Note on holonomic constants

Errata: efficient accelero-summation of holonomic functions

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There were several problems with the proofs of [2, Proposition 4.7(c) and (d)]. In this note, we present corrected proofs (in the case of [2, Proposition 4.7(c)], we slightly modified the statement), as well as a theorem that the class $\mathbb{K}^{\text{rhol}}$ regular singular holonomic constants is essentially the same as the class of $\mathbb{K}^{\text{hol}}$ of ordinary holonomic constants. Until the very last subsection, we will assume that $\mathbb{K} = \mathbb{Q}^{\text{alg}}$ is the field of algebraic numbers. In the last subsection, we also discuss a few related questions and results from [1]; we are grateful to Marc Mezzarobba for this reference. Throughout our note, we will freely use notations and references from [2].

Other Errata for [2]

- Remark 7.4 is wrong. In fact, thanks to optimizations by Marc Mezzarobba [4], the top two entries of the rightmost column of Table 7.1 can now be replaced by $O(\mathsf{M}(n)\log^2 n)$.
- The last sentence of the first paragraph of Appendix A is slightly misleading; it should be "Indeed, for any $N \in \mathbb{N}$, we may compute $(\operatorname{sum}_N \tilde{f})(z)$ using ..."; the N that we take is generally smaller than the maximal N allowed by summation up to the least term. In fact, if $|\log z| \geqslant \log n$ and $\operatorname{size}(z) = O(\log z)$, then $(\operatorname{sum}_N \tilde{f})(z)$ can be computed in time $O(\mathsf{M}(n)\log n)$ through a refined analysis of the binary splitting algorithm in this context.
- In Theorem A.1, we need to require that $\operatorname{size}(z) = O(n)$. The proof in the case when $\operatorname{size}(z) > \log n$ is not completely trivial. It is given in [3, Proposition 7] for expeditosummation. If $\operatorname{size}(z) = O(\log n)$, then the bound actually becomes $O(\mathsf{M}(n)\log n)$, as noted above.

Notations

Let $\mathcal{L}^{\text{hol}}$ and $\mathcal{L}^{\text{shol}}$ denote for the sets of monic $L \in \mathbb{K}(z)[\partial]$ whose coefficients are respectively defined on $\bar{\mathcal{D}}_{0,1}$ and $\mathcal{D}_{0,1}$. Let $\mathcal{L}^{\text{rhol}}$ be the set of $L \in \mathcal{L}^{\text{shol}}$ such that L is at worst regular singular at $z = 1$. We define $\mathcal{F}^{\text{hol}}$, $\mathcal{F}^{\text{rhol}}$, and $\mathcal{F}^{\text{shol}}$ to be the sets of solutions $f \in \mathbb{K}\{\{z\}\}$ to an equation $Lf = 0$, where $L \in \mathcal{L}^{\text{hol}}$, $L \in \mathcal{L}^{\text{rhol}}$, or $L \in \mathcal{L}^{\text{shol}}$, respectively, and such that $\lim_{z \to 1} f(z)$ exists. We recall that $\mathbb{K}^{\text{hol}} = \{f(1) : f \in \mathcal{F}^{\text{hol}}\}$, $\mathbb{K}^{\text{rhol}} = \{\lim_{z\to 1} f(z) : f \in \mathcal{F}^{\text{rhol}}\}$, and $\mathbb{K}^{\text{shol}} = \{\lim_{z\to 1} f(z) : f \in \mathcal{F}^{\text{shol}}\}$.

It will be convenient to also introduce the variants $\mathcal{L}^{\text{hola}}$, $\mathcal{L}^{\text{rhola}}$, and $\mathcal{L}^{\text{shola}}$ of $\mathcal{L}^{\text{hol}}$, $\mathcal{L}^{\text{rhol}}$, and $\mathcal{L}^{\text{shol}}$ for which we allow L to be at most regular singular at $z = 0$. For instance, $\mathcal{L}^{\text{hola}}$ consists of monic operators $L \in \mathbb{K}(z)[\partial]$ whose coefficients are defined on $\bar{\mathcal{D}}_{0,1} \setminus \{0\}$ and such that L is at worst regular singular at $z = 0$. The counterparts $\mathcal{F}^{\text{hola}}$, $\mathcal{F}^{\text{rhola}}$, and $\mathcal{F}^{\text{shola}}$ are defined in a similar way as before; we still require analytic solutions $f \in \mathbb{K}\{\{z\}\}$ of $Lf = 0$ at $z = 0$. We again set $\mathbb{K}^{\text{hola}} = \{f(1) : f \in \mathcal{F}^{\text{hola}}\}$, $\mathbb{K}^{\text{rhol}} = \{\lim_{z\to 1} f(z) : f \in \mathcal{F}^{\text{rhola}}\}$, and $\mathbb{K}^{\text{shola}} = \{\lim_{z\to 1} f(z) : f \in \mathcal{F}^{\text{shola}}\}$.

