbb{K}[\partial]$. Using twisting, it is possible to replace the operator ∂ at the right hand side of (3.1) by any first order differential operator:

PROPOSITION 3.5. *Let $\mathbb{K}$ be a differential field for ∂. For any $L \in \mathbb{K}[\partial]$ and $\varphi \in \mathbb{K}$, there exists a*

$$P_{L,\varphi}(u, f) \in (\mathbb{K}[\partial] \otimes \mathbb{K}[\partial])(u \otimes f)$$

such that

$$u\, L(f) - (L_{\ltimes(-\varphi)})^*(u)\, f \;=\; (\partial + \varphi)(P_{L,\varphi}(u, f)).$$

Proof. Let $E \in \hat{\mathbb{K}}^{\neq}$ be a formal solution of $E^\dagger = \varphi$ in some extension field $\hat{\mathbb{K}}$ of $\mathbb{K}$. Let $g = Ef$ and $K = L_{\ltimes(-\varphi)} \in \mathbb{K}[\partial]$. Then Proposition 3.1 provides us with $P_K(u,g) \in (\mathbb{K}[\partial] \otimes \mathbb{K}[\partial])(u \otimes g)$ such that

$$\begin{aligned} u\, L(f) - (L_{\ltimes(-\varphi)})^*(u)\, f &= u\, L(E^{-1} g) - K^*(u)\, E^{-1} g \\ &= E^{-1}(u\, K(g) - K^*(u)\, g) \\ &= E^{-1}\partial(P_K(u,g)) \\ &= E^{-1}\partial\, (P_K(u, Ef)). \end{aligned}$$

Let $Q \in (\mathbb{K}[\partial] \otimes \mathbb{K}[\partial])(u \otimes f)$ be the twist of P_K in f by φ with $P_K(u, Ef) = E\, Q(u, f)$. Then we conclude by taking $P_{L,\varphi} = Q$. □

This proposition again admits a matrix generalization: for $L \in \mathbb{K}^{r\times r}[\partial]$, it suffices to replace φ and E by matrices $\Phi, E \in \mathbb{K}^{r\times r}$ with E invertible and $\partial E = \Phi\, E$. We also need to assume that E commutes with each coefficient L_i of L.

One interesting special case occurs as follows: let $L \in \mathbb{K}[\partial]$, let $R \in \mathbb{K}[\partial]$ be monic of order r and assume that we wish to obtain a generalization of (3.1) with R instead of ∂ in the right hand side. Then we lift L into an operator $L^{[r]} \in \mathbb{K}^{r\times r}[\partial]$ by sending each coefficient $c \in \mathbb{K}$ to $c\,\mathrm{Id}_r \in \mathbb{K}^{r\times r}$ and take

$$\Phi \;=\; \begin{pmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -R_0 & -R_1 & \cdots & -R_{r-1} \end{pmatrix}.$$

We next apply Proposition 3.5 to obtain a $P_{L,\varphi}(u, f) \in (\mathbb{K}^{r\times r}[\partial]^* \otimes \mathbb{K}^{r\times r}[\partial])(u \otimes f)$ with

$$u\, L^{[r]}(f) - (L^{[r]}_{\ltimes(-\varphi)})^*(u)\, f \;=\; (\partial + \Phi)(P_{L,\Phi}(u, f)).$$

This equation can be considered as the "desired" generalization of (3.1) with R instead of ∂ in the right hand side.

As usual, Proposition 3.5 also admits a difference version. Let $\mathbb{K}$ be a differential field for σ and let φ, E be such that $\Delta(E) = \varphi\, E$. The twist of an operator $L = L_r \sigma^r + \cdots + L_0$ by E is again defined as $L_{\ltimes\varphi} = E^{-1} L\, E$, so that $\sigma_{\ltimes\varphi} = (\varphi + 1)\,\sigma$.

PROPOSITION 3.6. *Let $\mathbb{K}$ be a differential field for σ. For any $L \in \mathbb{K}[\sigma]$ and $\varphi \in \mathbb{K}$, there exists a*

$$P_{L,\varphi}(u, f) \in (\mathbb{K}[\sigma^{-1}] \otimes \mathbb{K}[\sigma])(u \otimes f)$$

such that

$$u\,L(f)-(L_{\ltimes(-\varphi)})^*(u)\,f \;=\; (\Delta+\varphi)(P_{L,\varphi}(u,f)). \qquad \square$$

3.5. Another twisted variant

There are many more variants and generalizations of Lagrange's identity. Let us describe one more variant in the differential case. Let $\mathbb{A}$ be a possibly non-commutative $\mathbb{K}$-algebra, where $\mathbb{K}$ is a field.

PROPOSITION 3.7. *Let $L=\partial-A$ be a monic first order operator in $\mathbb{A}[\partial]$ and let $R\in\mathbb{K}[\partial]$ be a monic operator of order r. Then there exist differential operators $\Psi_0,\ldots,\Psi_r\in\mathbb{K}[A][\partial]\subseteq\mathbb{A}[\partial]$ such that*

$$R(uf) \;=\; \sum_{i=0}^{r}\Psi_i(u)\,(\partial-A)^i(f).$$

Proof. Writing $K'=t\,K_t\,\partial^{t-1}+\cdots+2\,K_2\,\partial+K_1$ for the derivative of a differential operator $K=K_t\,\partial^t+\cdots+K_0$, and using the expansion

$$\partial^k=(\partial-A+A)^k=\sum_{i=0}^{k}\binom{k}{i}A^{k-i}\,(\partial-A)^i,$$

we notice that

$$\begin{aligned}R(uf) &= \sum_{k=0}^{r}\frac{1}{k!}R^{(k)}(u)\,\partial^k f\\ &= \sum_{k=0}^{r}\sum_{i=0}^{k}\frac{1}{k!}\binom{k}{i}A^{k-i}R^{(k)}(u)\,(\partial-A)^i(f)\\ &= \sum_{i=0}^{r}\left[\sum_{k=i}^{r}\frac{1}{k!}\binom{k}{i}A^{k-i}R^{(k)}(u)\right](\partial-A)^i(f)\end{aligned}$$

Taking $\Psi_i=\sum_{k=i}^{r}\frac{1}{k!}\binom{k}{i}A^{k-i}R^{(k)}(u)$ for $i=0,\ldots,r$, the result follows. $\square$

We typically read the above identity as

$$\begin{aligned}\Xi((\partial-A)(f))+\Psi_0(u)\,f &= R(uf)\\ \Xi &= \Psi_1(u)+\cdots+\Psi_r(u)\,(\partial-A)^{r-1},\end{aligned} \tag{3.5}$$

where $\Psi_0(u)$ plays a similar role as $-L^*(u)$ in (3.1), where uf plays the role of $P_L(u,f)$, and where the derivation ∂ got replaced by a general monic operator $R\in\mathbb{K}[\partial]$.

4. DIFFERENTIAL REDUCTIONS

4.1. Partial fraction decomposition and Hermite reduction

Let $\mathbb{K}$ be a field with algebraic closure $\bar{\mathbb{K}}$. Any rational function $f\in\bar{\mathbb{K}}(z)$ admits a partial fraction decomposition

$$f \;=\; f_{(\star)}+\sum_{\alpha\in\bar{\mathbb{K}}\cup\{\infty\}}f_{(\alpha)}, \tag{4.1}$$

where $f_{(\star)} \in \mathbb{K}(z)_{(\star)} = \bar{\mathbb{K}}$,

$$\begin{aligned} f_{(\infty)} &= \sum_{k=1}^{\text{ord}_\infty f} f_{(\infty),k} z^k \in \bar{\mathbb{K}}(z)_{(\infty)} \\ \bar{\mathbb{K}}(z)_{(\infty)} &= z\,\bar{\mathbb{K}}[z], \end{aligned}$$

and, for each $\alpha \in \bar{\mathbb{K}}$,

$$\begin{aligned} f_{(\alpha)} &= \sum_{k=1}^{\text{ord}_\alpha f} \frac{f_{(\alpha),k}}{(z-\alpha)^k} \in \bar{\mathbb{K}}(z)_{(\alpha)} \\ \bar{\mathbb{K}}(z)_{(\alpha)} &= \frac{1}{z-\alpha}\,\bar{\mathbb{K}}\left[\frac{1}{z-\alpha}\right]. \end{aligned}$$

Given $\alpha \in \bar{K} \cup \{\infty\}$, we will write $\pi_{(\alpha)}$ for the projection $f \mapsto f_{(\alpha)}$ and we call $f_{(\alpha)}$ the *polar part* of f at α. The number $\text{ord}_\alpha f \in \mathbb{N}$ stands for the *order* of the pole of f. We also write $\pi_{(\star)}$ for the projection $f \mapsto f_{(\star)}$ and call $f_{(\star)}$ the *constant part* of f.

The decomposition (4.1) has the advantage of being symmetric with respect to all points in $\bar{\mathbb{K}} \cup \{\infty\}$. Often, the polar part at infinity is combined with the constant part, in which case (4.1) becomes

$$f = f_{(\infty,\star)} + \sum_{\alpha \in \bar{\mathbb{K}}} f_{(\alpha)}, \tag{4.2}$$

where $f_{(\infty,\star)} = f_{(\infty)} + f_{(\star)} \in \mathbb{K}_{(\infty,\star)} = \mathbb{K}[z]$. More generally, for subsets $\mathrm{A} \subseteq \bar{\mathbb{K}} \cup \{\infty, \star\}$, it is convenient to denote

$$\begin{aligned} f_{(\mathrm{A})} &= \sum_{\alpha \in \mathrm{A}} f_{(\alpha)} \in \bar{\mathbb{K}}(z)_{(\mathrm{A})} \\ \bar{\mathbb{K}}(z)_{(\mathrm{A})} &= \bigoplus_{\alpha \in \mathrm{A}} \bar{\mathbb{K}}(z)_{(\alpha)}. \end{aligned}$$

We also denote $\bar{\mathrm{A}} = \bar{\mathbb{K}} \cup \{\infty, \star\} \setminus \mathrm{A}$ and write $\pi_{(\mathrm{A})}$ for the projection $f \mapsto f_{(\mathrm{A})}$. Notice that $\bar{\mathbb{K}}(z)_{(\mathrm{A},\star)}$ forms a ring for any $\mathrm{A} \subseteq \bar{\mathbb{K}} \cup \{\infty\}$.

The *Hermite reduction* $[f]$ of $f \in \bar{\mathbb{K}}(z)$ with respect to the ordinary differentiation $\partial = \partial_z$ with respect to z is defined by

$$[f] = \sum_{\alpha \in \bar{\mathbb{K}}} \frac{f_{(\alpha),1}}{z-\alpha}.$$

It is not hard to check that this indeed defines a normal reduction on $\bar{\mathbb{K}}(z)$ with respect to ∂. Its restriction to $\mathbb{K}(z)$ determines a normal reduction on $\mathbb{K}(z)$ with respect to ∂, by Proposition 2.3.

4.2. Local reduction with respect to differential operators

Let us now consider an arbitrary non-zero linear differential operator $L \in \mathbb{K}[z][\partial]$ instead of ∂. For any $\alpha \in \bar{\mathbb{K}}$, it is well-known [55] that there exists an *indicial polynomial* $\text{ind}_\alpha \in \bar{\mathbb{K}}[\rho]$ at α and a *shift* $\tau_\alpha \in \mathbb{Z}$ at α with

$$L((z-\alpha)^{-\rho}) = (z-\alpha)^{-\rho-\tau_\alpha}(\text{ind}_\alpha(\rho) + o(1)) \qquad (z \to \alpha),$$

for all $\rho \in \mathbb{Z}$. Similarly, at infinity there exist $\text{ind}_\infty \in \bar{\mathbb{K}}[\rho]$ and $\tau_\infty \in \mathbb{Z}$ with

$$L(z^\rho) = z^{\rho+\tau_\infty}(\text{ind}_\infty(\rho) + o(1)) \qquad (z \to \infty).$$

Given $\alpha \in \bar{\mathbb{K}}$, let us now construct a local reduction $[]_\alpha: \bar{\mathbb{K}}(z) \to \bar{\mathbb{K}}(z)$ at α. Given $f \in \bar{\mathbb{K}}(z)$, let $\rho \in \mathbb{Z}$ and $c \in \bar{\mathbb{K}}$ be such that

$$f = (z-\alpha)^{-\rho}(c+o(1)) \qquad (z \to \alpha).$$

Setting $B_\alpha := \max(0, \tau_\alpha)$, we define $[f]_\alpha$ by induction over ρ, as follows:

$$[f]_\alpha = \begin{cases} f & \text{if } \rho \leqslant B_\alpha \\ c(z-\alpha)^{-\rho} + [f - c(z-\alpha)^{-\rho}]_\alpha & \text{if } \rho > B_\alpha \text{ and } \operatorname{ind}_\alpha(\rho - \tau_\alpha) = 0 \\ \left[f - L\left(\frac{c}{\operatorname{ind}_\alpha(\rho-\tau_\alpha)}(z-\alpha)^{-\rho+\tau_\alpha}\right)\right]_\alpha & \text{if } \rho > B_\alpha \text{ and } \operatorname{ind}_\alpha(\rho-\tau_\alpha) \neq 0. \end{cases} \tag{4.3}$$

In a similar way, we may construct a local reduction $[]_\infty: \bar{\mathbb{K}}(z) \to \bar{\mathbb{K}}(z)$ at infinity. Given $f \in \bar{\mathbb{K}}(z)$, let $\rho \in \mathbb{Z}$ and $c \in \bar{\mathbb{K}}$ be such that

$$f = z^\rho(c+o(1)) \qquad (z \to \infty).$$

Setting $B_\infty := \max(0, \tau_\infty)$, we define $[f]_\infty$ by induction over ρ:

$$[f]_\infty = \begin{cases} f & \text{if } \rho \leqslant B_\infty \\ cz^\rho + [f - cz^\rho]_\infty & \text{if } \rho > B_\infty \text{ and } \operatorname{ind}_\infty(\rho-\tau_\infty) = 0 \\ \left[f - L\left(\frac{c}{\operatorname{ind}_\infty(\rho-\tau_\infty)} z^{\rho-\tau_\alpha}\right)\right]_\infty & \text{if } \rho > B_\infty \text{ and } \operatorname{ind}_\infty(\rho-\tau_\infty) \neq 0. \end{cases} \tag{4.4}$$

Notice that $[f]_\infty \in \mathbb{K}(z)$ for any $f \in \mathbb{K}(z)$.

PROPOSITION 4.1. *Let $f \in \bar{\mathbb{K}}(z)$ and $\alpha \in \bar{\mathbb{K}} \cup \{\infty\}$. The mapping $[]_\alpha$ is a reduction with respect to L such that:*

- *i. The reduction $[]_\alpha$ is local for $\pi_{(\alpha)}$.*
- *ii. We have $\{f\}_\alpha \in \bar{\mathbb{K}}(z)_{(\alpha,\infty,\star)}$.*
- *iii. The reductions $[]_\alpha$ and $[]_\beta$ are independent for any $\alpha, \beta \in \bar{\mathbb{K}}$ with $\alpha \neq \beta$.*
- *iv. If $\alpha \in \bar{\mathbb{K}}$, then $(\{f\}_\infty)_{(\alpha)} = 0$.*

Proof. We adopt the notations from above. Whenever $\rho > B_\alpha$, we let $h = c(z-\alpha)^{-\rho}$ if $\alpha \in \bar{\mathbb{K}}$ and $h = cz^\rho$ if $\alpha = \infty$. Whenever $\operatorname{ind}_\alpha(\rho - \tau_\alpha) \neq 0$, we also denote $g = c(z-\alpha)^{-\rho+\tau_\alpha}/\operatorname{ind}_\alpha(\rho-\tau_\alpha)$ if $\alpha \in \bar{\mathbb{K}}$ and $g = cz^{\rho-\tau_\alpha}/\operatorname{ind}_\alpha(\rho-\tau_\alpha)$ if $\alpha = \infty$.

By induction on ρ, let us first show that there exists a $u \in \bar{\mathbb{K}}(z)_{(\alpha,\infty,\star)}$ with $\operatorname{ord}_\alpha u \leqslant \max(\rho - \tau_\alpha, 0)$ and $\{f\}_\alpha = L(u)$; notice that this in particular implies *ii*, as well as $\operatorname{ord}_\alpha [f]_\alpha \leqslant \rho$. If $\rho \leqslant B_\alpha$, then we may take $u = 0$. If $\rho > B_\alpha$ and $\operatorname{ind}_\alpha(\rho-\tau_\alpha) = 0$, then there exists a $u \in \bar{\mathbb{K}}(z)_{(\alpha,\infty,\star)}$ with $\operatorname{ord}_\alpha u \leqslant \max(\rho - 1 - \tau_\alpha, 0)$ and $\{f - h\}_\alpha = L(u)$, whence $\{f\}_\alpha = \{f-h\}_\alpha = L(u)$. If $\rho > B_\alpha$ and $\operatorname{ind}_\alpha(\rho-\tau_\alpha) \neq 0$, then there exists a $v \in \bar{\mathbb{K}}(z)_{(\alpha,\infty,\star)}$ with $\operatorname{ord}_\alpha v \leqslant \max(\rho - 1 - \tau_\alpha, 0)$ and $\{f - L(g)\}_\alpha = L(v)$. Hence $\{f\}_\alpha = L(g) + \{f - L(g)\}_\alpha = L(g+v)$, so we may take $u = g + v \in \bar{\mathbb{K}}(z)_{(\alpha,\infty,\star)}$ with $\operatorname{ord}_\alpha u \leqslant \max(\rho - \tau_\alpha, 0)$.

Again by induction on ρ, let us next show that $[[f]_\alpha]_\alpha = [f]_\alpha$. If $\rho \leqslant B_\alpha$, then we have $[f]_\alpha = f = [[f]_\alpha]_\alpha$. If $\rho > B_\alpha$ and $\operatorname{ind}_\alpha(\rho-\tau_\alpha) = 0$, then $\operatorname{ord}_\alpha [f-h]_\alpha \leqslant \rho - 1$ implies $[h + [f-h]_\alpha]_\alpha = h + [[f-h]_\alpha]_\alpha$. Consequently, $[[f]_\alpha]_\alpha = [h + [f-h]_\alpha]_\alpha = h + [[f-h]_\alpha]_\alpha = h + [f-h]_\alpha = [f]_\alpha$. If $\rho > B_\alpha$ and $\operatorname{ind}_\alpha(\rho-\tau_\alpha) \neq 0$, then $[[f]_\alpha]_\alpha = [[f - L(g)]_\alpha]_\alpha = [f - L(g)]_\alpha = [f]_\alpha$. This completes the proof that $[]_\alpha$ is a reduction with respect to L.

Decomposing $f = \varphi + \psi$ with $\varphi = f_{(\alpha)}$, let us next show by induction on ρ that $[f]_\alpha = [\varphi]_\alpha + \psi$. If $\rho \leqslant B_\alpha$, then we have $\varphi = 0 = [\varphi]_\alpha$ and $[f]_\alpha = f = \psi$. If $\rho > B_\alpha$ and $\operatorname{ind}_\alpha(\rho-\tau_\alpha) = 0$, then we may decompose $f = h + \tilde{f}$ and $\varphi = h + \tilde{\varphi}$ with $\tilde{\varphi} = \tilde{f}_{(\alpha)}$, after which $[f]_\alpha = h + [\tilde{f}]_\alpha = h + [\tilde{\varphi}]_\alpha + \psi = [\varphi]_\alpha + \psi$. If $\rho > B_\alpha$ and $\operatorname{ind}_\alpha(\rho-\tau_\alpha) \neq 0$, then $\varphi = (z-\alpha)^{-\rho}(c+o(1))$ implies $[\varphi]_\alpha = [\varphi - L(g)]_\alpha$. It follows that $[f]_\alpha = [f - L(g)]_\alpha = [f - \varphi]_\alpha + [\varphi - L(g)]_\alpha = [\varphi - L(g)]_\alpha + \psi = [\varphi]_\alpha + \psi$. This shows *i*.

Now let $\beta \in \bar{\mathbb{K}}$ be such that $\beta \neq \alpha$. The projections $\pi_{(\alpha)}$ and $\pi_{(\beta)}$ are clearly orthogonal and it follows from *ii* that $(\{f\}_\alpha)_{(\beta)} = 0$ and $(\{f\}_\beta)_{(\alpha)} = 0$; this shows *iii*. The last fact *iv* also follows from *ii*. □

PROPOSITION 4.2. *For each* $\alpha \in \bar{\mathbb{K}} \cup \{\infty\}$, *let*

$$\mathcal{F}_\alpha = \{(z-\alpha)^{-\rho} : 0 < \rho \wedge (\rho \leqslant B_\alpha \vee \operatorname{ind}_\alpha(\rho - \tau_\alpha) = 0)\} \qquad (\alpha \in \bar{\mathbb{K}})$$

$$\mathcal{F}_\infty = \{z^\rho : 0 < \rho \wedge (\rho \leqslant B_\infty \vee \operatorname{ind}_\infty(\rho - \tau_\infty) = 0)\} \qquad (\alpha = \infty)$$

Then $([\bar{\mathbb{K}}(z)]_\alpha)_{(\alpha)}$ *is included in the vector space* $\operatorname{Vect}(\mathcal{F}_\alpha)$ *spanned by* $\mathcal{F}_\alpha$. *We also have* $|\mathcal{F}_\alpha| \leqslant 2r$ *for* $\alpha \in \bar{\mathbb{K}}$ *and* $|\mathcal{F}_\infty| \leqslant r + \deg_z L$.

Proof. Adopt the same notations as in the proof of Proposition 4.1. Given $f \in \bar{\mathbb{K}}(z)$, let us prove by induction on $\rho = \operatorname{ord}_\alpha f$ that $([f]_\alpha)_{(\alpha)} \in \operatorname{Vect}(\mathcal{F}_\alpha)$. This is clear if $\rho \leqslant B_\alpha$. If $\rho > B_\alpha$ and $\operatorname{ind}_\alpha(\rho - \tau_\alpha) = 0$, then the induction hypothesis yields $([f-h]_\alpha)_{(\alpha)} \in \operatorname{Vect}(\mathcal{F}_\alpha)$, whence $([f]_\alpha)_{(\alpha)} = ([f-h]_\alpha)_{(\alpha)} + h_{(\alpha)} = ([f-h]_\alpha)_{(\alpha)} + h \in \operatorname{Vect}(\mathcal{F}_\alpha)$. If $\rho > B_\alpha$ and $\operatorname{ind}_\alpha(\rho - \tau_\alpha) \neq 0$, then the induction hypothesis directly yields $([f]_\alpha)_{(\alpha)} = ([f - L(g)]_\alpha)_{(\alpha)} \in \operatorname{Vect}(\mathcal{F}_\alpha)$. This completes the proof that $([\bar{\mathbb{K}}(z)]_\alpha)_{(\alpha)} \subseteq \operatorname{Vect}(\mathcal{F}_\alpha)$. If $\alpha \in \bar{\mathbb{K}}$, then we also have $B_\alpha \leqslant r$ and $B_\infty = \deg_z L$. Since $\operatorname{ind}_\alpha$ admits at most r roots, the cardinality bounds follow. □

Remark 4.3. Proposition 4.2 shows that $[\,]_\alpha$ is locally confined for all $\alpha \in \bar{\mathbb{K}} \cup \{\infty\}$. This makes it possible to define a locally normal reduction $[\![\,]\!]_\alpha$ using Proposition 2.10. In order to make this construction fully effective, we still have to determine the space $E = ([\operatorname{im} L]_\alpha)_{(\alpha)}$. Let

$$\mathcal{F}_\alpha^\# = \{(z-\alpha)^{-\rho} : 0 < \rho \leqslant \max \mathcal{F}_\alpha\} \qquad (\alpha \in \bar{\mathbb{K}})$$

$$\mathcal{F}_\infty^\# = \{z^\rho : 0 < \rho \leqslant \max \mathcal{F}_\infty\} \qquad (\alpha = \infty)$$

We claim that $E = ([L(\operatorname{Vect}(\mathcal{F}_\alpha^\#))]_\alpha)_{(\alpha)}$. Assume for contradiction that there exists a function $f \in \bar{\mathbb{K}}(z)$ with $([L(f)]_\alpha)_{(\alpha)} \in E \setminus ([L(\operatorname{Vect}(\mathcal{F}_\alpha^\#))]_\alpha)_{(\alpha)}$ and chose f such that $\rho = \operatorname{ord}_\alpha f$ is minimal with this property. Let g as in the proof of Proposition 4.1. Since we must have $\rho > \max \mathcal{F}_\alpha$, it follows that $[f]_\alpha = [f - L(g)]_\alpha$. Therefore $f - L(g)$ provides a counterexample with $\operatorname{ord}_\alpha (f - L(g)) < \rho$, contracting our minimality hypothesis.

Having proved our claim, one may use linear algebra to determine elements $f_1, \ldots, f_e \in \mathcal{F}_\alpha^\#$ with $e = \dim_{\bar{\mathbb{K}}} E$ and such that $([L(f_1)]_\alpha)_{(\alpha)}, \ldots, ([L(f_e)]_\alpha)_{(\alpha)}$ form a basis for E. Setting $\operatorname{ord}_\alpha E = \{k > 0 : \exists \psi \in E, \operatorname{ord}_\alpha \psi = k\}$ and $E^\perp = \operatorname{Vect}((z-\alpha)^{-\rho} : \rho \notin \operatorname{ord}_\alpha E)$, we define π_E to be the "distinguished" projection of $\bar{\mathbb{K}}(z)_{(\alpha)}$ on E such that $\varphi - \pi_E(\varphi) \in E^\perp$ for all $\varphi \in \bar{\mathbb{K}}(z)_{(\alpha)}$. In order to apply Proposition 2.10, we finally construct the mapping ξ by sending each $([L(f_i)]_\alpha)_{(\alpha)}$ to $[L(f_i)]_\alpha$ and then extend ξ to $\bar{\mathbb{K}}(z)_{(\alpha)}$ by setting $\xi(\varphi) = \xi(\pi_E(\varphi))$.

Since $\max \mathcal{F}_\alpha$ can be arbitrarily large, we notice that the determination of $f_1, \ldots, f_e$ can be very expensive. In particular, there exists no polynomial time algorithm to compute them. We also notice that the construction of $[\![\,]\!]_\alpha$ is symmetric in the sense that $[\![\sigma(f)]\!]_{\sigma(\alpha)} = \sigma([\![f]\!]_\alpha)$ for all automorphisms σ of $\bar{\mathbb{K}}$ over $\mathbb{K}$.

4.3. Global reduction with respect to differential operators

From Propositions 4.1 and 2.6, it follows that we can glue the local reductions at the finite points $\alpha \in \bar{\mathbb{K}}$ together into a reduction $[\,]_{\bar{\mathbb{K}}} : \bar{\mathbb{K}}(z) \to \bar{\mathbb{K}}(z)$, defined by

$$[f]_{\bar{\mathbb{K}}} = f_{(\infty,\star)} + \sum_{\alpha \in \bar{\mathbb{K}}} [f_{(\alpha)}]_\alpha. \qquad (4.5)$$

Notice that Proposition 2.3 implies that $[f]_{\bar{\mathbb{K}}} \in \mathbb{K}(z)$ whenever $f \in \mathbb{K}(z)$. Using Propositions 4.1-*iv* and 2.8, we may next define the *global reduction* $[\,]: \bar{\mathbb{K}}(z) \to \bar{\mathbb{K}}(z)$ with respect to L by

$$[f] \;=\; [[f]_{\bar{\mathbb{K}}}]_{\infty}. \tag{4.6}$$

Notice that $[f] \in \mathbb{K}(z)$ whenever $f \in \mathbb{K}(z)$.

Remark 4.4. Applying the above construction to the local reductions $[\![\,]\!]_\alpha$ from Remark 4.3 instead of $[\,]_\alpha$, we claim that we obtain a normal reduction. Indeed, given $f \in \operatorname{im} L$, we get $([\![f]\!]_\alpha)_{(\alpha)} = 0$ for all $\alpha \in \bar{\mathbb{K}}$, whence $([\![f_{(\alpha)}]\!]_\alpha)_{(\alpha)} = ([\![f]\!]_\alpha)_{(\alpha)} - (f_{(\bar{\alpha})})_{(\alpha)} = 0$ and $[\![f]\!]_{\bar{\mathbb{K}}} \in \operatorname{im} L \cap \bar{\mathbb{K}}[z]$. Since $[\![\,]\!]_\infty$ is locally normal, we conclude that $[\![f]\!] = 0$. We notice that $[\![\,]\!]$ is again symmetric in the sense that $[\![\mathbb{K}(z)]\!] \subseteq \mathbb{K}(z)$. It is also easily checked that $[\![\,]\!]$ is locally confined for each projection π_α.

It is interesting to consider the restriction of the global reduction $[\,]$ to certain differential subrings of $\bar{\mathbb{K}}(z)$ and $\mathbb{K}(z)$. More precisely, given a subset $\mathrm{A} \subseteq \bar{\mathbb{K}}$ of poles, the set $\bar{\mathbb{K}}(z)_{(\mathrm{A},\infty,\star)}$ forms a differential $\bar{\mathbb{K}}[z]$-subalgebra of $\bar{\mathbb{K}}(z)$. From Proposition 4.1-*ii*, it follows that $\bar{\mathbb{K}}(z)_{(\mathrm{A},\infty,\star)}$ is stable under reduction. If A is the set of zeros of a monic separable polynomial $\psi \in \mathbb{K}[z]$, then we observe that $\bar{\mathbb{K}}(z)_{(\mathrm{A},\infty,\star)}$ coincides with $\bar{\mathbb{K}}[z, \psi^{-1}]$. In that case, Proposition 2.3 implies that $\mathbb{K}[z, \psi^{-1}]$ is stable under reduction as well. We notice that $\mathbb{K}[z, \psi^{-1}]$ is finitely generated by ψ^{-1} as a $\mathbb{K}[z][\partial]$-submodule of $\mathbb{K}(z)$; for this reason we call it a *narrow* submodule of $\mathbb{K}(z)$.

THEOREM 4.5. *Let $L \in \mathbb{K}[z][\partial]$ be of order $r \geqslant 0$ and let $[\,]: \bar{\mathbb{K}}(z) \to \bar{\mathbb{K}}(z)$ be the global reduction with respect to L. Assume that* A *is the set of zeros of a monic separable polynomial $\psi \in \mathbb{K}[z]$ of degree d. Then the restriction $[\,]_{\mathbb{M}}$ of $[\,]$ to $\mathbb{M} = \mathbb{K}[z, \psi^{-1}]$ is a confined reduction with respect to L such that*

$$\dim_{\mathbb{K}} [\mathbb{M}]_{\mathbb{M}} \;\leqslant\; (2d+1)\, r + \deg_z L + 1.$$

Proof. Given $u \in \bar{\mathbb{K}}(z)$ and $\alpha \in \bar{\mathbb{K}} \setminus \mathrm{A}$ with $\operatorname{ord}_\alpha u > 0$, we notice that $\operatorname{ord}_\alpha L(u) = \operatorname{ord}_\alpha u + r$, whence $L(u) \notin \bar{\mathbb{K}}[z, \psi^{-1}]$. Given $f \in \bar{\mathbb{K}}(z)$ with $\{f\} = L(u)$, it follows that $u \in \bar{\mathbb{K}}[z, \psi^{-1}]$. In other words, $\bar{\mathbb{K}}[z, \psi^{-1}]$ is stable under reduction and so is $\mathbb{M}$, since the reduction commutes with all automorphisms of $\bar{\mathbb{K}}$ over $\mathbb{K}$.

