Introduction[†]

Dans cette thèse, nous présenterons la construction de corps munis des fonctions plus rapides que toute itération d'une fonction exponentielle. Le but de cette introduction est de décrire la signification des mots "construction", "plus vite" et "fonction exponentielle". Nous le faisons espérant de donner aux lecteurs une bonne idée de ce à quoi ils peuvent attendre de cette thèse ; nous espérons également fournir une motivation de l'étude présentée, et finalement nous pensons qu'elle sert comme guide pour aider les lecteurs à traverser les différentes parties de la thèse.

Nous commençons en expliquant quelques concepts de base et en présentant les principaux résultats. Ensuite, nous résumerons ce que l'on savait déjà sur les fonctions super-exponentielles. Dans la troisième partie de l'introduction, nous donnerons les motivations pour la construction présentée. Il suit une section qui constitue la "carte routière" de la thèse : nous résumerons chaque chapitre en quelques mots pour munir les possibles lecteurs d'un guide d'orientation. Il y a un intérêt particulier à faire cela, car quelques chapitres sont assez techniques, et il existe un danger réel de se perdre dans les détails qui sont néanmoins nécessaires. Finalement, nous donnerons une liste des notations les plus fréquemment utilisées.

Les résultats principaux

Le but principal de la thèse présentée est l'étude de la possibilité d'existence des fonctions avec croissance rapide sur des corps de séries généralisées.

Pour tout corps C et tout groupe multiplicatif $\mathfrak{M}$, totalement ordonnés, une fonction

$$f : \mathfrak{M} \longrightarrow C$$

est une série généralisée, si l'ensemble des $\mathfrak{m} \in \mathfrak{M}$ tels que $f(\mathfrak{m}) \neq 0$ (désormais appellé le support de f) est bien-ordonné dans $\mathfrak{M}$. Pour $C, \mathfrak{M}$ fixés, l'ensemble $\mathbb{S} = C[[\mathfrak{M}]]$ des séries généralisées $f : \mathfrak{M} \to C$ admet une multiplication et une addition telles que $\mathbb{S}$ est un corps. Donc, chaque polynôme $P \in \mathbb{S}[X]$ avec des coefficients dans $\mathbb{S}$ peut être associé d'une façon canonique à une fonction $f_P : \mathbb{S} \to \mathbb{S}$.

De plus, puisque C et $\mathfrak{M}$ sont totalement ordonnés, il est possible d'introduire un ordre total sur $\mathbb{S}$. Il existe donc une interpretation naturelle de "croissance" dans $\mathbb{S}$. En effet, pour deux polynômes $P, Q \in \mathbb{S}[X]$, on dit que P est plus rapide que Q, s'il y a une série $s \in \mathbb{S}$ avec

$$|f_Q(t)| < |f_P(t)|$$

[†]A translation of this introduction can be found at the end of the thesis.

pour $t > s$. Des résultats classiques concernant les séries généralisées impliquent que deux polynômes distincts peuvent être comparés dans ce sens. D'une manière analogue aux fonctions réelles, on peut se demander s'il y a des fonctions sur $\mathbb{S}$ ou au moins sur un interval $(f, +\infty)$ qui est plus rapide que tout polynôme dans $\mathbb{S}[X]$. En continuant l'analogie on pourrait même s'interesser à l'existence d'une fonction exponentielle sur $\mathbb{S}$.[1]

Il n'y a *a priori* aucune raison pour le corps $\mathbb{S} = C[[\mathfrak{M}]]$ d'avoir plus de structure que celle mentionée au-dessus. Donc, pour admettre des fonctions exponentielles et logarithmiques, les objets de base C et $\mathfrak{M}$ doivent satisfaire des conditions supplémentaires. Sans les préciser ici, nous remarquons que les corps de transséries constituent le bon cadre pour introduire des fonctions exponentielles et logarithmiques.[2] Dans ce qui suit, les corps de transséries seront notés $\mathbb{T}$ plutôt que $\mathbb{S}$.

Il y a une propriété importante particulière aux corps de transséries $\mathbb{T}$, notammnet que la fonction logarithmique est totalement définie sur l'ensemble de séries positives, mais que par contre la fonction exponentielle n'est pas totale. Pour surmonter ce problème, on utilise un processus introduit par Dahn qui étend le corps $\mathbb{T}$ à un corps de transséries $\mathbb{T}_{\exp}$, et on construit donc une tour des corps de transsŕies

$$\begin{array}{c} \vdots \\ \cup \\ \mathbb{T}_{\exp,\exp} \\ \cup \\ \mathbb{T}_{\exp} \\ \cup \\ \mathbb{T} \end{array}$$

tel que le logarithme et la fonction exponentielle peuvent être totalement définis sur la partie positive de la réunion. Nous montrerons comment on peut continuer cette construction au-delà pour construire un corps $\mathbb{T}_\alpha$ pour chaque ordinal α. De plus, l'ordre sur la réunion des corps sera de la façon que pour chaque série f suffisamment large, l'exponentiel de f est plus large que chaque f^i $(i \in \mathbb{N})$.

De nouveau, vu le corps des réels, il est raisonable de se demander si les corps des séries généralisées possèdent plus de structure que just les propriétés des corps ou – comme dans le cas des corps des transséries – une fonction logarithmique. En particulier, est-il possible d'introduire des sommes infinies, des dérivations ou compositions dans de tels corps ?

Comme c'était le cas pour la fonction exponentielle, il est d'avantage nécessaire de donner

[1] Evidemment faut il spécifier la signification de c'est-ce que c'est une fonction exponentielle. Dans l'intérêt de cette section – notamment de présenter les principaux résultats – il suffit de penser à une fonction exponentielle comme une fonction non-constante F telle que $F(x + y) = F(x) \cdot F(y)$ si les deux termes sont définis.

[2] Il mérite d'être mentioné qu'il y a des approches différents de l'approche présentée dans la thèse. Elles se ressemblent pourtant dans le sens qu'elles ont le même obstacle à surmonter. Les fonctions exponentielles et logarithmiques ne sont pas simultanement totalement définies sur l'ensemble des séries positives. Ce fait a été établi indépendamment par S. and F.-V. Kuhlmann et S. Shelah dans [KKS97] et J. van der Hoeven dans [vdH97].

une signification aux telles notions. Soient $\mathbb{S} = C[[\mathfrak{M}]]$ et $\mathcal{F} = (f_i)_{i\in I} \in \mathbb{S}^I$. Tout notion d'une somme infinie devrait coincider avec les opérateurs du corps, si I est un ensemble fini. Cette condition est satisfaite, si $\mathcal{F}$ est une famille noethérienne, i.e. si la réunion des supports de toutes les séries f_i est bien-ordonnée dans $\mathfrak{M}$ et si pour chaque $\mathfrak{m} \in \mathfrak{M}$ l'ensemble des indices $i \in I$ tel que $f_i(\mathfrak{m}) \neq 0$ est fini. Si c'est le cas, nous noterons par $\sum \mathcal{F}$ la série dans $\mathbb{S}$ avec $\sum \mathcal{F}(\mathfrak{m}) = \sum_I f_i(\mathfrak{m})$.

De même, la notion canonique d'une dérivation ∂ sur $\mathbb{S}$ devrait satisfaire les propriétés suivantes :

- ∂ est constamment 0 sur C
- pour toute $f, g \in \mathbb{S}$ on a $\partial(fg) = \partial(f) \cdot g + f \cdot \partial(g)$
- si $\mathcal{F} = (f_i)_{i\in I}$ est une famille noethérienne, alors la famille $\partial(\mathcal{F}) = (\partial(f_i))_{i\in I}$ l'est aussi et $\partial(\sum \mathcal{F}) = \sum \partial(\mathcal{F})$.

De plus, si $\mathbb{S}$ est un corps de transséries, la condition

- si $0 < f$, alors $\partial(f) = f \cdot \partial(\log f)$

doit être vraie. Nous annonçons notre premier résultat.

Résultat 1 *Si ∂ est une dérivation sur $\mathbb{T}$, alors pour chaque ordinal α il y a une unique dérivation ∂_α sur $\mathbb{T}_\alpha$ qui étend ∂.*

D'une manière identique on définit une notion de composition. Soient $\mathbb{T}_1$ et $\mathbb{T}_2$ des corps de transséries, alors une fonction $\vartriangle : \mathbb{T}_1 \to \mathbb{T}_2$ est une composition à droite si les conditions suivantes sont satisfaites :

- $\vartriangle$ est injective et $\forall c \in C : \vartriangle(c) = c$
- $\vartriangle$ est multiplicative
- si $\mathcal{F} = (f_i)_{i\in I}$ est une famille noethérienne (dans $\mathbb{T}_1$), alors $\vartriangle(\mathcal{F}) = (\vartriangle(f_i))_{i\in I}$ l'est aussi (dans $\mathbb{T}_2$) et $\vartriangle(\sum \mathcal{F}) = \sum \vartriangle(\mathcal{F})$
- pour toute $f \in \operatorname{dom} \exp$ dans $\mathbb{T}_1$: $\vartriangle(\exp f) = \exp \vartriangle(f)$.