Ring structure

PROPOSITION 1. $\mathbb{K}^{\mathrm{hol}}$, $\mathbb{K}^{\mathrm{rhol}}$, $\mathbb{K}^{\mathrm{shol}}$, $\mathbb{K}^{\mathrm{hola}}$, $\mathbb{K}^{\mathrm{rhola}}$, *and* $\mathbb{K}^{\mathrm{shola}}$ *are all subrings of* $\mathbb{C}$.

Proof. This is proved in a similar way as Proposition 4.6(*a*). For instance, in order to see that $\mathbb{K}^{\mathrm{rhol}}$ is closed under multiplication, consider solutions f and g of $Kf=0$ and $Lg=0$ with initial conditions in $\mathbb{K}^m$ resp. $\mathbb{K}^n$, where the coefficients of K and L are defined on $\mathcal{D}_{0,1}$, where K and L are regular singular at $z=1$, and such that the limits of f and g at $z=1$ exist. Then Corollary 4.4 implies that $K \boxtimes L$ is defined on $\mathcal{D}_{0,1}$ and Corollary 4.5 implies that $K \boxtimes L$ is regular singular at $z=1$. Consequently, $\lim_{z\to 1}(fg)(z) = (\lim_{z\to 1} f(z))\,(\lim_{z\to 1} g(z))$ belongs to $\mathbb{K}^{\mathrm{rhol}}$. □

This proposition also allows us to consider initial conditions in $\mathbb{K}^{\mathrm{hol}}$ instead of $\mathbb{K}$ in many circumstances. For instance, by definition, the value of a function $f \in \mathcal{F}^{\mathrm{hol}}$ at a point in $\bar{\mathcal{D}}_{0,1} \cap \mathbb{K}$ lies in $\mathbb{K}^{\mathrm{hol}}$. Thanks to the proposition, this even holds for solutions $f \in \mathbb{K}^{\mathrm{hol}}\{\{z\}\}$ to an equation $Lf=0$ with $f \in \mathcal{L}^{\mathrm{hol}}$. Indeed, given $z \in \mathbb{K}$, we have $F(z) = \Delta_{0\to z} F(0)$; since $\Delta_{0\to z}$ and $F(0)$ both have coefficients in $\mathbb{K}^{\mathrm{hol}}$, the same holds for $F(z)$.

Regular singular transition matrices

LEMMA 2. *Let* $L \in \mathcal{L}^{\mathrm{hola}}$, $\alpha \in \mathbb{K}$, *and consider a solution* $f \in z^{\alpha_i}\,\mathbb{K}\{\{z\}\}[\log z]$ *of the equation* $Lf=0$. *Then* $f \in z^{\alpha_i}\,\mathcal{F}^{\mathrm{hola}}[\log z]$.

Proof. Without loss of generality, we may assume that $\alpha=0$. Now write $f = f_d\,(\log z)^d + \cdots + f_0$ with $f_0, \ldots, f_d \in \mathbb{K}\{\{z\}\}$. Then $f(z\,\mathrm{e}^{2\pi\mathrm{i}}) = f_d\,(\log z + 2\pi\mathrm{i})^d + \cdots + f_0 \in \mathbb{K}\{\{z\}\}[\log z][2\pi\mathrm{i}]$ is also annihilated by L. Since $2\pi\mathrm{i}$ is transcendental, each of the coefficients of $f(z\,\mathrm{e}^{2\pi\mathrm{i}})$ as a polynomial in $2\pi\mathrm{i}$ is again annihilated by L; these coefficients are

$$f_d, \quad d f_d \log z + f_{d-1}, \quad \ldots, \quad f_d\,(\log z)^d + \cdots + f_0.$$

It follows that

$$f_d \in \mathcal{F}^{\mathrm{hola}}, \quad f_{d-1} \in \mathcal{F}^{\mathrm{hola}} + \mathcal{F}^{\mathrm{hola}} \log z, \quad \ldots, \quad f_0 \in \mathcal{F}^{\mathrm{hola}} + \cdots + \mathcal{F}^{\mathrm{hola}}\,(\log z)^d,$$

whence $f \in \mathcal{F}^{\mathrm{hola}}[\log z]$. □

PROPOSITION 3. *Let L be a linear differential operator of order n in* $\mathbb{K}(z)[\partial]$. *Then* $\Delta_\gamma \in \mathsf{Mat}_n(\mathbb{K}^{\mathrm{hola}})$ *for any regular singular broken-line path γ as in section 4.3.2.*