Let us now turn to the dimension bound. Since $[\,]$ is $\bar{\mathbb{K}}$-linear, it suffices to show that $\dim_{\bar{\mathbb{K}}} [\bar{\mathbb{K}}[z, \psi^{-1}]] \leqslant (2d+1)\, r + \deg_z L + 1$. Let $\mathcal{F}_\alpha$ be defined as in Proposition 4.2 for all $\alpha \in \mathrm{A} \cup \{\infty\}$. Then the proposition implies that

$$\mathcal{F} = \{1\} \cup \bigcup_{\alpha \in \mathrm{A} \cup \{\infty\}} \mathcal{F}_\alpha$$

contains at most $|\mathcal{F}| \leqslant (2d+1)\, r + \deg_z L + 1$ elements. Now let $f \in \bar{\mathbb{K}}[z, \psi^{-1}]$, so that $f = f_{(\infty,\star)} + \sum_{\alpha \in \mathrm{A}} f_{(\alpha)}$. We have

$$[f]_{\bar{\mathbb{K}}} \;=\; f_{(\infty,\star)} + \sum_{\alpha \in \mathrm{A}} [f_{(\alpha)}]_\alpha.$$

For any $\alpha \in \mathrm{A}$, we have $\{f_{(\alpha)}\}_\alpha \in \bar{\mathbb{K}}(z)_{(\alpha,\infty,\star)}$ by Proposition 4.1-*ii*, whence $[f_{(\alpha)}]_\alpha \in \bar{\mathbb{K}}(z)_{(\alpha,\infty,\star)}$. Since $([f_{(\alpha)}]_\alpha)_{(\alpha)} \in \operatorname{Vect}(\mathcal{F}_\alpha)$ by Proposition 4.2, it follows that $[f_{(\alpha)}]_\alpha \in \bar{\mathbb{K}}[z] + \operatorname{Vect}(\mathcal{F}_\alpha)$, whence

$$[f]_{\bar{\mathbb{K}}} \;\in\; \bar{\mathbb{K}}[z] + \operatorname{Vect}(\mathcal{F}).$$

Now $[f] = [[f]_{\bar{\mathbb{K}}}]_\infty \in [f]_{\bar{\mathbb{K}}} + \bar{\mathbb{K}}[z]$ by Proposition 4.1-*ii* and $[f]_{(\infty)} \in \operatorname{Vect}(\mathcal{F}_\infty)$ by Proposition 4.2. We conclude that $[f] \in \operatorname{Vect}(\mathcal{F})$, with $\dim_{\bar{\mathbb{K}}} \operatorname{Vect}(\mathcal{F}) \leqslant (2d+1)\, r + \deg_z L + 1$. □

Remark 4.6. A bit of flexibility is possible regarding the choice of B_α in the definitions of the local reductions $[]_\alpha$. Taking $B_\alpha = 0$ for $\alpha \in \bar{\mathbb{K}}$ as in [16] may lead to slightly more confined reductions in Theorem 4.5. Conversely, by taking B_α somewhat larger, one may ensure that $\{f\}_\alpha \in \bar{\mathbb{K}}(z)_{(\alpha)}$ for all $f \in \bar{\mathbb{K}}(z)$ in Proposition 4.1-*ii*, so that $[]_\alpha$ and $[]_\beta$ commute for all $\alpha, \beta \in \bar{\mathbb{K}} \cup \{\infty\}$ in Proposition 4.1-*iii*. This makes it possible to treat the singularity at infinity in a more symmetric way and replace definition (4.6) by

$$[f] = f_{(\star)} + \sum_{\alpha \in \bar{\mathbb{K}} \cup \{\infty\}} [f_{(\alpha)}]_\alpha.$$

Remark 4.7. Since the reduction $[\![\,]\!]$ form Remark 4.4 is both normal and locally confined for each projection π_α with $\alpha \in \mathrm{A} \cup \{\infty\}$, its restriction to $(\mathbb{K}(z))_{(\mathrm{A},\infty,\star)}$ yields a normal confined reduction.

4.4. Back to reductions with respect to ∂

Assume still that $L \in \mathbb{K}[z][\partial]$ is a non-zero differential operator. Let $r \geqslant 0$ be its order, let $\phi = L_r \in \mathbb{K}[z]$ be its dominant coefficient, and let $\psi \in \mathbb{K}[z]$ be an arbitrary monic separable polynomial such that ϕ^{-1} is contained in

$$\mathbb{M} = \mathbb{K}[z, \psi^{-1}].$$

Given formal indeterminates $f, \ldots, f^{(r-1)}$, the set

$$\mathbb{D} = \mathbb{M} f \oplus \cdots \oplus \mathbb{M} f^{(r-1)} \tag{4.7}$$

admits a natural $\mathbb{M}[\partial]$-module structure for the derivation ∂ with

$$\partial f^{(i)} = \begin{cases} f^{(i+1)} & \text{if } i < r-1 \\ -\frac{1}{\phi}(L_0 f + \cdots + L_{r-1} f^{(r-1)}) & \text{if } i = r-1. \end{cases}$$

By construction, f is a formal solution to $L(f) = 0$ in $\mathbb{D}$.

Let us show how to construct a confined reduction $[]_{\mathbb{D}}$ with respect to ∂ on $\mathbb{D}$. We start with the confined reduction $[]_{\mathbb{M}}: \mathbb{M} \to \mathbb{M}$ with respect L^* provided by Theorem 4.5. Now consider

$$w = w_0 f + \cdots + w_s f^{(s)} \in \mathbb{D}$$

with $s < r$ and let us define $[w]_{\mathbb{D}}$ by induction on s. If $s = 0$, then we take

$$[w_0 f]_{\mathbb{D}} = [w_0]_{\mathbb{M}} f.$$

By construction, we have $\{w_0\}_{\mathbb{M}} \in \operatorname{im} L^*$, whence there exists a $u \in \mathbb{M}$ such that $\{w_0\}_{\mathbb{M}} = L^*(u)$. Using Proposition 3.1, it follows that

$$\{w\}_{\mathbb{D}} = \{w_0\}_{\mathbb{M}} f = L^*(u) f = L^*(u) f - u L(f) \in \partial \mathbb{D}.$$

Assume now that $s > 0$. Then we define

$$\begin{aligned} [w]_{\mathbb{D}} &= [w_0 f]_{\mathbb{D}} + [\tilde{w}]_{\mathbb{D}} \\ \tilde{w} &= -((\partial w_1) f + \cdots + (\partial w_s) f^{(s-1)}). \end{aligned}$$

We notice that

$$w - w_0 f - \tilde{w} = \partial(w_1 f + \cdots + w_s f^{(s-1)}) \in \partial \mathbb{D}.$$

It follows that

$$\{w\}_{\mathbb{D}} = \{w_0 f\}_{\mathbb{D}} + w - w_0 f - \tilde{w} + \{\tilde{w}\}_{\mathbb{D}} \in \partial \mathbb{D}.$$

In fact, an easy induction on s shows that

$$[w]_{\mathbb{D}} = [w_0 - \partial w_1 + \partial^2 w_2 + \cdots + (-1)^s \partial^s w_s]_{\mathbb{M}} f. \tag{4.8}$$

In particular, the constructed reduction is confined with $\operatorname{im}[]_{\mathbb{D}} = (\operatorname{im}[]_{\mathbb{M}}) f$:

THEOREM 4.8. *Let $L \in \mathbb{K}[z][\partial]$ be of order $r \geqslant 0$ and let $\psi \in \mathbb{K}[z]$ be a monic separable polynomial of degree d such that (4.7) admits the structure of a $\mathbb{K}[z, \psi^{-1}][\partial]$-module. Then (4.8) defines a confined reduction $[]_{\mathbb{D}}: \mathbb{D} \to \mathbb{D}$ with respect to ∂ and we have*

$$\dim_{\mathbb{K}} [\mathbb{D}]_{\mathbb{D}} \leqslant (2d+1) r + \deg_z L + 1. \qquad \square$$

4.5. First order systems

Let us now turn to first order systems and study analogues for the theory and results from the previous subsections. This means that the operator L from section 4.4 is now a first order operator in $\mathbb{K}(z)^{r \times r}[\partial]$ and we need to adapt the theory of sections 4.2 and 4.3 to the adjoint operator $L^* \in \mathbb{K}(z)^{r \times r}[\partial]^*$; recall that L^* operates at the right on matrices in $\bar{\mathbb{K}}(z)^{r \times r}$. By Proposition 2.1, we also recall that the construction of a reduction $[]_{L^*}$ with respect to L^* is equivalent to the construction of a reduction $[]_{UL^*}$ with respect to UL^*, for any invertible matrix $U \in \bar{\mathbb{K}}(z)^{r \times r}$. In particular, for the construction of a local reduction at $\alpha \in \bar{\mathbb{K}}$, it suffices to consider operators of the form $\partial (z - \alpha) + A \in \bar{\mathbb{K}}(z)^{r \times r}[\partial]^*$ with $A \in \bar{\mathbb{K}}(z)^{r \times r}$.

As the analogue for the operator L from sections 4.2 and 4.3, we will therefore start with an operator $L = \partial (z - \alpha) + A \in \bar{\mathbb{K}}(z)^{r \times r}[\partial]^*$ and show how to construct the local reduction $[]_\alpha = []_{\alpha,L}$ at $\alpha \in \bar{\mathbb{K}}$ with respect to L. It will be convenient to first put L into an even simpler form through multiplication by a suitable invertible matrix $U \in \bar{\mathbb{K}}(z)^{r \times r}$. Let $\lambda = \operatorname{ord}_\alpha A$. In analogy with [50], we say that an invertible matrix $U \in \bar{\mathbb{K}}[z, (z - \alpha)^{-1}]^{r \times r}$ is a *tail chopper* for L at α if we either have $\lambda = 0$ and $A = \mathrm{Id}_r$, or $\lambda > 0$, $\operatorname{ord}_\alpha (UA) = \lambda$, and for some $k \in \{0, \ldots, n\}$:

T1. If $k > 0$, then $\operatorname{ord}_\alpha U^\sharp \leqslant \lambda - 1$ and $\operatorname{rank}(UA^\sharp)_{(\alpha),\lambda} = k$ for the matrix $U^\sharp$ formed by the first k rows of U.

T2. If $k < r$, then $\operatorname{ord}_\alpha U^\flat = \lambda$ and $\operatorname{rank} U^\flat_{(\alpha),\lambda} = r - k$ for the matrix $U^\flat$ formed by the last $r - k$ rows of U.

Tail choppers can be computed using the following algorithm:

Algorithm 4.1
INPUT: $L = \partial (z - \alpha) + A \in \bar{\mathbb{K}}(z)^{r \times r}[\partial]^*$
OUTPUT: a tail chopper for L at α

Let $k := 0$, $\lambda := \operatorname{ord}_\alpha A$
Let $U^\sharp = 0_{r \times 0} \in \bar{\mathbb{K}}\left[z, \frac{1}{z - \alpha}\right]^{r \times k}$, $U^\flat = \mathrm{Id}_{r \times r} \in \bar{\mathbb{K}}\left[z, \frac{1}{z - \alpha}\right]^{r \times (r-k)}$
while $k < r$ and $\operatorname{ord}_\alpha U^\flat < \lambda$ **do**

- Let $C := (UA)_{(\alpha),\lambda} \in \bar{\mathbb{K}}^{r \times r}$.
- Using row sweeping, we determine an invertible matrix $T \in \bar{\mathbb{K}}^{r \times r}$ such that the first k rows of TC and T are the same, the first $k' \geqslant k$ rows of TC have rank k', and the last $r - k'$ rows of TC are zero.
- Set $U := TU$ and next multiply the last $r - k'$ rows of U with $(z - \alpha)^{-1}$.
- Set $k := k'$ and let $U^\sharp$ and $U^\flat$ be the matrices formed by the first k and last $r - k$ rows of U.

return U

PROPOSITION 4.9. *Algorithm 4.1 terminates and computes a tail chopper U for L at α.*

Proof. At every iteration of this loop, we notice that $i=\operatorname{ord}_{\alpha} U^{\flat}$ increases by one (as long as $k<r$) and it is easily checked that **T1** and $\operatorname{rank}(U^{\flat})_{(\alpha),i}=r-k$ are loop invariants. □

Remark 4.10. The above algorithm for the computation of tail choppers is somewhat reminiscent of Abramov's EG-elimination method [1].

Given a tail chopper U as above, let $\psi\in\bar{\mathbb{K}}[z]$ be the denominator of A (so that $\operatorname{ord}_{\alpha}\psi=\lambda$), and consider the operator $K=\psi U L\in\bar{\mathbb{K}}[z]^{r\times r}[\partial]^{*}$. We have

$$K \;=\; ((\psi U A)_{(\star)}+o(1))+(\partial-\alpha)((\psi U)_{(\star)}+o(1)), \qquad (z\to\alpha) \tag{4.9}$$

We consider K as the "preconditioned" version of L with respect to which we will now construct the local reduction $[]_{\alpha}=[]_{\alpha,K}$. From (4.9) it follows that the shift of K at α is simply $\tau_{\alpha}=0$ and its indicial polynomial is given by

$$\operatorname{ind}_{\alpha}(\rho) \;=\; (A U\psi)_{(\star)}-(U\psi)_{(\star)}\,\rho.$$

By construction, $\operatorname{ind}_{\alpha}(\rho)$ is invertible as a matrix in $\bar{\mathbb{K}}(\rho)^{r\times r}$, whence there exist at most r values of ρ where $\det\operatorname{ind}_{\alpha}(\rho)=0$. We are now in a position to formulate a counterpart of (4.3).

Given $f\in\bar{\mathbb{K}}(z)^{1\times r}$, let $\rho\in\mathbb{Z}$ and $c\in\bar{\mathbb{K}}^{1\times r}$ be such that

$$f \;=\; (z-\alpha)^{-\rho}(c+o(1)) \qquad (z\to\alpha).$$

We define $[f]_{\alpha}$ by induction over ρ, as follows:

$$[f]_{\alpha}=\begin{cases} f & \text{if } \rho\leqslant 0\\ c(z-\alpha)^{-\rho}+[f-c(z-\alpha)^{-\rho}]_{\alpha} & \text{if } \rho>0 \text{ and } \det\operatorname{ind}_{\alpha}(\rho)=0\\ [f-K(c\operatorname{ind}_{\alpha}(\rho)^{-1}(z-\alpha)^{-\rho})]_{\alpha} & \text{if } \rho>0 \text{ and } \det\operatorname{ind}_{\alpha}(\rho)\neq 0.\end{cases} \tag{4.10}$$

Although this definition works, it is slightly suboptimal in the sense that values of ρ for which $\det\operatorname{ind}_{\alpha}(\rho)=0$ are "skipped" altogether even though the system $y\operatorname{ind}_{\alpha}(\rho)=c$ might actually be solvable. Given an arbitrary scalar matrix $M\in\bar{\mathbb{K}}^{r\times r}$, let $M^{\dagger}\in\bar{\mathbb{K}}^{r\times r}$ denote a pseudo-inverse such that $cM^{\dagger}M=c$ for any $c\in\bar{\mathbb{K}}^{1\times r}M$. Also let $M^{\ddagger}\in\bar{\mathbb{K}}^{r\times r}$ denote a matrix with $c=cM^{\ddagger}+cM^{\dagger}M$ for all $c\in\bar{\mathbb{K}}^{1\times r}$. We may now replace (4.10) by

$$[f]_{\alpha}=\begin{cases} f & \text{if } \rho\leqslant 0\\ \frac{cP}{(z-\alpha)^{\rho}}+\left[f-\frac{cP}{(z-\alpha)^{\rho}}-K\left(\frac{cU}{(z-\alpha)^{\rho}}\right)\right]_{\alpha} & \text{if } \rho>0 \text{ and } P=\operatorname{ind}_{\alpha}(\rho)^{\ddagger}, U=\operatorname{ind}_{\alpha}(\rho)^{\dagger}.\end{cases} \tag{4.11}$$

This "more confined" definition ensures that $d=\dim_{\mathbb{K}}\pi_{(\alpha)}([\bar{\mathbb{K}}^{1\times r}(z)]_{\alpha})\leqslant r$ instead of $d\leqslant r^2$. The local reduction $[]_{\infty}$ at infinity can be defined in a similar fashion, *via* the change of variables $z\mapsto z^{-1}$.

Given an arbitrary first order operator $L\in\mathbb{K}(z)^{r\times r}[\partial]^{*}$, this completes our construction of the local reductions $[]_{\alpha}$ and $[]_{\infty}$ with respect to L. Proposition 4.1 and the theory from section 4.3 can be adapted *mutatis mutandis*. In particular, $[]_{\alpha}$ and $[]_{\infty}$ are indeed local reductions for $\pi_{(\alpha)}$ and $\pi_{(\infty)}$ that can be glued together into a global reduction $[]:\bar{\mathbb{K}}(z)^{1\times r}\to\bar{\mathbb{K}}(z)^{1\times r}$ with respect to L. Assuming that the mappings $M\mapsto M^{\dagger}$ and $M\mapsto M^{\ddagger}$ were taken to commute with all automorphisms of $\bar{\mathbb{K}}$ over $\mathbb{K}$, we again have $[f]\in\mathbb{K}(z)^{1\times r}$ for all $f\in\mathbb{K}(z)^{1\times r}$. The analogue of Theorem 4.5 is:

THEOREM 4.11. *Let* $L\in\mathbb{K}[z]^{r\times r}[\partial]^*$ *be of order one with invertible* L_1 *and let* $[]:\bar{\mathbb{K}}(z)^{1\times r}\to\bar{\mathbb{K}}(z)^{1\times r}$ *be the global reduction with respect to L. Assume that* A *is the set of zeros of a monic separable polynomial* $\psi\in\mathbb{K}[z]$ *of degree d. Then the restriction* $[]_{\mathbb{M}}$ *of the reduction* $[]$ *to* $\mathbb{M}=\mathbb{K}[z,\psi^{-1}]^{1\times r}$ *is a confined reduction with respect to L such that*

$$\dim_{\mathbb{K}}[\mathbb{M}]_{\mathbb{M}} \leqslant (2d+\deg_z L+2)\,r. \qquad \square$$

As to the counterpart for section 4.4, assume that $L=\partial-A\in\mathbb{K}(z)^{r\times r}[\partial]$ with $A\in\mathbb{K}(z)^{r\times r}$. Let $\psi\in\mathbb{K}[z]$ be an arbitrary monic separable polynomial such that $A\in\mathbb{A}^{r\times r}$, where $\mathbb{A}=\mathbb{K}[z,\psi^{-1}]$. Given a formal column vector F with entries $f_1,\ldots,f_r$, the set

$$\mathbb{D} = \mathbb{A}\,f_1\oplus\cdots\oplus\mathbb{A}\,f_r \tag{4.12}$$

admits the natural structure of a $\mathbb{K}[z][\partial]$-module for the derivation ∂ with $\partial F=AF$. Given a row vector $w\in\mathbb{M}=\mathbb{A}^{1\times r}$, we define the reduction on $\mathbb{D}$ by

$$[wF]_{\mathbb{D}} = [w]_{\mathbb{M}}F,$$

where $[]_{\mathbb{M}}$ is the confined reduction on $\mathbb{M}$ with respect to L^* as constructed above. By construction, there exists a $u\in\mathbb{M}$ with $\{w\}_{\mathbb{M}}=L^*(u)$, whence

$$\{wF\}_{\mathbb{D}} = \{w\}_{\mathbb{M}}F = (-\partial u-uA)F = (-\partial u-uA)F-u(\partial F-AF) = \partial(-uF).$$

The further results from section 4.4 now adapt *mutatis mutandis*.

Remark 4.12. Starting from a reduction $[]_{\mathbb{D}}:\mathbb{D}\to\mathbb{D}$ for ∂ with $[\mathbb{D}]_{\mathbb{D}}\subseteq\mathbb{M}f$, one may also define a reduction $[]_{\mathbb{M}}:\mathbb{M}\to\mathbb{M}$ for L^*: given $w\in\mathbb{M}$, there exists a $v\in\mathbb{M}$ with $[wF]_{\mathbb{D}}=vF$, and we take $[w]_{\mathbb{M}}=v$. Then there exists a $u\in\mathbb{M}$ with $\{wF\}_{\mathbb{D}}=\partial(uF)$. Using Lagrange's identity the other way around, we obtain $\{wF\}_{\mathbb{D}}=(w-v)F=\partial(uF)=(\partial u+uA)F=L^*(u)F$, whence $\{w\}_{\mathbb{M}}=w-v=L^*(u)$. This shows that $[]_{\mathbb{M}}$ is indeed a reduction with respect to L^*.

4.6. Reductions on D-modules with respect to general operators

We have seen how to construct reductions on $\mathbb{K}(z)$ with respect to general operators and how to construct reductions on D-modules with respect to ∂. It is natural to ask whether we can construct reductions on D-modules with respect to more general operators.

Let us first show how to do this in the setting of first order systems from the previous subsection. More precisely, assume that $\mathbb{A}$, $L=\partial-A\in\mathbb{K}(z)^{r\times r}[\partial]$ with $A\in\mathbb{A}^{r\times r}$ and $\mathbb{D}$ are as in the previous subsection. Given another matrix $\Phi\in\mathbb{A}^{s\times s}$, our aim is to construct a reduction on $\mathbb{D}^s$ with respect to $R=\partial-\Phi$. The idea is to use an explicit variant of Proposition 3.5. Given a matrix $U\in\mathbb{A}^{s\times r}$, we notice that

$$\begin{aligned}(\partial-\Phi)(UF) &= U\partial F+(\partial U)F-\Phi UF\\ &= U(\partial-A)(F)+(\partial U+UA-\Phi U)F.\end{aligned} \tag{4.13}$$

Now we may reinterpret $U\mapsto\partial U+UA-\Phi U$ as a first order differential operator of dimension rs on $\mathbb{M}:=\mathbb{A}^{s\times r}$, for which we may construct a confined reduction $[]_{\mathbb{M}}$ using the theory from the previous subsection. Given $W\in\mathbb{M}$, this means that there exists a $U\in\mathbb{M}$ with $\{W\}_{\mathbb{M}}=\partial U+UA-\Phi U$. Setting

$$[WF]_{\mathbb{D}^s} = [W]_{\mathbb{M}}F, \tag{4.14}$$

it follows from (4.13) and the defining equation $(\partial-A)(F)=0$ of F that

$$\{WF\}_{\mathbb{D}^s} = \{W\}_{\mathbb{M}}F = (\partial-\Phi)(UF) \in \operatorname{im}(\partial-\Phi).$$

This shows that (4.14) indeed defines a reduction with respect to $\partial-\Phi$ on $\mathbb{D}^s$.

The above construction admits several variants. Let us briefly sketch what happens if the first order matrix operator R is replaced by a monic operator $R \in \mathbb{A}[\partial]$. This time, we rather rely on Proposition 3.7 and more precisely on formula (3.5). Given a row matrix $u \in \mathbb{A}^{1 \times r}$ and using the fact that $(\partial - A)(F) = 0$, we have

$$R(uF) \;=\; \Xi((\partial - A)(F)) + \Psi_0(u)F \;=\; \Psi_0(u)F.$$

The idea is now to construct a confined reduction $[]_{\mathbb{M}}$ with respect to Ψ_0 on $\mathbb{M} = \mathbb{A}^{1 \times r}$. This can be done by generalizing the theory of sections 4.2-4.4 to allow for matrix coefficients, along the lines of section 4.5. We next define the confined reduction on $\mathbb{D}$ as usual by $[wF]_{\mathbb{D}} = [w]_{\mathbb{M}} F$. Given $u \in \mathbb{D}$, there then exists a $w \in \mathbb{M}$ with $\{w\}_{\mathbb{M}} = \Psi_0(u)$, whence

$$\{wF\}_{\mathbb{D}} \;=\; \{w\}_{\mathbb{M}} F \;=\; \Psi_0(u)F \;=\; R(uF) \;\in\; \operatorname{im} R.$$

We conclude that $[]_{\mathbb{D}}$ is a confined reduction with respect to R on $\mathbb{D}$.

5. DIFFERENCE REDUCTIONS

5.1. The field $\mathbb{K}(z)$ as a difference field

Let $\mathbb{K}$ be a field and consider a birational map $\tau \in \mathbb{K}(z)$ from the projective line over $\mathbb{K}$ into itself. Such an map τ is called a *homography* and is necessarily of the form

$$\tau = \tau_H = \tau_{H,z}, \qquad H = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \tau(z) = \frac{az + b}{cz + d},$$

where the matrix H is invertible. Given $\alpha \in \bar{\mathbb{K}} \cup \{\infty\}$ and $k \in \mathbb{Z}$, we will denote $\alpha^{(k)} = \tau^{-k}(\alpha)$.

Given a homography $\tau \in \mathbb{K}(z)$, the field $\mathbb{K}(z)$ becomes a difference field for the automorphism $\sigma = \sigma_H$ that postcomposes with τ:

$$\sigma(f) \;=\; f \circ \tau.$$

Notice that

$$\operatorname{ord}_{\alpha^{(k)}} \sigma^k(f) = \operatorname{ord}_{\alpha} f$$

for all $f \in \bar{\mathbb{K}}(z)$, $\alpha \in \bar{\mathbb{K}} \cup \{\infty\}$, and $k \in \mathbb{Z}$. More generally, if $\mathbb{K}$ was already equipped with an automorphism ω, then we may extend ω into an automorphism $\sigma = \sigma_{\omega,H}$ on $\mathbb{K}$ by taking

$$\sigma(f) \;=\; \omega(f) \circ \tau,$$

where ω acts coefficientwise on rational functions $f \in \mathbb{K}(z)$. Inversely, we must have $\tau = \sigma(z)$ for automorphisms σ of this kind, and we call τ the homography *associated* to σ.

Two important special cases are the shift operators S_η and the q-difference operator Q_q:

$$S_\eta \;=\; \sigma_H, \qquad H = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}, \qquad \eta \in \mathbb{K}$$

$$Q_q \;=\; \sigma_H, \qquad H = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}, \qquad q \in \mathbb{K}^{\neq}.$$

Notice that ∞ is a fixed point for τ for both of these examples.

In fact, we claim that the general case can essentially be reduced to one of these two special cases *via* a change of variables $z = \tau_T(u)$. Indeed, such a change of variables transforms difference equations in z and σ_H into difference equations in u and $\sigma_T^{-1} \circ \sigma_H \circ \sigma_T = \sigma_{T^{-1}HT}$. Assuming that we are allowed to extend $\mathbb{K}$ with the roots of the characteristic polynomial of H, we may first put H in Jordan normal form and then normalize it through division by d. After this, we have $H = S_\eta$ for some $\eta \in \mathbb{K}$ or $H = Q_q$ for some $q \in \mathbb{K}^{\neq}$.

For what follows, we will always assume that τ has infinite order. In the case of a shift operator S_η, this means that we should have $\eta \neq 0$; for a q-difference operator Q_q, the number q should not be a root of unity.

5.2. Difference modules

One particularity of automorphisms σ of $\mathbb{K}(z)$ as above with respect to the ordinary differentiation ∂ is that the sets $\mathbb{K}(z)_{(\alpha)}$ are no longer stable under σ. On the other hand, the set $\mathbb{K}(z)_{(\alpha\uparrow)}$ is stable under σ, where $\alpha\uparrow = \{\alpha, \alpha^{(1)}, \alpha^{(2)}, \ldots\}$ and $\mathbb{K}(z)_{(\alpha\uparrow)} = \bigoplus_{\beta \in \alpha\uparrow} \bar{\mathbb{K}}(z)_{(\beta)}$. It will be convenient to expand somewhat more on this observation and introduce a few more notations in addition to those from section 4.1.

Let us first introduce truncated analogues of the sets $\bar{\mathbb{K}}(z)_{(\alpha)}$ with $\alpha \in \bar{\mathbb{K}} \cup \{\infty\}$: given $\mu \in \mathbb{N}$, we define

$$\bar{\mathbb{K}}(z)_{(\alpha);\mu} = \bigoplus_{1 \leqslant k \leqslant \mu} \frac{\mathbb{K}}{(z-\alpha)^k} \qquad (\alpha \in \bar{\mathbb{K}})$$

$$\bar{\mathbb{K}}(z)_{(\infty);\mu} = \bigoplus_{1 \leqslant k \leqslant \mu} \mathbb{K} z^k.$$

The definitions extend to the case when $\mu = \infty$ by taking $\bar{\mathbb{K}}(z)_{(\alpha);\infty} = \bar{\mathbb{K}}(z)_{(\alpha)}$. Given $\mathrm{A} \subseteq \bar{\mathbb{K}} \cup \{\infty\}$ and a map $\mu: \bar{\mathbb{K}} \to \mathbb{N} \cup \{\infty\}$, we also denote

$$\bar{\mathbb{K}}(z)_{(\mathrm{A});\mu} = \bigoplus_{\alpha \in \mathrm{A}} \bar{\mathbb{K}}(z)_{(\alpha);\mu(\alpha)}.$$

Each of these vector spaces also come with projections similar to the $\pi_{(\alpha)}$ and $\pi_{(\mathrm{A})}$ that were already defined before. For example, $f_{(\alpha);\mu} = \pi_{(\alpha);\mu}(f)$ stands for truncation at order μ of $f_{(\alpha)} = \pi_{(\alpha)}(f)$. For $\mathrm{A} \subseteq \bar{\mathbb{K}} \cup \{\infty\}$, it is also convenient to introduce $\operatorname{ord}_{\mathrm{A}} f = \sup_{\alpha \in \mathrm{A}} \operatorname{ord}_\alpha f \in \mathbb{N}$.