Résultat 2 *Si $\vartriangle : \mathbb{T}_1 \to \mathbb{T}_2$ est une composition à droite, alors pour chaque ordinal α il y a une unique composition à droite $\vartriangle_\alpha : \mathbb{T}_{1,\alpha} \to \mathbb{T}_{2,\alpha}$ qui étend $\vartriangle$.*

A la lumière des résultats ci-dessus, une question s'impose immédiatement, notamment s'il y a une connection entre les dérivations et les compositions à droite. En particulier, les développements de Taylor, peuvent-t-ils être généralisés aux corps de transséries ? Cette question n'est pas purement académique ; en effet, il faut une réponse affirmative à cette question pour pouvoir continuer avec des investigations structurelles dans les corps des transséries.

La première étape est une généralisation de la notion d'une composition à droite à une composition en général. Soient $\mathbb{T}_i$ $(i = 1, 2, 3)$ des corps de transséries fixes avec des dérivations ∂^1, ∂^2 sur $\mathbb{T}_1$ et $\mathbb{T}_2$.[3] Une fonction partielle $\circ : \mathbb{T}_1 \times \mathbb{T}_3 \to \mathbb{T}_2$ est une composition compatible, si

- $\mathbb{T}_3 \subseteq \mathbb{T}_2$, et la restriction de ∂^2 à $\mathbb{T}_3$ est une dérivation

[3] Afin de facilier la lecture, nous écrivons désormais f' au lieu de $\partial^1(f)$ et $\partial^2(f)$. La n-ième dérivation de f sera notée $f^{(n)}$ dans les deux cas.

- pour chaque série $g \in \mathbb{T}_3$ avec $C < g$, la fonction $\Delta_g : \mathbb{T}_1 \to \mathbb{T}_2$ définie par $\Delta_g(f) = f \circ g$ est une composition à droite
- pour chaque $\mathfrak{m} \in \mathfrak{M}_1$ plus grande que 1, la fonction $\mathfrak{m} \circ \cdot : \{f \in \mathbb{T}_3 \,|\, C < f\} \to \mathbb{T}_2$ est strictement croissante
- la règle de chaîne est satisfaite par $\circ$, i.e. pour toute série $f \in \mathbb{T}_1$ et toute $g \in \mathbb{T}_3$ avec $g \in \mathrm{dom}\,(f \circ \cdot)$, on a $g \in \mathrm{dom}\,(f' \circ \cdot)$ et $(f \circ g)' = (f' \circ g) \cdot g'$
- soient $f \in \mathbb{T}_1$, $g \in \mathbb{T}_3$ et $(\varepsilon_i)_{i \in I}$ une famille noethérienne dans $\mathbb{T}_2$ tel que

$$\forall i \in I : \forall \mathfrak{m} \in \mathrm{supp}\, f : C < \left| \frac{\mathfrak{m} \circ g}{\mathfrak{m}' \circ g \cdot \varepsilon_i} \right|,$$

alors $g + \sum_I \varepsilon_i \in \mathrm{dom}\, f \circ \cdot$, $(f^{(n)} \circ g \cdot \varepsilon_i)_{0 \leqslant n, i \in I^n}$ est une famille noethérienne et

$$f \circ (g + \textstyle\sum_I \varepsilon_i) = \displaystyle\sum_{0 \leqslant n} \frac{1}{n!} f^{(n)} \circ g \cdot \sum_{i \in I^n} \varepsilon_i,$$

où $\varepsilon_i = \varepsilon_{i_1} \cdots \varepsilon_{i_n}$ pour $i = (i_1, \ldots, i_n) \in I^n$.

Résultat 3 *Si $\circ : \mathbb{T}_1 \times \mathbb{T}_3 \to \mathbb{T}_2$ est une composition compatible et α un ordinal, alors il y a une unique composition compatible $\circ_\alpha : \mathbb{T}_{1,\alpha} \times \mathbb{T}_3 \to \mathbb{T}_{2,\alpha}$ qui étend $\circ$.*

Une question qui découle naturellement de ces résultats concerne l'existence des corps de séries généralisées admettant non seulement des fonctions exponentielles, mais aussi d'autres fonctions à croissance supérieure à la croissance de chaque itération d'une fonction exponentielle. Par exemple, une fonction E satisfaisant l'équation fonctionnelle $E(x+1) = \exp \circ E(x)$ a cette propriété. On remarquera que – une fois construite une telle fonction E – une fonction $\mathcal{E}$ avec $\mathcal{E}(x+1) = E \circ \mathcal{E}(x)$ est aussi plus rapide que chaque $\exp_i$. Nous disons que e_{ω^i} est une fonction exponentielle de force $i \geqslant 0$, si $\mathrm{e}_{\omega^0} = \exp$ et

$$\begin{array}{rcl} \mathrm{e}_{\omega^1}(x+1) & = & \mathrm{e}_{\omega^0} \circ \mathrm{e}_{\omega^1}(x) \\ \mathrm{e}_{\omega^2}(x+1) & = & \mathrm{e}_{\omega^1} \circ \mathrm{e}_{\omega^2}(x) \\ \mathrm{e}_{\omega^3}(x+1) & = & \mathrm{e}_{\omega^2} \circ \mathrm{e}_{\omega^3}(x) \\ & \vdots & \end{array}$$

Nous appelons aussi les fonctions exponentielles de force 1 fonctions super-exponentielles. Comme il n'y avait pas de raison pour un corps de séries généralisées d'avoir une fonction exponentielle, un tel corps n'a pas non plus nécessairement des fonctions exponentielles de force $n \geqslant 0$. Nous appelons de force n de tels corps.

Résultat 4 *Pour chaque $n \in \mathbb{N}$, il y a des corps de force n.*

En généralisant le processus d'une extension exponentielle dû à Dahn, nous introduisons des e_{ω^n}-extensions pour étendre un corps $\mathbb{T}$ de force n à un corps $\mathbb{T}_{\mathrm{e}_{\omega^n}}$ qui est également de force n. Nous utilisons ces extensions pour montrer :

Résultat 5 *Soit $n \geqslant 0$. Il y a des corps $\mathcal{K}_n$ de séries généralisées admettant des fonctions exponentielles de force n tel que e_{ω^n} et sa fonction inverse l_{ω^n} sont totalement définies sur l'ensemble $\{f \in \mathcal{K}_n \,|\, C < f\}$.*

Fonctions super-exponentielles – une histoire brève

Les fonctions super-exponentielles et des problèmes associés ont été déjà étudiés à quelques occasions. Dans cette section, nous citons quelques résultats ; pourtant, nous ne prétendons pas donner un survol complet de l'histoire de ce sujet.

A la différence de notre construction, les fonctions super-exponentielles ont été utilisées soit dans la construction d'autres classes de fonctions (en particulier, dans la construction des itérées fractionnelles d'une fonction donnée) et elles ont donc servi plutôt comme outil, soit elles ont été considerées comme fonctions réelles (ou au moins des germes de telles fonctions). Nous ne connaissons pas d'article traitant des fonctions exponentielles de force supérieure à 1.

Les premiers pas vers la construction d'une fonction super-exponentielle remontent au 19-ième siècle, où P. du Bois-Raymond démontre qu'il n'y a pas de borne supérieure à la croissance pour le fonctions réelles. Plus précisement, soient$f_1 \prec f_2 \prec \cdots$[4] des fonctions définies sur un intervalle $(a, \infty) \subseteq \mathbb{R}$. Alors, il existe une fonction $F : (a, \infty) \to \mathbb{R}$ telle que $f_i \prec F$ pour tout i. Dans [Har10], G. H. Hardy montre ce fait de deux façons différentes ; et appliqué à l'ensemble des fonctions $\exp_i = \exp \circ \cdots \circ \exp$ (la i-ème itération de la fonction exponentielle) on en déduit l'existence d'une fonction F de croissance supérieure à celle de chaque itération de la fonction exp.