Proof. In view of (4.6), it suffices to prove the result for paths of the form $\sigma_\theta \to \sigma + z$ and for paths of the form $\sigma + z \to \sigma_\theta$. Without loss of generality we may assume that $\sigma = 0$. By what precedes, the entries of $\Delta_{0_\theta \to z}$ as functions in z are all in $\mathcal{F}^{\mathrm{hola}}[z^{\mathbb{K}}][\log z]$. Now values of functions $z^\alpha\,(\log z)^k$ with $\alpha \in \mathbb{K}$ and $k \in \mathbb{N}$ at points $z \in \mathbb{K}^{\neq}$ are in $\mathbb{K}^{\mathrm{hol}}$. Consequently, values of entries of $\Delta_{0_\theta \to z}$ at $z \in \mathbb{K}^{\neq}$ are in $\mathbb{K}^{\mathrm{hola}}$. Let $h_1, \ldots, h_r$ be the canonical basis of solutions of $Lf=0$ at the origin. Then we recall that

$$\Delta_{0_\theta \to z} = \begin{pmatrix} h_1(z) & \cdots & h_r(z) \\ \vdots & & \vdots \\ h_1^{(r-1)}(z) & \cdots & h_r^{(r-1)}(z) \end{pmatrix}.$$

The determinant $W = W_{h_1,\ldots,h_r}$ of this matrix satisfies the equation $W' + L_{r-1} W = 0$ and its inverse W^{-1} satisfies $(W^{-1})' - L_{r-1} W^{-1} = 0$. Since L is at worst regular singular at $z = 0$, we have $L_{r-1} = \frac{\alpha}{z} + Q$, where $\alpha \in \mathbb{K}$ and $Q \in \mathbb{K}(z)$ is analytic at $z = 0$. It follows that $W = c^{-1} z^{-\alpha} \mathrm{e}^{-\int Q}$ and $W^{-1} = c z^{\alpha} \mathrm{e}^{\int Q}$ for some $c \in \mathbb{K}$, where $(\int Q)(0) = 0$ (here $c \in \mathbb{K}$ follows from the fact that the coefficients of all $h_i^{(j)}$ are in $\mathbb{K}$ as oscillatory transseries). Given $z \in \mathbb{K}^{\neq}$ where W^{-1} is defined, it follows that $W^{-1}(z) \in \mathbb{K}^{\mathrm{hol}}$. Since $\mathbb{K}^{\mathrm{hola}}$ is a ring, it follows that the coefficients of

$$\Delta_{z \to 0_\theta} = \Delta_{0_\theta \to z}^{-1} = W^{-1}(z)\, \mathrm{adj}(\Delta_{0_\theta \to z})$$

are in $\mathbb{K}^{\mathrm{hola}}$. □

COROLLARY 4. *We have* $\mathbb{K}^{\mathrm{shol}} \subseteq \mathbb{K}^{\mathrm{hola}}$.

Proof. Given $c \in \mathbb{K}^{\mathrm{shol}}$, let $L \in \mathcal{L}^{\mathrm{shol}}$ and $f \in \mathcal{F}^{\mathrm{shol}}$ such that $Lf = 0$ and $c = \lim_{z \to 1} f(z)$. Let $\lambda_1, \ldots, \lambda_r \in \mathbb{K}^{\mathrm{hola}}$ be the entries of $\Delta_{0 \to 1} F(0)$. Then $f = \lambda_1 h_1 + \cdots + \lambda_r h_r$, where $h_1, \ldots, h_r$ is the canonical fundamental system of solutions of $Lf = 0$. Since $\lim_{z \to 1} f(z)$ exists, we must have $h_i = O(1)$ whenever $\lambda_i \neq 0$ and $f(1) = \sum_{i, \lambda_i \neq 0} \lambda_i h_i(0) \in \mathbb{K}^{\mathrm{hola}}$. □

Alien operators

Given an analytic function f defined on a neighbourhood of the origin on the Riemann surface of the logarithm, we define $(\nabla f)(z) = f(z) - f(z \mathrm{e}^{-2\pi \mathrm{i}})$. Setting $\Lambda = (2\pi \mathrm{i})^{-1} \log z$, the operator $f(z) \mapsto f(z \mathrm{e}^{-2\pi \mathrm{i}})$ acts on $\mathbb{C}\{\{z\}\}[\Lambda]$ by sending Λ to $\Lambda - 1$, whence $\nabla = 1 - \mathrm{e}^{-\partial_\Lambda} = \partial_\Lambda - {}^1\!/_2\, \partial_\Lambda + \cdots$. Given $d \in \mathbb{N}$, let $\mathbb{C}\{\{z\}\}[\Lambda]_{<d}$ be the set of $f \in \mathbb{C}\{\{z\}\}[\Lambda]$ of degree $<d$ in Λ. For $0 \leqslant i \leqslant j$, we note that the operator

$$\mathrm{Z}^{i;j} := (i - \Lambda \nabla) \cdots (j - 1 - \Lambda \nabla)$$

sends $\mathbb{C}\{\{z\}\}[\Lambda]_{<j}$ into $\mathbb{C}\{\{z\}\}[\Lambda]_{<i}$. We also note that $\mathrm{Z}^{i;j}(\Lambda^{i-1}) \sim (i-j)!\, \Lambda^{i-1}$.