Given $\alpha, \beta \in \bar{\mathbb{K}} \cup \{\infty\}$, we write $\alpha \sim \beta$ if there exists an $\ell \in \mathbb{Z}$ such that $\beta = \alpha^{(\ell)}$. This clearly defines an equivalence relation. We also write $\alpha \preccurlyeq \beta$ if there exists an $\ell \in \mathbb{N}$ such that $\beta = \alpha^{(\ell)}$. Notice that $\preccurlyeq$ is a partial ordering on $\bar{\mathbb{K}} \cup \{\infty\}$ by our assumption that τ has infinite order. Given $\alpha \in \bar{\mathbb{K}} \cup \{\infty\}$, we introduce the subsets $\alpha\uparrow$, $\alpha\downarrow$ and $\alpha\updownarrow$ of $\bar{\mathbb{K}} \cup \{\infty\}$ by

$$\alpha\uparrow = \{\alpha^{(k)} : k \in \mathbb{N}\}$$

$$\alpha\downarrow = \{\alpha^{(-k)} : k \in \mathbb{N}\}$$

$$\alpha\updownarrow = \{\alpha^{(k)} : k \in \mathbb{Z}\}.$$

For $\mathrm{A} \subseteq \bar{\mathbb{K}} \cup \{\infty\}$, we understand that $\mathrm{A}\uparrow = \{\alpha\uparrow : \alpha \in \mathrm{A}\}$, $\mathrm{A}\downarrow = \{\alpha\downarrow : \alpha \in \mathrm{A}\}$, etc. We notice that $\bar{\mathbb{K}}(z)_{(\mathrm{A}\uparrow)}$ is a $\bar{\mathbb{K}}[\sigma]$-module, whereas $\bar{\mathbb{K}}(z)_{(\mathrm{A}\downarrow)}$ is a $\bar{\mathbb{K}}[\sigma^{-1}]$-module. Since ∞ is a fixed point for τ, then we notice that $\bar{\mathbb{K}}(z)_{(\mathrm{A}\uparrow,\infty,\star)}$ is even a $\bar{\mathbb{K}}[z][\partial]$-module.

More generally, let $\mu: \mathrm{A}\uparrow \to \mathbb{N} \cup \{\infty\}$ be an increasing mapping in the sense that $\mu(\alpha^{(1)}) \geqslant \mu(\alpha)$ for all $\alpha \in \mathrm{A}\uparrow$. Then $\bar{\mathbb{K}}(z)_{(\mathrm{A}\uparrow);\mu}$ is a $\bar{\mathbb{K}}[\sigma]$-module. If $\mathrm{A} \subseteq \bar{\mathbb{K}}$, then the set $\bar{\mathbb{K}}[z] \oplus \bar{\mathbb{K}}(z)_{(\mathrm{A}\uparrow);\mu}$ is actually a $\bar{\mathbb{K}}[z][\sigma]$-module. We say that A is *symmetric* if it is stable under all automorphisms of $\bar{\mathbb{K}}$ over $\mathbb{K}$. Similarly, μ is *symmetric* if A is symmetric and μ commutes with all automorphisms of $\bar{\mathbb{K}}$ over $\mathbb{K}$. In that case, the intersection $\mathbb{K}(z)_{(\mathrm{A}\uparrow);\mu} = \bar{\mathbb{K}}(z)_{(\mathrm{A}\uparrow);\mu} \cap \mathbb{K}(z)$ is a $\mathbb{K}[\sigma]$-module and $\mathbb{K}[z] \oplus \mathbb{K}(z)_{(\mathrm{A}\uparrow);\mu}$ a $\mathbb{K}[z][\partial]$-module, whenever $\mathrm{A} \subseteq \bar{\mathbb{K}}$.

Assume that $\mathrm{A} \subseteq \bar{\mathbb{K}}$ and consider a $\bar{\mathbb{K}}[z][\sigma]$-submodule $\mathbb{M}$ of $\bar{\mathbb{K}}(z)$ with $1 \in \mathbb{M}$. It is not hard to check that such a module is necessarily of the form $\mathbb{M} = \bar{\mathbb{K}}[z] \oplus \bar{\mathbb{K}}(z)_{(\mathrm{A}\uparrow);\mu}$ for some increasing μ. We say that $\mathbb{M}$ is *narrow* if it is a finitely generated as a $\bar{\mathbb{K}}[z][\sigma]$-module. More generally, a finitely generated $\mathbb{K}[z][\sigma]$-module of $\mathbb{K}(z)$ of the form $\mathbb{K}[z] \oplus \mathbb{K}(z)_{(\mathrm{A}\uparrow);\mu}$ is said to be *narrow*. Such a narrow submodule of $\mathbb{K}(z)$ is always the intersection of $\mathbb{K}(z)$ with a narrow submodule $\bar{\mathbb{K}}[z] \oplus \bar{\mathbb{K}}(z)_{(\mathrm{A}\uparrow);\mu}$ of $\bar{\mathbb{K}}(z)$ for some symmetric μ.

Consider a narrow submodule $\bar{\mathbb{K}}[z] \oplus \bar{\mathbb{K}}(z)_{(\mathrm{A}\uparrow);\mu}$ that is finitely generated by $\mathcal{F} \subseteq \bar{\mathbb{K}}(z)$. Without loss of generality, we may replace each $f \in \mathcal{F}$ by the finite set $\{(z-\alpha)^{-\nu} : \alpha \in \bar{\mathbb{K}}, \operatorname{ord}_\alpha f = \nu > 0\}$ and remove any elements $(z-\alpha)^{-\nu'}$ such that $(z-\alpha)^{-\nu} \in \mathcal{F}$ for some $\nu > \nu'$. This means that we may assume that A is finite and $\mathcal{F} = \{(z-\alpha)^{-\nu(\alpha)} : \alpha \in \mathrm{A}\}$ for some function $\nu : \mathrm{A} \to \mathbb{N}^{>}$. Now consider the map $\nu\uparrow : \mathrm{A}\uparrow \to \mathbb{N}$ defined by

$$(\nu\uparrow)(\alpha) \;=\; \max_{\beta \preccurlyeq \alpha} \nu(\beta).$$

Then $\bar{\mathbb{K}}[z] \oplus \bar{\mathbb{K}}(z)_{(\mathrm{A}\uparrow);\nu\uparrow}$ coincides with the $\bar{\mathbb{K}}[z][\sigma]$-module generated by $\mathcal{F}$, so that $\mu = \nu\uparrow$.

5.3. Local and semi-local reduction with respect to shift operators

Let us now move our attention to a shift operator $L \in \mathbb{K}[z][\sigma^{-1}]$, where $\sigma = S_\eta$ with $\eta \in \mathbb{K}^{\neq}$. We assume that $L = L_r \sigma^{-r} + \cdots + L_0$ with both $L_r \neq 0$ and $L_0 \neq 0$. We took an operator with respect to σ^{-1} rather than σ since, for the applications that we have in mind (see Proposition 3.2), the operator L will actually be the adjoint of an operator in $\mathbb{K}[z][\sigma]$.

Since $\sigma^{-1} = \mathrm{e}^{-\eta\partial}$, the action of L at infinity can be approximated to any order by a differential operator in $\mathbb{K}[z][\partial]$. In particular, there still exist an indicial polynomial $\operatorname{ind}_\infty \in \mathbb{K}[\rho]$ and a shift $\tau_\infty \in \mathbb{Z}$ (in fact, $\tau_\infty \in \mathbb{N}$, since L admits coefficients in $\mathbb{K}[z]$) such that

$$L(z^\rho) \;=\; z^{\rho+\tau_\infty}(\operatorname{ind}_\infty(\rho) + o(1)) \qquad (z \to \infty).$$

Consequently, we may define the local reduction $[\,]_\infty : \bar{\mathbb{K}}(z) \to \bar{\mathbb{K}}(z)$ at infinity with respect to L in a similar way as in subsection 4.2.

Our next aim is to define a "semi-local" reduction $[\,]_\alpha : \bar{\mathbb{K}}(z) \to \bar{\mathbb{K}}(z)$ at $\alpha \in \bar{\mathbb{K}}$. Since $f \in \bar{\mathbb{K}}(z)$ admits a singularity at α if and only if $\sigma^k(f)$ admits a singularity at $\alpha^{(k)}$ for all $k \in \mathbb{N}$, we also need to take into account the behaviour at each of these shifted singularities. For this reason, the reduction $[\,]_\alpha$ will only be "semi-local".

Given $f \in \mathbb{K}(z)$, we define the *α-span* of f to be the largest integer $\rho = \operatorname{span}_\alpha f \in \mathbb{N}$ such that $f_{(\alpha^{(\rho)})} \neq 0$. If no such integer exists, then we set $\rho = \operatorname{span}_\alpha f = -\infty$. Assuming that $\rho \geqslant 0$, consider

$$g \;=\; \left(\frac{f}{L_0}\right)_{(\alpha^{(\rho)})} \;=\; \left(\frac{f_{(\alpha^{(\rho)})}}{L_0}\right)_{(\alpha^{(\rho)})}.$$

If $L_0(\alpha^{(\rho)}) \neq 0$, then the orders of f and g at $\alpha^{(\rho)}$ coincide,

$$L(g)_{(\alpha^{(\rho)})} \;=\; f_{(\alpha^{(\rho)})},$$

and $L(g) \in \bar{\mathbb{K}}[z] \oplus \bar{\mathbb{K}}(z)_{(\alpha^{(\rho-r)}, \ldots, \alpha^{(\rho)})}$. If $\rho \geqslant r$, then it follows that

$$(f - L(g))_{(\alpha\uparrow)} \;\in\; \bar{\mathbb{K}}(z)_{(\alpha, \ldots, \alpha^{(\rho-1)})}.$$

This allows us to define $[f]_\alpha$ by induction on $\rho = \operatorname{span}_\alpha f$ as follows:

$$[f]_\alpha \;=\; \begin{cases} f & \text{if } \rho < r \\ f_{(\alpha^{(\rho)})} + [f - f_{(\alpha^{(\rho)})}]_\alpha & \text{if } \rho \geqslant r \text{ and } L_0(\alpha^{(\rho)}) = 0 \\ \left[f - L\left(\left(\frac{f}{L_0}\right)_{(\alpha^{(\rho)})}\right)\right]_\alpha & \text{if } \rho \geqslant r \text{ and } L_0(\alpha^{(\rho)}) \neq 0. \end{cases} \tag{5.1}$$

We call $[\,]_\alpha$ the *downward semi-local reduction* with respect to L at α.

PROPOSITION 5.1. *Let $f \in \bar{\mathbb{K}}(z)$ and $\alpha \in \bar{\mathbb{K}} \cup \{\infty\}$. The mapping $[\,]_\alpha$ is a reduction with respect to L such that:*

i. The reduction $[\,]_\alpha$ is local for $\pi_{(\alpha\uparrow)}$.

ii. We have $\{f\}_\alpha \in \bar{\mathbb{K}}(z)_{(\alpha\uparrow, \infty, \star)}$.

iii. *The reductions* $[\,]_\alpha$ *and* $[\,]_\beta$ *are independent for any* $\alpha,\beta\in\bar{\mathbb{K}}$ *with* $\alpha\nsim\beta$.

iv. *If* $\alpha\in\bar{\mathbb{K}}$, *then* $(\{f\}_\infty)_{(\alpha\uparrow)}=0$.

Proof. If $\alpha=\infty$, then we have already noticed that $[\,]_\alpha$ is a reduction with respect to L such that *i* and *ii* are satisfied. If $\alpha\in\bar{\mathbb{K}}$, then the proof follows the same scheme as in Proposition 4.1; for the sake of completeness, we give it in full. Throughout the proof, we denote $h=f_{(\alpha^{(\rho)})}$ and $g=(f/L_0)_{(\alpha^{(\rho)})}$.

By induction on $\rho=\operatorname{span}_\alpha f$, let us first show that there exists a $u\in\sigma^r(\bar{\mathbb{K}}(z)_{(\alpha\uparrow)})$ with $\{f\}_\alpha=L(u)$; notice that this in particular implies *ii* and also $\operatorname{span}_\alpha [f]_\alpha\leqslant\rho$. If $\rho<r$, then we may take $u=0$. If $\rho\geqslant r$ and $L_0(\alpha^{(\rho)})=0$, then there exists a $u\in\sigma^r(\bar{\mathbb{K}}(z)_{(\alpha\uparrow)})$ with $\{f-h\}_\alpha=L(u)$, whence $\{f\}_\alpha=\{f-h\}_\alpha=L(u)$. If $\rho\geqslant r$ and $L_0(\alpha^{(\rho)})\neq 0$, then there exists a $v\in\sigma^r(\bar{\mathbb{K}}(z)_{(\alpha\uparrow)})$ with $\{f-L(g)\}_\alpha=L(v)$, where $g\in\sigma^r(\bar{\mathbb{K}}(z)_{(\alpha\uparrow)})$. Hence $\{f\}_\alpha=L(g)+\{f-L(g)\}_\alpha=L(g+v)$, so we may take $u=g+v\in\sigma^r(\bar{\mathbb{K}}(z)_{(\alpha\uparrow)})$.

Again by induction on ρ, let us next show that $[[f]_\alpha]_\alpha=[f]_\alpha$. If $\rho<r$, we have $[f]_\alpha=f=[[f]_\alpha]_\alpha$. If $\rho\geqslant r$ and $L_0(\alpha^{(\rho)})=0$, then $\operatorname{span}_\alpha [f-h]_\alpha\leqslant\rho-1$ implies $[h+[f-h]_\alpha]_\alpha=h+[[f-h]_\alpha]_\alpha$. It follows that $[[f]_\alpha]_\alpha=[h+[f-h]_\alpha]_\alpha=h+[[f-h]_\alpha]_\alpha=h+[f-h]_\alpha=[f]_\alpha$. If $\rho\geqslant r$ and $L_0(\alpha^{(\rho)})\neq0$, then we have $[[f]_\alpha]_\alpha=[[f-L(g)]_\alpha]_\alpha=[f-L(g)]_\alpha=[f]_\alpha$. This completes the proof that $[\,]_\alpha$ is a reduction with respect to L.

Decomposing $f=\varphi+\psi$ with $\varphi=f_{(\alpha\uparrow)}$, let us next show by induction on ρ that $[f]_\alpha=[\varphi]_\alpha+\psi$. If $\rho<r$, then we have $\varphi=0=[\varphi]_\alpha$ and $[f]_\alpha=f=\psi$. If $\rho\geqslant r$ and $L_0(\alpha^{(\rho)})=0$, then we may decompose $f=h+\tilde{f}$ and $\varphi=h+\tilde{\varphi}$ with $\tilde{\varphi}=\tilde{f}_{(\alpha\uparrow)}$, after which $[f]_\alpha=h+[\tilde{f}]_\alpha=h+[\tilde{\varphi}]_\alpha+\psi=[\varphi]_\alpha+\psi$. If $\rho\geqslant r$ and $L_0(\alpha^{(\rho)})\neq0$, then $(\varphi/L_0)_{(\alpha^{(\rho)})}=g$ implies $[\varphi]_\alpha=[\varphi-L(g)]_\alpha$. It follows that $[f]_\alpha=[f-L(g)]_\alpha=[f-\varphi]_\alpha+[\varphi-L(g)]_\alpha=[\varphi-L(g)]_\alpha+\psi=[\varphi]_\alpha+\psi$. This shows *i*.

Now let $\beta\in\bar{\mathbb{K}}$ be such that $\alpha\nsim\beta$. Then the projections $\pi_{(\alpha\uparrow)}$ and $\pi_{(\beta\uparrow)}$ are clearly orthogonal and it follows from *ii* that $(\{f\}_\alpha)_{(\beta)}=0$ and $(\{f\}_\beta)_{(\alpha)}=0$; this shows *iii*. The next fact *iv* also follows from *ii*. □

PROPOSITION 5.2. *Let* $f\in\bar{\mathbb{K}}(z)$ *and* $\alpha\in\bar{\mathbb{K}}$.

i. *We have* $([f]_\alpha)_{(\alpha\uparrow)}\in\bar{\mathbb{K}}(z)_{(\mathcal{F}_\alpha)}$, *where*

$$\mathcal{R}_\alpha=\{\alpha,\alpha^{(1)},\ldots,\alpha^{(r-1)}\}\cup\{\alpha^{(k)}:k\geqslant r,L_0(\alpha^{(k)})=0\}. \tag{5.2}$$

ii. *For some* $u\in\bar{\mathbb{K}}(z)_{(\alpha\uparrow)}$, *we have* $\{f\}_\alpha=L(\sigma^r(u))$, $\operatorname{span}_\alpha\sigma^r(u)\leqslant\operatorname{span}_\alpha f$, *and* $\operatorname{ord}_{\alpha\uparrow}u\leqslant\operatorname{ord}_{\alpha\uparrow}f$.

iii. *We have* $\operatorname{span}_\alpha [f]_\alpha\leqslant\operatorname{span}_\alpha f$.

iv. *We have* $\operatorname{ord}_{\alpha\uparrow}[f]_\alpha\leqslant\operatorname{ord}_{\alpha\uparrow}f$.

Proof. Assume the notations from the previous proposition. Let us prove *i* by induction on ρ. If $\rho<r$, then $([f]_\alpha)_{(\alpha\uparrow)}=0$. If $\rho\geqslant r$ and $L_0(\alpha^{(\rho)})=0$, then $h\in\bar{\mathbb{K}}(z)_{(\mathcal{R}_\alpha)}$; since the induction hypothesis implies $([f-h]_\alpha)_{(\alpha\uparrow)}\in\bar{\mathbb{K}}(z)_{(\mathcal{R}_\alpha)}$, we obtain $([f]_\alpha)_{(\alpha\uparrow)}\in\bar{\mathbb{K}}(z)_{(\mathcal{R}_\alpha)}$. If $\rho\geqslant r$ and $L_0(\alpha^{(\rho)})\neq0$, then the induction hypothesis yields $([f]_\alpha)_{(\alpha\uparrow)}=([f-L(g)]_\alpha)_{(\alpha\uparrow)}\in\bar{\mathbb{K}}(z)_{(\mathcal{R}_\alpha)}$.

Let us next prove *ii*, again by induction over ρ. If $\rho<r$, then we may take $u=0$. If $\rho\geqslant r$ and $L_0(\alpha^{(\rho)})=0$, then let $u\in\bar{\mathbb{K}}(z)_{(\alpha\uparrow)}$ be such that $\{f-h\}_\alpha=L(\sigma^r(u))$, $\operatorname{span}_\alpha\sigma^r(u)\leqslant\operatorname{span}_\alpha(f-h)$, and $\operatorname{ord}_{\alpha\uparrow}u\leqslant\operatorname{ord}_{\alpha\uparrow}(f-h)$. Then we have $\{f\}_\alpha=L(\sigma^r(u))$, $\operatorname{span}_\alpha\sigma^r(u)\leqslant\operatorname{span}_\alpha(f-h)\leqslant\rho-1<\operatorname{span}_\alpha f$, and $\operatorname{ord}_{\alpha\uparrow}u\leqslant\operatorname{ord}_{\alpha\uparrow}(f-h)\leqslant\operatorname{ord}_{\alpha\uparrow}f$. If $\rho\geqslant r$ and $L_0(\alpha^{(\rho)})\neq0$, then let $v\in\bar{\mathbb{K}}(z)_{(\alpha\uparrow)}$ be such that $\{f-L(g)\}_\alpha=L(\sigma^r(v))$, $\operatorname{span}_\alpha\sigma^r(v)\leqslant\operatorname{span}_\alpha(f-L(g))\leqslant\rho-1$, and $\operatorname{ord}_{\alpha\uparrow}v\leqslant\operatorname{ord}_{\alpha\uparrow}(f-L(g))$. We have $\{f\}_\alpha=\{f-L(g)\}_\alpha+L(g)$. Since $\rho\geqslant r$, we have $\sigma^{-r}(g)\in\bar{\mathbb{K}}(z)_{(\alpha\uparrow)}$, $\operatorname{span}_\alpha\sigma^r(\sigma^{-r}(g))=\rho$, and $\operatorname{ord}_{\alpha\uparrow}\sigma^{-r}(g)=\operatorname{ord}_{\alpha^{(\rho)}\uparrow}g=\operatorname{ord}_{\alpha^{(\rho)}}f\leqslant\operatorname{ord}_{\alpha\uparrow}f$. It thus suffices to take $u=v+\sigma^{-r}(g)$.

With u as above, we have $\operatorname{span}_\alpha\{f\}_\alpha=\operatorname{span}_\alpha L(\sigma^r(u))\leqslant\operatorname{span}_\alpha\sigma^r(u)\leqslant\operatorname{span}_\alpha f$, which implies *iii*. Similarly, $\operatorname{ord}_{\alpha\uparrow}\{f\}_\alpha=\operatorname{ord}_{\alpha\uparrow}(L\sigma^r)(u)\leqslant\operatorname{ord}_{\alpha\uparrow}u\leqslant\operatorname{ord}_{\alpha\uparrow}f$ yields *iv*. □

5.4. Aligned downward reduction with respect to shift operators

Let us now turn to the problem of combining the semi-local reductions $[]_\alpha$ where $\alpha \in \mathrm{A}$ for some subset $\mathrm{A} \subseteq \bar{\mathbb{K}}$. Given a general subset $\mathrm{A} \subseteq \bar{\mathbb{K}}$, the set $\mathrm{A}\uparrow$ admits the partitioning

$$\mathrm{A}\uparrow = \coprod_{\alpha \in \mathrm{A}} \mathrm{I}_\alpha, \qquad \mathrm{I}_\alpha = \alpha\uparrow \setminus \{\beta \in \mathrm{A} : \alpha \prec \beta\}\uparrow.$$

The notion of α-span can also be generalized:

$$\mathrm{span}_\mathrm{A} f = \max\{\mathrm{span}_\alpha f_{(\mathrm{I}_\alpha)} : \alpha \in \mathrm{A}\} \in \mathbb{N} \cup \{-\infty\}.$$

In this subsection we start with the case when A is *aligned* in the sense that $\mathrm{A} = \{\alpha_1, \ldots, \alpha_t\}$ with $\alpha_1 \prec \cdots \prec \alpha_t$. In that case, we notice that

$$\begin{aligned} \mathrm{I}_{\alpha_i} &= \{\beta \in \bar{\mathbb{K}} : \alpha_i \preccurlyeq \beta \prec \alpha_{i+1}\} \qquad (i = 1, \ldots, t-1) \\ \mathrm{I}_{\alpha_t} &= \{\beta \in \bar{\mathbb{K}} : \alpha_t \preccurlyeq \beta\}. \end{aligned}$$

Although the construction from the previous subsection provides us with a local reduction $[]_{\alpha_1}$ for the projection $\pi_{(\mathrm{A}\uparrow)} = \pi_{(\alpha_1\uparrow)}$, this reduction is sometimes suboptimal due to the fact that it may take a long time to reduce functions for which $\mathrm{span}_\mathrm{A} f$ is small. For this reason, we introduce an alternative semi-local reduction $[]_\mathrm{A} : \bar{\mathbb{K}}(z) \to \bar{\mathbb{K}}(z)$ by

$$[f]_\mathrm{A} = f_{(\overline{\mathrm{A}\uparrow})} + \sum_{\alpha \in \mathrm{A}} [f_{(\mathrm{I}_\alpha)}]_\alpha.$$

We call $[]_\mathrm{A}$ the *downward reduction* with respect to L for A. It is "less confined" than $[]_{\alpha_1}$ but relies on the reduction $[]_{\alpha_i}$ in order to reduce the polar parts for the singularities in I_{α_i}.

PROPOSITION 5.3. *Let $\mathrm{A} = \{\alpha_1, \ldots, \alpha_t\} \subseteq \bar{\mathbb{K}}$ be aligned with $\alpha_1 \prec \cdots \prec \alpha_t$ and $f \in \bar{\mathbb{K}}(z)$. Then the mapping $[]_\mathrm{A}$ is a reduction with respect to L that satisfies:*

i. The reduction $[]_\mathrm{A}$ is local for $\pi_{(\mathrm{A}\uparrow)}$.

ii. We have $\{f\}_\mathrm{A} \in \bar{\mathbb{K}}(z)_{(\mathrm{A}\uparrow, \infty, \star)}$.

iii. The reductions $[]_\mathrm{A}$ and $[]_\mathrm{B}$ are independent for any aligned $\mathrm{B} \subseteq \bar{\mathbb{K}}$ with $\mathrm{A}\updownarrow \neq \mathrm{B}\updownarrow$.

iv. We have $(\{f\}_\infty)_{(\mathrm{A}\uparrow)} = 0$.

Proof. Let us denote $f_\# = f_{(\overline{\mathrm{A}\uparrow})}$ and $f_i = f_{(\mathrm{I}_{\alpha_i})}$ for $i = 1, \ldots, t$, so that $f = f_\# + f_1 + \cdots + f_t$. Also let $\varphi_i = [f_i]_{\alpha_i} = g_i + h_i$ with $g_i = (\varphi_i)_{(\overline{\mathrm{A}\uparrow})}$ and $h_i = (\varphi_i)_{(\mathrm{A}\uparrow)}$ for $i = 1, \ldots, t$.

Let us first show that $[]_\mathrm{A}$ is a reduction with respect to L. We clearly have $\{f\}_\mathrm{A} = \{f_1\}_{\alpha_1} + \cdots + \{f_t\}_{\alpha_t} \in \mathrm{im}\, L$. By Propositions 5.1-*ii* and 5.2-*iii*, we notice that $\varphi_i \in \bar{\mathbb{K}}(z)_{(\mathrm{I}_i, \infty, \star)}$ for $i = 1, \ldots, t$, whence $h_i = (\varphi_i)_{(\mathrm{I}_{\alpha_i})}$ and $[g_i + h_i]_{\alpha_i} = g_i + [h_i]_{\alpha_i} = g_i + h_i$. Then $[f]_\mathrm{A} = f_\# + g_1 + \cdots + g_t + h_1 + \cdots + h_t$ equals $[[f]_\mathrm{A}]_\mathrm{A} = f_\# + g_1 + \cdots + g_t + [h_1]_{\alpha_1} + \cdots + [h_t]_{\alpha_t}$.

The reduction $[]_\mathrm{A}$ is also local for $\pi_{(\mathrm{A}\uparrow)}$ since $[f]_\mathrm{A} = f_\# + \varphi_1 + \cdots + \varphi_t = f_\# + [f_1 + \cdots + f_t]_\mathrm{A}$. The relation $\{f\}_\mathrm{A} = \{f_1\}_{\alpha_1} + \cdots + \{f_t\}_{\alpha_t}$ shows that $\{f\}_\mathrm{A} \in \bar{\mathbb{K}}(z)_{(\mathrm{A}\uparrow, \infty, \star)}$. The properties *iii* and *iv* are shown in a similar way as for Proposition 5.1. □

PROPOSITION 5.4. *Let $\mathrm{A} = \{\alpha_1, \ldots, \alpha_t\} \subseteq \bar{\mathbb{K}}$ be aligned with $\alpha_1 \prec \cdots \prec \alpha_t$ and $f \in \bar{\mathbb{K}}(z)$.*

i. We have $([f]_\mathrm{A})_{(\mathrm{A}\uparrow)} \in \bar{\mathbb{K}}(z)_{(\mathcal{R}_\mathrm{A})}$, where $\mathcal{R}_\mathrm{A} = \mathcal{R}_{\alpha_1} \cup \cdots \cup \mathcal{R}_{\alpha_t}$, using the notation from (5.2).

ii. For some $u \in \bar{\mathbb{K}}(z)_{(\mathrm{A}\uparrow)}$, we have $\{f\}_\mathrm{A} = L(\sigma^r(u))$ and $\mathrm{ord}_{\mathrm{I}_{\alpha_i}} u \leqslant \mathrm{ord}_{\mathrm{I}_{\alpha_i}} f$ $(i = 1, \ldots, t)$.

iii. We have $\mathrm{span}_\mathrm{A} [f]_\mathrm{A} \leqslant \mathrm{span}_\mathrm{A} f$.

iv. We have $\mathrm{ord}_{\mathrm{I}_{\alpha_i}} [f]_\mathrm{A} \leqslant \mathrm{ord}_{\mathrm{I}_{\alpha_i}} f$ $(i = 1, \ldots, t)$.

Proof. With the same notations as in the previous proof, Proposition 5.2-*i* implies $h_i \in \bar{\mathbb{K}}(z)_{(\mathcal{R}_{\alpha_i})}$ for $i=1,\ldots,t$, whence $([f]_{\mathrm{A}})_{(\mathrm{A}\uparrow)} = h_1 + \cdots + h_t \in \bar{\mathbb{K}}(z)_{(\mathcal{R}_{\mathrm{A}})}$. This proves *i*.

Proposition 5.2-*ii* implies the existence of $u_1 \in \bar{\mathbb{K}}(z)_{(\alpha_1\uparrow)}, \ldots, u_t \in \bar{\mathbb{K}}(z)_{(\alpha_t\uparrow)}$ with $\{f_i\}_{\alpha_i} = L(\sigma^r(u_i))$, $\operatorname{span}_{\alpha_i} \sigma^r(u_i) \leqslant \operatorname{span}_\alpha f_i$, and $\operatorname{ord}_{\alpha_i\uparrow} u_i \leqslant \operatorname{ord}_{\alpha_i\uparrow} f_i$, for $i=1,\ldots,t$. In particular, this yields $u_i \in \bar{\mathbb{K}}(z)_{(\mathrm{I}_{\alpha_i})}$. Now consider $u = u_1 + \cdots + u_t$ with $\{f\}_{\mathrm{A}} = L(\sigma^r(u))$. For $i=1,\ldots,t$, we conclude that $\operatorname{ord}_{\mathrm{I}_{\alpha_i}} u = \operatorname{ord}_{\mathrm{I}_{\alpha_i}} u_i \leqslant \operatorname{ord}_{\mathrm{I}_{\alpha_i}} f_i = \operatorname{ord}_{\mathrm{I}_{\alpha_i}} f$.

As to *iii*, we have $\operatorname{span}_{\mathrm{A}} [f]_{\mathrm{A}} = \max\{\operatorname{span}_{\alpha_1} h_1, \ldots, \operatorname{span}_{\alpha_t} h_t\} = \max\{\operatorname{span}_{\alpha_1} \varphi_1, \ldots, \operatorname{span}_{\alpha_t} \varphi_t\} \leqslant \max\{\operatorname{span}_{\alpha_1} f_1, \ldots, \operatorname{span}_{\alpha_t} f_t\} = \operatorname{span}_{\mathrm{A}} f$, thanks to Proposition 5.2-*iii*. Proposition 5.2-*iv* similarly implies *iv*, since $\operatorname{ord}_{\mathrm{I}_{\alpha_i}} [f]_{\mathrm{A}} = \operatorname{ord}_{\mathrm{I}_{\alpha_i}} h_i = \operatorname{ord}_{\alpha_i\uparrow} h_i = \operatorname{ord}_{\alpha_i\uparrow} \varphi_i \leqslant \operatorname{ord}_{\alpha_i\uparrow} f_i = \operatorname{ord}_{\mathrm{I}_{\alpha_i}} f_i = \operatorname{ord}_{\mathrm{I}_{\alpha_i}} f$ for $i=1,\ldots,t$. □

COROLLARY 5.5. *Let* $\mathrm{A} = \{\alpha_1, \ldots, \alpha_t\} \subseteq \bar{\mathbb{K}}$ *be aligned with* $\alpha_1 \prec \cdots \prec \alpha_t$, *let* $\nu: \mathrm{A} \to \mathbb{N}$, *and consider the narrow* $\bar{\mathbb{K}}[z][\sigma]$*-module* $\bar{\mathbb{M}} = \bar{\mathbb{K}}[z] \oplus \bar{\mathbb{K}}(z)_{(\mathrm{A}\uparrow);\nu\uparrow}$. *Then* $\bar{\mathbb{M}}$ *is stable under* $[]_{\mathrm{A}}$.