L'existence d'une telle fonction F ne nous donne aucune information sur le comportement de cette fonction. Pour au moins restreindre la croissance de telles fonctions, nous avons introduit la notion de fonctions super-exponentielles comme solutions de l'équation fonctionnelle

$$\exp E(x) = E(x+1). \tag{1}$$

Nous remarquons que si E est une solution de l'équation (1) et si g est une fonction de période 1, alors $E^*(x) = E(x+g(x))$ est aussi une solution. C'est-à-dire, les fonctions super-exponentielles sont loin d'être uniquement déterminées par l'équation fonctionnelle qui restreint leur croissance.

C'est H. Kneser dans les années quarante qui a contribué au prochain progrès significatif de l'étude des solutions E. Dans [Kne50], il construit une fonction super-exponentielle analytique en utilisant un point fixe complexe de la fonction e^x et des transformations conformes. En effet, il utilise sa solution afin de définir une demi-itérée analytique de la fonction exponentielle, i.e. une fonction φ avec

$$\varphi \circ \varphi(x) = e^x.$$

De plus, il définit un ensemble des fonctions analytiques $\exp_r$ (où $r \in \mathbb{R}$), appelées les itérations fractionnelles de exp, avec

$$\begin{aligned} \exp_1(x) &= e^x \\ \exp_{r+s}(x) &= \exp_r \circ \exp_s(x) \end{aligned}$$

[4] A l'instar de Hardy, nous noterons $\prec$ la relation de domination entre fonctions réelles, i.e. $f \prec g$ ssi $\lim_{x\to\infty} f(x)/g(x) = 0$.

pour $r, s \in \mathbb{R}$. En construisant E et son unique inverse L, il obtient les fonctions recherchées par $\exp_r(x) = E(L(x) + r)$.[5]

Après la parution de l'article de Kneser, l'étude des itérations fractionnelles s'intensifie, et la fonction exponentielle est de temps en temps remplacée par d'autres fonctions. Plus éminent, dans les travaux de G. Szekeres et K. W. Morris [Sze58], [Sze62],[SM62], les fonctions à croissance exponentielle ont été considerées, i.e. des fonctions f telles que

$$\exp_{k-1}(x) \prec f_k \prec \exp_{k+1}(x)$$

pour tout $k \in \mathbb{N}^+$. Les fonctions e^x et $e^x - 1$ en sont des exemples. Les itérations fractionnelles[6] de f peuvent être construites comme solutions de l'équation fonctionnelle

$$B(f(x)) = B(x) + 1,$$

aussi appelée équation d'Abel, et puis définissant les fonctions $f_r(x) = B_{-1}(B(x) + r)$. Pour la fonction $f(x) = e^x$, une solution B est une fonction super-logarithmique. Parmi les travaux de Szekeres, nous signalons un résultat intéressant concernant l'unicité de B pour une large classe des fonctions f. Si B est loin d'être uniquement déterminé pour $f(x) = e^x$, ce n'est pas le cas pour les fonctions f réelles analytiques (pour $x \geqslant 0$) avec $x < f(x)$ et $0 < f'(x)$ (pour $x > 0$) qui admettent un développement

$$f(x) = x + ax^2 + \cdots \qquad \text{où } a > 0.$$

Ceci est le cas pour $f(x) = e^x - 1$ et $a = 1/2$. Puis, il n'y a qu'une seule fonction b telle que

$$\lim_{x \to 0^+} x^2 b(x) = \frac{1}{a}$$

avec $b = B'$ pour une solution B de l'équation d'Abel. En d'autre termes, B est uniquement déterminé à une constante près.

Finalement, les fonctions super-exponentielles se retrouvent aussi dans les travaux de M. Boshernitzan sur les fonctions trans-exponentielles. Dans [Bos86], il considère des solutions E de l'équation fonctionnelle

$$h(E(x)) = E(x + 1),$$

où $h(x)$ est une des fonctions e^x où $e^x - 1$. Compte tenu de leurs propriétés de croissance, il baptise les solutions E trans-exponentielles, et il montre qu'il y a des germes de solutions qui appartiennent aux champs de Hardy. Comme résultat intermédaire, il montre que si E est une C^1-solution, alors $E' \prec E^3$. On reviendra à cette observation plus tard.

[5] Bien que l'article remonte à 1950, Kneser remarque que l'existence des fonctions analytiques $\exp_{\frac{1}{2}}$ avait été discutée pendant la conférence annuelle de la Société Mathématique Allemande en 1941. Les discussions étaient motivées par le besoin d'une solution "raisonnable" dans l'industrie. En considerant le temps et le lieu et aussi la situation politique en Allemagne, on pourrait rétrospectivement se demander quel était l'intérêt de l'industrie allemande dans les demi-itérées analytiques de e^x ou pour quelle raison Kneser le fait-il remarquer au juste.

[6] i.e. un ensemble de fonctions f_r tel que $f_1 = f$ et $f_{r+s} = f_r \circ f_s$ pour tout $r, s \in \mathbb{R}$

Motivations

Après notre brève revue de l'histoire des développements autour des fonctions super-exponentielles, nous allons dédier quelques remarques à nos motivations pour la construction présentée. Comme on a déjà vu, les fonctions super-exponentielles ont jusqu'à maintenant essentiellement servi comme outil pour obtenir des ensembles d'itérations fractionnelles. Néanmoins, il existe encore d'autres raisons pour poursuivre la construction ; on en mentionnera deux ici.

La première motivation est l'étude modèle-théorique du corps réel. Soit $\bar{\mathbb{R}}$ le corps totalement ordonné des réels et $\mathcal{L}$ le langage des anneaux ordonnés. Une observation bien connue et due à A. Tarski [Tar51] dit que chaque sous-ensemble définissable de $\bar{\mathbb{R}}$ est la réunion finie des intervalles dans $\mathbb{R} \cup \{\pm\infty\}$. Autrement dit, les ensembles définissables peuvent être décrits en n'utilisant que la rélation d'ordre $\leqslant$ et des paramètres dans $\mathbb{R} \cup \{\pm\infty\}$. Une question qui découle du résultat de Tarski concerne la possibilité d'ajouter des fonctions à $\bar{\mathbb{R}}$ (et de même des symboles de fonction à $\mathcal{L}$) sans perdre cette propriété pour les sous-ensembles définissables (voir par exemple [vdD84]). Plus précisement, si $\mathcal{F}$ est un ensemble de fonctions réelles et $\mathcal{L}_{\mathcal{F}}$ est le langage des anneaux ordonnés augmenté par un symbole de fonction pour chaque fonction dans $\mathcal{F}$, alors, chaque sous-ensemble définissable dans $\langle\bar{\mathbb{R}}, \mathcal{F}\rangle$ est-t-il réunion finie des intervalles ?

Pour des $\mathcal{F}$ avec cette propriété on dit que $\langle\bar{\mathbb{R}}, \mathcal{F}\rangle$ est o-minimale. Pendant les années quatre-vingt et quatre-vingt-dix, les structures o-minimales ont été étudiées intensivement.[7] De plus, les structures o-minimales possèdent beaucoup de propriétés intéressantes d'un point de vue topologique (décomposition cellulaire, stratification, triangularisation etc.) qui ont été étudiées en détail par L. van den Dries [vdD98].

Nous, par contre, sommes plus intéressés par des ensembles $\mathcal{F}$ qu'on peut ajouter à $\bar{\mathbb{R}}$ tels que la structure résultante est o-minimale. Plus précisément, nous nous occuppons de la croissance d'une fonction définissable dans cette structure. Comme premier résultat important, on a constaté qu'il est possible d'augmenter $\bar{\mathbb{R}}$ par l'ensemble de fonctions analytiques restreintes (voir [vdD86]). Dans cette structure, la croissance d'une fonction définissable est polynômialement bornée à l'infinie. Ensuite, A. Wilkie [Wil96] montre qu'il est possible d'ajouter la fonction exponentielle et d'obtenir une structure o-minimale. Ce résultat était généralisé à plusieurs occasions [vdDMM94], [Res93], mais les structures o-minimales sont toujours de sorte que les fonctions définissables peuvent être bornées par des itérations de la fonction exponentielle. D'où la question s'il y a des structures o-minimales $\langle\bar{\mathbb{R}}, \mathcal{F}\rangle$ telles qu'il existe des fonctions définissables là-dedans avec une croissance qui n'est pas bornée par une fonction $\exp_k$. Certes, la structure $\langle\bar{\mathbb{R}}, E\rangle$ est un candidat.