For each $\alpha \in \mathbb{K}$, let $\nabla_\alpha := z^\alpha \nabla z^{-\alpha}$ and $\mathbb{L}_\alpha := z^\alpha\, \mathbb{C}\{\{z\}\}[\log z]$. Then $\nabla_\alpha\, \mathbb{L}_\beta \subseteq \mathbb{L}_\beta$ for all β and ∇_α acts like multiplication by $1 - \mathrm{e}^{-2\pi \mathrm{i}(\beta - \alpha)}$ on $z^\beta\, \mathbb{C}\{\{z\}\}$. Moreover, given $\varphi \in \mathbb{L}_\alpha$ of degree $<d$ in $\log z$, we have $\nabla_\alpha^d\, \varphi = 0$. We define $\mathcal{X}$ to be the monoid of power products $(1 - \mathrm{e}^{-2\pi \mathrm{i} \alpha_1})^{k_1} \cdots (\mathrm{e}^{1 - 2\pi \mathrm{i} \alpha_\ell})^{k_\ell}$ with $\alpha_1, \ldots, \alpha_\ell \in (\mathbb{K} \cap \mathbb{R}) \setminus \mathbb{Q}$ and $k_1, \ldots, k_\ell \in \mathbb{N}$. For $0 \leqslant i \leqslant j$, we also define

$$\mathrm{Z}_\alpha^{i;j} = z^\alpha\, \mathrm{Z}^{i;j} z^{-\alpha} = (i - \Lambda \nabla_\alpha) \cdots (j - 1 - \Lambda \nabla_\alpha).$$

We note that $\mathrm{Z}_\alpha^{i;j}$ sends $z^\alpha\, \mathbb{C}\{\{z\}\}[\Lambda]_{<j}$ into $z^\alpha\, \mathbb{C}\{\{z\}\}[\Lambda]_{<i}$.

Let $\mathcal{H}$ be the space of holonomic functions f on $\bar{\mathcal{D}}_{0,1} \setminus \{0\}$ that are regular singular at the origin and such that $F(1) = (f(1), \ldots, f^{(r-1)}(1)) \in (\mathbb{K}^{\mathrm{hol}})^r$. Such a function f satisfies an equation $Lf = 0$ with $L \in \mathcal{L}^{\mathrm{hola}}$. We regard $F(1)$ as a column vector, as usual, and recall that $f(z \mathrm{e}^{-2\pi \mathrm{i}})$ is another solution of $Lf = 0$ with $F(\mathrm{e}^{-2\pi \mathrm{i}}) = \Delta_{1 \circlearrowleft 1} F(1) \in (\mathbb{K}^{\mathrm{hol}})^r$. Indeed, the monodromy matrix $\Delta_{1 \circlearrowleft 1}$ of L around $z = 0$ with end-points at $z = 1$ has coefficients in $\mathbb{K}^{\mathrm{hol}}$. It follows that $\nabla f \in \mathcal{H}$. Moreover, $\mathcal{H}$ is a ring with $z^{\mathbb{K}} \subseteq \mathcal{H}$ and $\log z \in \mathcal{H}$. It follows that $\nabla_\alpha f \in \mathcal{H}$ and $\mathrm{Z}_\alpha^{i;j} f \in \mathcal{H}$ for all $\alpha \in \mathbb{K}$ and $0 \leqslant i \leqslant j$. Note that $\nabla_\alpha f$ always satisfies the same equation as f, contrary to $\mathrm{Z}_\alpha^{i;j} f$.

Eschewing regular singularities

THEOREM 5. *We have*

$$\mathbb{K}^{\mathrm{hol}} \subseteq \mathbb{K}^{\mathrm{shol}} \subseteq \mathbb{K}^{\mathrm{hola}} \subseteq \mathcal{X}^{-1}\, \mathbb{K}^{\mathrm{hol}}.$$

Proof. The inclusion $\mathbb{K}^{\mathrm{hol}} \subseteq \mathbb{K}^{\mathrm{shol}}$ is trivial and we already proved that $\mathbb{K}^{\mathrm{shol}} \subseteq \mathbb{K}^{\mathrm{hola}}$, so we focus on the remaining inclusion $\mathbb{K}^{\mathrm{hola}} \subseteq \mathcal{X}^{-1}\,\mathbb{K}^{\mathrm{hol}}$.