Proof. Let $f \in \bar{\mathbb{K}}(z)_{(\mathrm{A}\uparrow,\infty,\star)}$ and decompose it as $f = f_\# + f_1 + \cdots + f_t$ as above. From Proposition 5.3-*ii*, it follows that $[f]_{\mathrm{A}} \in \bar{\mathbb{K}}(z)_{(\mathrm{A}\uparrow,\infty,\star)}$. Notice that $f \in \bar{\mathbb{M}}$ if and only if $\operatorname{ord}_{\mathrm{I}_{\alpha_i}} f_i \leqslant (\nu\uparrow)(\alpha_i)$ for $i=1,\ldots,n$. Assuming that that $f \in \bar{\mathbb{M}}$, it follows from Proposition 5.4-*iv* that $\operatorname{ord}_{\mathrm{I}_{\alpha_i}} ([f]_{\mathrm{A}})_{(\mathrm{I}_{\alpha_i})} = \operatorname{ord}_{\mathrm{I}_{\alpha_i}} [f]_{\mathrm{A}} \leqslant \operatorname{ord}_{\mathrm{I}_{\alpha_i}} f = \operatorname{ord}_{\mathrm{I}_{\alpha_i}} f_i \leqslant (\nu\uparrow)(\alpha_i)$ for $i=1,\ldots,n$, whence $[f]_{\mathrm{A}} \in \bar{\mathbb{M}}$. □

Remark 5.6. In analogy with Remark 4.3, one may define a local normalization $[\![]\!]_{\mathrm{A}}$ of $[]_{\mathrm{A}}$. This time, it follows from Proposition 5.4 that the space $E = ([\operatorname{im} L]_\alpha)_{(\alpha)}$ is spanned by $\{(z-\alpha)^{-k} : \alpha \in \mathrm{A}\uparrow \cap \mathrm{A}\downarrow, 0 < k \leqslant (\nu\uparrow)(\alpha)\}$. Again the computation of a basis of E can be expensive and there exists no general polynomial time algorithm for doing so.

5.5. General downward reduction with respect to shift operators

Let us now assume that $\mathrm{A} \subseteq \bar{\mathbb{K}}$ is such that $\alpha\updownarrow \cap \mathrm{A}$ is finite for every $\alpha \in \mathrm{A}$. The equivalence relation $\sim$ restricted to A then leads to a natural partitioning

$$\mathrm{A} = \coprod_{i \in I} \mathrm{A}_i, \qquad I = \mathrm{A}/\sim, \qquad \mathrm{A}_{\alpha/\sim} = \alpha\updownarrow \cap \mathrm{A}, \tag{5.3}$$

where each component A_i is aligned, and $\mathrm{A}_i\updownarrow \neq \mathrm{A}_j\updownarrow$ whenever $i \neq j$. This allows us to define

$$\begin{aligned} [f]_{\mathrm{A}} &= f_{(\overline{\mathrm{A}\uparrow})} + \sum_{i \in I} [f_{(\mathrm{A}_i\uparrow)}]_{\mathrm{A}_i} \\ [f]_{\mathrm{A},\infty} &= [[f]_{\mathrm{A}}]_\infty. \end{aligned}$$

By Propositions 5.3-*i*, 5.3-*iii* and 2.6, we get that $[]_{\mathrm{A}}: \bar{\mathbb{K}}(z) \to \bar{\mathbb{K}}(z)$ is a local reduction for $\pi_{(\mathrm{A}\uparrow)}$ with respect to L. From Propositions 5.3-*iv* and 2.8, we deduce that $[]_{\mathrm{A},\infty}: \bar{\mathbb{K}}(z) \to \bar{\mathbb{K}}(z)$ is a reduction with respect to L. We call $[]_{\mathrm{A}}$ and $[]_{\mathrm{A},\infty}$ the *downward reductions* with respect to L for A and $\mathrm{A} \cup \{\infty\}$. If A is symmetric, then $[\mathbb{K}(z)]_{\mathrm{A}} \subseteq \mathbb{K}(z)$ and $[\mathbb{K}(z)]_{\mathrm{A},\infty} \subseteq \mathbb{K}(z)$, by Proposition 2.3.

Remark 5.7. When taking A to contain a section of $\bar{\mathbb{K}}$ with respect to the equivalence relation $\sim$, we ensure that $\bar{\mathbb{K}} = \coprod_{i \in I} \mathrm{A}_i\updownarrow$. Nevertheless, each $\mathrm{A}_i\uparrow$ can only contain "half" of the set $\mathrm{A}_i\updownarrow$, since A_i is assumed to be finite. This means that $\coprod_{i \in I} \mathrm{A}_i\uparrow$ cannot cover $\bar{\mathbb{K}}$, so $[]_{\mathrm{A},\infty}$ cannot truly qualify as a "global reduction". In order to obtain a global reduction, one needs to combine downward and "upward" reductions (constructed from downward reductions with respect to the operator $L\sigma^r \in \mathbb{K}[z][\sigma]$). Even then, the obtained "global" reduction crucially depends on the choice of A. In fact, this kind of reductions will not be useful for our application to creative telescoping; we rather need reductions on suitable narrow submodules.

In order to obtain a counterpart for Theorem 4.5, let us now turn our attention to restrictions of downward reductions to suitable narrow submodules of $\mathbb{K}(z)$.

THEOREM 5.8. *Let $L\in\mathbb{K}[z][\sigma]$ be of order $r\geqslant 0$. Consider a narrow $\mathbb{K}[z][\sigma]$-submodule*

$$\mathbb{M} \;=\; \mathbb{K}[z]\oplus\mathbb{K}(z)_{(\mathrm{A}\uparrow);\mu}$$

of $\mathbb{K}(z)$, where $\mathrm{A}\subseteq\bar{\mathbb{K}}$ is a symmetric finite set and $\mu=\nu\uparrow$ for some symmetric function $\nu:\mathrm{A}\to\mathbb{N}^{>}$. Let $\mathrm{Z}=\{\alpha\in\mathrm{A}\uparrow:L_0(\alpha)=0\}$, $\mu(\mathrm{A})=\sum_{\alpha\in\mathrm{A}}\mu(\alpha)$, and $\mu(\mathrm{Z})=\sum_{\alpha\in\mathrm{Z}}\mu(\alpha)$. Then the restriction $[]_{\mathbb{M}}$ of $[]_{\mathrm{A},\infty}$ to $\mathbb{M}$ is a confined reduction with respect to L with

$$\dim_{\mathbb{K}}[\mathbb{M}]_{\mathbb{M}} \;\leqslant\; r\,\mu(\mathrm{A})+\mu(\mathrm{Z})+\deg_z L+r.$$

Given $f\in\mathbb{M}$, there exists a $u\in\mathbb{M}$ with $\{f\}_{\mathrm{A},\infty}=L(\sigma^r(u))$.

Proof. Let A be partitioned as in (5.3). Let us first show that the narrow $\bar{\mathbb{K}}[z][\sigma]$-submodule

$$\bar{\mathbb{M}} \;=\; \bar{\mathbb{K}}[z]\oplus\bar{\mathbb{K}}(z)_{(\mathrm{A}\uparrow);\mu}$$

of $\bar{\mathbb{K}}(z)$ is stable under downward reduction $[]_{\mathrm{A},\infty}$. The module $\bar{\mathbb{M}}$ is clearly stable under the projection $\pi_{(S)}$ for any subset $S\subseteq\bar{\mathbb{K}}\cup\{\infty,\star\}$. Since $[]_\infty$ is local for $\pi_{(\infty)}$, we also have stability under the reduction $[]_\infty$. Now given $f\in\bar{\mathbb{M}}$ and $i\in I$, we have $[f]_{\mathrm{A}_i}=[f_{(\mathrm{A}_i\uparrow)}]_{\mathrm{A}_i}+f_{(\overline{\mathrm{A}_i\uparrow})}$ with $f_{(\mathrm{A}_i\uparrow)},f_{(\overline{\mathrm{A}_i\uparrow})}\in\bar{\mathbb{M}}$ and $[f_{(\mathrm{A}_i\uparrow)}]_{\mathrm{A}_i}$ by Corollary 5.5. For any $f\in\bar{\mathbb{M}}$, it follows that $[f]_{\mathrm{A}}\in\bar{\mathbb{M}}$ and $[f]_{\mathrm{A},\infty}\in\bar{\mathbb{M}}$. Since $[]_{\alpha,\infty}$ commutes with all automorphisms of $\bar{\mathbb{K}}$ over $\mathbb{K}$, we also get the stability of $\mathbb{M}$ under $[]_{\mathrm{A},\infty}$.

Given $f\in\bar{\mathbb{M}}$, let us next prove the existence of some $u\in\bar{\mathbb{M}}$ with $\{f\}_{\mathrm{A},\infty}=L(\sigma^r(u))$. For each $i\in I$, setting $f_i=f_{(\mathrm{A}_i\uparrow)}$, Proposition 5.4-*ii* yields an element $u_i\in\bar{\mathbb{K}}(z)_{(\mathrm{A}_i\uparrow)}$ with $\{f_i\}_{\mathrm{A}_i}=L(\sigma^r(u_i))$ and $\operatorname{ord}_\alpha u\leqslant\mu(\alpha)$ for all $\alpha\in\mathrm{A}_i\uparrow$, whence $u_i\in\bar{\mathbb{M}}$. At infinity, there also exists a $v\in\bar{\mathbb{K}}[z]$ with $\{[f]_{\mathrm{A}}\}_\infty=L(v)$, and $\sigma^{-r}(v)\in\bar{\mathbb{K}}[z]$. Taking $u=\sum_{i\in I}u_i+\sigma^{-r}(v)$, it follows that $u\in\bar{\mathbb{M}}$ and

$$\{f\}_{\mathrm{A},\infty} \;=\; \{f\}_{\mathrm{A}}+\{[f]_{\mathrm{A}}\}_\infty \;=\; \sum_{i\in I}\{f_i\}_{\mathrm{A}_i}+\{[f]_{\mathrm{A}}\}_\infty \;=\; \sum_{i\in I}L(\sigma^r(u_i))+L(v) \;=\; L(\sigma^r(u)).$$

If $f\in\mathbb{M}$ and u admits coefficients in an extension field $\mathbb{L}$ of $\mathbb{K}$ of degree s, then we may replace u by $s^{-1}\operatorname{Tr}_{\mathbb{L}/\mathbb{K}}u\in\mathbb{M}$, since $\phi(u)\in\bar{\mathbb{M}}$ and $\{\phi(f)\}_{\mathrm{A},\infty}=L(\sigma^r(\phi(u)))$ for every automorphism ϕ of $\bar{\mathbb{K}}$ over $\mathbb{K}$. Notice that this shows in particular that $[]_{\mathbb{M}}$ is a reduction on $\mathbb{M}$.

Let us finally show that

$$\dim_{\bar{\mathbb{K}}}[\bar{\mathbb{M}}]_{\mathrm{A},\infty} \;\leqslant\; r\,\mu(\mathrm{A})+\mu(\mathrm{Z})+\deg_z L+r, \tag{5.4}$$

which also implies the bound on $\dim_{\mathbb{K}}[\mathbb{M}]_{\mathrm{A},\infty}$. For each $\alpha\in\mathrm{A}$, let $\mathcal{R}_\alpha$ be as in (5.2) and notice that $\mathcal{R}=\bigcup_{\alpha\in\mathrm{A}}\mathcal{R}_\alpha$ satisfies

$$\mathcal{R} \;\subseteq\; \mathrm{A}\cup\mathrm{A}^{(1)}\cup\cdots\cup\mathrm{A}^{(r-1)}\cup\{\alpha\in\bar{\mathbb{K}}:L_0(\alpha)=0\}.$$

Given $\alpha\in\mathcal{R}$, let

$$\mathcal{F}_\alpha \;=\; \{(z-\alpha)^{-k}:0<k\leqslant\mu(\alpha)\}.$$

We also define $\mathcal{F}_{\mathrm{A}}=\bigcup_{\alpha\in\mathrm{A}}\mathcal{F}_\alpha$,

$$\mathcal{F}_\infty \;=\; \{z^\rho:0<\rho\wedge(\rho\leqslant\deg_z L\vee\operatorname{ind}_\infty(\rho-\tau_\infty)=0)\},$$

and $\mathcal{F}=\{1\}\cup\mathcal{F}_\infty\cup\mathcal{F}_{\mathrm{A}}$. Now given $f\in\bar{\mathbb{M}}$, Proposition 5.3 implies $([f_i]_{\mathrm{A}_i})_{(\mathrm{A}_i\uparrow)}\in\operatorname{Vect}(\mathcal{F}_{\mathrm{A}_i})\subseteq\operatorname{Vect}(\mathcal{F}_{\mathrm{A}})$ for all $i\in I$, where $f_i=f_{(\mathrm{A}_i\uparrow)}$. It follows that $[f]_{\mathrm{A}}\in\bar{\mathbb{K}}[z]\oplus\operatorname{Vect}(\mathcal{F}_{\mathrm{A}})$. We also have $[([f]_{\mathrm{A}})_{(\infty)}]_\infty\in\operatorname{Vect}(\{1\}\cup\mathcal{F}_\infty)$, whence $[f]_{\mathrm{A},\infty}=([f]_{\mathrm{A}})_{(\bar{\infty})}+[([f]_{\mathrm{A}})_{(\infty)}]_\infty\in\operatorname{Vect}(\mathcal{F})$. Since $|\mathcal{F}|\leqslant r\,\mu(\mathrm{A})+\mu(\mathrm{Z})+\deg_z L+r$, we conclude that (5.4) indeed holds. □

Remark 5.9. When using the locally normal reductions $[\![]\!]_{\mathrm{A}_i}$ from Remark 5.6 instead of the reductions $[]_{\mathrm{A}_i}$ and similarly for the local reduction at infinity, it can be checked that the resulting reduction $[\![]\!]_{\mathbb{M}}$ in Theorem 5.8 is both confined and normal.

5.6. Back to reductions with respect to Δ

Assume now that $L \in \mathbb{K}[z][\sigma]$ is a difference operator of order $r \geqslant 0$ in $\sigma = S_\eta$ with $L_0 \neq 0$. Consider a narrow $\mathbb{K}[z][\sigma]$-submodule $\mathbb{M} = \mathbb{K}[z] \oplus \mathbb{K}(z)_{(\mathrm{A}\uparrow);\mu}$ of $\mathbb{K}(z)$, where $\mathrm{A} \subseteq \bar{\mathbb{K}}$ is a symmetric finite set, $\nu: \mathrm{A} \to \mathbb{N}^>$ a symmetric function, and $\mu = \nu \uparrow$. Assume that for every zero $\alpha \in \bar{\mathbb{K}}$ of L_r of multiplicity $\varrho(\alpha)$, we have

$$\mu(\alpha) \;\geqslant\; \mu(\alpha^{(-1)}) + \varrho(\alpha), \tag{5.5}$$

where we understand that $\mu(\alpha^{(-1)}) = 0$ if $\alpha^{(-1)} \notin \mathrm{A}\uparrow$.

Given formal indeterminates $f, \ldots, f^{[r-1]}$, we claim that the set

$$\mathbb{D} \;=\; \mathbb{M} f \oplus \sigma(\mathbb{M}) f^{[1]} \oplus \cdots \oplus \sigma^{r-1}(\mathbb{M}) f^{[r-1]} \tag{5.6}$$

admits the natural structure of a difference module over $\mathbb{K}[z]$ for the shift operator σ with

$$\sigma f^{[i]} \;=\; \begin{cases} f^{[i+1]} & \text{if } i < r-1 \\ -\frac{1}{L_r}(L_0 f + \cdots + L_{r-1} f^{[r-1]}) & \text{if } i = r-1. \end{cases}$$

Indeed, given $w_0 \in \mathbb{M}, w_1 \in \sigma(\mathbb{M}), \ldots, w_{r-1} \in \sigma^{r-1}(\mathbb{M})$, this forces us to take

$$\sigma(w_0 f + \cdots + w_{r-1} f^{[r-1]}) \;=\; \sigma(w_0) f^{[1]} + \cdots + \sigma(w_{r-2}) f^{[r-1]} - \frac{\sigma(w_{r-1})}{L_r}(L_0 f + \cdots + L_{r-1} f^{[r-1]}),$$

and it suffices to check that $\sigma(w_{r-1}) / L_r \in \mathbb{M}$. Now for any root α of L_r of multiplicity $\varrho(\alpha)$, we have $\mathrm{ord}_\alpha\, \sigma(w_{r-1}) \leqslant \mu(\alpha^{(-1)})$, whence $\mathrm{ord}_\alpha\, \sigma(w_{r-1}) / L_r \leqslant \mu(\alpha^{(-1)}) + \varrho(\alpha) \leqslant \mu(\alpha)$, thanks to our assumption (5.5). We conclude that $\sigma(w_{r-1}) / L_r \in \mathbb{M}$, as desired. Notice that f is a formal solution to the equation $L(f) = 0$ in $\mathbb{D}$.

Let us now show how to construct a confined reduction $[\,]_{\mathbb{D}}$ with respect to Δ on $\mathbb{D}$. Let $[\,]_{\mathbb{M}}$ denote the restriction of the directed reduction at A with respect to L^* as constructed in the previous subsection. Now consider

$$w = w_0 f + \cdots + w_s f^{[s]} \in \mathbb{D}, \qquad w_i \in \sigma^i(\mathbb{M}),$$

with $s < r$ and let us show how to define $[w]_{\mathbb{D}}$ by induction on s. If $s = 0$, then we take

$$[w_0 f]_{\mathbb{D}} \;=\; [w_0]_{\mathbb{M}} f.$$

By construction, we have $\{w_0\}_{\mathbb{M}} \in \mathrm{im}\, L^*$, whence there exists a $u \in \mathbb{M}$ such that $\{w_0\}_{\mathbb{M}} = L^*(u)$. By Theorem 5.8, we actually have $u \in \sigma^r(\mathbb{M})$. Using Proposition 3.2, it follows that

$$\{w\}_{\mathbb{D}} \;=\; \{w_0\}_{\mathbb{D}} f \;=\; L^*(u) f \;=\; L^*(u) f - u L(f) \;\in\; \Delta \mathbb{D}.$$

Assume now that $s > 0$. Then we define

$$\begin{aligned} [w]_{\mathbb{D}} &\;=\; [w_0 f]_{\mathbb{D}} + [\tilde{w}]_{\mathbb{D}} \\ \tilde{w} &\;=\; \sigma^{-1}(w_1) f + \cdots + \sigma^{-1}(w_s) f^{[s-1]}. \end{aligned}$$

We notice that

$$w - w_0 f - \tilde{w} \;=\; \Delta(\tilde{w}) \;\in\; \Delta \mathbb{D}.$$

By what precedes, it follows that

$$\{w\}_{\mathbb{D}} \;=\; \{w_0 f\}_{\mathbb{D}} + w - w_0 f - \tilde{w} + \{\tilde{w}\}_{\mathbb{D}} \;\in\; \Delta \mathbb{D}.$$

In fact, an easy induction on s shows that

$$[w]_{\mathbb{D}} \;=\; [w_0 + \sigma^{-1}(w_1) + \sigma^{-2}(w_2) + \cdots + \sigma^{-s}(w_s)]_{\mathbb{M}} f. \tag{5.7}$$

In particular, the constructed reduction is confined with $\mathrm{im}\, [\,]_{\mathbb{D}} = [\mathbb{M}]_{\mathbb{M}} f$:

THEOREM 5.10. *Let $L \in \mathbb{K}[z][\sigma]$ be of order $r \geqslant 0$ with $\deg L_0 \geqslant 0$ and consider* A, μ, $\mathbb{M}$ *and* $\mathbb{D}$ *as above. Let* $\mathrm{Z} = \{\alpha \in \mathrm{A}\uparrow : L_0(\alpha) = 0\}$, $\mu(\mathrm{A}) = \sum_{\alpha \in \mathrm{A}} \mu(\alpha)$, *and* $\mu(\mathrm{Z}) = \sum_{\alpha \in \mathrm{Z}} \mu(\alpha)$. *Then* $[]_{\mathbb{D}}$ *is a confined reduction such that*

$$\dim_{\mathbb{K}} [\mathbb{D}]_{\mathbb{D}} \leqslant r\,\mu(\mathrm{A}) + \mu(\mathrm{Z}) + \deg_z L + r. \qquad \square$$

Remark 5.11. The smallest set $\mathrm{A} \subseteq \bar{\mathbb{K}}$ and map $\nu : \mathrm{A} \to \mathbb{N}$ that satisfy our hypotheses are given by $\mathrm{A} = \{\alpha \in \bar{\mathbb{K}} : L_r(\alpha) = 0\}$ and $\nu(\alpha) = \sum_{\beta \leqslant \alpha} \varrho(\beta)$. Assume that A and ν are taken this way. If $\alpha \neq \beta \Rightarrow \alpha \nsim \beta$ for all $\alpha, \beta \in \mathrm{A}$, then it follows that $\mu(\alpha) = \nu(\alpha) = \varrho(\alpha)$ for all $\alpha \in \mathrm{A}$, whence $\mu(\mathrm{A}) = d$. However, in the other extreme case when A is aligned, we only have $\mu(\alpha) \leqslant d$ for all $\alpha \in \mathrm{A}$ and the growth of $\mu(\mathrm{A})$ can be quadratic in d. This larger growth is due to the fact that we required functions of small span to be reducible fast in section 5.4. If we drop this requirement, then linear growth can be restored by changing the definition (5.1) to

$$[f]_\alpha = \begin{cases} f & \text{if } \rho < r \\ f_{(\beta)} - f_{(\beta);k} + \left[f_{(\bar{\beta})} + f_{(\beta);k} - L\left(\left(\frac{f_{(\beta);k}}{L_0} \right)_{(\beta)} \right) \right]_\alpha & \text{if } \rho \geqslant r, \end{cases} \tag{5.8}$$

where $\beta = \alpha^{(\rho)}$ and $k = \max(\mu(\alpha^{(\rho - r)}) - \operatorname{ord}_\beta L_0^{-1}, 0)$. Compromises between both definitions are also possible, which should ultimately make it possible to systematically gain a factor d with respect to some of the complexity bounds later in this paper. However, the details are technical, so we reserve them for a future work.

5.7. The case of q-difference operators

The theory of the previous section naturally adapts to q-difference operators $\sigma = Q_q$ (where $q \in \mathbb{K}^{\neq}$ is not a root of unity), except that both zero and infinity now need to be treated apart.

So let us start with the construction of the local reductions $[]_0$ and $[]_\infty$ for $L \in \mathbb{K}[z][\sigma^{-1}]$. This time, the behaviour of L at zero and at infinity is given by

$$\begin{aligned} L(z^{-\rho}) &= z^{-\rho - \tau_0} (\operatorname{ind}_0(q^\rho) + o(1)) \qquad (z \to 0) \\ L(z^\rho) &= z^{\rho + \tau_\infty} (\operatorname{ind}_\infty(q^\rho) + o(1)) \qquad (z \to \infty), \end{aligned}$$

where $\operatorname{ind}_0, \operatorname{ind}_\infty \in \mathbb{K}[q^\rho]$ and $\tau_0, \tau_\infty \in \mathbb{Z}$ (in fact, we have $\tau_0 = -\operatorname{val}_z L \in -\mathbb{N}$ and $\tau_\infty = \deg_z L \in \mathbb{N}$). Since q is not a root of unity, we notice that $q^\rho = \alpha$ admits at most one solution for any $\alpha \in \bar{\mathbb{K}}^{\neq}$.

Now consider $f \in \bar{\mathbb{K}}(z)$. Let $\rho \in \mathbb{Z}$ and $c \in \bar{\mathbb{K}}$ be such that

$$f = z^{-\rho} (c + o(1)) \qquad (z \to 0).$$

Setting $B_0 := \max(0, \tau_0) = 0$, we define $[f]_0$ by induction over ρ:

$$[f]_0 = \begin{cases} f & \text{if } \rho \leqslant B_0 \\ c z^{-\rho} + [f - c z^{-\rho}]_0 & \text{if } \rho > B_0 \text{ and } \operatorname{ind}_0(q^{\rho - \tau_0}) = 0 \\ \left[f - L\left(\frac{c}{\operatorname{ind}_0(q^{\rho - \tau_0})} z^{-\rho + \tau_0} \right) \right]_0 & \text{if } \rho > B_0 \text{ and } \operatorname{ind}_0(q^{\rho - \tau_0}) \neq 0. \end{cases} \tag{5.9}$$

Similarly, assume now that $\rho \in \mathbb{Z}$ and $c \in \bar{\mathbb{K}}$ are such that

$$f = z^\rho (c + o(1)) \qquad (z \to \infty).$$

Setting $B_\infty := \max(0, \tau_\infty) = \deg_z L$, we define $[f]_\infty$ by induction over ρ:

$$[f]_\infty = \begin{cases} f & \text{if } \rho \leqslant B_\infty \\ c z^\rho + [f - c z^\rho]_\infty & \text{if } \rho > B_\infty \text{ and } \operatorname{ind}_\infty(q^{\rho - \tau_\infty}) = 0 \\ \left[f - L\left(\frac{c}{\operatorname{ind}_\infty(q^{\rho - \tau_\infty})} z^{\rho - \tau_\infty} \right) \right]_\infty & \text{if } \rho > B_\infty \text{ and } \operatorname{ind}_\infty(q^{\rho - \tau_\infty}) \neq 0. \end{cases} \tag{5.10}$$

Notice that $[f]_0, [f]_\infty \in \mathbb{K}(z)$ for any $f \in \mathbb{K}(z)$.

The local reductions $[]_\alpha$ at other points $\alpha \in \bar{\mathbb{K}}^{\neq}$ are constructed in a similar way as in subsection 5.3. It is not hard to check that the $[]_\alpha$ with $\alpha \in \bar{\mathbb{K}} \cup \{\infty\}$ are indeed local reductions that satisfy similar properties as in Proposition 5.1. Given a subset $\mathrm{A} \subseteq \mathbb{K}^{\neq}$, this allows us to glue them together into a reduction $[]_{\mathrm{A},0,\infty}$ in a similar way as in subsections 5.4 and 5.5. *Mutatis mutandis*, this leads to the following analogue of Theorem 5.8:

THEOREM 5.12. *Let $L \in \mathbb{K}[z][\sigma]$ be of order $r \geqslant 0$. Consider a narrow $\mathbb{K}[z][\sigma]$-submodule*

$$\mathbb{M} \;=\; \mathbb{K}[z, z^{-1}] \oplus \mathbb{K}(z)_{(\mathrm{A}\uparrow);\mu}$$

of $\mathbb{K}(z)$, where $\mathrm{A} \subseteq \bar{\mathbb{K}}^{\neq}$ is a symmetric finite set and $\mu = \nu\uparrow$ for some symmetric function $\nu: \mathrm{A} \to \mathbb{N}^{>}$. Let $\mathrm{Z} = \{\alpha \in \mathrm{A}\uparrow : L_0(\alpha) = 0\}$, $\mu(\mathrm{A}) = \sum_{\alpha \in \mathrm{A}} \mu(\alpha)$, and $\mu(\mathrm{Z}) = \sum_{\alpha \in \mathrm{Z}} \mu(\alpha)$. Then the restriction $[]_{\mathbb{M}}$ of $[]_{\mathrm{A},0,\infty}$ to $\mathbb{M}$ is a confined reduction with respect to L with

$$\dim_{\mathbb{K}} [\mathbb{M}]_{\mathrm{A},0,\infty} \;\leqslant\; r\,\mu(\mathrm{A}) + \mu(\mathrm{Z}) + \deg_z L + 2r.$$

Given $f \in \mathbb{M}$, there exists a $u \in \mathbb{M}$ with $\{f\}_{\mathrm{A},0,\infty} = L(\sigma^r(u))$. □

The results from subsection 5.6 also naturally adapt to the q-difference setting.

5.8. First order systems

Let us now outline how to adapt the theory of this section to first order systems. We assume that $\sigma = S_\eta$ for $\eta \in \mathbb{K}^{\neq}$; the case of q-difference operators can be treated in a similar way. Concerning the analogue of section 5.3, let us start with an operator $L = \sigma^{-1}\psi + A \in \mathbb{K}[z]^{r\times r}[\sigma]^*$, where $\psi \in \mathbb{K}[z]$ is a monic separable polynomial and where $A \in \mathbb{K}(z)^{r\times r}$ is an invertible matrix. Recall that L operates at the right on matrices in $\bar{\mathbb{K}}(z)^{r\times r}$.