En vue de la démonstration de J.P. Ressayre du théorème de Wilkie, il est d'intérêt d'avoir des modèles non-standards de $\mathrm{Th}(\bar{\mathbb{R}}, \exp, E)$. Nous ignorons pour l'instant si notre construction est une vraie contribution à la solution de cette question, mais des résultats récents dûs à Ressayre [Res99] suggèrent qu'au moins notre modèle est un outil pour mieux comprendre le comportement d'une fonction super-exponentielle dans un modèle non-standard. De plus, une fois la o-minimalité du corps réel avec une fonction super-exponentielle est démontrée, on posera la question des bornes pour les fonctions définissables à nouveau. Il est donc raisonnable de

[7] Pour une introduction aux structures o-minimales, voir [PS86], [KPS86].

poursuivre la construction à une force arbitraire plutôt que de se restreindre à la force 1.[8]

Notre deuxième motivation est un programme de travail de J. van der Hoeven dans lequel il cherche à construire un corps de transséries tel que chaque équation algébrique, fonctionnelle ou différentielle avec des paramètres dans le corps admet une solution dans ce même corps, s'il y a des solutions. Dans ce cadre, ajouter une fonction super-exponentielle peut être vu comme la clôture du corps sous l'action des solutions de l'équation fonctionnelle $E(x+1) = \exp E(x)$.

Nous nous permettons en connection avec cette motivation le remarque que la Section 2.5 concernant les monômes et séries imbriqués font aussi partie de ce programme. Effectivement, on a pas besoin des ces objets pour la construction des fonction E ou L (ni, d'ailleurs, dans la construction des fonctions à force superieure), mais ces objets représentent des solutions des équations fonctionnelles. Les travaux de van der Hoeven se poursuivent en ce moment, et nous espérons que notre travail est une contribution valable pour une conclusion de son programme.

La structure de la thèse

Dans cette section, nous donnons une description de la structure de la thèse en résumant chaque chapitre.

Chapitre 1 : Le premier chapitre introduit les fondements. Bien qu'il n'est pas notre but de présenter cette thèse comme étude independante, nous commençons avec quelques rappelles concernant des idées et résultats bien connus.

Notre point de départ est la définition d'un ordre comme une rélation binaire anti-symétrique, réflexive et transitive sur un ensemble P. En connection avec des ordres, nous introduisons les notions de comparabilité, des ordres totals, des anti-chaînes, des chaînes décroissantes et des ordres bien-fondés. Nous rappelons que tous ces objets sont des notions généralement connues dans les mathématiques, et nous ne prétrendrons pas d'originalité particulière en les introduisant. De même pour la généralisation du concept des ensembles bien-ordonnés dans des ordres totaux au cadre d'ordres en général : un ordre est appelé noethérien s'il admet ni des chaînes strictement décroissantes ni des anti-chaînes infinies.

La théorie des ordres noethériens est amplement étudiée, et nous donnons des formulations équivalentes que nous utilisons fréquemment dans le reste de la thèse. Ensuite, nous introduisons des mots sur un ensemble donné P, où nous établisons une distinction entre des mots commutatifs et non-commutatifs $P^{\diamondsuit}$ et $P^{\star}$. De plus, s'il existe un ordre $\leqslant$ sur P, alors nous introduisons respectivement des ordres $\leqslant_{P^{\diamondsuit}}$ et $\leqslant_{P^{\star}}$ sur les ensembles $P^{\diamondsuit}$ et $P^{\star}$. Nous rappelons le résultat de Higman que $(P^{\diamondsuit}, \leqslant_{P^{\diamondsuit}})$ et $(P^{\star}, \leqslant_{P^{\star}})$ sont noethériens, si l'ordre $(P, \leqslant)$ l'est aussi.

Après remettre en mémoire la notion d'un corps archimédien et de comment généraliser cette notion aux modules, nous introduisons enfin les principaux objets de notre étude, les séries généralisées. En effet, à ce point là, nous définissons l'ensemble $\mathbb{S} = C[[\mathfrak{M}]]$ de séries généralisées sur $\mathfrak{M}$ avec des coefficients dans C d'une façon plutôt générale, car nous permettons $\mathfrak{M}$ d'être

[8] A ce propos, nous remarquons qu'une réponse affirmative à la question de o-minimalité de la structure $\langle \bar{\mathbb{R}}, E \rangle$ a aussi des implications concernant des structures o-minimales nivelées étudiées par D. Marker et Ch. Miller [MM97].

un semi-groupe ordonné et C d'être un anneau. Puis $f \in C[[\mathfrak{M}]]$, si $f : \mathfrak{M} \to C$ est une fonction avec un support noethérien dans $\mathfrak{M}$. En général, cependant, nous allons prendre des groupes abeliens multiplicatifs et ordonnés pour $\mathfrak{M}$. C'est le bon endroit d'introduire une collection de notation utiles. Nous commençons avec quelques sous-ensembles de $\mathfrak{M}$. Soit $\succcurlyeq$ l'ordre sur $\mathfrak{M}$, alors

$$\begin{aligned}
\mathfrak{M}^{\uparrow} &= \{\mathfrak{m} \in \mathfrak{M} \mid \mathfrak{m} \succ 1\},\\
\mathfrak{M}^{\underline{\uparrow}} &= \{\mathfrak{m} \in \mathfrak{M} \mid \mathfrak{m} \succcurlyeq 1\},\\
\mathfrak{M}^{\downarrow} &= \{\mathfrak{m} \in \mathfrak{M} \mid 1 \succ \mathfrak{m}\},\\
\mathfrak{M}^{\overline{\downarrow}} &= \{\mathfrak{m} \in \mathfrak{M} \mid 1 \succcurlyeq \mathfrak{m}\}.
\end{aligned}$$

De plus, nous mettons $\mathbb{S}^{\uparrow} = C[[\mathfrak{M}^{\uparrow}]]$, et nous définissons les ensembles $\mathbb{S}^{\underline{\uparrow}}, \mathbb{S}^{\downarrow}, \mathbb{S}^{\overline{\downarrow}}$ d'une manière semblante. Nous utlitisons les flèches aussi comme opérateurs qui agissent sur l'ensemble de séries en définissant $f^{\uparrow} \in \mathbb{S}$ par

$$f^{\uparrow}(\mathfrak{m}) = \begin{cases} f(\mathfrak{m}) & \text{si } \mathfrak{m} \in \mathfrak{M}^{\uparrow},\\ 0 & \text{sinon.} \end{cases}$$

D'une façon pareille, nous définissons les séries $f^{\underline{\uparrow}}, f^{\downarrow}, f^{\overline{\downarrow}}$ qui sont respectivement éléments de $\mathbb{S}^{\underline{\uparrow}}, \mathbb{S}^{\downarrow}, \mathbb{S}^{\overline{\downarrow}}$. Nous écrivons $f_{\mathfrak{m}}$ au lieu de $f(\mathfrak{m})$ pour exprimer l'idée que f devrait être vu plutôt comme série (d'où le nom) que comme une fonction, alors qu'il s'agit de $f_{\mathfrak{m}}$ du coefficient lié au monôme $\mathfrak{m}$. En utilisant cette convention, nous écrivons $f = \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}$.

Nous introduisons une addition et une multiplication sur $\mathbb{S}$ par

$$\begin{aligned}
f + g &= \sum_{\mathfrak{m} \in \mathfrak{M}} (f_{\mathfrak{m}} + g_{\mathfrak{m}}) \mathfrak{m},\\
f \cdot g &= \sum_{\mathfrak{m} \in \mathfrak{M}} \Big(\sum_{\mathfrak{a}\mathfrak{b} = \mathfrak{m}} f_{\mathfrak{a}} g_{\mathfrak{b}}\Big) \mathfrak{m}.
\end{aligned}$$

Avec ces opérations, l'ensemble $\mathbb{S}$ est un anneau. Il y a aussi des plongements canoniques de C et $\mathfrak{M}$ dans $\mathbb{S}$. De plus, l'anneau $\mathbb{S}$ est un corps si et seulement si C est un corps.

Afin de montrer la dernière propriété, il est nécessaire d'introduire la notion d'une somme infinie qui étend l'addition d'un nombre fini de séries. Bien sûr, on ne trouve pas nécessairement une expression raisonnable $f_1 + f_2 + \cdots$ pour chaque séquence $(f_1, f_2, \ldots)$ de séries dans $\mathbb{S}$. Pourtant, si la séquence $F = (f_i)_{i \in I} \in \mathbb{S}^I$ est telle que $\bigcup_{i \in I} \operatorname{supp} f_i$ est noethérien dans $\mathfrak{M}$ et telle que pour chaque $\mathfrak{m} \in \mathfrak{M}$ il n'y a qu'un nombre fini d'indices $i \in I$ avec $\mathfrak{m} \in \operatorname{supp} f_i$, alors nous pouvons définir

$$\sum F = \textstyle\sum_I f_i = \displaystyle\sum_{\mathfrak{m} \in \mathfrak{M}} \sum_{i \in I} f_{i,\mathfrak{m}} \mathfrak{m}.$$

Des séquences F avec les propriétés au-dessus sont appelées familles noethériennes, et nous montrons que les familles noethériennes admettent des propriétés algébriques agréables.