Consider a monic $L \in \mathcal{L}^{\mathrm{hola}}$ of order r. Then $Lh = 0$ has a canonical fundamental system of solutions

$$h_{i,j} \in z^{\alpha_i}\,\mathbb{K}\{\{z\}\}[\log z]_{\leqslant j}, \qquad h_{i,j} \sim z^{\alpha_i}(\log z)^j, \qquad i = 1, \ldots, \ell, \quad j = 0, \ldots, \nu_i - 1.$$

In particular, we have $r = \nu_1 + \cdots + \nu_\ell$. For each $i \in \{1, \ldots, \ell\}$, let

$$\Pi_i := \nabla_{\alpha_1}^{\nu_1} \cdots \nabla_{\alpha_{i-1}}^{\nu_{i-1}} \nabla_{\alpha_{i+1}}^{\nu_{i+1}} \cdots \nabla_{\alpha_\ell}^{\nu_\ell},$$

so that $\Pi_i f \in \mathbb{L}_{\alpha_i}$ for any solution f of $Lf = 0$. We also define

$$u_i := \prod_{i' \neq i} (1 - \mathrm{e}^{-2\pi \mathrm{i}(\alpha_{i'} - \alpha_i)})^{\nu_{i'}},$$

so that $\Pi_i f = u_i f$ whenever $f \in z^{\alpha_i}\,\mathbb{C}\{\{z\}\}$.

Let f be a solution of $Lf = 0$ with $F(1) = (f(1), \ldots, f^{(r-1)}(1)) \in \mathbb{K}^{\mathrm{hol}}$ and let $\lambda_{i,j} \in \mathbb{C}$ be such that $f = \sum_{i,j} \lambda_{i,j} h_{i,j}$. We need to show that $\lambda_{i,j} \in \mathcal{X}^{-1}\,\mathbb{K}^{\mathrm{hol}}$ for all i and j. Given $i \in \{1, \ldots, \ell\}$, let us show by induction on j that $\lambda_{i,j} \in u_i^{-1}\,\mathbb{K}^{\mathrm{hol}}$. To this effect, given $j \in \{0, \ldots, \nu_i - 1\}$, assume that $\lambda_{i,j+1}, \ldots, \lambda_{i,\nu_i - 1} \in u_i^{-1}\,\mathbb{K}^{\mathrm{hol}}$, and let us show that $\lambda_{i,j} \in u_i^{-1}\,\mathbb{K}^{\mathrm{hol}}$.

Let $g := \mathrm{Z}_\alpha^{j+1;\nu_i} f$ and let $f_\alpha = \lambda_{i,0} h_{i,0} + \cdots + \lambda_{i,\nu_i - 1} h_{i,\nu_i - 1}$ and $g_\alpha = \mathrm{Z}_\alpha^{j+1;\nu_i} f_\alpha$ be the components of f and g in $z^\alpha\,\mathbb{C}\{\{z\}\}[\log z]$. By construction, g_α has degree at most j in $\log z$ and the coefficient $(g_\alpha)_j$ of degree j is of the form

$$(g_\alpha)_j = (\nu_i - j - 1)!\,\lambda_{i,j} + c_{j+1} \lambda_{i,j+1} + \cdots + c_{\nu_i - 1} \lambda_{i,\nu_i - 1} + o(1)$$

for constants $c_{j+1}, \ldots, c_{\nu_i - 1} \in \mathbb{K}[(2\pi \mathrm{i})^{-1}]$ that can be computed explicitly.

We next consider the function $\varphi = z^{-\alpha_i} \Pi_i \nabla_\alpha^j g$. By construction, $\varphi \in \mathbb{C}\{\{z\}\}$ and

$$\varphi = j!\,u_i\,((\nu_i - j - 1)!\,\lambda_{i,j} + c_{j+1} \lambda_{i,j+1} + \cdots + c_{\nu_i - 1} \lambda_{i,\nu_i - 1} + o(1)).$$

Moreover, both g and φ belong to $\mathcal{H}$, so the value of the contour integral

$$\varphi(0) \;=\; \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} \frac{\varphi(z)}{z} \mathrm{d}z$$

actually lies in $\mathbb{K}^{\mathrm{hol}}$. By our assumption that $\lambda_{i,j+1}, \ldots, \lambda_{i,\nu_i - 1} \in u_i^{-1}\,\mathbb{K}^{\mathrm{hol}}$, it follows that

$$\varphi(0) - j!\,u_i\,(c_{j+1} \lambda_{i,j+1} + \cdots + c_{\nu_i - 1} \lambda_{i,\nu_i - 1}) \in \mathbb{K}^{\mathrm{hol}},$$

whence $u_i \lambda_{i,j} \in \mathbb{K}^{\mathrm{hol}}$. By induction on j, this shows that $\lambda_{i,j} \in u_i^{-1}\,\mathbb{K}^{\mathrm{hol}}$, for all j. □

Remark 6. In the special case when $\ell = 1$ or when $\alpha_i - \alpha_j \in \mathbb{Q}$ for all i, j, we note that the numbers u_i are all in $\mathbb{K}$. It follows that $\Delta_{0^\theta \to z}$ and $\Delta_{z \to 0^\theta}$ have coefficients in $\mathbb{K}^{\mathrm{hol}}$.