The local reduction at infinity $[]_\infty$ is constructed in a similar way as for differential operators. The semi-local reductions $[]_\alpha$ at other points $\alpha \in \bar{\mathbb{K}}$ are somewhat easier to construct in the sense that there is no need for any auxiliary tail choppers. Given $f \in \bar{\mathbb{K}}(z)^{1\times r}$, we define $[f]_\alpha$ by induction over $\rho = \operatorname{span}_\alpha f$:

$$[f]_\alpha \;=\; \begin{cases} f & \text{if } \rho \leqslant 0\\ f_{(\alpha^{(\rho)})} + [f - f_{(\alpha^{(\rho)})}]_\alpha & \text{if } \rho > 0 \text{ and } \det A(\alpha^{(\rho)}) = 0\\ [f - L((fA^{-1})_{(\alpha^{(\rho)})})]_\alpha & \text{if } \rho > 0 \text{ and } \det A(\alpha^{(\rho)}) \neq 0. \end{cases} \tag{5.11}$$

As in section 4.5, this definition can be further optimized through the use of pseudo-inverses:

$$[f]_\alpha \;=\; \begin{cases} f & \text{if } \rho \leqslant 0\\ f_{(\alpha^{(\rho)})} H^{\ddagger} + [f - f_{(\alpha^{(\rho)})} H^{\ddagger} - L(f_{(\alpha^{(\rho)})} H^{\dagger})]_\alpha & \text{if } \rho > 0 \text{ and } H = A(\alpha^{(\rho)}). \end{cases} \tag{5.12}$$

The remainder of the theory and result from sections 5.3, 5.4, 5.5, and 5.6 can be adapted *mutatis mutandis*. In particular, for every symmetric finite set $\mathrm{A} \subseteq \bar{\mathbb{K}}$, we obtain a reduction $[]_{\mathrm{A},\infty}$ that satisfies the following analogue of Theorem 5.8:

THEOREM 5.13. *Let $L \in \mathbb{K}[z]^{r\times r}[\sigma]^*$ be of order be of order one. Assume that L_0 is invertible and that $L_1 = \psi\,\mathrm{Id}_r$ for some monic $\psi \in \mathbb{K}[z]$. Consider a narrow $\mathbb{K}[z][\sigma]$-submodule*

$$\mathbb{A} \;=\; \mathbb{K}[z] \oplus \mathbb{K}(z)_{(\mathrm{A}\uparrow);\mu}$$

of $\mathbb{K}(z)$, where $\mathrm{A} \subseteq \bar{\mathbb{K}}$ is a symmetric finite set and $\mu = \nu\uparrow$ for some symmetric function $\nu: \mathrm{A} \to \mathbb{N}^{>}$. Let $\mathrm{Z} = \{\alpha \in \mathrm{A}\uparrow : \det L_0(\alpha) = 0\}$, $\mu(\mathrm{A}) = \sum_{\alpha \in \mathrm{A}} \mu(\alpha)$, and $\mu(\mathrm{Z}) = \sum_{\alpha \in \mathrm{Z}} \mu(\alpha)$. Then the restriction $[]_{\mathbb{M}}$ of $[]_{\mathrm{A},\infty}$ to $\mathbb{M} = \mathbb{A}^{1\times r}$ is a confined reduction with respect to L with

$$\dim_{\mathbb{K}} [\mathbb{M}]_{\mathbb{M}} \;\leqslant\; (\mu(\mathrm{A}) + \mu(\mathrm{Z}) + \deg_z L + 2)\,r.$$

Given $f \in \mathbb{M}$, there exists a $u \in \mathbb{M}$ with $\{f\}_{\mathbb{M}} = L(\sigma(u))$. □

6. CREATIVE TELESCOPING

6.1. D-finite ideals of DD-operator algebras

Let $\mathbb{K}$ be a field that is equipped with n pairwise commuting operators $\theta_1, \ldots, \theta_n: \mathbb{K} \to \mathbb{K}$, together with derivations $\partial_1, \ldots, \partial_n: \mathbb{K} \to \mathbb{K}$ and automorphisms $\sigma_1, \ldots, \sigma_n: \mathbb{K} \to \mathbb{K}$ such that for each $i \in \{1, \ldots, n\}$ we have either $\theta_i = \partial_i$ and $\sigma_i = 1$ or $\theta_i = \sigma_i$ and $\partial_i = 0$. Given such a field $\mathbb{K}$, we will say that the skew polynomial ring

$$\mathbb{A} = \mathbb{K}[\theta_1, \ldots, \theta_n]$$

is a *DD-operator algebra*. In practice, we usually have $\mathbb{K} = \mathbb{k}(u_1, \ldots, u_n)$ for some constant field $\mathbb{k}$, and each θ_i is given by either one of the formulas

$$\begin{array}{llll} \theta_i = \partial_i, & \partial_i = \frac{\partial}{\partial u_i}, & \sigma_i = \mathrm{Id}; & \text{or} \\ \theta_i = \sigma_i, & \partial_i = 0, & \sigma_i = \sigma_{H_i, u_i}, \; H_i \in \mathbb{k}^{2 \times 2}. & \end{array}$$

Modulo a homographic change of variables (and quadratic algebraic extensions), we have seen in section 5.1 that we may further reduce to the case when each σ_i is either a shift or a q-difference operator. DD-operator algebras of this special type will be called *standard*.

A (left) ideal I of $\mathbb{A}$ is said to be *D-finite* if $\mathbb{A}/I$ is finite dimensional as a vector space over $\mathbb{K}$. Given an $\mathbb{A}$-module $\mathcal{F}$ and a "function" $f \in \mathcal{F}$, we say that f is *D-finite* if its annihilating ideal

$$\operatorname{ann} f = \operatorname{ann}_{\mathbb{A}} f = \{\omega \in \mathbb{A} : \omega(f) = 0\}$$

is D-finite. We say that an ideal I of $\mathbb{A}$ is *reflexive* if for every $i \in \{1, \ldots, n\}$ and $\omega \in \mathbb{A}$, we have $\sigma_i \omega \in I \Rightarrow \omega \in I$.

It is well known [40, 60, 71, 64, 26, 70] that the theory of Gröbner bases generalizes to skew polynomial rings such as $\mathbb{A}$. As usual, this assumes a total ordering $\preccurlyeq$ on the set of "monomials"

$$\Theta_{\mathbb{A}} = \theta_1^{\mathbb{N}} \cdots \theta_n^{\mathbb{N}} = \{\theta_1^{k_1} \cdots \theta_n^{k_n} : k_1, \ldots, k_n \in \mathbb{N}\}$$

that refines the divisibility relation on $\Theta_{\mathbb{A}}$. Let $\preccurlyeq$ denote this divisibility relation. Given a finite subset of $\mathbb{A}$, a Gröbner basis G for the left ideal generated by this subset can be computed using a non-commutative version of Buchberger's algorithm. The initial segment $\{\omega_1, \ldots, \omega_r\} \subseteq \Theta_{\mathbb{A}}$ of reduced monomials for $\preccurlyeq$ under the Gröbner stairs for G admits the usual property that $\{\omega_1 + I, \ldots, \omega_r + I\}$ forms a basis for the vector space $\mathbb{A}/I$. Given $f \in \mathbb{A}$, the reduction of f with respect to G yields $c_1, \ldots, c_r \in \mathbb{K}$ with $f - c_1 \omega_1 - \cdots - c_r \omega_r \in I$.

6.2. Matrix representations of operators

Let I be a reflexive D-finite ideal of $\mathbb{A}$. Consider any basis $\{f_1, \ldots, f_r\}$ of $\mathbb{A}/I$ and let $F \in (\mathbb{A}/I)^{r \times 1}$ be the column vector with entries $f_1, \ldots, f_r$ (it will be convenient to call such a column vector F a basis of $\mathbb{A}/I$ as well). Given an operator $\xi \in \mathbb{A}$, the action of ξ on $\mathbb{A}/I$ gives rise to a matrix $M_\xi = M_{\xi, F} \in \mathbb{K}[\theta_1, \ldots, \theta_n]^{r \times r}$ with

$$\xi F = M_\xi F. \tag{6.1}$$

In particular, for each $i \in \{1, \ldots, r\}$ there is a matrix $M_i = M_{\theta_i} \in \mathbb{K}[\theta_1, \ldots, \theta_n]^{r \times r}$ with

$$\theta_i F = M_i F. \tag{6.2}$$

If $\omega_1, \ldots, \omega_r \in \mathbb{K} \rho_1 \oplus \cdots \oplus \mathbb{K} \rho_r$ are such that $f_1 = \omega_1 + I, \ldots, f_r = \omega_r + I$, then we may compute the j-th column of M_ξ by reducing ξf_j with respect to G and writing the result as a linear combination of $\omega_1, \ldots, \omega_r$.

PROPOSITION 6.1. *If $\sigma \in \mathbb{A}$ is an automorphism, then M_σ is invertible.*

Proof. Assume for contradiction that there exists a non-zero row vector $V \in \mathbb{K}^{1 \times r}$ with $V M_i = 0$. Then we would get $\sigma(\sigma^{-1}(V) F) = V \sigma(F) = V M_\sigma F = 0$. Writing $\Omega \in \mathbb{A}^{r \times 1}$ for the column matrix with entries $\omega_1, \ldots, \omega_r$, we thus obtain $\sigma(\sigma^{-1}(V) \Omega) \in I$ and $\sigma^{-1}(V) \Omega \in I$, which yields a non-trivial relation between the f_i. □

PROPOSITION 6.2. *For any $i, j \in \{1, \ldots, n\}$ with $i \neq j$, we have:*

$$\begin{array}{rcll} \partial_j M_i + M_i M_j &=& \partial_i M_j + M_j M_i & (\theta_i = \partial_i,\ \theta_j = \partial_j) \\ \partial_j M_i + M_i M_j &=& \sigma_i(M_j) M_i & (\theta_i = \sigma_i,\ \theta_j = \partial_j) \\ \sigma_j(M_i) M_j &=& \sigma_i(M_j) M_i & (\theta_i = \sigma_i,\ \theta_j = \sigma_j) \end{array}$$

Proof. If $\theta_i = \partial_i$ and $\theta_j = \partial_j$, then (6.2) yields

$$\begin{array}{rcccl} \partial_i \partial_j F &=& \partial_i(M_j F) &=& (\partial_i M_j + M_j M_i) F \\ \partial_j \partial_i F &=& \partial_j(M_i F) &=& (\partial_j M_i + M_i M_j) F. \end{array}$$

Since both derivations commute and the entries of F form a basis of $\mathbb{A}/I$, this yields the first relation. The two other relations are proved in a similar way. □

Given $i \in \{1, \ldots, r\}$, we say that $f \in \mathbb{A}/I$ is a *cyclic vector* for θ_i if $f, \theta f, \ldots, \theta^{r-1} f$ form a basis for $\mathbb{A}/I$. If $\theta_i = \partial_i$ and $\mathbb{K}$ contains an element x with $\partial_i x \neq 0$, then it is well known that such a cyclic vector f always exists and can be computed [3, 25, 15]. These results can naturally be adapted to the case when θ_i is a shift operator or a q-difference operator such that $\mathbb{K}$ contains an element u_i such that $\theta_i^{\mathbb{Z}}(u_i)$ is infinite. We will say that θ_i *acts non trivially* on $\mathbb{K}$ if we are in either one of these situations. In the terminology of [6, section 5.3], this implies that $\mathbb{K}[\theta_i]$ is simple, whence [6, Corollary 5.3.6] yields a way to compute cyclic vectors. Given a cyclic vector f, the left ideal of $\mathbb{K}[\theta_i]$ of operators that annihilate f is principal and its unique monic generator has order r. In terms of matrices, these properties may be restated as follows:

PROPOSITION 6.3. *Let $i \in \{1, \ldots, n\}$ be such that θ_i acts non trivially on $\mathbb{K}$. Then there exists a basis F of $\mathbb{A}/I$ with entries $f_1, \ldots, f_r$ such that*

$$M_{\theta_i, F} = \begin{pmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -L_0 & -L_1 & \cdots & -L_{r-1} \end{pmatrix}.$$

The operator $L = \theta_i^r + L_{r-1} \theta_i^{r-1} + \cdots + L_0$ is the unique monic operator of order r with $L(f_1) = 0$. □

6.3. Narrowness

Let us now consider a reflexive D-finite ideal I of a DD-operator algebra

$$\mathbb{B} = \mathbb{K}(z)[\theta_1, \ldots, \theta_n, \theta_{n+1}]$$

with one additional operator $\theta = \theta_{n+1}$. We abbreviate $\partial = \partial_{n+1}$, $\sigma = \sigma_{n+1}$, $\Theta_{\mathbb{B}} = \Theta_{\mathbb{A}} \theta^{\mathbb{N}}$, and assume that either

$$\begin{array}{rclrcll} \partial &=& \frac{\partial}{\partial z}, & \sigma &=& 1, & \text{or} \\ \partial &=& 0, & \sigma &=& \sigma_{H,z},\ H \in \Bbbk^{2 \times 2}, & \end{array}$$

where $\Bbbk = \{f \in \mathbb{K} : \forall i \in \{1, \ldots, n\}, \partial_i f = 0, \sigma_i f = f\}$ is the constant field of $\mathbb{K}$. Moreover, if $\theta = \sigma$, then we require that σ is either a shift operator S_η with $\eta \in \Bbbk^{\neq}$ or a q-difference operator Q_q, where $q \in \Bbbk^{\neq}$ is not a root of unity. As usual, we set $\tau = \tau_{H,z}$, where we understand that $H = \mathrm{Id}_2 \in \Bbbk^{2 \times 2}$ if $\sigma = 1$. We also write $\delta = \partial$ if $\theta = \partial$ and $\delta = \Delta = \sigma - 1$ if $\theta = \sigma$.

Recall that a $\mathbb{K}[z]$-submodule $\mathbb{M}\ni 1$ of $\mathbb{K}(z)$ is said to be *narrow* in θ if it is stable under θ and finitely generated as a $\mathbb{K}[z][\theta]$-module. Given such a module $\mathbb{M}$, we have shown in section 5.2 that there exists a finite symmetric set A and a symmetric $\nu\colon \bar{\mathbb{K}}\to\mathbb{N}\cup\{\infty\}$ such that $\mathbb{M}=\mathbb{K}[z]\oplus\mathbb{K}(z)_{(\mathrm{A}\uparrow);\nu\uparrow}$. Let $F\in(\mathbb{B}/I)^{r\times 1}$ be a fixed basis of $\mathbb{B}/I$. A $\mathbb{K}[z]$-submodule $\mathbb{D}$ of $\mathbb{K}(z)^{1\times r}F$ is said to be *narrow* in θ if it is of the form $\mathbb{D}=\mathbb{M}^{1\times r}F$ for a narrow submodule $\mathbb{M}$ of $\mathbb{K}(z)$.

Given a narrow submodule $\mathbb{M}$ of $\mathbb{K}(z)$ as above and $\mathbb{D}=\mathbb{K}(z)^{1\times r}F$, we have seen in sections 4.5 and 5.8 how to construct a computable confined reduction $[\,]_{\mathbb{M}}\colon\mathbb{M}\to\mathbb{M}$ with respect to the first order operator $(\theta-M_\theta)^*$ as well as a corresponding computable confined reduction $[\,]_{\mathbb{D}}\colon\mathbb{D}\to\mathbb{D}$ with respect to $\delta=\partial$. Alternatively, one may rely on the constructions of reductions with respect to scalar linear differential and difference operators $L\in\mathbb{K}[z][\theta]^*$, as we will detail now.

Let $f_1,\ldots,f_r$ be the entries of the basis F. We say that F is *cyclic* if $f_k=\theta^{k-1}f_1$ for $k=1,\ldots,r$. Such a basis always exists by Proposition 6.3 and it comes with a unique monic operator $L_{\mathrm{mon}}\in\mathbb{K}(z)[\theta]$ of order r with $L_{\mathrm{mon}}(f_1)=0$. Multiplying L_{mon} with the lcm of the denominators of its coefficients, we obtain a new operator $L=L_r\theta^r+\cdots+L_0\in\mathbb{K}[z][\theta]$ with $\gcd(L_0,\ldots,L_r)=1$ and $L(f_1)=0$.

Assume that $\theta=\partial$ and that we are given a narrow submodule $\mathbb{M}$ in θ as above with respect to a cyclic basis F. Then we have shown in sections 4.2, 4.3 and 4.4 how to construct a computable confined reduction $[\,]_{\mathbb{M}}\colon\mathbb{M}\to\mathbb{M}$ with respect to L^* as well as a corresponding computable confined reduction $[\,]_{\mathbb{D}}\colon\mathbb{D}\to\mathbb{D}$ with respect to $\delta=\partial$.

Assume now that $\theta=\sigma$ and that we are given a narrow submodule $\mathbb{M}$ in θ as above with respect to a cyclic basis F. Assume also that $\sum_{\beta\leqslant\alpha}\mathrm{ord}_\beta L_r^{-1}\leqslant\nu(\alpha)$ for all $\alpha\in\bar{\mathbb{K}}\setminus\mathrm{Fix}\,\tau$, where $\mathrm{Fix}\,\tau\subseteq\bar{\mathbb{K}}\cup\{\infty\}$ denotes the set of fixed points of τ. Then we have shown in section 5 how to construct a computable confined reduction $[\,]_{\mathbb{M}}\colon\mathbb{M}\to\mathbb{M}$ with respect to L^* as well as a corresponding computable confined reduction $[\,]_{\mathbb{D}'}\colon\mathbb{D}'\to\mathbb{D}'$ with respect to $\delta=\Delta$, where $\mathbb{D}'=\mathbb{M}f_1\oplus\sigma(\mathbb{M})f_2\oplus\cdots\oplus\sigma^{r-1}(\mathbb{M})f_r$. For $i=0,\ldots,r-1$, we notice that $\mathbb{M}=\sigma^i(\mathbb{M})\oplus\mathbb{V}_i$ for some $\mathbb{K}$-vector space $\mathbb{V}_i$ of dimension at most $\sum_\alpha(\nu(\alpha)-\nu(\alpha^{(-i)}))$. Given $w=w_1f_1+\cdots+w_sf_s$ with $w_1,\ldots,w_s\in\mathbb{M}$ and $s\leqslant r$, we may write $w_i=u_i+\sigma(v_i)$ with $u_i\in\mathbb{V}_1$ and $v_i\in\mathbb{M}$ for $i=1,\ldots,s$. By induction on s, we may then define a confined reduction on $\mathbb{D}$ by $[w]_{\mathbb{D}}=[w_1f_1]_{\mathbb{D}_1}+[v_2f_1+\cdots+v_sf_{s-1}]_{\mathbb{D}}+u_2f_2+\cdots+u_sf_s$. We have $\dim_{\mathbb{K}}[\mathbb{D}]_{\mathbb{D}}\leqslant\dim_{\mathbb{K}}[\mathbb{D}']_{\mathbb{D}'}+(r-1)\dim_{\mathbb{K}}\mathbb{V}_1$.

Assume that there exists a $\mathbb{K}[z][\theta_1,\ldots,\theta_n,\theta]$-submodule $\mathbb{D}$ of $\mathbb{K}(z)^{1\times r}F$ that comes with a (computable) confined reduction $[\,]_{\mathbb{D}}\colon\mathbb{D}\to\mathbb{D}$ with respect to δ. Then we say that I is *telescopable* in θ for the chosen basis F, and we call $[\,]_{\mathbb{D}}$ the *associated reduction*. By what precedes, this is the case whenever there exists a $\mathbb{K}[z][\theta_1,\ldots,\theta_n,\theta]$-submodule $\mathbb{D}$ of $\mathbb{K}(z)^{1\times r}F$ that is narrow in θ.

6.4. Creative telescoping

With the notations from the previous subsection, given a function φ in some $\mathbb{B}$-module, the set $\mathcal{T}_{\varphi,\delta}$ of *telescopers* for φ with respect to δ is defined by

$$\mathcal{T}_{\varphi,\delta} = (\mathrm{ann}_{\mathbb{B}}\,\varphi+\delta\,\mathbb{B})\cap\mathbb{A}.$$

From this definition it is immediately apparent that $\mathcal{T}_{\varphi,\delta}$ forms an ideal of $\mathbb{A}$. Given a telescoper $g\in\mathcal{T}_{\varphi,\delta}$, an element $h\in\mathbb{B}$ such that $(g-\delta h)(\varphi)=0$ is called a *certificate* for g.

Assume now that I is telescopable in θ, with associated reduction $[\,]_{\mathbb{D}}\colon\mathbb{D}\to\mathbb{D}$. We say that a function $\varphi\in\mathbb{D}$ is *primitive* if for each $\psi\in\mathbb{D}$, there exists an $\omega\in\mathbb{B}$ with $\psi=\omega\varphi$. We write $\mathbb{D}^{\mathrm{prim}}$ for the set of such functions. Given $\varphi\in\mathbb{D}^{\mathrm{prim}}$ and an operator $\omega\in\mathbb{A}$ with $\omega\varphi\in\mathbb{D}$, we notice that $[\omega\varphi]_{\mathbb{D}}=0$ implies the existence of some $\omega'\in\mathbb{B}$ with $\omega\varphi=\delta\,\omega'\varphi$, whence $\omega-\delta\,\omega'\in\mathrm{ann}_{\mathbb{B}}\,\varphi$ and $\omega\in(\mathrm{ann}_{\mathbb{B}}\,\varphi+\delta\,\mathbb{B})\cap\mathbb{A}=\mathcal{T}_{\varphi,\delta}$. In particular, given an operator $c_1\omega_1+\cdots+c_k\omega_k\in\mathbb{A}$ with $c_1,\ldots,c_k\in\mathbb{K}$ and $\omega_1,\ldots,\omega_k\in\Theta_{\mathbb{A}}$, we have

$$c_1[\omega_1\varphi]_{\mathbb{D}}+\cdots+c_k[\omega_k\varphi]_{\mathbb{D}}=[(c_1\omega_1+\cdots+c_k\omega_k)\varphi]_{\mathbb{D}}=0 \implies c_1\omega_1+\cdots+c_k\omega_k\in\mathcal{T}_{\varphi,\delta}. \tag{6.3}$$

Since $[(c_1 \omega_1 + \cdots + c_k \omega_k) \varphi]_{\mathbb{D}}$ lives in the finite dimensional $\mathbb{K}$-vector space $[\mathbb{D}]_{\mathbb{D}}$, this yields a way to determine telescopers through linear algebra, simply by searching $\mathbb{K}$-linear relations between the $[\omega \varphi]_{\mathbb{D}}$ where ω runs over $\Theta_{\mathbb{A}}$. For instance, taking $\omega = 1, \theta_i, \theta_i^2, \ldots$, one obtains a telescoper in $(\mathrm{ann}_{\mathbb{B}} \varphi + \delta \mathbb{B}) \cap \mathbb{K}[\theta_i]$, which generalizes the introductory example (1.4). Doing this for each $i \in \{1, \ldots, n\}$ shows in particular that $\mathcal{T}_{\varphi,\delta}$ is D-finite.

In our DD-algebra setting, it is also natural to perform the linear algebra incrementally following the term order $\preccurlyeq$ on $\Theta_{\mathbb{A}}$, as in the FGLM algorithm [37]. This leads to the following algorithm to compute a D-finite ideal contained in $\mathcal{T}_{\varphi,\delta}$.

Algorithm 6.1

INPUT: a telescopable D-finite ideal $I \subseteq \mathbb{B}$ with associated reduction $[\,]_{\mathbb{D}}$ and $\varphi \in \mathbb{D}^{\mathrm{prim}}$

OUTPUT: the reduced Gröbner basis $\mathcal{G}$ for a D-finite ideal $(\mathcal{G})$ included in $\mathcal{T}_{\varphi,\delta}$

$\mathcal{L} := \{1\}$ — monomials remaining to be treated

$\mathcal{G} := \emptyset$ — Gröbner basis being constructed

$\mathcal{R} := \emptyset$ — monomials under stairs

while $\mathcal{L} \neq \emptyset$ **do**

  Let $\omega \in \mathcal{L}$ be minimal for $\preccurlyeq$ and set $\mathcal{L} := \mathcal{L} \setminus \{\omega\}$

  if ω is not reducible with respect to $\mathcal{G}$ **then**

    Compute $R_\omega := [\omega \varphi]_{\mathbb{D}}$

    (If $\omega = \theta_i \omega'$, then we may also take $R_\omega := [\theta_i R_{\omega'}]_{\mathbb{D}}$)

    if $\exists (c_\rho)_{\rho \in \mathcal{R}} \in \mathbb{K}^{\mathcal{R}}$, $R_\omega = \sum_{\rho \in \mathcal{R}} c_\rho R_\rho$ **then**

      $\mathcal{G} := \mathcal{G} \cup \{\omega - \sum_{\rho \in \mathcal{R}} c_\rho \rho\}$

    else

      $\mathcal{R} := \mathcal{R} \cup \{\omega\}$

      $\mathcal{L} := \mathcal{L} \cup \{\theta_1 \omega, \ldots, \theta_n \omega\}$

return the reduced Gröbner basis for the ideal generated by $\mathcal{G}$

THEOREM 6.4. *Algorithm 6.1 is correct, terminates, and returns the Gröbner basis $\mathcal{G}$ of a D-finite ideal $(\mathcal{G}) \subseteq \mathcal{T}_{\varphi,\delta}$ with*

$$\dim_{\mathbb{K}} \mathbb{A} / (\mathcal{G}) \leqslant \dim_{\mathbb{K}} [\mathbb{D}]_{\mathbb{D}}. \tag{6.4}$$

Moreover, if $[\,]_{\mathbb{D}}$ is normal, then $(\mathcal{G}) = \mathcal{T}_{\varphi,\delta}$.

Proof. Throughout the algorithm, we observe that the elements $R_\rho \in [\mathbb{D}]_{\mathbb{D}}$ with $\rho \in \mathcal{R}$ are $\mathbb{K}$-linearly independent. Consequently, $|\mathcal{R}|$ never exceeds $\dim_{\mathbb{K}} [\mathbb{D}]_{\mathbb{D}}$, which ensures termination of the algorithm. By (6.3), it is also clear that we only insert telescopers in $\mathcal{T}_{\varphi,\delta}$ to $\mathcal{G}$. Let us now prove the bound (6.4).

Throughout the algorithm, it is easily verified that $\mathcal{R}$ contains only monomials that are smaller than ω. It follows that the leading monomial of $\omega - \sum_{\rho \in \mathcal{R}} c_\rho \rho$ is ω, when inserting a new element into $\mathcal{G}$. This in turn implies that, at the start of each iteration of the main loop, the set $\mathcal{R}$ contains precisely those monomials below ω that cannot be reduced with respect to $\mathcal{G}$. At the end of the loop, this means that $\mathcal{R}$ consists exactly of those monomials that cannot be reduced with respect to $\mathcal{G}$. Consequently, $\dim_{\mathbb{K}} \mathbb{A} / (\mathcal{G}) \leqslant |\mathcal{R}| \leqslant \dim_{\mathbb{K}} [\mathbb{D}]_{\mathbb{D}}$.

Assume finally that $[\,]_{\mathbb{D}}$ is a normal reduction. Then it is straightforward to verify that (6.3) becomes an equivalence. In particular, given a telescoper of the form $\sum_{\rho \in \mathcal{R}} c_\rho \rho$, we get that $\sum_{\rho \in \mathcal{R}} c_\rho [\rho \varphi]_{\mathbb{D}} = 0$, whence $c_\rho = 0$ for all $\rho \in \mathcal{R}$, since the elements R_ρ with $\rho \in \mathcal{R}$ are linearly independent. In other words, $\dim_{\mathbb{K}} \mathbb{A} / \mathcal{T}_{\varphi,\delta} \geqslant |\mathcal{R}| \geqslant \dim_{\mathbb{K}} \mathbb{A} / (\mathcal{G})$, whence $\mathcal{T}_{\varphi,\delta} \subseteq (\mathcal{G})$. At the end of the main loop, this also means that the S-polynomials of any two elements in $\mathcal{G}$ necessarily reduce to zero; in other words, $\mathcal{G}$ is already a reduced Gröbner basis, so we may directly return it after the main loop. □

Remark 6.5. The important property of R_ω is that $R_\omega - \omega\varphi \in \delta\mathbb{B}$. If $\omega = \theta_i \omega'$, then it follows that we may also take $R_\omega := [\theta_i R_{\omega'}]_{\mathbb{D}}$ instead of $R_\omega := [\omega\varphi]_{\mathbb{D}}$, since

$$[\theta_i R_{\omega'}]_{\mathbb{D}} - \omega\varphi = [\theta_i R_{\omega'}]_{\mathbb{D}} - \theta_i R_{\omega'} + \theta_i (R_{\omega'} - \omega'\varphi),$$

where $[\theta_i R_{\omega'}]_{\mathbb{D}} - \theta_i R_{\omega'} \in \delta\mathbb{B}$ and $\theta_i (R_{\omega'} - \omega'\varphi) \in \theta_i \delta\mathbb{B} = \delta\theta_i \mathbb{B} \subseteq \delta\mathbb{B}$.

Remark 6.6. Reduction-based algorithms for creative telescoping do not require the computation of certificates. Nevertheless, it is not hard to modify the algorithms such that certificates are computed along with the telescopers themselves.

Remark 6.7. It is possible to generalize the theory of this section to the case when δ is replaced by a more general operator $R \in \mathbb{K}[\delta]$ that commutes with $\theta_1, \ldots, \theta_n$. In the differential case, it essentially suffices to replace the confined reduction $[\,]_{\mathbb{D}}$ with respect to δ by a confined reduction with respect to R, as constructed in section 4.6. In the difference case, such reductions can be constructed along similar lines.

7. D-FINITENESS TESTS FOR TELESCOPING IDEALS

7.1. Location of singularities

Adopt the same notations as in section 6.3. Let $F \in (\mathbb{B}/I)^{r\times 1}$ be a fixed basis of $\mathbb{B}/I$ with entries $f_1, \ldots, f_r$. Given an operator $\omega \in \mathbb{B}$, we define its *order* $\operatorname{ord}_{\alpha,F}\omega$ at $\alpha \in \bar{\mathbb{K}} \cup \{\infty\}$ in z and with respect to the basis F by

$$\operatorname{ord}_\alpha \omega \;=\; \operatorname{ord}_{\alpha,F}\omega \;=\; \operatorname{ord}_\alpha M_{\omega,F}$$

and its (finite) set of *singularities* in z by

$$\operatorname{Sing}(\omega) = \operatorname{Sing}_F(\omega) = \{\alpha \in \bar{\mathbb{K}} \cup \{\infty\} : \operatorname{ord}_{\alpha,F}\omega > 0\}.$$

Thanks to Proposition 6.2, it turns out that $\operatorname{Sing}(\theta_i)$ is closely related to $\operatorname{Sing}(\theta_{n+1})$ for all $i \in \mathbb{N}$.