La sommation des familles noethériennes peut être vue dans le carde plus général des algèbres fortes. Sans donner les détails ici, nous remarquons que des corps de séries généralisées $C[[\mathfrak{M}]]$

sont des C-algèbres fortes par rapport à la sommation $\sum_I$ au-dessus. Une propriété clef, que sera utilisée dans les constructions effectuées dans la thèse, est la suivante. Soient $C[[\mathfrak{M}]], C[[\mathfrak{N}]]$ des anneaux de séries généralisées. Soit $\varphi : \mathfrak{M} \longrightarrow C[[\mathfrak{N}]]$ une application telle que l'image d'un ensemble noethérien dans $\mathfrak{M}$ est une famille noethérienne dans le corps $C[[\mathfrak{N}]]$. Alors, il y a une unique application $\hat{\varphi} : C[[\mathfrak{M}]] \longrightarrow C[[\mathfrak{N}]]$ qui étend φ telle que pour chaque famille noethérienne $(f_i)_{i\in I}$ dans $C[[\mathfrak{M}]]$ nous avons

$$\textstyle\sum_I \hat{\varphi}(f_i) = \hat{\varphi}(\sum_I f_i).$$

De plus, si φ est multiplicatif, l'opération $\hat{\varphi}$ l'est aussi. De même, si pour tout $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$ l'équation $\varphi(\mathfrak{m}\mathfrak{n}) = \mathfrak{m} \cdot \varphi(\mathfrak{n}) + \varphi(\mathfrak{m}) \cdot \mathfrak{n}$ est satisfaite, alors l'application $\hat{\varphi}$ hérite cette propriété aussi, i.e. $\hat{\varphi}(fg) = f \cdot \hat{\varphi}(g) + \hat{\varphi}(f) \cdot g$ pour toutes les séries $f, g \in C[[\mathfrak{M}]]$.

Le reste de ce chapitre ne cosidere que des corps de séries généralisées $C[[\mathfrak{M}]]$ où C et $\mathfrak{M}$ sont totalement ordonnés. Comme conséquence nous obtenons plusieurs représentations canoniques d'une série $f \in \mathbb{S}$. D'abord, nous remarquons que dans ce cas nous avons $\mathfrak{M} = \mathfrak{M}^{\uparrow} \cup \{1\} \cup \mathfrak{M}^{\downarrow}$, donc qu'il y a une unique constante $f^{=} = f_{1_{\mathfrak{M}}} \in C$ telle que

$$\begin{aligned} f &= f^{\uparrow} + f^{=} + f^{\downarrow} \\ &= f^{\uparrow} + f^{\overline{\downarrow}} \\ &= f^{\underline{\uparrow}} + f^{\downarrow}. \end{aligned}$$

Au-delà, le support de f est bien-ordonné dans $(\mathfrak{M}, \succcurlyeq)$ et il admet donc un élément minimal que nous appelons le monôme minimal, symbolisé par $\mathfrak{d}_f$. Le valeur de f dans $\mathfrak{d}_f$ est c_f, le coefficient dominant. Soit $\tau_f = c_f \mathfrak{d}_f$ le terme dominant de f, alors il y a des séries R_f, δ_f avec

$$\begin{aligned} f &= \tau_f + R_f \\ &= \tau_f(1 + \delta_f). \end{aligned}$$

De plus, les ordres de C et $\mathfrak{M}$ induissent un ordre total sur l'ensemble $\mathbb{S}$ qui est défini par

$$\begin{aligned} 0 < f \quad &\Leftrightarrow \quad 0 < c_f \\ g < f \quad &\Leftrightarrow \quad 0 < f - g. \end{aligned}$$

Nous finissons le premier chapitre avec quelques considerations générales sur des troncatures et le comportement de supprtts pour des séquences de séries. Pour une série donnée $f \in \mathbb{S}$, la série g est une troncature de f si le support de g est un ségment initial du support de f et si les deux séries coincident sur le support de g. En autre mots, il existe un monôme $\mathfrak{m}_g$ tel que

$$g = \sum_{\mathfrak{m} \succ \mathfrak{m}_g} f_{\mathfrak{m}} \mathfrak{m}.$$

Nous utilisons des troncatures et leur propriétés dans des nombreuses démonstrations. D'une manière pareille, nous introduissons les cardinalités cofinales comme un outil. Pour un ordre total $P = (P, \leqslant)$ et un cardinal κ nous disons que P admet une cardinalité cofinale $< \kappa$ si la cardinalité de chaque sous-ensemble bien-ordonné dans P admet une cardinalité inferieur à κ.

Par exemple, les nombres réels avec leur ordre naturel ont une cardinalité cofinale $< \aleph_1$. Nous montrons que si C et $\mathfrak{M}$ ont respectivement les cofinalités cardinales $< \kappa_1$ et $< \kappa_2$, alors pour toute séquence $(f_\alpha)_{\alpha<\tau}$, strictement décroissante dans $\mathbb{S}$, nous avons $|\tau| < \max(\kappa_1, \kappa_2)$.

Chapitre 2 : Jusqu'à ici, les corps de séries généralisées sont munis de très peu de structure. Pourtant, par demander quelques propriétés bies choisies, nous pouvons characteriser des classes de corps de séries généralisées qui admettent au moins des fonctions exponentielles et logarithmiques. Afin de le faire, nous commençons le deuxiem chapitre en fixant les conditions pour qu'une fonction soit une fonction exponentielle.

En fait, une fonction exp qui est partiellement définie sur un corps totalement ordonné C est une fonction exponentielle si elle est strictement croissante, si $a+1 \leqslant \exp a$ pour tout $a \in C$ dans le domain de exp, et si

$$\exp(a+b) = (\exp a)(\exp b),$$

si les deux termes sont définis. Dans ce cas, les corps C est appelé un corps exponentiel.

Si C est un corps exponentiel tel que $C = \operatorname{dom} \exp$, alors nous pouvons définir une fonction exp sur $C[[\mathfrak{M}]]^{\updownarrow}$ par

$$\exp f = \exp(f^{=}) \cdot e(f^{\downarrow}),$$

où $e(x) = \sum_{\mathbb{N}} \frac{1}{n!} x^n$. L'image de exp est l'ensemble $\mathbb{S}^{\updownarrow,+}$ de séries positives et non-infinies. Dans ce sens, chaque $\mathbb{S}$ admet une structure de base comme corps exponentiel. La fonction inverse de exp est notée par log, elle satisfait

$$\log f = \log c_f + l(f^{\downarrow})$$

pour tout $0 < f$, où $l(x) = \sum_{1 \leqslant n} \frac{(-1)^{n+1}}{n} x^n$. Des propriétés supplementaires du corps $\mathbb{S}$ sont nécessaires, si on veut que log est défini sur l'ensemble de ses séries positives. Un corps $C[[\mathfrak{M}]]$ est corps de transséries, si C est un exp-log corps avec $C = \operatorname{dom} \exp$ et si log étend partiellement à $\mathbb{T} = C[[\mathfrak{M}]]$ tel que

T1. $\operatorname{dom} \log = \mathbb{T}^{+}$

T2. $\log \mathfrak{M} \subseteq \mathbb{T}^{\uparrow}$

T3. $\log(1+f) = l(f)$, pour tout $f \in \mathbb{T}^{\downarrow}$

T4. pour chaque séquence $(\mathfrak{m}_i)_{0 \leqslant i} \subseteq \mathfrak{M}$ telle que $\mathfrak{m}_{i+1} \in \operatorname{supp} \log \mathfrak{m}_i$ pour tout $0 \leqslant i$, il existe un entier $n_0 \in \mathbb{N}$ tel que

$$\forall n_0 \leqslant n : \forall \mathfrak{n} \in \operatorname{supp} \log \mathfrak{m}_n : \mathfrak{n} \succcurlyeq \mathfrak{m}_{n+1} \ \wedge \ (\log \mathfrak{m}_n)_{\mathfrak{m}_{n+1}} = \pm 1.$$

Les conditions **T1** – **T3** nous permettent de poursuivre le processus d'extension dû à Dahn [Dah84], et la condition **T4** est essentielle pour le traitement des expressions imbriquées. Les expressions exponentielles et imbriquées sont au centre de l'intérêt de ce chapitre.