Irregular singularities

PROPOSITION 7. *Let L be a linear differential operator of order n in $\mathbb{K}(z)[\partial]$. Then $\Delta_\gamma \in \mathsf{Mat}_n(\mathbb{K}^{\mathrm{shola}})$ for any singular broken-line path γ as in section 4.3.2.*

Proof. In view of (4.6), it suffices to prove the result for paths of the form $\sigma_{k,\theta} \to \sigma + z$ or $\sigma + z \to \sigma_{k,\theta}$. In fact, it suffices to consider paths of the form $\sigma_{k,\theta} \to \sigma + z$, by using a similar argument as in the regular singular case, based on the Wronskian. Without loss of generality we may assume that $\sigma = 0$.

Now, as shown in detail in section 7.3, the matrix $\Delta_{0_{k,\theta}\to z}$ can be expressed as a product of matrices whose entries are either evaluations of regular singular Borel transforms $\check{f}_1^{(m)}(a_1)$ at a point $a_1 \in \mathbb{K}^{\neq}$ near the origin, or of the form

$$\int_{b_i}^{e^{i\theta_i}\infty} \hat{\varphi}_i(\zeta_i)\, \check{K}_{k_i,k_{i+1}}^{(m)}(\zeta_i, a_{i+1})\, \mathrm{d}\zeta_i, \tag{1}$$

or

$$\int_{b_p}^{e^{i\theta_p}\infty} \hat{\varphi}_p(\zeta_p)\, (e^{-\zeta_p/z_p})^{(m)}\, \mathrm{d}\zeta_i, \tag{2}$$

where $a_{i+1}, b_i, z_p \in \mathbb{K}$, $m \in \mathbb{N}$ and $\hat{\varphi}_i$ is holonomic with initial conditions in $\mathbb{K}^{\mathrm{shola}}$. Moreover, b_i and b_p may be chosen as large as desired.

Let us first consider the evaluations $\check{f}_1^{(m)}(a_1)$ near the origin. The functions $\hat{f}_1$ and $\check{f}_1$ satisfy holonomic equations $\hat{L}_1\, \hat{f}_1 = 0$ and $\check{L}_1\, \check{f}_1$ that are regular singular at the origin. In particular, the transition matrices between 0 and a_1 for these equations have entries in $\mathbb{K}^{\mathrm{hola}}$. Thanks to the explicit formulas of $(\tilde{\mathcal{B}}_{z_1} z_1^{\sigma} \log^r z_1)(\zeta_1)$ in section 2.1, we see that $\hat{f}_1$ can be expressed as a $\mathbb{K}[\gamma^{(\mathbb{N})}(\mathbb{K})]$-linear combination of the canonical solutions of $\hat{L}_1$ at the origin, where $\mathbb{K}[\gamma^{(\mathbb{N})}(\mathbb{K})]$ is the smallest $\mathbb{K}$-algebra that contains all constants of the form $\gamma^{(m)}(\sigma)$ with $\gamma(z) = 1/\Gamma(z)$, $m \in \mathbb{N}$, and $\sigma \in \mathbb{K}$. Now

$$\gamma^{(m)}(z) \;=\; \frac{i}{2\pi} \int_{\mathcal{H}} (-\log(-t))^m (-t)^{-z} e^{-t}\, \mathrm{d}t,$$

for all $m \in \mathbb{N}$, where $\mathcal{H}$ is a Hankel contour from ∞ around 0 and then back to ∞. Such integrals can be evaluated using the technique from section 6, so $\mathbb{K}[\gamma^{(\mathbb{N})}(\mathbb{K})] \subseteq \mathbb{K}^{\mathrm{shol}}$. Using the explicit formula for majors of functions of the form $\varphi(\zeta)\, \zeta^{\sigma} \log^k \zeta$ in section 2.2, we next deduce that $\check{f}_1$ is a $\mathbb{K}\big[\gamma^{(\mathbb{N})}(\mathbb{K}), (1 - e^{-2\pi i \mathbb{K}^{\neq}})^{-1}\big]$-linear combination of the canonical solutions of $\check{L}_1$ at the origin. As we will see in the section below, we again have $(1 - e^{-2\pi i \mathbb{K}^{\neq}})^{-1} \subseteq \mathbb{K}^{\mathrm{shol}}$. It follows that $\check{f}_1^{(m)}(a_1) \subseteq \mathbb{K}^{\mathrm{shola}}$ for $a_1 \in \mathbb{K}^{\neq}$ near the origin and all $m \in \mathbb{N}$.