PROPOSITION 7.1. *Assume that $\theta = \partial$ and let $i \in \{1, \ldots, n\}$. Then*

$$\begin{aligned} \operatorname{Sing}(\partial_i) &\subseteq \operatorname{Sing}(\partial) && (\theta_i = \partial_i)\\ \operatorname{Sing}(\sigma_i) &\subseteq \operatorname{Sing}(\partial) \cup \sigma_i(\operatorname{Sing}(\partial)) && (\theta_i = \sigma_i)\end{aligned}$$

Proof. Assuming for contradiction that $\theta_i = \partial_i$ and $\alpha \in \operatorname{Sing}(\partial_i) \setminus \operatorname{Sing}(\partial)$, we have $\operatorname{ord}_\alpha M_{n+1} = \operatorname{ord}_\alpha \partial_i M_{n+1} = 0$, whence Proposition 6.2 implies

$$\begin{aligned} \partial M_i &= \partial_i M_{n+1} + M_{n+1} M_i - M_i M_{n+1}\\ \operatorname{ord}_\alpha M_i + 1 &= \operatorname{ord}_\alpha \partial M_i \leqslant \max(\operatorname{ord}_\alpha \partial_i M_{n+1}, \operatorname{ord}_\alpha M_{n+1} + \operatorname{ord}_\alpha M_i) = \operatorname{ord}_\alpha M_i.\end{aligned}$$

Similarly, if $\theta_i = \sigma_i$ and $\alpha \in \operatorname{Sing}(\sigma_i) \setminus (\operatorname{Sing}(\partial) \cup \sigma_i(\operatorname{Sing}(\partial)))$, then we have $\operatorname{ord}_\alpha M_{n+1} = \operatorname{ord}_\alpha \sigma_i(M_{n+1}) = 0$, whence

$$\begin{aligned} \partial M_i &= \sigma_i(M_{n+1}) M_i - M_i M_{n+1}\\ \operatorname{ord}_\alpha M_i + 1 &\leqslant \max(\operatorname{ord}_\alpha \sigma_i(M_{n+1}) + \operatorname{ord}_\alpha M_i, \operatorname{ord}_\alpha M_i + \operatorname{ord}_\alpha M_{n+1}) = \operatorname{ord}_\alpha M_i. \qquad \square\end{aligned}$$

PROPOSITION 7.2. *Assume that $\theta = \sigma$ and let $i \in \{1, \ldots, n\}$. Let*

$$\begin{aligned} S &= \operatorname{Sing}(\sigma) \cup \operatorname{Sing}(\sigma^{-1})^{(1)} && (\theta_i = \delta_i)\\ S &= \operatorname{Sing}(\sigma) \cup \operatorname{Sing}(\sigma^{-1})^{(1)} \cup \sigma_i(\operatorname{Sing}(\sigma)) \cup \sigma_i(\operatorname{Sing}(\sigma^{-1})^{(1)}) && (\theta_i = \sigma_i)\end{aligned}$$

Then for any $\alpha \in \operatorname{Sing}(\theta_i)$ *with* $\alpha \neq \alpha^{(1)}$*, there exist integers* $k \leqslant 0$ *and* $l > 0$ *with*

$$\alpha^{(k)} \in S, \qquad \alpha^{(k+1)}, \ldots, \alpha^{(l-1)} \in \operatorname{Sing}(\theta_i), \qquad \alpha^{(l)} \in S.$$

Proof. Since $M_{\sigma^{-1}} = \sigma^{-1}(M_{n+1})^{-1}$, we notice that

$$\begin{aligned}\operatorname{Sing}(\sigma^{-1}) &= \{\alpha \in \bar{\mathbb{K}} \cup \{\infty\} : \operatorname{ord}_\alpha \sigma^{-1}(M_{n+1}^{-1}) > 0\}\\ &= \{\alpha \in \bar{\mathbb{K}} \cup \{\infty\} : \operatorname{ord}_{\alpha^{(1)}} M_{n+1}^{-1} > 0\}\\ &= \{\alpha \in \bar{\mathbb{K}} \cup \{\infty\} : \operatorname{ord}_\alpha M_{n+1}^{-1} > 0\}^{(-1)}.\end{aligned}$$

Assume that $\theta_i = \partial_i$ and $\alpha \in \bar{\mathbb{K}} \cup \{\infty\} \setminus S$. Then Proposition 6.2 implies

$$\begin{aligned}M_i &= M_{n+1}^{-1} \sigma(M_i) M_{n+1} - M_{n+1}^{-1} \partial_i M_{n+1}\\ \operatorname{ord}_\alpha M_i &\leqslant \operatorname{ord}_\alpha \sigma(M_i) = \operatorname{ord}_{\alpha^{(-1)}} M_i,\end{aligned}$$

as well as

$$\begin{aligned}\sigma(M_i) &= (\partial_i M_{n+1}) M_{n+1}^{-1} + M_{n+1} M_i M_{n+1}^{-1}\\ \operatorname{ord}_{\alpha^{(-1)}} M_i &\leqslant \operatorname{ord}_\alpha M_i,\end{aligned}$$

whence $\alpha \in \operatorname{Sing}(\partial_i) \Leftrightarrow \alpha^{(-1)} \in \operatorname{Sing}(\partial_i)$. Assuming that $\alpha \in \operatorname{Sing}(\partial_i)$, it follows by induction on $m \in \mathbb{N}$, we get that $\alpha, \ldots, \alpha^{(-m+1)} \notin S$ implies $\alpha^{(-1)}, \ldots, \alpha^{(-m)} \in \operatorname{Sing}(\partial_i)$. Since $\operatorname{Sing}(\partial_i)$ is finite, we must have $\alpha^{(k)} \in S$ for some $k \leqslant 0$. If $\alpha \in \operatorname{Sing}(\partial_i)$, it follows similarly that $\alpha^{(1)}, \ldots, \alpha^{(l)} \notin S$ implies $\alpha^{(1)}, \ldots, \alpha^{(l)} \in \operatorname{Sing}(\partial_i)$, whence the existence of some $l > 0$ with $\alpha^{(l)} \in S$.

Let us next consider the case when $\theta_i = \sigma_i$ and $\alpha \in \bar{\mathbb{K}} \cup \{\infty\} \setminus S$. Then Proposition 6.2 implies

$$\begin{aligned}M_i &= \sigma_i(M_{n+1})^{-1} \sigma(M_i) M_{n+1}\\ \sigma(M_i) &= \sigma_i(M_{n+1}) M_i M_{n+1}^{-1},\end{aligned}$$

whence $\operatorname{ord}_\alpha M_i = \operatorname{ord}_{\alpha^{(-1)}} M_i$ and $\alpha \in \operatorname{Sing}(\sigma_i) \Leftrightarrow \alpha^{-1} \in \operatorname{Sing}(\sigma_i)$. We conclude in a similar way as above. □

7.2. Orders at singularities

Given a set of operators $\Omega \subseteq \mathbb{B}$, we define

$$\operatorname{ord}_\alpha \Omega = \operatorname{ord}_{\alpha,F} \Omega = \sup_{\omega \in \Omega} \operatorname{ord}_{\alpha,F} \omega \in \mathbb{N} \cup \{\infty\}.$$

Let $\mathcal{D}_{\mathbb{A}}$ be the monoid generated by the θ_i with $i \in \{1, \ldots, n\}$ and $\theta_i = \partial_i$. Similarly, let $\Sigma_{\mathbb{A}}$ denote the multiplicative monoid generated by the θ_i such that $\theta_i = \sigma_i$.

PROPOSITION 7.3. *Assume that* $\theta = \sigma$ *and let* $\alpha \in \bar{\mathbb{K}}$ *with* $\tau(\alpha) \neq \alpha$ *be such that* $\partial_i(\alpha) = 0$ *for* $i = 1, \ldots, n$ *with* $\theta_i = \partial_i$*. Then we have*

$$\operatorname{ord}_\alpha \mathcal{D}_{\mathbb{A}} \leqslant \sum_{k \geqslant 1} \operatorname{ord}_{\alpha^{(k)}} M_\sigma + \sum_{k \geqslant 1} \operatorname{ord}_{\alpha^{(k)}} M_\sigma^{-1}.$$

Proof. Let $\ell \in \mathbb{N}$ be sufficiently large such that $\operatorname{ord}_{\alpha^{(\ell)}} \partial_i = 0$ for all $i \in \{1, \ldots, n\}$ with $\theta_i = \partial_i$. This implies that $\operatorname{ord}_{\alpha^{(\ell)}} \mathcal{D}_{\mathbb{A}} = 0$. Let $\omega \in \mathcal{D}_{\mathbb{A}}$. Since $M_{\sigma^\ell \omega} = \sigma^\ell(M_\omega) M_{\sigma^\ell}$ and M_{σ^ℓ} is invertible, we get

$$\operatorname{ord}_{\alpha^{(\ell)}} (\sigma^\ell \omega) \geqslant \operatorname{ord}_{\alpha^{(\ell)}} \sigma^\ell(M_\omega) - \operatorname{ord}_{\alpha^{(\ell)}} M_{\sigma^\ell}^{-1} = \operatorname{ord}_\alpha \omega - \operatorname{ord}_{\alpha^{(\ell)}} M_{\sigma^\ell}^{-1}. \tag{7.1}$$

On the other hand, there are constants $c_{\omega',\omega''} \in \mathbb{N}$ with

$$(\omega \sigma^\ell)(F) = \omega(M_{\sigma^\ell} F) = \sum_{\omega = \omega' \omega''} c_{\omega',\omega''} (\omega'(M_{\sigma^\ell})) (\omega''(F)).$$

Since $\partial_i(\alpha^{(\ell)}) = (\partial_i \alpha)^{(\ell)} = 0$ for $i = 1, \ldots, n$, we have $\operatorname{ord}_{\alpha^{(\ell)}} \omega'(M_{\sigma^\ell}) \leqslant \operatorname{ord}_{\alpha^{(\ell)}} M_{\sigma^\ell}$ in the above formula. By our choice of ℓ, we also have $\operatorname{ord}_{\alpha^{(\ell)}} \omega'' = 0$, whence

$$\operatorname{ord}_{\alpha^{(\ell)}} (\omega \sigma^\ell) \leqslant \operatorname{ord}_{\alpha^{(\ell)}} M_{\sigma^\ell}.$$

Combined with (7.1), this yields

$$\operatorname{ord}_\alpha \omega \leqslant \operatorname{ord}_{\alpha^{(\ell)}} M_{\sigma^\ell} + \operatorname{ord}_{\alpha^{(\ell)}} M_{\sigma^\ell}^{-1}.$$

Since $M_{\sigma^\ell} = \sigma^{\ell-1}(M_\sigma) \cdots \sigma(M_\sigma) M_\sigma$, we have $\operatorname{ord}_{\alpha^{(\ell)}} M_{\sigma^\ell} \leqslant \operatorname{ord}_{\alpha^{(1)}} M_\sigma + \cdots + \operatorname{ord}_{\alpha^{(\ell)}} M_\sigma$ and similarly $\operatorname{ord}_{\alpha^{(\ell)}} M_{\sigma^\ell}^{-1} \leqslant \operatorname{ord}_{\alpha^{(1)}} M_\sigma^{-1} + \cdots + \operatorname{ord}_{\alpha^{(\ell)}} M_\sigma^{-1}$. □

PROPOSITION 7.4. *Assume that $\theta = \sigma$. Let $\omega \in \Sigma_{\mathbb{A}}$ and $\alpha \in \bar{\mathbb{K}}$ with $\tau(\alpha) \neq \alpha$. Then*

$$\operatorname{ord}_\alpha M_\omega \leqslant \sum_{k \geqslant 1} \operatorname{ord}_{\omega^{-1}(\alpha^{(k)})} M_\sigma + \sum_{k \geqslant 1} \operatorname{ord}_{\alpha^{(k)}} M_\sigma^{-1}.$$

Proof. Let $\ell \in \mathbb{N}$ be sufficiently large such that $\operatorname{ord}_{\alpha^{(\ell)}} \omega = 0$. Since $M_{\sigma^\ell \omega} = \sigma^\ell(M_\omega) M_{\sigma^\ell}$ and M_{σ^ℓ} is invertible, we have

$$\operatorname{ord}_\alpha M_\omega = \operatorname{ord}_{\alpha^{(\ell)}} \sigma^\ell(M_\omega) \leqslant \operatorname{ord}_{\alpha^{(\ell)}} M_{\sigma^\ell \omega} + \operatorname{ord}_{\alpha^{(\ell)}} M_{\sigma^\ell}^{-1}.$$

Since $M_{\sigma^\ell \omega} = M_{\omega \sigma^\ell} = \omega(M_{\sigma^\ell}) M_\omega$, we also have

$$\operatorname{ord}_{\alpha^{(\ell)}} M_{\sigma^\ell \omega} \leqslant \operatorname{ord}_{\alpha^{(\ell)}} \omega(M_{\sigma^\ell}) + \operatorname{ord}_{\alpha^{(\ell)}} M_\omega = \operatorname{ord}_{\alpha^{(\ell)}} \omega(M_{\sigma^\ell}) = \operatorname{ord}_{\omega^{-1}(\alpha^{(\ell)})} M_{\sigma^\ell}.$$

The combination of both formulas yields

$$\operatorname{ord}_\alpha M_\omega \leqslant \operatorname{ord}_{\omega^{-1}(\alpha^{(\ell)})} M_{\sigma^\ell} + \operatorname{ord}_{\alpha^{(\ell)}} M_{\sigma^\ell}^{-1}.$$

We conclude in a similar way as in the proof of Proposition 7.3. □

7.3. Explicit telescopability

PROPOSITION 7.5. *Assume that $\theta = \partial$. If $\Sigma_{\mathbb{A}}(\operatorname{Sing}(\partial))$ is finite, then I is telescopable in ∂.*

Proof. Assume that $A = \Sigma_{\mathbb{A}}(\operatorname{Sing}(\partial)) \setminus \{\infty\} \subseteq \bar{\mathbb{K}}$ is finite and let $\mathbb{M} = \bar{\mathbb{K}}[z, \psi^{-1}]$, where $\psi = \prod_{\alpha \in A} (z - \alpha)$. Notice that $\mathbb{M}$ is stable under both ∂ and $\theta_1, \ldots, \theta_n$. Given $i \in \{1, \ldots, n+1\}$, Proposition 7.1 implies that $\operatorname{Sing}(\theta_i) \subseteq A$. Given $\varphi \in \mathbb{M}^{1 \times r}$, this means that

$$\partial_i(\varphi F) = (\partial_i \varphi + \varphi M_i) F \in \mathbb{M}^{1 \times r} F \qquad (\theta_i = \partial_i)$$
$$\sigma_i(\varphi F) = (\sigma_i \varphi) M_i F \in \mathbb{M}^{1 \times r} F \qquad (\theta_i = \sigma_i)$$

This proves that the $\bar{\mathbb{K}}[z][\partial]$-module $\mathbb{D}$ is narrow, as required. □

Example 7.6. If $\mathbb{A}$ is a standard DD-operator algebra with $\mathbb{K} = \Bbbk(u_1, \ldots, u_n)$, then $\Sigma_{\mathbb{A}}(\operatorname{Sing}(\partial))$ is finite if and only if $\operatorname{Sing}(\partial)$ does not depend on u_i for every $i \in \{1, \ldots, n\}$ with $\theta_i = \sigma_i$. In particular, we may take $A = \operatorname{Sing}(\partial) \setminus \{\infty\}$ in the proof.

PROPOSITION 7.7. *Assume that $\theta = \sigma$ and let $S = \operatorname{Sing}(\sigma) \cup \operatorname{Sing}(\sigma^{-1})^{(1)}$. Assume that the following holds for every $i \in \{1, \ldots, n\}$:*

- *if $\theta_i = \partial_i$, then for all $\alpha \in \operatorname{Sing}(\sigma)$, we have $\partial_i \alpha = 0$;*
- *if $\theta_i = \sigma_i$, then for all $\alpha \in S$, there exist integers $p, q > 0$ with $\sigma_i^q(\alpha) \subseteq \alpha^{(p)}$.*

Then I is telescopable in σ.

Proof. We define the function $\delta: \bar{\mathbb{K}} \cup \{\infty\} \to \mathbb{N}$ by

$$\delta(\alpha) = \max_{1 \leqslant i \leqslant n+1, \theta_i = \sigma_i} \operatorname{ord}_\alpha \sigma_i.$$

Since $\{\alpha \in \bar{\mathbb{K}} \cup \{\infty\} : \delta(\alpha) \neq 0\}$ is finite, we may also define $\mu : \bar{\mathbb{K}} \setminus \operatorname{Fix} \tau \to \mathbb{N}$ by

$$\hat{\mu}(\alpha) = \begin{cases} 0 & \text{if } \alpha \notin S\updownarrow \\ \sum_{\beta \in \Sigma_{\mathbb{A}}^{-1}(\alpha^{(-\mathbb{N})})} \delta(\alpha^{(\lambda)}) & \text{if } \alpha \in S\updownarrow. \end{cases}$$

By construction, this function is increasing. We also notice that

$$\mathrm{A} = \{\alpha \in \bar{\mathbb{K}} : \hat{\mu}(\alpha) > \hat{\mu}(\alpha^{(-1)})\}$$

is finite, with $\{\alpha \in \bar{\mathbb{K}} : \hat{\mu}(\alpha) \neq 0\} \subseteq \mathrm{A}\uparrow$, and the restrictions $\tilde{\mu} = \hat{\mu}_{|\mathrm{A}\uparrow}$ and $\tilde{\nu} = \hat{\mu}_{|\mathrm{A}}$ of μ to $\mathrm{A}\uparrow$ and A satisfy $\tilde{\mu} = \tilde{\nu}\uparrow$. It follows that $\tilde{\mathbb{M}} = \mathbb{K}(z)_{(\operatorname{Fix}\tau,\star)} \oplus \mathbb{K}(z)_{(\mathrm{A}\uparrow);\mu}$ is a narrow $\mathbb{K}[z][\sigma]$-module of $\mathbb{K}(z)$. In particular, the $\mathbb{K}[z][\sigma]$-module $\tilde{\mathbb{D}} = \tilde{\mathbb{M}}^{r \times 1} F$ comes with a reduction $[\,]_{\tilde{\mathbb{D}}}$ with respect to Δ.

We claim that $\tilde{\mathbb{D}}$ is stable under $\Sigma_{\mathbb{B}}$. The stability under σ is clear. Let $\varphi \in \tilde{\mathbb{M}}^{r \times 1}$, $\alpha \in \bar{\mathbb{K}} \setminus \operatorname{Fix} \tau$ and $i \in \{1, \ldots, n\}$ with $\theta_i = \sigma_i$, so that $\sigma_i(\varphi F) = \sigma_i(\varphi) M_{\sigma_i} F$. Then for any $\alpha \in \mathrm{A}\uparrow \setminus \operatorname{Fix} \tau$, we have

$$\begin{aligned} \operatorname{ord}_\alpha (\sigma_i(\varphi) M_{\sigma_i}) &\leqslant \operatorname{ord}_\alpha \sigma_i(\varphi) + \operatorname{ord}_\alpha \sigma_i \\ &= \operatorname{ord}_{\sigma_i^{-1}(\alpha)} \varphi + \operatorname{ord}_\alpha \sigma_i \\ &\leqslant \hat{\mu}(\sigma_i^{-1}(\alpha)) + \delta(\alpha) \\ &\leqslant \hat{\mu}(\alpha). \end{aligned}$$

In other words, $\sigma_i(\varphi F) \in \tilde{\mathbb{D}}$, which completes the proof of our claim.

Given an iterated derivative $\omega \in \mathcal{D}_{\mathbb{A}}$ and $\varphi \in \tilde{\mathbb{M}}^{r \times 1}$, we next observe that there exist integers $c_{\omega',\omega''} \in \mathbb{N}$ with

$$\begin{aligned} \omega(\varphi F) &= \sum_{\omega = \omega'\omega''} c_{\omega',\omega''}\, \omega'(\varphi)\, \omega''(F), \\ &= \left(\sum_{\omega = \omega'\omega''} c_{\omega',\omega''}\, \omega'(\varphi)\, M_{\omega''} \right)(F), \end{aligned}$$

where we notice that $\omega'(\varphi) \in \tilde{\mathbb{M}}^{r \times 1}$ for all $\omega' \in \mathcal{D}_{\mathbb{A}}$. For any $\alpha \in \bar{\mathbb{K}} \setminus \operatorname{Fix} \tau$, it follows that

$$\operatorname{ord}_\alpha \left(\sum_{\omega = \omega'\omega''} c_{\omega',\omega''}\, \omega'(\varphi)\, M_{\omega''} \right) \leqslant \hat{\mu}(\alpha) + \operatorname{ord}_\alpha \mathcal{D}_{\mathbb{A}}.$$

By Proposition 7.3, there are only a finite number of points $\alpha \in \bar{\mathbb{K}} \setminus \operatorname{Fix} \tau$ where $\operatorname{ord}_\alpha \mathcal{D}_{\mathbb{A}} > 0$ and $\operatorname{ord}_\alpha \mathcal{D}_\alpha \in \mathbb{N}$ at these points. In other words, setting $\mathbb{D} = \mathbb{K}[\mathcal{D}_{\mathbb{A}}](\tilde{\mathbb{D}})$, there exists a finite dimensional $\mathbb{K}$-vector space $\mathbb{V} \subseteq \mathbb{D}$ with $\mathbb{D} = \tilde{\mathbb{D}} \oplus \mathbb{V}$. We extend $[\,]_{\tilde{\mathbb{D}}}$ to a confined reduction $[\,]_{\mathbb{D}} : \mathbb{D} \to \mathbb{D}$ with respect to Δ by setting $[g + h]_{\mathbb{D}} = [g]_{\tilde{\mathbb{D}}} + h$ for all $g \in \tilde{\mathbb{D}}$ and $h \in \mathbb{V}$. Notice that $\mathbb{D}$ is still stable under $\Sigma_{\mathbb{B}}$, since $\Sigma_{\mathbb{B}}$ commutes with $\mathcal{D}_{\mathbb{A}}$. □

Example 7.8. Consider the case when $\mathbb{A}$ is a standard DD-operator algebra with $\mathbb{K} = \Bbbk(u)$. If $\sigma = S_{1,z}$ and $\theta_1 = S_{1,u}$, then it follows that the denominator of M_σ as a rational function in z should be a power product of linear forms $kz + lu - \alpha$ with $k \in \mathbb{N}^>$, $l \in \mathbb{N}$ and $\alpha \in \bar{\mathbb{K}}$. If this is not the case, then one may try replacing the operator $\theta_1 = S_{1,u}$ by its inverse $\theta_1 = S_{-1,u}$, for which one might have more luck. If either σ or $\theta = Q_{q,u}$ is a q-difference operator, and the other one is a shift operator, then the denominator of M_σ should be a polynomial in $\Bbbk[z]$. If both $\sigma = Q_{q,z}$ and $\theta = Q_{q,u}$ are q-difference operators, then the denominator of M_σ should be a power product of terms $z^k u^l - \alpha$ with $k \in \mathbb{N}^>$, $l \in \mathbb{N}$, and $\alpha \in \bar{\mathbb{K}}$.

We say that I is *explicitly telescopable* in θ with respect to the basis F if it satisfies the sufficient conditions from Propositions 7.5 or 7.7. These conditions present the advantage that they are formulated exclusively in terms of the singularities of θ. For a fixed basis F, they are also easy to check. It remains to investigate how to compute a basis with respect to which I is explicitly telescopable, whenever such a basis exists.

We will restrict our attention to the case when $\theta = \partial$; an algorithm can probably be developed for the difference case as well, but we will leave this for a future work. Given a singularity $\alpha \in \mathrm{Sing}_F(\partial) \cap \bar{\mathbb{K}}$, let us show how to compute a basis G with $\mathrm{Sing}_G(\partial) \subseteq \mathrm{Sing}_F(\partial) \setminus \{\alpha\}$ whenever such a basis exists. We outline the algorithm over $\bar{\mathbb{K}}$, but it is not hard to adapt it to work over $\mathbb{K}$. The computed basis is no longer cyclic; with additional effort, it can be shown that there also exist cyclic bases G with $\mathrm{Sing}_G(\partial) \subseteq \mathrm{Sing}_F(\partial) \setminus \{\alpha\}$.

We start with the computation of a basis of formal local transseries solutions of the system $\partial F = M_{\partial,F} F$ at α, given in the form of a fundamental matrix Y. If there exist local solutions that are not Laurent series, then the singularity α clearly cannot be removable. Otherwise, the truncation $\tilde{Y}$ of Y at a sufficiently high order $O((z-\alpha)^k)$ gives rise to a basis $G = \tilde{Y}^{-1} F = 1 + O(z-\alpha)$ such that $M_{\partial,G} = (\tilde{Y}^{-1} Y)' \, (Y^{-1} \tilde{Y})$ admits no singularity at α.

Unfortunately, $\tilde{Y}^{-1}$ may introduce new singularities, so this algorithm needs to be tweaked a little more. We replace $\tilde{Y}$ with a new approximation at a order $O((z-\alpha)^k)$ that can be written as a product of diagonal matrices with entries of the form $(z-\alpha)^i$ with $i \in \mathbb{Z}$, invertible constant matrices in $\mathbb{K}^{r\times r}$, and upper/lower triangular matrices of the form $\mathrm{Id}_r + O(z-\alpha)$. It is not hard to compute such an approximation. By construction, its inverse admits a factorization of the same form, whence it introduces no new singularities.

Even small equations such as $(z-\alpha)\,\partial f = kf$ may admit solutions with a high valuation k. It follows that the above algorithm for changing bases may be very expensive and its cost is generally not polynomial in the input size.

7.4. Testing D-finiteness

If I is explicitly telescopable in θ with respect to the chosen basis F and associated reduction $[\,]_{\mathbb{D}}$, then the telescoping ideal $\mathcal{T}_{\varphi,\delta}$ is certainly D-finite for any $\varphi \in \mathbb{D}$. Moreover, Algorithm 6.1 allows us to efficiently compute a Gröbner basis for a D-finite subideal of $\mathcal{T}_{\varphi,\delta}$; with more computational effort, one may even compute a Gröbner basis for $\mathcal{T}_{\varphi,\delta}$ itself.

If I is not explicitly telescopable in θ, even after a suitable change of basis, then telescoping ideals $\mathcal{T}_{\varphi,\delta}$ are generally not D-finite, although they exceptionally might be. In this subsection, we will show how to modify Algorithm 6.1 in order to test D-finiteness of $\mathcal{T}_{\varphi,\delta}$ and to compute a Gröbner basis for $\mathcal{T}_{\varphi,\delta}$ if so. We will restrict our attention to the case when $\theta = \partial$, but a similar approach is likely to work in the difference case as well. It will also be convenient to assume that for any $\omega \in \Sigma_{\mathbb{A}}$ and $c \in \mathbb{K}$ with $\omega(c) \neq c$, we have $\omega^k(c) \neq c$ for all $k \in \mathbb{Z}^{\neq}$. This is the case for standard DD-operator algebras.

Assume that we have fixed a cyclic basis F of $\mathbb{B}/I$ with entries $f_1, \ldots, f_r$ and let L be the vanishing operator of F_1 in ∂. Denote $\mathrm{A} = \mathrm{Sing}(\partial)$, $\mathrm{B} = \Sigma_{\mathbb{A}}(\mathrm{A})$, $\mathbb{M} = \mathbb{K}(z)_{(\mathrm{B},\infty,\star)}$, and $\mathbb{D} = \mathbb{M}^{1\times r} F$. Then $\mathbb{D}$ is a $\mathbb{K}[z, \theta_1, \ldots, \theta_n, \theta]$-submodule of $\mathbb{B}/I$ that contains $f_1, \ldots, f_r$, but that is not necessarily narrow in θ.

Let $[\![\,]\!]$ be the normal global reduction on $\mathbb{K}(z)$ with respect to L^* as constructed in Remark 4.7. The construction from section 4.4 still works and leads to a corresponding normal reduction $[\![\,]\!]$ on $\mathbb{B}/I$. The restrictions $[\![\,]\!]_{\mathbb{M}}$ and $[\![\,]\!]_{\mathbb{D}}$ of these reductions to $\mathbb{M}$ and $\mathbb{D}$ are also normal. We call $[\![\,]\!]_{\mathbb{D}}$ the *associated normal reduction* to I (and for the chosen basis F). For any subset $\Gamma \subseteq \bar{\mathbb{K}}$, we notice that $[\![\bar{\mathbb{K}}(z)_{(\Gamma,\infty,\star)}]\!] \subseteq \bar{\mathbb{K}}(z)_{(\Gamma,\infty,\star)}$. In particular, $\mathrm{Sing}([\![u]\!]_{\mathbb{M}}) \subseteq \mathrm{Sing}(u) \cup \{\infty\}$ for all $u \in \mathbb{M}$.

LEMMA 7.9. *Let $\omega \in \Sigma_{\mathbb{A}}^{\mathbb{Z}}$ and $u \in \mathbb{M}$. Let $\lambda_1, \ldots, \lambda_r$ be the entries of the first row of M_ω. Then $[\![\omega(u f_1)]\!]_{\mathbb{D}} = [\![\omega\{u\}\, f_1]\!]_{\mathbb{D}}$, where*

$$\omega\{u\} \;=\; \omega(u)\,\lambda_1 - \partial\,(\omega(u)\,\lambda_2) + \cdots + (-1)^{r-1}\,\partial^{r-1}\,(\omega(u)\,\lambda_r)$$

and $\mathrm{Sing}(\omega\{u\}) \subseteq \omega(\mathrm{Sing}(u)) \cup \mathrm{Sing}(\omega)$.

Proof. Let $\hat{u} \in \mathbb{M}^{1\times r}$ be the row vector with entries $u, 0, \ldots, 0$. Then $\omega(\hat{u} F) = \omega(\hat{u})\, M_\omega F$ implies $\omega(u f_1) = \omega(u)\,(\lambda_1 f_1 + \cdots + \lambda_r f_r)$. The result now follows from (4.8). □

PROPOSITION 7.10. *Let Ξ be the set of all $\alpha \in \mathrm{B}$ such that there exists an $i \in \{1,\ldots,n\}$ with $\sigma_i(\alpha) \neq \alpha$ and $\sigma_i^{\mathbb{N}}(\alpha) \cap (\mathrm{A} \cup \mathrm{Sing}(\sigma_i) \cup \mathrm{Sing}(\sigma_i^{-1})) = \emptyset$. Let $u f_1 \in [\![\mathbb{D}]\!]_{\mathbb{D}}$ with $u \in [\![\mathbb{M}]\!]_{\mathbb{M}}$ be such that $u_{(\Xi)} \neq 0$. Then the $\mathbb{K}$-vector space $\mathrm{Vect}(\Sigma_{\mathbb{A}}(u f_1))$ generated by all $\omega(u f_1)$ with $\omega \in \Sigma_{\mathbb{A}}$ has infinite dimension.*

Proof. Let $\alpha \in \Xi$ be such that $u_{(\alpha)} \neq 0$ and let $i \in \{1,\ldots,n\}$ be such that $\sigma_i(\alpha) \neq \alpha$ and $\sigma_i^{\mathbb{N}}(\alpha) \cap (\mathrm{A} \cup \mathrm{Sing}(\sigma_i) \cup \mathrm{Sing}(\sigma_i^{-1})) = \emptyset$. Modulo the replacement of α by $\sigma_i^k(\alpha)$, we may assume without loss of generality that $u_{(\sigma_i^k(\alpha))} = 0$ for all $k > 0$. Recall that $\alpha, \sigma_i(\alpha), \sigma_i^2(\alpha), \ldots$ are pairwise distinct.

Consider the sequence $u_0, u_1, \ldots \in \mathbb{M}$ with $u_0 = u$ and $u_{k+1} = [\![\sigma_i\{u_k\}]\!]_{\mathbb{M}}$ for all $k \in \mathbb{N}$. Let us show by induction on k that $(u_k)_{(\sigma_i^k(\alpha))} \neq 0$ and $(u_k)_{(\sigma_i^l(\alpha))} = 0$ for all $l > k$. These assertions hold by assumption for $k = 0$, so assume that $k > 0$. By Lemma 7.9, we have

$$\mathrm{Sing}(u_{k+1}) \subseteq \sigma_i(\mathrm{Sing}(u_k)) \cup \mathrm{Sing}(\sigma_i).$$

For each $l > k$, the assumption that $\sigma_i^l(\alpha) \notin \mathrm{Sing}(u_k)$ therefore implies $\sigma_i^{l+1}(\alpha) \notin \mathrm{Sing}(u_{k+1})$. Assume for contradiction that $(u_{k+1})_{(\sigma_i^{k+1}(\alpha))} = 0$. Let $g \in \mathbb{B}/I$ be such that

$$u_{k+1} f_1 = \sigma_i(u_k f_1) + \partial(g).$$

Then we have

$$[\![\sigma_i^{-1}\{u_{k+1}\} f_1]\!] = [\![\sigma_i^{-1}(u_{k+1} f_1)]\!] = [\![u_k f_1 + \partial(\sigma_i^{-1}(g))]\!] = [\![u_k f_1]\!]$$

and Lemma 7.9 implies

$$\mathrm{Sing}(\sigma_i^{-1}\{u_{k+1}\}) \subseteq \sigma_i^{-1}(\mathrm{Sing}(u_{k+1})) \cup \mathrm{Sing}(\sigma_i^{-1}).$$

Since $\sigma_i^{k+1}(\alpha) \notin \mathrm{Sing}(u_{k+1})$ and $\sigma_i^k(\alpha) \notin \mathrm{Sing}(\sigma_i^{-1})$, it follows that $\sigma_i^k(\alpha) \notin \mathrm{Sing}(\sigma_i^{-1}\{u_{k+1}\})$. From $[\![(u_k - \sigma_i^{-1}\{u_{k+1}\}) f_1]\!] = 0$, $\sigma_i^k(\alpha) \notin \mathrm{Sing}(\partial)$, and the normality of $[\![\,]\!]$, we also get that $(u_k - \sigma_i^{-1}\{u_{k+1}\})_{(\sigma_i^k(\alpha))} = 0$. We conclude that $\sigma_i^k(\alpha) \notin \mathrm{Sing}(u_k)$, which contradicts the induction hypothesis.