Pour distinguer les corps de transséries des corps de séries généralisées au sense usuel, nous utilisons le symbol $\mathbb{T}$ au lieu de $\mathbb{S}$. Un exemple simple d'un corps de transséries est $\mathbb{L} = \mathbb{R}[[\log^{\mathbb{Z}^\star} x]]$, où

$$\log^{\mathbb{Z}^\star} x = \{\log^a x = x^{a_0} \log^{a_1} x \cdots \log_n^{a_n} x \mid a \in \mathbb{Z}^\star\}.$$

Du fait qu'il n'y a pas de corps de transséries $\mathbb{T}$ tel que les fonctions exp et log sont respectivement totalement définies sur $\mathbb{T}$ et $\mathbb{T}^+$, il suit la nécessité d'élargier $\mathbb{T}$. C'est le processus dû a Dahn qui joue la rôle principale ici. Nous mettons $\mathbb{T}_{\exp} = C[[\exp \mathbb{T}^\uparrow]]$, l'extension exponentielle de $\mathbb{T}$. Soit α un nombre ordinal, alors nous définissons le corps de transseries $\mathbb{T}_\alpha = C[[\mathfrak{M}_\alpha]]$ par

$$\mathbb{T}_\alpha = \begin{cases} \mathbb{T} & \text{si } \alpha = 0 \\ \mathbb{T}_{\beta,\exp} & \text{si } \alpha = \beta + 1 \\ C[[\bigcup\limits_{\beta<\alpha} \mathfrak{M}_\beta]] & \text{si } \alpha \text{ est un ordinal limit.} \end{cases}$$

Nous appelons des corps de la forme $\mathbb{T}_\alpha$ aussi extensions exponentielles transfinies de $\mathbb{T}$. Il y a deux façons différentes d'obtenir des corps de séries généralisées tels que exp et log sont totalement définis. Premiérement, l'ensemble $\bigcup_{\alpha<\lambda} \mathbb{T}_\alpha$ admet cette propriété, si λ est un ordinal limit ; mais ce corps ne plus de la forme $C[[\mathfrak{N}]]$. La deuxiéme possibilité consiste à ajouter une condition concernant la cardinalité du support à la définition des séries généralisées, notamment on exige pour que f soit une série généralisée que la cardinalité de $\operatorname{supp} f$ est inferieure ou égale à un cardinal fixé. Dans ce dernier cas, le processus d'extension est stabilisant.

Une raison particulière pour regarder le processus d'extension exponentielle en détail est le fait que beaucoup de principes qui font marcher ce processus vont réapparaître dans une forme semblante pendant la construction des extensions exponentielles de force positive. Effectivement, le programme à suivre est le suivant :

- étendre le groupe de monômes à un ensemble $\hat{\mathfrak{M}} \supseteq \mathfrak{M}$,
- définire une structure d'un groupe multiplicatif sur $\hat{\mathfrak{M}}$,
- définir un ordre sur $\hat{\mathfrak{M}}$ qui est compatible avec la multiplication,
- définir un logarithme sur $\hat{\mathfrak{M}}$ et $\hat{\mathbb{T}} = C[[\hat{\mathfrak{M}}]]$ tel que $\hat{\mathbb{T}}$ est un corps de transséries.

Il y a deux résultats généraux concernant les corps de transséries que nous voulons mentioner ici. Premièrement, ces corps admettent une composition avec les éléments du corps $\mathbb{L} = C[[\log^{\mathbb{Z}^\star} x]]$. Plus précisement, nous montrons que pour chaque série $f \in \mathbb{L}$ et chaque $g \in \mathbb{T}^+_\infty$ (l'ensemble d'éléments positifs avec $\mathfrak{d}_g \in \mathfrak{M}^\uparrow$), nous pouvons remplacer x par g. En autre mots, nous montrons que l'application $\varphi : \log^{\mathbb{Z}^\star} x \longrightarrow \mathbb{T}$ définie par

$$\mathfrak{m} = \log^a x \longmapsto \mathfrak{m} \circ g = g^{a_0} \log^{a_1} g \cdots \log_n^{a_n} g$$

est noethérienne et admet donc une extension unique $\hat{\varphi} : \mathbb{L} \longrightarrow \mathbb{T}$ avec $f \circ g = \sum_{\mathfrak{m}} f_{\mathfrak{m}} \mathfrak{m} \circ g$. Deuxièment, nous continuons l'étude du comportement de supports sous le processus d'extension. Comme avant, nous supposons que C et $\mathfrak{M}$ ont respectivement les cardinalités cofinales $< \kappa_1$ et $< \kappa_2$. Alors, nous montrons que

$$|\operatorname{supp} f| < \max(\kappa_1, \kappa_2)$$

pour toute série $f \in \mathbb{T}_{\exp}$.

La deuxième partie du chapitre montre qu'il est possible d'introduire des expressions monômielles imbriquées. Par cela nous voulons dire qu'il y a des transmonômes comme

$$e^{x^2 + e^{\log_2^2 x + e^{\log_4^2 x + e^{\cdot^{\cdot^{\cdot}}}}}}. \tag{2}$$

L'expression (2) nous fournis d'une solution canonique à l'équation fonctionelle $f(x) = \exp(x^2 + f(\log_2 x))$. Des expressions de cette forme apparaissent d'une façon naturelle dans la characterisation des intervalles de transséries. Plus d'information sur ce sujet on trouve dans [vdH97].

Des expressions comme (2) n'ont *a priori* aucune raison d'appartenir à un corps de transséries donné. On peut, par exemple, facilement vérifier qu'elle est ni un élément de $\mathbb{L}$, ni dans une extension exponentielle transfinie $\mathbb{L}_\alpha$. Nous mettons à la disposition un outil pour étendre un corps de transséries $\mathbb{T}$ par des monômes imbriqués, qui nous fournis d'un moyen pour clore le corps $\mathbb{T}$ sous l'action des équations fonctionnelles qui induissent de telles expressions. Plus précisement, pour des séquences $\varphi = (\varphi_0, \varphi_1, \dots)$ et $\sigma = (\sigma_0, \sigma_1, \dots) \in \{-1, +1\}^{\mathbb{N}}$ avec

- $\forall i \geqslant 0 : \varphi_i \in \mathbb{T}^{\uparrow} \wedge 0 < \varphi_{i+1}$,
- $\forall i \geqslant 0 : \forall \mathfrak{m} \in \operatorname{supp} \varphi_i : \exists j > i : \forall \psi \in \mathbb{T}^{\uparrow}$:

$$\operatorname{supp} \varphi_j \succ \psi \Rightarrow \mathfrak{m} \succ \sigma_i \, e^{\varphi_{i+1} + \sigma_{i+1} \, e^{\cdot^{\cdot^{\cdot \sigma_{j-1} \, e^{\varphi_j + \psi}}}}},$$

nous montrons comment construire un corps de transséries $\mathbb{T}_{\mathrm{nest}}$ qui contient $\mathbb{T}$ et l'expression

$$e^{\varphi_0 + \sigma_0 \, e^{\varphi_1 + \sigma_1 \, e^{\cdot^{\cdot^{\cdot}}}}}.$$

Remarques sur les notations

Contents

Chapter 1

Generalized power series and Noetherian families

1.1 Noetherian orders

Let $(P, \leqslant)$ be a **(partial) order**, i.e. P is a set and $\leqslant \subseteq P^2$ is a relation such that for all $a, b, c \in P$ we have

PO1. $a \leqslant b \wedge b \leqslant a \Rightarrow a = b$,
PO2. $a \leqslant a$ and
PO3. $a \leqslant b \wedge b \leqslant c \Rightarrow a \leqslant c$.

We will henceforth call $\leqslant$ the **ordering** of the order $(P, \leqslant)$, and we will speak of P as the order if the ordering $\leqslant$ is clear from the context. To every order $(P, \leqslant)$, we can define the **inverse order** $(P, \leqslant^*)$ by letting $a \leqslant^* b$ iff $b \leqslant a$ for all $a, b \in P$. If we let $\geqslant = \leqslant^*$, then $a \leqslant^* b$ if and only if $a \geqslant b$. Hence $(P, \geqslant)$ is the inverse order of $(P, \leqslant)$. The distinction between an ordering $\leqslant$ and its inverse ordering $\geqslant$ will have advantages for formulating properties of subsets of the underlying set P, but one should always be aware of the ordering with which the property is defined, since otherwise confusion may arise. It is common practice to distinguish between partial and total orders: **total orders** are orders $(P, \leqslant)$ such that

TO. $\forall a, b \in P : a \leqslant b \vee b \leqslant a$.