By the results from section 4.2, the kernels $\check{K}_{k_i,k_{i+1}}^{(m)}(\zeta_i, a_{i+1})$, $(e^{-\zeta_p/z_p})^{(m)}$ and the integrands or (1) and (2) are all holonomic, with initial conditions in $\mathbb{K}^{\mathrm{shola}}$ at b_i. Note that the m-th derivatives are taken with respect to a_{i+1} and z_p, so they amount to multiplying the integrands with a polynomial in ζ_i or ζ_p of degree m. Let us focus on the integrals of type (1); the integrals of type (2) are treated similarly. The function $\hat{\varphi}_i$ satisfies a holonomic equation (i.e. a monic linear differential equation with coefficients in $\mathbb{K}(\zeta_i)$) of which all solutions have a growth bounded by $B\, e^{C|\zeta_i|^{k_i/(k_i - k_{i+1})}}$, for some fixed constant $C > 0$ and a constant B that depends on the solution. Likewise, as shown in section 7.1, $\check{K}_{k_i,k_{i+1}}^{(m)}(\zeta_i, a_{i+1})$ satisfies a holonomic equation of which all solutions are bounded by $B\, e^{-C|\zeta_i|^{k_i/(k_i - k_{i+1})}}$ for a fixed constant $C > 0$ that can be made arbitrarily large (by taking b_i large). By Lemma 4.3(*b*), it follows that the same holds for the integrand $I(\zeta) := \hat{\varphi}_i(\zeta_i)\, \check{K}_{k_i,k_{i+1}}^{(m)}(\zeta_i, a_{i+1})$.

Given such a holonomic equation satisfied by I, consider the canonical fundamental basis $h_1, \ldots, h_s$ of solutions to this equation at $\zeta_i = b_i$. For each j, the function h_j has initial conditions in $\mathbb{K}$ at b_i and the integral

$$\int_{b_i}^{e^{i\theta_i}\infty} h_j(\zeta_i)\, \mathrm{d}\zeta_i,$$

converges. Taking b_i sufficiently large and applying a change of variables of the form $(\zeta_i/b_i)^c = (1-\xi)^{-1}$, we see that the value of the integral lies in $\mathbb{K}^{\text{shol}}$. Since I can be rewritten as a $\mathbb{K}^{\text{shola}}$-linear combination of $h_1,\ldots,h_s$, we conclude that (1) also takes a value in $\mathbb{K}^{\text{shola}}$. □

Invertible elements

Given an integral domain R, let $R^\times$ be its subgroup of invertible elements. An interesting question is to determine the sets $(\mathbb{K}^{\text{hol}})^\times$, $(\mathbb{K}^{\text{rhol}})^\times$, etc. Obviously, $\mathbb{K}^{\neq} \subseteq (\mathbb{K}^{\text{hol}})^\times$ and $\mathrm{e}^{\mathbb{K}} \subseteq (\mathbb{K}^{\text{hol}})^\times$. We also know that $\pi^{\mathbb{Z}} \subseteq (\mathbb{K}^{\text{hol}})^\times$, since

$$\begin{aligned}\pi &= 4\left(\arctan\frac{1}{2}+\arctan\frac{1}{3}\right) \in \mathbb{K}^{\text{hol}}\\ \frac{1}{\pi} &= \frac{2\sqrt{2}}{9801}\sum_{k\in\mathbb{N}}\frac{(4k)!\,(1103+26390k)}{(k!)^4\,396^{4k}} \in \mathbb{K}^{\text{hol}},\end{aligned}$$

and $\pi^{1/2\mathbb{Z}} \subseteq (\mathbb{K}^{\text{shol}})^\times$, since

$$\sqrt{\pi} = \Gamma\left(\frac{1}{2}\right).$$

It would be interesting to know whether $\pi^\alpha \in (\mathbb{K}^{\text{shol}})^\times$ for other rational numbers $\alpha \in \mathbb{Q}\setminus(1/2\,\mathbb{Z})$. From

$$\begin{aligned}\Gamma(z) &= \int_0^\infty x^{z-1}\mathrm{e}^{-x}\,\mathrm{d}x\\ \frac{1}{\Gamma(z)} &= \frac{\mathrm{i}}{2\pi}\int_{\mathcal{H}}(-t)^{-z}\mathrm{e}^{-t}\,\mathrm{d}t,\end{aligned}$$