We claim that $u_0, u_1, \ldots$ are $\mathbb{K}$-linearly independent. Indeed, given a non-trivial relation $\lambda_0 u_0 + \cdots + \lambda_k u_k = 0$ with $\lambda_k \neq 0$, we would obtain the contradiction

$$0 = (\lambda_0 u_0 + \cdots + \lambda_k u_k)_{(\sigma_i^k(\alpha))} = \lambda_k (u_k)_{(\sigma_i^k(\alpha))} \neq 0.$$

We conclude by noticing that $u_0, u_1, \ldots \in \mathrm{Vect}(\Sigma_{\mathbb{A}}(u f_1))$. □

We are now in a position to state the adaptation of Algorithm 6.1 to the case when the ideal I is not necessarily telescopable.

Algorithm 7.1
INPUT: a D-finite ideal $I \subseteq \mathbb{B}$ with associated normal reduction $[\![\,]\!]_{\mathbb{D}}$ and $\varphi \in \mathbb{D}^{\mathrm{prim}}$
OUTPUT: the reduced Gröbner basis $\mathcal{G}$ for $\mathcal{T}_{\varphi,\delta}$ if $\mathcal{T}_{\varphi,\delta}$ is D-finite and $\perp$ otherwise

$\mathcal{L} := \{1\}$, $\mathcal{G} := \emptyset$, $\mathcal{R} := \emptyset$
while $\mathcal{L} \neq \emptyset$ **do**
  Let $\omega \in \mathcal{L}$ be minimal for $\leqslant$ and set $\mathcal{L} := \mathcal{L} \setminus \{\omega\}$
  if ω is not reducible with respect to $\mathcal{G}$ **then**
    if $\omega = 1$ **then** let $R_\omega := [\![\varphi]\!]_{\mathbb{D}}$
    else decompose $\omega = \theta_i \omega'$ and let $R_\omega := [\![\theta_i R_{\omega'}]\!]_{\mathbb{D}}$
    Let $u \in \mathbb{M}$ be such that $\varphi = u f_1$ and let Ξ be as in Proposition 7.10
    if $u_{(\Xi)} \neq 0$ **then return** $\perp$
    if $\exists (c_\rho)_{\rho \in \mathcal{R}} \in \mathbb{K}^{\mathcal{R}}$, $R_\omega = \sum_{\rho \in \mathcal{R}} c_\rho R_\rho$ **then** $\mathcal{G} := \mathcal{G} \cup \{\omega - \sum_{\rho \in \mathcal{R}} c_\rho \rho\}$
    else $\mathcal{R} := \mathcal{R} \cup \{\omega\}$, $\mathcal{L} := \mathcal{L} \cup \{\theta_1 \omega, \ldots, \theta_n \omega\}$
return $\mathcal{G}$

THEOREM 7.11. *Algorithm 7.1 is correct and terminates.*

Proof. Let us first show that $\mathrm{B} \setminus \Xi$ is finite. Given $\alpha \in \mathrm{B} \setminus \Xi$, we either have $\Sigma_{\mathbb{A}}(\alpha) = \{\alpha\}$ and $\alpha \in \mathrm{A}$, or $\sigma_i(\alpha) \neq \alpha$ for some i. In the latter case, our assumption $\alpha \in \mathrm{B} \setminus \Xi$ implies the existence of a maximal $k \in \mathbb{N}$ with $\sigma_i^k(\alpha) \in \mathrm{A} \cup \mathrm{Sing}(\sigma_i) \cup \mathrm{Sing}(\sigma_i^{-1})$. We claim that there also exists a maximal $l \in \mathbb{N}$ with $\sigma_i^{-l}(\alpha) \notin \Xi$. Our claim clearly implies that $\mathrm{B} \setminus \Xi$ is finite, since we are left with a finite number of possible α for each i.

Assume for contradiction that the claim does not hold. Then $\sigma_i^{-1}(\alpha) \in \Sigma_{\mathbb{A}}(\mathrm{A})$, whence there exists a $j \in \{1, \ldots, n\}$ with $\sigma_j^{\mathbb{N}}(\alpha) \cap \sigma_i^{\mathbb{N}}(\alpha) = \{\alpha\}$. For each $l \in \mathbb{N}$, there exists an $m \in \mathbb{N}$ with $\sigma_j^m(\sigma_i^{-l}(\alpha)) \in \mathrm{A} \cup \mathrm{Sing}(\sigma_j) \cup \mathrm{Sing}(\sigma_j^{-1})$. For some $l' \neq l$ and $m' \neq m$, it follows that $\sigma_j^m(\sigma_i^{-l}(\alpha)) = \sigma_j^{m'}(\sigma_i^{-l'}(\alpha))$. Since $\sigma_j^{\mathbb{N}}(\alpha) \cap \sigma_i^{\mathbb{N}}(\alpha) = \{\alpha\}$, we have $\varrho = (l' - l) / (m - m') > 0$ and σ_j acts on $(\sigma_i^{\mathbb{Z}} \sigma_j^{\mathbb{Z}})(\alpha)$ as $\sigma_i^{-\varrho}$. In particular, there exist minimal $s \in \varrho\, \mathbb{Z}$ and maximal $t \in \varrho\, \mathbb{Z}$ with $\sigma_i^s(\alpha)$, $\sigma_i^t(\alpha) \in \mathrm{A} \cup \mathrm{Sing}(\sigma_i) \cup \mathrm{Sing}(\sigma_i^{-1}) \cup \mathrm{Sing}(\sigma_j) \cup \mathrm{Sing}(\sigma_j^{-1})$. For $l'' > -s$, we conclude that there cannot exist an $m'' \in \mathbb{N}$ with $\sigma_j^{m''}(\sigma_i^{-l''}(\alpha)) = \sigma_i^{-l'' - \varrho m''}(\alpha) \in \mathrm{A} \cup \mathrm{Sing}(\sigma_j) \cup \mathrm{Sing}(\sigma_j^{-1})$.

Having shown that $\mathrm{B} \setminus \Xi$ is finite, we either hit some $u \in \mathbb{M}$ with $u_{(\Xi)} \neq 0$, in which case the correctness follows from Proposition 7.10, or all computations take place in the narrow submodule $\mathbb{K}(z)_{(\mathrm{B} \setminus \Xi, \infty, \star)}$ of $\mathbb{K}(z)$, and the correctness and termination are proved in a similar way as for Theorem 6.4. □

8. COMPLEXITY OF RATIONAL FUNCTION ARITHMETIC

8.1. Complexity of arithmetic in $\mathbb{K}$

Consider a standard DD-operator algebra $\mathbb{A} = \mathbb{K}[\theta_1, \ldots, \theta_n]$ with $\mathbb{K} = \Bbbk(u_1, \ldots, u_n)$, as in section 6.1. We will use the algebraic complexity model: running times are measured in terms of the required number of field operations in $\Bbbk$ and space complexity in terms of the required number of coefficients in $\Bbbk$.

We use a dense representation for polynomials in $\Bbbk[u_1, \ldots, u_n]$. A polynomial $P \in \Bbbk[u_1, \ldots, u_n]$ of total degree $d = \deg P$ thus requires space $\binom{n+d}{n} = O(d^n)$. From now on, we assume that the dimension n is a fixed constant, whereas d may become large. Given $P, Q \in \Bbbk[u_1, \ldots, u_n]$ of total degrees $\deg P$ and $\deg Q$, it is well known for that their product can be computed in quasi-linear time $\tilde{O}((\deg(PQ))^n)$; see [53] for a particularly efficient algorithm. Partial derivatives and q-differences can clearly be computed in linear time $O((\deg P)^n)$. It is also well-known [11] that partial shifts reduce to coefficientwise univariate multiplications, so they can be computed in time $\tilde{O}((\deg P)^n)$.

The deterministic computation of the greatest common divisor of two polynomials $P, Q \in \Bbbk[u_1, \ldots, u_n]$ is more expensive. The best currently known algorithms first select a principal variable, say u_n, and reinterpret P and Q as univariate polynomials in u_n. They next recursively factor out the content and then compute the univariate gcd using a division-free algorithm for the computation of subresultants such as Berkowitz' algorithm [8]; see also [69]. This leads to the complexity bound $\tilde{O}(\max(\deg P, \deg Q)^{\gamma n})$ with $\gamma \leqslant 4$. When using randomized algorithms of Las Vegas type, one may take $\gamma = 1$ instead; since $\deg \gcd(P, Q) \leqslant \max(\deg P, \deg Q)$, this means that gcds can be computed in quasi-linear time (more precisely: after a generic linear change of variables, both P and Q can be assumed to be monic in u_n, and using evaluation/interpolation at $O(d^{n-1})$ points, one reduces to fast univariate gcds [18]).

Rational functions in $\mathbb{K}$ are represented as fractions P, Q with $P, Q \in \Bbbk[u_1, \ldots, u_n]$, $\gcd(P, Q) = 1$, and Q monic with respect to a suitable term ordering. We write $\deg(P/Q) = \max(\deg P, \deg Q)$ for the *degree* of such a rational function. Given $f, g \in \mathbb{K}$ and $\square \in \{+, -, \times, /\}$, we notice that $\deg(f \square g) \leqslant \deg f + \deg g$. One may compute $f \square g$ in non simplified form in quasi-linear time $\tilde{O}((\deg f + \deg g)^n)$, whereas the simplification of the resulting fraction requires $\tilde{O}((\deg f + \deg g)^{\gamma n})$ further operations. Of course, it is possible to delay the simplification of fractions in many cases.

When computing with vectors of matrices with entries in $\mathbb{K}$, we always assume that denominators have been factored out. In other words, a matrix $M\in\mathbb{K}^{r\times c}$ is represented as a matrix $N\in\mathbb{k}[u_1,\ldots,u_n]^{r\times c}$ divided by a monic polynomial $D\in\mathbb{k}[u_1,\ldots,u_n]$ such that the gcd of D and all entries of N equals one (here D is understood to be monic with respect to a suitable monomial ordering on $u_1^{\mathbb{N}}\cdots u_n^{\mathbb{N}}$). We also define $\deg M=\max(\max_{i,j}\deg N_{i,j},\deg D)$. Simplifying a general fraction N/D can be done in time $\tilde{O}(r\,c\max(\deg N,\deg D)^{\gamma n})$. Ignoring the cost of the final simplification, two matrices $M,N\in\mathbb{K}^{r\times c}$ can be added in time $\tilde{O}(r^2(\deg M+\deg N)^n)$ and multiplied in time $\tilde{O}(r^{\omega}(\deg M+\deg N)^n)$ if $r=c$. Here $\omega\leqslant 3$ denotes the "exponent of matrix multiplication", i.e. two matrices in $\mathbb{k}^{r\times r}$ can be multiplied using $O(r^{\omega})$ operations in $\mathbb{k}$.

When allowing for probabilistic algorithms (of Las Vegas type), it is interesting to consider the alternative straight line program (SLP) representation for polynomials and rational functions [20]. In this model, the length $\#f$ of the SLP that evaluates $f\in\mathbb{K}$ is a natural mesure for the complexity of f. Using sparse interpolation [7, 58, 57, 51, 5], the actual coefficients of a polynomial $P\in\mathbb{k}[u_1,\ldots,u_n]$ of degree bounded by d can be retrieved from its SLP representation in expected time $\tilde{O}((\#P)\,d^n)$. Similarly, the dense representation of a rational function $f\in\mathbb{K}$ can be retrieved in expected time $\tilde{O}((\#f)\,(\deg f)^n)$; see [59].

Algorithms that rely on SLP representations and sparse interpolation are particularly interesting when intermediate expressions swell in size, but the end-result is small in comparison. It turns out that the sizes of SLPs for creative telescoping remain reasonably small, which makes this approach relevant in our context. For this reason, we will also analyze the complexity of our algorithms in this model.

8.2. Complexity of arithmetic in $\mathbb{K}(z)$

Now consider a standard DD-operator algebra extension $\mathbb{B}=\mathbb{K}(z)[\theta_1,\ldots,\theta_n,\theta]$ of $\mathbb{A}$ as in section 6.3. Polynomials in $\mathbb{K}[z]$ and rational fractions in $\mathbb{K}(z)$ can be represented in a similar way as in the previous subsection, but for our complexity analysis, it is useful to distinguish between the degree in z and the total degree in u. As in the case of vectors and matrices, we also assume that the denominators of polynomials in $\mathbb{K}[z]$ have been factored out.

With the above conventions, the monic gcd G of two polynomials $P,Q\in\mathbb{K}[z]$ can be computed in time $\tilde{O}(\max(\deg_u P,\deg_u Q)^{\gamma n}\max(\deg_z P,\deg_z Q))$ and we have

$$\deg_u G \leqslant \max(\deg_u P,\deg_u Q)\max(\deg_z P,\deg_z Q). \tag{8.1}$$

In fact, for any monic polynomial $A\in\mathbb{K}[z]$ that divides P, we even have

$$\deg_u A \leqslant \deg_u P\cdot\deg_z A. \tag{8.2}$$

It is also possible to compute the corresponding cofactors $U,V\in\mathbb{K}[z]$ with

$$G = UP+VQ,$$

still with the same complexity. These cofactors satisfy similar degree bounds

$$\deg_u U \leqslant \max(\deg_u P,\deg_u Q)\max(\deg_z P,\deg_z Q) \tag{8.3}$$

$$\deg_u V \leqslant \max(\deg_u P,\deg_u Q)\max(\deg_z P,\deg_z Q). \tag{8.4}$$

in addition to the usual bounds $\deg_z U<\deg_z Q-\deg_z G$ and $\deg_z V<\deg_z P-\deg_z G$ in z.

Similarly, given two polynomials $A,B\in\mathbb{K}[z]$ with monic B, the Euclidean division of A by B yields a relation

$$A = QB+R,$$

in which the quotient Q and the remainder R satisfy the degree bounds

$$\deg_u Q \leqslant \deg_u A+\max(\deg_z A-\deg_z B+1,0)\deg_u B \tag{8.5}$$

$$\deg_u R \leqslant \deg_u A+\max(\deg_z A-\deg_z B+1,0)\deg_u B. \tag{8.6}$$

Setting $d = \deg_u A + \max(\deg_z A - \deg_z B + 1, 0) \deg_u B$, the division itself can be done in time $\tilde{O}(d^{\gamma n} \max(\deg_z A, \deg_z B))$.

8.3. Complexity of arithmetic in $\bar{\mathbb{K}}$

The computation of the local reductions $[]_\alpha$ in sections 4 and 5 requires us to compute with roots $\alpha \in \bar{\mathbb{K}}$ of the leading coefficient $P = L_r$ of our differential or difference operator L. If the factorization of P is known, then this comes down to computations over the extension field $\mathbb{L} = \mathbb{K}[z]/(H(z))$, where H is the monic irreducible factor of P with $H(\alpha) = 0$.

Let us first examine the cost of arithmetic in such an extension field $\mathbb{L}$ and denote $d = \deg_z H$. Elements λ in $\mathbb{L}$ are represented by their pre-images $\check{\lambda} \in \mathbb{K}[z]$ with $\lambda = \check{\lambda} + (H)$ and $\deg_z \check{\lambda} < \deg_z H$. It is natural to define the total degree of λ in u by $\deg_u \lambda = \deg_u \check{\lambda}$. Given $\lambda, \mu \in \mathbb{L}$, it follows from (8.6) that

$$\deg_u \lambda \mu \leqslant \deg_u \lambda + \deg_u \mu + (\deg_u H) d,$$

where we recall from (8.2) that $\deg_u H \leqslant \deg_u P \cdot \deg_z P$. Similarly, the inverse of a non-zero $\lambda \in \mathbb{L}$ satisfies

$$\deg_u \lambda^{-1} \leqslant \max(\deg_u \lambda, \deg_u H) d,$$

thanks to (8.3). Since additions and subtraction do not require any Euclidean divisions, we also have

$$\deg_u (\lambda \pm \mu) \leqslant \deg_u \lambda + \deg_u \mu.$$

The computations of $\lambda \mu$ and $\lambda \pm \mu$ can respectively be performed in time $\tilde{O}((\deg_u \lambda + \deg_u \mu + (\deg_u H) d)^{\gamma n} d)$ and $\tilde{O}((\deg_u \lambda + \deg_u \mu)^{\gamma n} d)$, whereas λ^{-1} can be computed in time $\tilde{O}(\max(\deg_u \lambda, \deg_u H)^{\gamma n} d)$. One may take $\gamma = 1$ in the first two complexity bounds if we only require unsimplified results.

For simplicity, we will assume that the irreducible factorization of P is known in what follows. In order to be complete, we still have to discuss the cost of computing such a factorization. Since this task can be rather expensive, it is actually better to avoid it by relying on the strategy of "dynamic evaluation" [31].

Starting with the square-free part H of P, the idea is to directly work in the extension $\mathbb{L} = \mathbb{K}[z]/(H(z))$ as if it were a field. Whenever we hit a non-trivial zero-divisor λ during the computation of an inverse in $\mathbb{L}$, we simply abort all computations and resume with $\gcd(\check{\lambda}, H)$ and $H / \gcd(\check{\lambda}, H)$ in the role of H. This can happen at most $\deg_z H$ times, so all complexity bounds that will be proved in the sequel have to be multiplied by $\deg_z H$ in order to take into account the cost of the factorization of P. In fact, the $\deg_z P$ overhead can be reduced at the expense of technical complications; since we merely want to establish polynomial complexity bounds, we can spare ourselves this effort.

Computations with algebraic functions in $\bar{\mathbb{K}}$ are often more efficient in the SLP model. In that case, at every evaluation point $u \in \mathbb{k}^n$, we work in the algebraic extension $\mathbb{l} = \mathbb{k}[z]/(H\langle u \rangle)$ of $\mathbb{k}$ instead of $\mathbb{L}$. Here we used angular brackets for the evaluation of H at u in order to avoid confusion with evaluation in z. Now SLPs over this algebraic extension $\mathbb{l}$ of $\mathbb{k}$ can be reinterpreted as SLPs over $\mathbb{k}$: additions/subtractions in $\mathbb{l}$ translate into d additions/subtractions in $\mathbb{k}$, whereas each multiplication or division in $\mathbb{l}$ give rise to $\tilde{O}(d)$ ring operations in $\mathbb{k}$. When implementing dynamic evaluation, one also must be careful to "take the same branch of the computation tree" at each evaluation point: whenever we hit a non-trivial zero-divisor λ of H, we store the corresponding SLP for λ, and first relaunch the computation with $\gcd(\check{\lambda}, H)$ in the role of H (for all evaluation points), and then once more with $H / \gcd(\check{\lambda}, H)$ in the role of H (again for all evaluation points).

8.4. Partial fraction decomposition

Consider a fraction $f=N/D\in\mathbb{K}(z)$ with $N,D\in\mathbb{K}[z]$ and $\gcd(N,D)=1$. Assume that D is monic and that its factorization over $\mathbb{K}$ is known, say $D=H_1^{\mu_1}\cdots H_l^{\mu_l}$. Let us analyze the cost of computing the partial fraction decomposition of f.

The partial fraction decomposition can actually be computed over $\mathbb{K}$ as well as over its algebraic closure. When working over $\mathbb{K}$, the partial fraction decomposition can be written at choice in one of the following two forms

$$f = f_{(\infty,\star)}+\sum_{i=1}^{l}\frac{C_i}{H_i^{\mu_i}}$$
$$f = f_{(\infty,\star)}+\sum_{i=1}^{l}\sum_{j=1}^{\mu_i}\frac{c_{i,j}}{H_i^j},$$

where $\deg_z C_i<\mu_i\deg_z H_i$ and $\deg_z c_{i,j}<\deg_z H_i$. It is well known [65] that partial fraction decompositions of these kinds can be computed in quasi-linear time $\tilde{O}(\deg_z f)$ in terms of operations over $\mathbb{K}$.

In this paper, we have chosen to systematically work over the algebraic closure $\bar{\mathbb{K}}$ of $\mathbb{K}$. Now given a root α of D and any automorphism ϕ of $\bar{\mathbb{K}}$ over $\mathbb{K}$, we notice that $f_{(\phi(\alpha))}=\phi(f_{(\alpha)})$. It therefore suffices to compute $f_{(\alpha)}$ for only one root α in each conjugacy class. We define the *partial fraction decomposition of f over $\bar{\mathbb{K}}$* to consist of $f_{(\star)}$ and the collection of coefficients $f_{(\alpha),k}$, where α runs over ∞ and one element in each conjugacy class among the set of roots of D, and where $0<k\leqslant\operatorname{ord}_\alpha f$. The "tangling" and "untangling" morphisms from [52, Section 4] allow for conversions between partial fraction decompositions over $\mathbb{K}$ and $\bar{\mathbb{K}}$ in quasi-linear time. It will be convenient to write

$$\deg_u^* f = \max\left(\deg_u f_{(\star)},\max_{\alpha,k}\deg_u f_{(\alpha),k}\right)$$

for the maximal degree in u of a coefficient in the partial fraction decomposition of f over $\bar{\mathbb{K}}$.

We have the following deterministic (but rather crude) bounds for the cost of partial fraction decomposition and the resulting degree swell in u.

PROPOSITION 8.1. *Let $f=N/D\in\mathbb{K}(z)$ be as above. Then*

$$\deg_u^* f = O((\deg_z f)^3\deg_u f).$$

If the factorization of D is known, then the partial fraction decomposition of f over $\bar{\mathbb{K}}$ can be computed in time

$$\tilde{O}(((\deg_z f)^4\deg_u f)^{\gamma n}\deg_z f).$$

Proof. From (8.5) and the fact that we may compute $f_{(\infty,\star)}$ as the quotient of the Euclidean division of N by D, we first notice that

$$\deg_u f_{(\infty,\star)} = O(\deg_z f\deg_u f).$$

Given a root $\alpha\in\bar{\mathbb{K}}$ of D of multiplicity ν, let us next prove that

$$\deg_u f_{(\alpha),k} = O((\deg_z f)^3\deg_u f),$$

for $k=1,\ldots,\nu$. Let $H\in\mathbb{K}[z]$ be the monic minimal polynomial of α. We first notice that $\deg_u\alpha^k\leqslant k\deg_u H$ for all $k\in\mathbb{N}$, whence $\deg_u N^{(i)}(\alpha)\leqslant\deg_z N\deg_u H+\deg_u N\leqslant\deg_z f\deg_u H+\deg_u f$ for all i and similarly for D. Let $N=N_0+N_1(z-\alpha)+\cdots$ and $D=D_\nu(z-\alpha)^\nu+D_{\nu+1}(z-\alpha)^{\nu+1}+\cdots$ be the power series expansions of N and D at $z\to\alpha$ with $D_\nu\neq0$. It follows that $\deg_u N_i$ and $\deg_u D_i$ are bounded by $\deg_z f\deg_u H+\deg_u f$ for all i, and $\deg_u D_\nu^{-1}\leqslant(\deg_z f\deg_u H+\deg_u f)\deg_z H$. Now consider the power series

$$Q=\frac{N}{D_\nu\left(1+\frac{D}{D_\nu(z-\alpha)^\nu}\right)}$$

in $z-\alpha$. By what precedes, we have $\deg_u Q_k = O(k(\deg_z f \deg_u H + \deg_u f) \deg_z H)$ and we notice that $Q_k = f_{(\alpha),\nu-k}$ for $k<\nu$. Since $\nu \leqslant \deg_z f / \deg_z H$ and $\deg_u H \leqslant \deg_z f \deg_u f$, we conclude that $\deg_u f_{(\alpha),k} = O((\deg_z f)^3 \deg_u f)$.

The quotient $f_{(\infty,\star)}$ of the Euclidean division of N and D can be computed in time $\tilde{O}((\deg_u f)^{\gamma n} \deg_z f)$. The naive expansion of the series N and D until order $O((z-\alpha)^{2\nu})$ takes time $\tilde{O}(((\deg_z f)^3 \deg_u f)^{\gamma n} \nu \deg_z f)$ and the division of the results with precision $O((z-\alpha)^\nu)$ can be performed in time $\tilde{O}(((\deg_z f)^3 \deg_u f \deg_z H)^{\gamma n} \nu)$. Doing this for one α in each conjugacy class yields the announced complexity bound. □

Partial fraction decomposition becomes much faster when carried out directly over $\mathbb{k}$. In the SLP model, we have the following complexity bound:

PROPOSITION 8.2. *Let $f = N/D \in \mathbb{K}(z)$ be as above and assume that the factorization $D = H_1^{\mu_1} \cdots H_l^{\mu_l}$ of D into irreducibles is known. Given a joint SLP for the evaluation of N, D, and $H_1, \ldots, H_l$ at points $u \in \mathbb{k}^n$, there exists an SLP for the evaluation of the partial fraction decomposition of f of length at most $\#(N, D, H_1, \ldots, H_l) + \tilde{O}(\deg_z f)$.*

Proof. Direct consequence of [65] and the quasi-linear time algorithm for the "untangling" morphism from [52, Sections 4.3 and 4.4]. □

Since the local reductions really operate on partial fraction decompositions, it is also interesting to study the inverse operation that recovers a rational function from its partial fraction decomposition over $\bar{\mathbb{K}}$.

PROPOSITION 8.3. *Let the partial fraction decomposition of $f = N/D \in \mathbb{K}(z)$ over $\bar{\mathbb{K}}$ be given and define $\deg_u^{\mathrm{sf}} f = \max\{\deg_u H : H$ is an irreducible factor of $D\}$. Then*

$$\deg_u f = O(\deg_z f (\deg_u^* f + \deg_u^{\mathrm{sf}} f))$$

and the standard representation $f = N/D$ can be computed in time

$$\tilde{O}((\deg_z f (\deg_u^* f + \deg_u^{\mathrm{sf}} f))^{\gamma n} \deg_z f).$$

Proof. Let $\alpha \in \bar{\mathbb{K}}$ be one the poles in the partial fraction decomposition of f and let $H \in \mathbb{K}[z]$ be its monic minimal polynomial. Denote $\nu = \deg_z H$, $\mathbb{L} = \mathbb{K}[z]/(H(z))$, and $\varphi = H/(z-\alpha) \in \mathbb{L}[z]$. Consider the polar part $f_{(\alpha)} = C/(z-\alpha)^\nu$ in the partial fraction decomposition of f with $C \in \mathbb{L}[z]$ and $\deg_z C < \nu$. This components gives rise to a corresponding component

$$\mathrm{Tr}_{\mathbb{L}/\mathbb{K}} \frac{C}{(z-\alpha)^\nu} = \mathrm{Tr}_{\mathbb{L}/\mathbb{K}} \frac{C\varphi^\nu}{H^\nu} = \frac{\mathrm{Tr}_{\mathbb{L}/\mathbb{K}}(C\varphi^\nu)}{H^\nu}$$

in the partial fraction decomposition of f over $\mathbb{K}$. Now $\deg_u \varphi \leqslant \deg_z H \deg_u H$, whence $\deg_u \varphi^\nu \leqslant (2\nu-1) \deg_z H \deg_u H$ and $\deg_u (C\varphi^\nu) \leqslant \deg_u^* f + 2\nu \deg_z H \deg_u H$. The traces of $1, \alpha, \ldots, \alpha^{\nu-1}$ can be computed using Newton–Girard's formulas and $\deg_u \mathrm{Tr}_{\mathbb{L}/\mathbb{K}} \alpha^k \leqslant 2k \deg_z H \deg_u H$ for all k. Since $\mathrm{Tr}_{\mathbb{L}/\mathbb{K}}$ is linear, it follows that

$$\deg_u \mathrm{Tr}_{\mathbb{L}/\mathbb{K}}(C\varphi^\nu) \leqslant \deg_u^* f + O(\nu \deg_z H \deg_u H).$$

Summing over all poles, the bound for $\deg_u f$ follows. The time complexity bound is proved in a similar way. □

PROPOSITION 8.4. *Let $f \in \mathbb{K}(z)$ and assume that we are given an SLP of length ℓ for the evaluation of the partial fraction decomposition of f over $\bar{\mathbb{K}}$ at points $u \in \Bbbk^n$ (this includes the evaluation of the roots of the denominator of f). Then there exists an SLP for the evaluation of the numerator and the denominator of f of length at most $\ell + \tilde{O}(\deg_z f)$.*

Proof. We first convert the partial fraction decomposition over $\bar{\mathbb{K}}$ back into a partial fraction decomposition over $\mathbb{K}$; using the tangling algorithm from [52, Section 4.5], this can be done in quasi-linear time. We next recover the numerator and denominator using traditional rational function arithmetic, again in quasi-linear time. □

9. COMPLEXITY OF CREATIVE TELESCOPING

9.1. Complexity of creative telescoping

Let $\mathbb{B}$ be a DD-algebra as in the previous section. For all our complexity bounds, we recall that $n = O(1)$ is assumed to be constant. Consider a telescopable D-finite ideal $I \subseteq \mathbb{B}$ with associated reduction $[\,]_{\mathbb{D}}$. Let $B \in \mathbb{D}^{s \times 1}$ with entries $b_1, \ldots, b_s$ be a basis for the finite dimensional vector space $[\mathbb{D}]_{\mathbb{D}}$. In Algorithm 6.1, it often occurs that around s iterations are necessary before the main loop terminates. In that case, it may be profitable to use the alternative way $R_\omega := [\theta_i R_{\omega'}]_{\mathbb{D}}$ to compute the R_ω and to further optimize the reduction process by precomputing $[\theta_i b_j]_{\mathbb{D}}$ for all basis elements b_j. For each $i \in \{1, \ldots, n\}$, this leads to a matrix $\Omega_i \in \mathbb{K}^{s \times s}$ such that

$$[\theta_i B]_{\mathbb{D}} \;=\; \Omega_i B.$$

Given $\varphi \in \mathbb{K}^{1 \times s}$, we then get $[\theta_i(\varphi B)]_{\mathbb{D}} = \varphi \Omega_i B$. We call $\Omega_1, \ldots, \Omega_n$ the *reduction matrices* for I and $[\,]_{\mathbb{D}}$. With these optimizations, Algorithm 6.1 becomes:

Algorithm 9.1
INPUT: reduction matrices $\Omega_1, \ldots, \Omega_n \in \mathbb{K}^{s \times s}$ for I and $\varphi \in \mathbb{K}^{1 \times s}$ with $\varphi B \in \mathbb{D}^{\text{prim}}$
OUTPUT: D-finite generators $\mathcal{G}$ of a D-finite ideal $(\mathcal{G})$ included in $\mathcal{T}_{\varphi B, \delta}$

$\mathcal{L} := \{1\}$, $\mathcal{G} := \emptyset$, $\mathcal{R} := \emptyset$
while $\mathcal{L} \neq \emptyset$ **do**
  Let $\omega \in \mathcal{L}$ be minimal for $\leqslant$ and set $\mathcal{L} := \mathcal{L} \setminus \{\omega\}$
  if ω is not reducible with respect to $\mathcal{G}$ **then**
    if $\omega = 1$ **then** $R_\omega := \varphi$
    else decompose $\omega = \theta_i \omega'$ and let $R_\omega := R_{\omega'} \Omega_i$
    if $\exists (c_\rho)_{\rho \in \mathcal{R}} \in \mathbb{K}^{\mathcal{R}}$, $R_\omega = \sum_{\rho \in \mathcal{R}} c_\rho R_\rho$ **then** $\mathcal{G} := \mathcal{G} \cup \{\omega - \sum_{\rho \in \mathcal{R}} c_\rho \rho\}$
    else $\mathcal{R} := \mathcal{R} \cup \{\omega\}$, $\mathcal{L} := \mathcal{L} \cup \{\theta_1 \omega, \ldots, \theta_n \omega\}$
return $\mathcal{G}$

Remark 9.1. Finding sharp bounds for the complexity of the Gröbner basis computation in the final step of Algorithm 6.1 is a problem that is somewhat independent from the cost of creative telescoping itself. For this reason, we removed this step here. The returned set $\mathcal{G}$ still admits the property that the set of monomials that are not divisible by a leading monomial of an element of $\mathcal{G}$ is finite; for this reason we call $\mathcal{G}$ a *D-finite* set of generators.