In the sequel, orders will be partial. If an order is total, it will be explicitly mentioned. From the ordering $\leqslant$ we obtain a **strict ordering** $<$ by letting $a < b$ iff $a \leqslant b \wedge a \neq b$. For every subset $S \subseteq P$, the ordering on P induces an ordering on S: if $a, b \in S$, then $a \leqslant_S b$ iff $a \leqslant b$. In other words, we have $\leqslant_S = \leqslant \cap S^2$. We call $\leqslant_S$ the **restriction** of $\leqslant$ to S. In general, the restriction of $\leqslant$ to a subset of P will also be denoted by $\leqslant$ since for all $S, T \subseteq P$: $\leqslant_S \cap \leqslant_T = \leqslant_{S \cap T}$.

A subset $S \subseteq P$ is called a **chain** in P iff $(S, \leqslant)$ is a total order. A chain S is said to be **strictly increasing** iff for every $s \in S$ there is a $t \in S$ such that $s < t$. Similarly, we say that the chain S is **strictly decreasing** iff for every $s \in S$ there is a $t \in S$ such that $t < s$. We say that the order $(P, \leqslant)$ has the **decreasing chain property** iff there are no strictly decreasing chains $S \subseteq P$. The order $(P, \leqslant)$ has the **increasing chain property** iff there are no strictly increasing chains in P. From the above it follows that an order has the increasing chain property if and only if its inverse order has the decreasing chain property.

A total order with the decreasing chain property is called a **well-order**. A (partial) order is **well-founded** iff it has the decreasing chain property. Equivalently, we say that P is **anti-well-founded** iff it has the increasing chain property. Again it follows from these notations that $(P, \leqslant)$ is well-founded if and only if $(P, \geqslant)$ is anti-well-founded. Hence, if we want to express that $(P, \leqslant)$ is anti-well-founded, we say that $(P, \geqslant)$ is well-founded.

Let $A, B \subseteq P$ and $a \in P$. We write $a \leqslant B$ iff $a \leqslant b$ for all $b \in B$. Similarly we define $a < B$, $B \leqslant a$ and $B < a$. We let $A < B$ and $A \leqslant B$ iff $a < B$ and $a \leqslant B$, respectively, for all $a \in A$. Two distinct elements $a, b \in P$ are said to be **incomparable** in the order $(P, \leqslant)$ (in symbols $a \perp_{\leqslant} b$ or simply $a \perp b$, if the ordering $\leqslant$ is clear from the context) iff neither $a \leqslant b$ nor $b \leqslant a$. We say that a, b are **comparable** iff they are not incomparable, i.e. if $a \not\perp b$ then either $a \leqslant b$ or $b \leqslant a$. Hence $a \not\perp a$ for all $a \in P$. Moreover, if $a \perp b$, then $b \perp a$ and $a \perp_{\leqslant} b$ if and only if $a \perp_{\geqslant} b$. We say that $a \in P$ is **incomparable** to a subset $S \subseteq P$, in symbols $a \perp S$, if P is incomparable to every element of S, i.e. $\forall s \in S : P \perp s$.

A set $A \subseteq P$ is an **anti-chain** in the order P iff any two distinct elements a, b of A are incomparable, i.e. $\forall a, b \in A : a \neq b \Rightarrow a \perp b$. An anti-chain in $(P, \leqslant)$ is also an anti-chain in $(P, \geqslant)$. Note that anti-chains in total orders can only have one element. An order is called **Noetherian** iff it has the decreasing chain property and if it has no infinite anti-chains. It should be noticed that every subset of a Noetherian order is again Noetherian.

Every subset of a well-ordered set has a minimal element, i.e. an element such that no other element of the subset is smaller than this element. To extend this notion to orders, we introduce minimal sets. Let $(P, \leqslant)$ be an order, then we call $\Delta = \Delta(P) \subseteq P$ a **minimal set** of P iff

MS1. $\forall p \in P : \exists q \in \Delta : q \leqslant p$,

MS2. $\forall q \in \Delta : \forall p \in P : \neg(p < q)$.

From **MS2** it follows that every minimal set is an anti-chain in the order P. This anti-chain is maximal: assume the contrary, then for some $p \in P$ we have $p \perp \Delta$, and by **MS1** we can find a $q \in \Delta$ with $q \leqslant p$, contradiction. Moreover, if $(P, \leqslant)$ has a minimal set, then this set is uniquely determined: Suppose that $\Delta_1 \neq \Delta_2$ have both the properties **MS1** and **MS2**. Then one of $\Delta_1 \setminus \Delta_2$ or $\Delta_2 \setminus \Delta_1$ is nonempty. Say, $q_1 \in \Delta_1 \setminus \Delta_2$, but then there must be an element $q_2 \in \Delta_2$ with $q_2 < q_1$. Contradiction.

Let $(P, \leqslant)$ be an ordering (not necessarily Noetherian). Then a subset $S \subseteq P$ is called a **final segment** of P iff $\forall p \in P : \forall s \in S : s \leqslant p \Rightarrow p \in S$. For $G \subseteq P$ we let

$$(G) := \{p \in P \mid \exists g \in G : g \leqslant p\}.$$

The set (G) is then a final segment of P, and we say that (G) is the final segment **generated** by G, or equivalently that G is a **generator** of that final segment.

Proposition 1.1.1 *Let $(P, \leqslant)$ be an order. Then the following are equivalent:*

(1) P is Noetherian.

(2) Every subset has a finite minimal set.
(3) Every final segment is finitely generated.
(4) From every sequence in P one can extract an increasing sub-sequence.

Proof: (1)⇒(2): Let $\mathcal{M} = \{\Delta \subseteq P \mid \forall q \in \Delta : \forall p \in P : \neg(p < q)\}$. This set is nonempty, for if it was not, then we could find a strictly decreasing sequence in P. The set $\mathcal{M}$ can be (partially) ordered by set inclusion. Let $(\Delta_i)_{i \in I}$ be a chain in $\mathcal{M}$. We put $\Delta = \bigcup_{i \in I} \Delta_i$. Let now $\delta \in \Delta$ and $p \in P$, then $q \in \Delta_i$ for some $i \in I$, hence $\neg(p < q)$. This shows $\Delta \in \mathcal{M}$. By Zorn's lemma, the set $\mathcal{M}$ has a $\subseteq$-maximal element Δ^*. Now suppose that there is a $p \in P$ such that for no $q \in \Delta^* : q \leqslant p$. Since Δ^* has property **MS2**, the element p is incomparable to Δ^*. Since P is Noetherian, there must be an element $q^* \in P$ such that $q^* \leqslant p$ and for no $q \in P$ we have $q < q^*$. But then $\Delta^* \cup \{q^*\} \in \mathcal{M}$, a contradiction. Since Δ^* is an anti-chain, it is finite.

(2)⇒(1): A strictly decreasing sequence cannot have a minimal set. An infinite anti-chain cannot have a finite minimal set.

(1)⇒(3): Let P be a Noetherian order and $S \subseteq P$ a final segment of P. Then S has a minimal set $\Delta(S)$. The finite segment generated by $\Delta(S)$ is S, and $\Delta(S)$ is thus a generator of S.

(3)⇒(4): Suppose that $a = (a_i)_{1 \leqslant i}$ is a sequence in P. We show that there is an increasing sub-sequence $(a_{i_n})_{1 \leqslant n}$ of a. Let S_1 be the final segment generated by the set $A_1 = \{a_i \mid 1 \leqslant i\}$. Then S_1 is finitely generated, i.e. there is a finite subset B_1 of A_1 such that $S_1 = (B_1)$. For at least one element a_{i_1} of B_1 there are infinitely many elements $a_i \in A_1$ such that $a_{i_1} \leqslant a_i$, let $A_2 = \{a_i \mid i_1 \leqslant i \wedge a_{i_1} \leqslant a_i\}$. Inductively, we may assume that for $n \geqslant 1$ we already have constructed an increasing sequence $(a_{i_j})_{j \leqslant n}$ and an infinite set $A_{n+1} = \{a_i \mid i_n \leqslant i \wedge a_{i_n} \leqslant a_i\}$. Then we take $S_{n+1} = (A_{n+1})$ and a finite $B_{n+2} \subset A_{n+1}$ with $(B_{n+1}) = S_{n+1}$. Now there has to be at least one element b of B_{n+1} such that for infinitely many elements $a \in A_{n+1}$ we have $b \leqslant a$. Let a_I be one such element, then we let $i_{n+1} = I$ and $A_{n+2} = \{a_i \mid i_{n+1} \leqslant i \ \wedge \ a_{i_{n+1}} \leqslant a_i\}$. Then the sequence $(a_{i_n})_{1 \leqslant n}$ is increasing.