we deduce that $\Gamma(\mathbb{K}\setminus\mathbb{Z}) \subseteq (\mathbb{K}^{\text{shol}})^\times$, where $\mathcal{H}$ is a Hankel contour from ∞ around 0 and then back to ∞. From the above facts and Euler's reflection formula

$$\Gamma(1-z)\,\Gamma(z) = \frac{\pi}{\sin(\pi z)},$$

we also deduce that $\sin(\pi(\mathbb{K}\setminus\mathbb{Z})) \subseteq (\mathbb{K}^{\text{shol}})^\times$. This is noteworthy, since $\sin z$ is a well known example of a holonomic function whose inverse $\frac{1}{\sin z}$ is *not* holonomic.

Apart from the invertible elements that directly follow from the above list of examples, the author is not aware of any other invertible holonomic constants. In particular, the precise status of $\mathcal{X}$ is unclear. From $\sin(\pi(\mathbb{K}\setminus\mathbb{Z})) \subseteq \mathbb{K}^{\text{shol}}$, it follows that $\mathcal{X}^{-1} \subseteq \mathbb{K}^{\text{shol}}$, whence

$$\mathbb{K}^{\text{shol}} = \mathbb{K}^{\text{shola}}.$$

In combination with Proposition 7, this actually provides a correct proof of [2, Proposition 4.7 (*d*)]. If $\mathcal{X}^{-1} \subseteq \mathbb{K}^{\text{hol}}$, then this would also imply $\mathbb{K}^{\text{hol}} = \mathbb{K}^{\text{rhol}} = \mathbb{K}^{\text{hola}} = \mathbb{K}^{\text{rhola}}$. It seems plausible though that $\mathcal{X} \cap \mathbb{K}^{\text{hol}} = \{1\}$ and $\mathbb{K}^{\text{hol}} = \mathbb{K}^{\text{rhol}} = \mathbb{K}^{\text{hola}} = \mathbb{K}^{\text{rhola}}$ both hold.

Further comments

For simplicity, we have assumed that $\mathbb{K} = \mathbb{Q}^{\text{alg}}$ is the field of algebraic numbers, throughout our exposition. Most results go through without much change for arbitrary algebraically closed fields $\mathbb{K}$. Only in the proof of Lemma 2, we used the assumption that $2\pi\mathrm{i} \notin \mathbb{K}$; if $2\pi\mathrm{i} \in \mathbb{K}$, then the same conclusion can be obtained by induction on d, by applying the induction hypothesis on $f(z\,\mathrm{e}^{2\pi\mathrm{i}}) - f(z)$, when $d > 0$.

Another interesting direction of generalization would be to consider holonomic functions that are completely defined over $\mathbb{Q}^{\mathrm{alg}}$, but to consider values at points in larger fields $\mathbb{K}$. Such classes of constants contain numbers like $\mathrm{e}^{\mathrm{e}^{\pi}}$, $\sin \Gamma(\sqrt{2})$, etc.

In [1], the authors consider values of so-called Siegel G-functions, which are a particular type of Fuchsian holonomic functions. They prove an analogue of Theorem 5 in this setting. Their proof is significantly simpler, thanks to special properties of G-functions [1, Theorem 3], and based on similar arguments as our proof of Proposition 3. The paper [1] also contains several results about fraction fields of fields of values of G-functions. It would be interesting to investigate analogues of these results in our setting.

Still in [1], the authors study the case when $\mathbb{K} \subsetneq \mathbb{Q}^{\mathrm{alg}}$ is an algebraic number field that is strictly contained in $\mathbb{Q}^{\mathrm{alg}}$. They showed that any real algebraic number can be obtained as the value at $z=1$ of a G-function over $\mathbb{Q}$ that is defined on $\bar{\mathcal{D}}_{0,1}$. In our setting, this immediately implies that $\mathbb{Q}^{\mathrm{alg}} \subseteq \mathbb{Q}^{\mathrm{hol}}[\mathrm{i}]$, whence $(\mathbb{Q}^{\mathrm{alg}})^{\mathrm{hol}} = \mathbb{Q}^{\mathrm{hol}}[\mathrm{i}]$. More generally, for any algebraic number field $\mathbb{K}$, we obtain $(\mathbb{Q}^{\mathrm{alg}})^{\mathrm{hol}} = \mathbb{K}^{\mathrm{hol}}[\mathrm{i}]$ if $\mathbb{K} \subseteq \mathbb{R}$ and $(\mathbb{Q}^{\mathrm{alg}})^{\mathrm{hol}} = \mathbb{K}^{\mathrm{hol}}$ if $\mathbb{K} \subsetneq \mathbb{R}$.

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