THEOREM 9.2. *Algorithm 9.1 is correct and terminates. Assuming that $\deg_u \Omega_i \leqslant d$ for $i = 1, \ldots, s$ and $\deg_u \varphi \leqslant d$, its running time is bounded by*

$$\tilde{O}((s^2 d)^{\gamma n} s^4).$$

Proof. Since each R_ω is obtained as the product of at most $s+1$ matrices of total degree d, we have $\deg_u R_\omega = O(d s)$ for all ω encountered in the algorithm. The number of iterations of the main loop is always bounded by $s n = O(s)$. It follows that the computation of the R_ω can be done in time $\tilde{O}((s d)^{\gamma n} s)$. The linear algebra of testing the existence of a relation $(c_\rho)_{\rho \in \mathcal{R}} \in \mathbb{K}^{\mathcal{R}}$ (and computing such a relation if it exists) reduces to the computation of the kernel of a matrix in $\mathbb{K}^{i \times s}$ of total degree $O(d s)$ in u and rank $\geqslant i-1$, where $i \leqslant s$. This can be done in time $\tilde{O}((s^2 d)^{\gamma n} s^3)$ through evaluation-interpolation of the numerator at $O((s^2 d)^n)$ points, using the fact that the determinants of all minors of this numerator have total degree $O(s^2 d)$. Multiplying with the maximal number of iterations $O(s)$, the main result follows. □

THEOREM 9.3. *Given an SLP for the joint evaluation of the reduction matrices $\Omega_1, \ldots, \Omega_n$ and φ, Algorithm 9.1 gives rise to an SLP for the joint evaluation of the polynomials in $\mathcal{G}$, whose length is bounded by*

$$\#(\Omega_1, \ldots, \Omega_n, \varphi) + O(s^3).$$

Proof. The evaluation of the input SLPs at a point $u \in \mathbb{k}^n$ takes $\#(\Omega_1, \ldots, \Omega_n, \varphi)$ operations and yields scalar matrices $\Omega_1\langle u\rangle, \ldots, \Omega_n\langle u\rangle \in \mathbb{k}^{s \times s}$ and a scalar vector $\varphi(u) \in \mathbb{k}^{1 \times s}$. Applying the algorithm to $\Omega_1\langle u\rangle, \ldots, \Omega_n\langle u\rangle, \varphi\langle u\rangle$ instead of $\Omega_1, \ldots, \Omega_n, \varphi$ produces $\mathcal{G}\langle u\rangle$ instead of $\mathcal{G}$. This time, each iteration amounts to one matrix-vector product of cost $O(s^2)$ and adding one row to the matrix of the R_ω; we incrementally put this matrix in echelon normal form in order to determine the relations c, again with cost $O(s^2)$. Since the algorithm uses at most $n s = O(s)$ iterations, the complexity bound follows. □

Remark 9.4. The cost $O(s^3)$ of the linear algebra in Theorem 9.3 can be further lowered if the matrices $\Omega_1, \ldots, \Omega_n$ are sparse or in the case when we are using a lexicographic admissible ordering. For details, see [38] and references therein.

It may happen that some of the basis elements are actually superfluous for the computation, in which case it is possible to replace the reduction matrices $\Omega_1, \ldots, \Omega_n$ by smaller "quasi-reduction" matrices. More precisely, assume that the vector space $\mathrm{Vect}(B')$ spanned by the basis elements $b_1, \ldots, b_{s'}$ is stable under the mappings $g \mapsto [\theta_i g]_{\mathbb{D}}$ for $i = 1, \ldots, n$. Then each Ω_i is a block matrix

$$\Omega_i = \begin{pmatrix} \Omega_i' & 0 \\ * & * \end{pmatrix},$$

where $\Omega_i' \in \mathbb{K}^{s' \times s'}$. If $f_1, \ldots, f_r \in \mathrm{Vect}(B')$, then we call $\Omega_1', \ldots, \Omega_n'$ *quasi-reduction matrices* for I and $[]_{\mathbb{D}}$. Given $\varphi' \in \mathbb{K}^{1 \times s'}$ with $\varphi' B' \in \mathbb{D}^{\mathrm{prim}}$, it is not hard to check that Algorithm 9.1 still applies to $\Omega_1', \ldots, \Omega_n', \varphi'$ instead of $\Omega_1, \ldots, \Omega_n, \varphi$, and that the above theorems generalize to this case.

9.2. Deterministic complexity of differential reduction

In this subsection, we assume that $\theta = \partial$ and that our D-finite ideal $I \subseteq \mathbb{B}$ is explicitly telescopable for some cyclic basis F. We write $[]_{\mathbb{D}}$ for the associated reduction. There also exists an operator $L \in \mathbb{K}[z][\partial]$ of order r than annihilates the first entry f_1 of F. We may write $\mathbb{D} = \mathbb{M}^{1 \times r} F$, where $\mathbb{M} = \mathbb{K}[z, \psi^{-1}]$ and $\psi \in \mathbb{K}[z]$ is the monic annihilator of the zero-set of L_r. We assume that the irreducible factors of L_r are all known.

In order to apply Theorem 9.2, we need to bound the dimension s of the space $[\mathbb{D}]_{\mathbb{D}}$ of reduced functions and to analyze the cost of computing the reduction matrices $\Omega_1, \ldots, \Omega_n$. Let us start with the dimension bound.

LEMMA 9.5. *Under the above assumptions, the reduction* $[]_{\mathbb{D}}$ *can be constructed in such a way that*

$$s \leqslant \dim_{\mathbb{K}} [\mathbb{D}]_{\mathbb{D}} \leqslant (2 \deg_z L + 1) r + \deg_z L + 1 = O(r \deg_z L).$$

Proof. We may take $A = \mathrm{Sing}(\partial) \setminus \{\infty\} = \{\alpha \in \bar{\mathbb{K}} : \mathrm{ord}_{\alpha,F} M_{\partial,F} > 0\} = \{\alpha \in \bar{\mathbb{K}} : L_r(\alpha) = 0\}$ in the proof of Proposition 7.5 (see Example 7.6). It follows that $|A| \leqslant \deg_z L$. When applying Theorem 4.5 to L^*, we thus have $d \leqslant \deg_z L$, whence the complexity bound, since $\dim_{\mathbb{K}} [\mathbb{D}]_{\mathbb{D}} = \dim [\mathbb{M}]_{\mathbb{M}}$. □

Given a vector $\lambda \in \mathbb{M}^{1 \times r}$ with entries $\lambda_1, \ldots, \lambda_r$, let us now analyze the cost to compute the reduction $[\lambda F]_{\mathbb{D}}$. By construction, we recall that $[\lambda F]_{\mathbb{D}} = [\Phi(\lambda) f_1]_{\mathbb{D}} = [\Phi(\lambda)]_{\mathbb{M}} f_1$, where $\Phi(\lambda) = \lambda_1 - \partial \lambda_2 + \cdots + (-1)^{r-1} \partial^{r-1} \lambda_r$ and $[]_{\mathbb{M}}$ denotes the reduction with respect to the adjoint operator L^*. It thus suffices to analyze the cost of the reduction $[]_{\mathbb{M}}$.

LEMMA 9.6. *Given* $\lambda \in \mathbb{M}$, *we have*

$$\begin{aligned}
\deg_u^* [\lambda]_{\mathbb{M}} &= O((\deg_z \lambda)^3 \deg_u \lambda + \deg_z \lambda (\deg_z L)^3 \deg_u L)\\
\deg_u [\lambda]_{\mathbb{M}} &= O((\deg_z \lambda)^4 \deg_u \lambda + (\deg_z \lambda)^2 (\deg_z L)^3 \deg_u L)
\end{aligned}$$

and $[\lambda]_{\mathbb{M}}$ *can be computed in time*

$$\tilde{O}(((\deg_z \lambda)^4 \deg_u \lambda + (\deg_z \lambda)^2 (\deg_z L)^3 \deg_u L)^{\gamma n} \deg_z \lambda).$$

Proof. Consider one of the poles $\alpha \in \bar{\mathbb{K}}$ with $\psi(\alpha) = 0$, let $H \in \mathbb{K}[z]$ be its monic minimal polynomial, and $\mathbb{L} = \mathbb{K}[z] / (H)$. Given $\lambda \in \mathbb{L}[z - \alpha, (z - \alpha)^{-1}]$, written as a Laurent polynomial, let $\deg_u^\alpha \lambda$ denote the maximal degree in u of one of its coefficients. Let us show by induction on $\rho = \mathrm{ord}_\alpha \lambda$ that

$$\begin{aligned}
\deg_u^\alpha [\lambda]_\alpha &\leqslant \deg_u^\alpha \lambda + \max(\mathrm{ord}_\alpha \lambda - B_\alpha, 0) \Delta_\alpha.\\
\Delta_\alpha &= \deg_u^\alpha L + \max(\deg_u \mathrm{ind}_\alpha, \deg_u H) \deg_z H + 2 \deg_u H.
\end{aligned}$$

This is clear for $\rho \leqslant B_\alpha$. Otherwise, consider the expansion $\lambda = (z - \alpha)^{-\rho} (c + o(1))$ for $z \to \alpha$. If $\mathrm{ind}_\alpha(\rho - \tau_\alpha) = 0$, then we get

$$\begin{aligned}
\deg_u^\alpha [\lambda]_\alpha &= \deg_u^\alpha (c (z - \alpha)^{-\rho} + [\lambda - c (z - \alpha)^{-\rho}]_\alpha)\\
&= \max(\deg_u c, \deg_u^\alpha [\lambda - c (z - \alpha)^{-\rho}]_\alpha)\\
&\leqslant \max(\deg_u^\alpha \lambda, \deg_u^\alpha \lambda + (\rho - 1 - B_\alpha) \Delta_\alpha)\\
&\leqslant \deg_u^\alpha \lambda + (\rho - B_\alpha) \Delta_\alpha.
\end{aligned}$$

If $\mathrm{ind}_\alpha(\rho - \tau_\alpha) \neq 0$ and $g = c (z - \alpha)^{-\rho + \tau_\alpha} / \mathrm{ind}_\alpha(\rho - \tau_\alpha)$, then the induction hypothesis yields

$$\deg_u^\alpha [\lambda - L(g)]_\alpha \leqslant \deg_u^\alpha (\lambda - L(g)) + (\rho - 1 - B_\alpha) \Delta_\alpha.$$

By construction, we have

$$\begin{aligned}
\deg_u^\alpha L(g) &\leqslant \deg_u^\alpha L + \deg_u^\alpha g + \deg_u H\\
\deg_u^\alpha g &\leqslant \deg_u c + \deg_u (\mathrm{ind}_\alpha(\rho - \tau_\alpha))^{-1} + \deg_u H\\
\deg_u (\mathrm{ind}_\alpha(\rho - \tau_\alpha))^{-1} &\leqslant \max(\deg_u \mathrm{ind}_\alpha, \deg_u H) \deg_z H.
\end{aligned}$$

Putting all these bounds together, we conclude that $\deg_u^\alpha [\lambda]_\alpha \leqslant \deg_u^\alpha \lambda + (\rho - B_\alpha) \Delta_\alpha$.

Let us now bound the quantity Δ_α. Rewriting L in terms of $z - \alpha$ and ∂ amounts to a Taylor shift from which it is also possible to read off the indicial polynomial ind_α. This yields

$$\begin{aligned}
\deg_u^\alpha L &\leqslant \deg_u L + \deg_u H \deg_z L\\
\deg_u \mathrm{ind}_\alpha &\leqslant \deg_u^\alpha L.
\end{aligned}$$

Since H divides L_r, we also have $\deg_z H \leqslant \deg_z L$ and $\deg_u H \leqslant \deg_z H \deg_u L$. It follows that

$$\Delta_\alpha \leqslant (1+o(1)) \deg_u L (\deg_z L)^3.$$

For $\lambda \in \mathbb{L}[z]$, a similar reasoning at infinity shows that

$$\begin{aligned} \deg_u [\lambda]_\infty &\leqslant \deg_u \lambda + (\operatorname{ord}_\infty \lambda - B_\infty) \Delta_\infty. \\ \Delta_\infty &= \deg_u L + \deg_u \operatorname{ind}_\infty \leqslant 2 \deg_u L. \end{aligned}$$

Let us now turn our attention to a general $\lambda \in \mathbb{M}$. Proposition 8.1 implies that $\deg_u^* \lambda = O((\deg_z \lambda)^3 \deg_u \lambda)$. From what precedes, we have $\deg_u^\alpha [\lambda]_\alpha \leqslant \deg_u^* \lambda + (\operatorname{ord}_\alpha \lambda - B_\alpha) \Delta_\alpha$ at every pole $\alpha \in \bar{\mathbb{K}}$ with $\psi(\alpha) = 0$, whence

$$\begin{aligned} \deg_u^\alpha [\lambda_{(\alpha)}]_\alpha &\leqslant \deg_u^* \lambda + O(\deg_z \lambda \deg_u L (\deg_z L)^3) \\ \deg_u^* [\lambda_{(\alpha)}]_\alpha &\leqslant \deg_u^\alpha [\lambda_{(\alpha)}]_\alpha + \deg_u H \deg_z ([\lambda_{(\alpha)}]_\alpha)_{(\infty,\star)} \\ &\leqslant \deg_u^\alpha [\lambda_{(\alpha)}]_\alpha + \deg_u H \deg_z L \\ &\leqslant \deg_u^* \lambda + O(\deg_z \lambda \deg_u L (\deg_z L)^3) \\ \deg_u^* [\lambda]_{\bar{\mathbb{K}}} &\leqslant \max (\max_\alpha \deg_u^* [\lambda_{(\alpha)}]_\alpha, \deg_u^* \lambda) \\ &\leqslant \deg_u^* \lambda + O(\deg_z \lambda \deg_u L (\deg_z L)^3) \\ \deg_u^* [\lambda]_{\mathbb{M}} &\leqslant \deg_u^* [\lambda]_{\bar{\mathbb{K}}} + O(\deg_u \lambda + \deg_z \lambda \deg_u L) \\ &\leqslant \deg_u^* \lambda + O(\deg_z \lambda \deg_u L (\deg_z L)^3) \\ &\leqslant O((\deg_z \lambda)^3 \deg_u \lambda + \deg_z \lambda (\deg_z L)^3 \deg_u L). \end{aligned}$$

From Proposition 8.3, we also have

$$\deg_u [\lambda]_{\mathbb{M}} = O(\deg_z \lambda (\deg_u^* [\lambda]_{\mathbb{M}} + \deg_u L)).$$

Combined with the previous bounds, this yields

$$\deg_u [\lambda]_{\mathbb{M}} = O((\deg_z \lambda)^4 \deg_u \lambda + (\deg_z \lambda)^2 (\deg_z L)^3 \deg_u L).$$

The time complexity bound is proved in a similar way. □

LEMMA 9.7. *Let*

$$\begin{aligned} d_u &= \max (\deg_u M_{\theta_1}, \ldots, \deg_u M_{\theta_n}) \\ d_z &= \max (\deg_z M_{\theta_1}, \ldots, \deg_z M_{\theta_n}). \end{aligned}$$

Then there exist quasi-reduction matrices $\Omega_1', \ldots, \Omega_n'$ for I and $[\,]_{\mathbb{D}}$ with

$$\deg_u [\Omega_i']_{\mathbb{M}} = O((r d_z \deg_z L)^5 (d_u + \deg_u L))$$

and we may compute them in time

$$\tilde{O}(((r d_z \deg_z L)^5 (d_u + \deg_u L))^{\gamma n} (r \deg_z L)^2 d_z).$$

Proof. Denote $\mathrm{A} = \operatorname{Sing}(\partial) \setminus \{\infty\}$. For each $\alpha \in \mathrm{A} \cup \{\infty\}$, let $\mu(\alpha) \in \mathbb{N}$ be minimal such that $\mu(\alpha) \geqslant B_\alpha + d_z$ and $\operatorname{ind}_\alpha(\rho - \tau_\alpha) \neq 0$ for $\rho = \mu(\alpha) - d_z - r, \mu(\alpha) - d_z - r + 1, \ldots, \mu(\alpha)$. Since $\operatorname{ind}_\alpha$ admits at most r roots and $B_\alpha \leqslant r$, we notice that $\mu(\alpha) \leqslant (r+1)(d_z + r)$. Consider the $\mathbb{K}$-subvector space $\mathbb{V} = [\mathbb{M}]_{\mathbb{M}} \cap \mathbb{K}(z)_{(\mathrm{A},\infty,\star);\mu}$ of $[\mathbb{M}]_{\mathbb{M}}$ and the corresponding $\mathbb{K}$-subvector space $\mathbb{W} = \mathbb{V} f_1$. The way we defined μ ensures that we actually have $\mathbb{V} \subseteq \mathbb{K}(z)_{(\mathrm{A},\infty,\star);\mu - d_z - r}$, whence $\theta_i \mathbb{W} \subseteq (\mathbb{K}(z)_{(\mathrm{A},\infty,\star);\mu - r})^{1 \times r} F$ and $\Phi(\theta_i \mathbb{W}) \subseteq \mathbb{K}(z)_{(\mathrm{A},\infty,\star);\mu} f_1$ for $i = 1, \ldots, r$. This means that $\mathbb{W}$ is stable under the mappings $g \mapsto [\theta_i g]_{\mathbb{D}}$. We construct $\Omega_1', \ldots, \Omega_n'$ as the quasi-reduction matrices with respect to a basis $b_1, \ldots, b_{s'}$ of $\mathbb{W}$.

The basis elements b_i can all be taken to be of the form z^ρ or $z^k/H^{-\rho}$, where H is the minimal polynomial of some $\alpha \in A$ and $k < \deg_z H$. For each basis element b_i, it then follows that $\deg_u b_i \leqslant \rho \deg_u H \leqslant \mu(\alpha) \deg_z L \deg_u L$. Writing b_i^* for the column vector with entries $b_i, 0, \ldots, 0$, we also have $\deg_u M_{\theta_j} b_i^* \leqslant d_u + \deg_u b_i$ and $\deg_u \Phi(M_{\theta_j} b_i^*) \leqslant \deg_u M_{\theta_j} b_i^* + r \deg_z L \deg_u L = d_u + O(r d_z \deg_z L \deg_u L)$ for $j = 1, \ldots, n$. Similarly, we have $\deg_z b_i \leqslant \rho \deg_z H \leqslant \mu(\alpha) \deg_z L$, $\deg_z M_{\theta_j} b_i^* \leqslant d_z + \deg_z b_i$, and $\deg_z \Phi(M_{\theta_j} b_i^*) \leqslant \deg_z M_{\theta_j} b_i^* + r = O(r d_z \deg_z L)$. Applying Lemma 9.6 to $\lambda = \Phi(M_{\theta_j} b_i^*)$, it follows that $\deg_u^* [\lambda]_{\mathbb{M}} = O((\deg_z \lambda)^3 \deg_u \lambda + \deg_z \lambda (\deg_z L)^3 \deg_u L) = O((r d_z \deg_z L)^4 (d_u + \deg_u L))$. Regrouping the coefficients of the partial fraction decomposition of $[\lambda]_{\mathbb{M}}$ by conjugate roots, we directly obtain the expression of $[\lambda]_{\mathbb{M}}$ in terms of the basis elements $b_1, \ldots, b_{s'}$. Using Proposition 8.3, it follows that $\deg_u [\Omega_j']_{\mathbb{M}} = O((r d_z \deg_z L)^5 (d_u + \deg_u L))$.

We need to compute $[\Phi(M_{\theta_j} b_i^*)]_{\mathbb{M}}$ for $i = 1, \ldots, s'$ and $j = 1, \ldots, n$, which means that we need $s' n = O(r \deg_z L)$ applications of Lemma 9.6, in view of Lemma 9.5. The complexity bound then follows from Lemma 9.6 and Proposition 8.3. □

THEOREM 9.8. *Assume that $\theta = \partial$ and that we are given an explicitly telescopable D-finite ideal $I \subseteq \mathbb{B}$ for some cyclic basis F with entries $f_1, \ldots, f_r$. Then there exists a polynomial time algorithm to compute a D-finite set of generators of a D-finite ideal that is contained in $\mathcal{T}_{f_1,\partial}$, as a function of the matrices $M_{\theta_1}, \ldots, M_{\theta_n}$ and the operator L associated to I.*

Proof. In view of the discussion at the end of section 8.3, this is a direct consequence of Lemmas 9.5 and 9.7, together with the straightforward generalization of Theorem 9.2 to the case of quasi-reduction matrices. □

Remark 9.9. We did not put a lot of effort in lowering the exponents in the complexity bounds in Lemmas 9.6 and 9.7. One idea that should allow for significant improvements is to avoid conversions between the default representation of rational functions and partial fraction decompositions: it should be possible to carry out most computations directly for the partial fraction representation; see [36] for a similar line of thought in a different context.

9.3. SLP complexity of differential reduction

With the same assumptions as in the previous subsection, let us now carry out the complexity analysis in the SLP model. Before stating the counterpart of Lemma 9.6, let us first show that the local reductions from section 4.2 can essentially be computed in quasi-linear time.

LEMMA 9.10. *Let $\alpha \in \bar{\mathbb{K}}$ with monic minimal polynomial $H \in \mathbb{K}[z]$, and $\mathbb{L} = \mathbb{K}[z]/(H)$. Given a Laurent polynomial $f \in \mathbb{L}[z-\alpha, (z-\alpha)^{-1}]$, we may compute $[f]_\alpha$ as in (4.3) using*

$$\tilde{O}(\deg_z H (\deg_z L + \deg_z f) r)$$

arithmetic operations in $\mathbb{K}$. Similarly, given a Laurent polynomial $f \in \mathbb{K}[z, z^{-1}]$, we may compute $[f]_\infty$ as in (4.4) using

$$\tilde{O}((\deg_z L + \deg_z f) r)$$

arithmetic operations in $\mathbb{K}$.

Proof. The operator L can be rewritten as an operator in $\mathbb{L}[z-\alpha][\partial]$ using a Taylor shift; this can be done in time $\tilde{O}(r \deg_z L \deg_z H)$. Rewritten in this way, the operator L naturally operates on Laurent polynomials in $\mathbb{L}[z-\alpha, (z-\alpha)^{-1}]$. We may also decompose $L = L^\# + L^\flat$ with $L^\#$, $L^\flat \in \mathbb{L}[z-\alpha][\partial]$ and, for all ρ,

$$\begin{aligned} L^\#((z-\alpha)^{-\rho}) &= \operatorname{ind}_\alpha(\rho) (z-\alpha)^{-\rho-\tau_\alpha} \\ L^\flat((z-\alpha)^{-\rho}) &= O((z-\alpha)^{-\rho-\tau_\alpha+1}), \qquad z \to \alpha. \end{aligned}$$

This decomposition can be done in linear time since both $L^{\#}$ and $L^{\flat}$ are of the form $\sum_{(i,j)\in\mathcal{S}} L_{i,j}\,(z-\alpha)^i\,\partial^j$ for suitable sets $\mathcal{S}$. Now consider the linear operators J, Π on $\mathbb{L}[z-\alpha,(z-\alpha)^{-1}]$ that act on monomials by

$$J((z-\alpha)^{-\rho}) \;=\; \begin{cases} \operatorname{ind}_\alpha(\rho)^{-1}\,(z-\alpha)^{-\rho}, & \text{if } \operatorname{ind}_\alpha(\rho)\neq 0\wedge\rho>B_\alpha\\ 0, & \text{if } \operatorname{ind}_\alpha(\rho)=0\vee\rho\leqslant B_\alpha.\end{cases}$$

$$\Pi((z-\alpha)^{-\rho}) \;=\; \begin{cases} 0, & \text{if } \operatorname{ind}_\alpha(\rho)\neq 0\wedge\rho>B_\alpha\\ (z-\alpha)^{-\rho}, & \text{if } \operatorname{ind}_\alpha(\rho)=0\vee\rho\leqslant B_\alpha.\end{cases}$$

Then we observe that the evaluation of (4.3) at u can be rewritten as

$$\begin{aligned}[f]_\alpha \;&=\; \Pi(g) \;=\; g-L^{\#}(J((z-\alpha)^{\tau_\alpha}g))\\ g \;&=\; f-L^{\flat}(J((z-\alpha)^{\tau_\alpha}g)).\end{aligned}$$

The second equation is "recursive", which allows us to compute its solution in quasi-linear time in terms of the size of the equation using relaxed evaluation [48, 47]. This directly implies the first complexity bound. The bound at infinity is proved in a similar way. □

LEMMA 9.11. *Given an SLP for the joint evaluation of L, the irreducible factors $H_1,\ldots,H_l$ of L_r and $\lambda\in\mathbb{M}$ at points $u\in\mathbb{k}^n$, we can compute an SLP for the evaluation of $[\lambda]_{\mathbb{M}}$, whose length is bounded by*

$$\#(L,H_1,\ldots,H_l,\lambda)+\tilde{O}((\deg_z\lambda+(\deg_z L)^2)\,r)$$

Proof. Denote $\mathrm{A}=\operatorname{Sing}(\partial)\setminus\{\infty\}$ and let $u\in\mathbb{k}^n$ be an evaluation point. By Proposition 8.2, we can compute the rational fraction decomposition of $\lambda\langle u\rangle$ over $\bar{\mathbb{k}}$ in quasi-linear time $\tilde{O}(\deg_z\lambda)$. For each $i\in\{1,\ldots,l\}$, let $\alpha_i\in\mathrm{A}$ be a root of H_i. Lemma 9.10 allows us to compute $[\lambda_{(\alpha_i)}]_{\alpha_i}\langle u\rangle$ in time $\tilde{O}(\deg_z H\,(\deg_z L+\operatorname{ord}_{\alpha_i}\lambda)\,r)$. Using the fact that

$$\deg_z\lambda=\sum_{\alpha\in\mathrm{A}\cup\{\infty\}}\operatorname{ord}_\alpha\lambda \;=\; \operatorname{ord}_\infty\lambda+\sum_{1\leqslant i\leqslant l}\deg_z H_i\operatorname{ord}_{a_i}\lambda,$$

it follows that all $[\lambda_{(\alpha_i)}]_{\alpha_i}\langle u\rangle$ for $i\in\{1,\ldots,l\}$ can be computed in time $\tilde{O}((\deg_z\lambda+(\deg_z L)^2)\,r)$. Using Proposition 8.4, these values allow us to compute $[\lambda_{(\bar{\mathbb{K}})}]_{\mathbb{K}}\langle u\rangle$ in quasi-linear time $\tilde{O}(\deg_z\lambda)$. Again by Lemma 9.10, we obtain $[\lambda]_{\mathbb{M}}\langle u\rangle$ using $\tilde{O}((\deg_z L+\deg_z\lambda)\,r)$ more operations in $\mathbb{k}$. □

LEMMA 9.12. *Let $d_z=\max(\deg_z M_{\theta_1},\ldots,\deg_z M_{\theta_n})$. Given an SLP for the joint evaluation of the coefficients of L, the irreducible factors $H_1,\ldots,H_l$ of L_r, and the matrices $M_{\theta_1},\ldots,M_{\theta_n}$ at points $u\in\mathbb{k}^n$, we can compute an SLP for the evaluation of quasi-reduction matrices $\Omega_1',\ldots,\Omega_n'$ for I and $[\,]_{\mathbb{D}}$, whose length is bounded by*

$$\#(L,H_1,\ldots,H_l,M_{\theta_1},\ldots,M_{\theta_n})+\tilde{O}((r\deg_z L)^3\,d_z).$$

Proof. Similarly as in the proof of Lemma 9.7, we apply Lemma 9.11 to $\lambda=\Phi(M_{\theta_j}b_i^*)$ for $i=1,\ldots,s'$ and $j=1,\ldots,n$. We again have $\deg_z\lambda=O(r\,d_z\deg_z L)$, whence each individual λ can be treated in time $\tilde{O}((r\deg_z L)^2\,d_z)$. Since there are $n\,s'=O(r\deg_z L)$ values of λ to consider, the conclusion follows. □

THEOREM 9.13. *Assume that $\theta=\partial$ and that we are given an explicitly telescopable D-finite ideal $I\subseteq\mathbb{B}$ for some cyclic basis F with entries $f_1,\ldots,f_r$. Let $d_z=\max(\deg_z M_{\theta_1},\ldots,\deg_z M_{\theta_n})$. Given an SLP for the joint evaluation of the coefficients of L, the irreducible factors $H_1,\ldots,H_l$ of L_r, and the matrices $M_{\theta_1},\ldots,M_{\theta_n}$ at points $u\in\mathbb{k}^n$, we can compute an SLP for the evaluation of a D-finite set of generators of a D-finite ideal that is contained in $\mathcal{T}_{f_1,\partial}$, whose length is bounded by*

$$\#(L,H_1,\ldots,H_l,M_{\theta_1},\ldots,M_{\theta_n})+\tilde{O}((r\deg_z L)^3\,d_z).$$

Proof. This is a direct consequence of Lemmas 9.5 and 9.12, together with the straightforward generalization of Theorem 9.3 to the case of quasi-reduction matrices. □

Remark 9.14. The bound $\deg_z \lambda = O(r\, d_z \deg_z L)$ in the proof of Lemma 9.12 is actually quite pessimistic, since it corresponds to highly unlucky values of the roots of the indicial polynomials. In practice, the better bound $\deg_z \lambda = O(d_z)$ is likely to hold. In that case, the time complexity per evaluation point in Theorem 9.13 drops to $\tilde{O}((r \deg_z L)^2 (d_z + r \deg_z L))$.

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