(4)⇒(1): Suppose that for an order $(P, \leqslant)$ and every sequence $a = (a_i)_{1 \leqslant i}$ in P it is possible to extract an increasing sub-sequence from a. If $(P, \leqslant)$ was not Noetherian, then we could find a sequence $(b_i)_{1 \leqslant i}$ which is either strictly decreasing or an anti-chain. But then we cannot find an increasing sub-sequence. This shows that P is Noetherian. □

More equivalent statements can be found in [Mil85] and [vdH97] which can also be taken as references for the rest of this section.

Let $(P, \leqslant)$ be an order. Then w is a **word** in P iff there is an integer $n \in \mathbb{N}$ such that $w \in P^n$. We call n the **length** of w. The only word with length 0 is called the **empty word**. By $P^\star := \bigcup_{n \in \mathbb{N}} P^n$ we denote the set of all words, and $P^\sharp$ denotes the set of non-empty words over P. Let $w \in P^\sharp$ be a word of length $n \geqslant 1$, then we write $w = [w_1, \ldots, w_n]$. Note that for every bijective $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ which is not the identity, $\pi(w) = [w_{\pi(1)}, \ldots, w_{\pi(n)}]$ is again a word of length n, but that $w \neq \pi(w)$. For this reason, $P^\star$ will also be called the set of **non-commutative words**.

We will work with such orders P where the order of a word does not play a role, i.e. we will in general not distinguish between w and $\pi(w)$. To this end, we introduce a relation $\sim_n$ on P^n

as follows. Let $a, b \in P^n$, then $a \sim_n b$ iff there is a bijective $\pi : \{1, \dots, n\} \to \{1, \dots, n\}$ with $a = \pi(b)$. Note that $\sim_n$ is an equivalence relation on P^n. We put

$$P^{\Diamond} := \bigcup_{n \in \mathbb{N}} P^n /_{\sim_n}$$

and call $P^{\Diamond}$ the set of **commutative words** over P. The set of non-empty commutative words is denoted by $P^{\dagger}$. We introduce the relation $\sim$ on $P^{\star}$ by $a \sim b$ iff there is an integer $n \in \mathbb{N}$ with $a, b \in P^n$ and $a \sim_n b$. We remark that $\sim$ is an equivalence relation on $P^{\star}$ and that we have $P^{\star}/_{\sim} = P^{\Diamond}$. For $a \in P^{\star}$ we let $a/_{\sim} = a/_{\sim_n}$, if a has length n.

Example 1.1.2 Let $\mathbb{Z} = (\mathbb{Z}, \leqslant)$, the integers with their usual ordering. Then $[]$, $[2, -56]$, $[45]$ and $[4, 1, 1973]$ are words in $\mathbb{Z}^{\star}$, but only the latter three are in $\mathbb{Z}^{\sharp}$. Although the words $[2, -56]$ and $[-56, 2]$ are distinct elements, we have $[2, -56] \sim_2 [-56, 2]$. Hence $[2, -56]/_{\sim_2} = [-56, 2]/_{\sim_2}$ in $\mathbb{Z}^{\dagger}$.

The ordering $\leqslant$ on P induces orderings $\leqslant_{P^{\star}}$ and $\leqslant_{P^{\Diamond}}$ on $P^{\star}$ and $P^{\Diamond}$ respectively: let $a, b \in P^{\star}$, then $a \leqslant_{P^{\star}} b$ iff $a \in P^n, b \in P^m$ and there is a strictly increasing $\pi : \{1, \dots, n\} \to \{1, \dots, m\}$ such that for all $1 \leqslant i \leqslant n : a_i \leqslant b_{\pi(i)}$. It follows form this definition that whenever $a \leqslant_{P^{\star}} b$, then the length of a is at most the length of b. Let $a/_{\sim}, b/_{\sim} \in P^{\Diamond}$, then $a/_{\sim} \leqslant_{P^{\Diamond}} b/_{\sim}$ iff there is are elements $c \in a/_{\sim}, d \in b/_{\sim}$ with $c \leqslant_{P^{\star}} d$. From this definition we obtain for all $a, b \in P^{\star} : a \leqslant_{P^{\star}} b \Rightarrow a/_{\sim} \leqslant_{P^{\Diamond}} b/_{\sim}$.

Example 1.1.3 We illustrate the above definitions with the following words from $\mathbb{Z}^{\star}$ and $\mathbb{Z}^{\Diamond}$. We denote elements $w/_{\sim}$ by w.

$$\begin{array}{rcl} [5, 4] & \not\leqslant_{\mathbb{Z}^{\star}} & [4, 1, 73] \\ [-11, 72] & \leqslant_{\mathbb{Z}^{\star}} & [4, 1, 73] \\ [9, 74] & \not\leqslant_{\mathbb{Z}^{\Diamond}} & [4, 1, 73] \\ [5, 4] & \leqslant_{\mathbb{Z}^{\Diamond}} & [4, 1, 73]. \end{array}$$

The following lemma is a classical result about Noetherian orders and due to Higman (see [Hig52]).

Lemma 1.1.4 (Higman) *If $(P, \leqslant)$ is an Noetherian order, then so are the orders $(P^{\star}, \leqslant_{P^{\star}})$ and $(P^{\Diamond}, \leqslant_{P^{\Diamond}})$.*

Proof: A concise proof which is based on a proof by Nash-Williams (see [NW63]) can be found in [vdH97]. □

Let P be an ordered multiplicative group with ordering $\leqslant$ and neutral element 1. We suppose that the multiplication is **compatible** with the ordering $\leqslant$, i.e. that for all $a, b, c \in P$ we have $a \leqslant b \Rightarrow ac \leqslant bc$. Since the same holds for the inverse of c, c^{-1}, we also have $ac \leqslant bc \Rightarrow a \leqslant b$.

Also, if $a, b, c, d \in P$ and $a \leqslant b$ and $c \leqslant d$, we obtain $ac \leqslant bc \leqslant bd$, hence $ac \leqslant bd$. In what follows, we will always assume that the multiplication is compatible with the ordering.

Let $\Delta \subseteq P$. If $w = [w_1, \dots, w_n] \in \Delta^n$ is a nonempty word over Δ, then $w_1 \cdots w_n$ is an element in P. We write $\prod w = w_1 \cdots w_n$ in that case. If P is abelian, then $\prod w$ does not depend on the order of the letters, i.e. for every $v \sim w$, we have $\prod w = \prod v$. Hence we can define $\prod$ for elements of $\Delta^\diamond$: if $a/_\sim \in \Delta^\diamond$ and $w \in a/_\sim$, then $\prod a/_\sim := \prod w$. The definition is independent from the choice of the representant w. If P is abelian and $a \in P^\star$, then $\prod a = \prod a/_\sim$. We put $\prod \Delta^\diamond := \{\prod w \mid w \in \Delta^\star\}$. Then $\prod \Delta^\diamond \subseteq P$, and we will write $\Delta^\diamond$ for $\prod \Delta^\diamond$. Note that with this notation we have $(\Delta^\diamond)^\diamond = \Delta^\diamond$.

If $a, b \in P^\star$ and $a \leqslant_{P^\star} b$, then we have a strictly increasing $\pi : \{1, \dots, n\} \to \{1, \dots, m\}$, where n and m are the lengths of a and b respectively, and $\prod a \leqslant b_{\pi(1)} \cdots b_{\pi(n)}$. Now consider the case where for all $i \in \{1, \dots, m\} : 1 \leqslant b_i$. Then $\prod a \leqslant \prod b$. Lemma 1.1.4 applied to this situation gives the following lemma.

Lemma 1.1.5 *Let P be an abelian group with ordering $\leqslant$. Let $S \subseteq P$ be such that $1 < S$ and such that $(S, \leqslant)$ is Noetherian. Then $1 \leqslant S^\diamond$ is Noetherian.*

We will need the following lemma in the next chapters.

Lemma 1.1.6 *Let $(P, \leqslant)$ be a multiplicative order and $A, B \subseteq P$ such that $(A, \leqslant)$ and $(B, \leqslant)$ are Noetherian.*

(1) Then $AB = \{ab \mid a \in A, b \in B\}$ is Noetherian. Note that the same holds for every finite set of Noetherian subsets of P.

(2) Let $q \in P$ and $P(q) = \{(a, b) \mid a \in A, b \in B, ab = q\}$. Then $P(q)$ is finite. The same holds for every finite set of Noetherian subsets of P.

Proof: (1) Let $(p_i)_{i \in I} \subset AB$, and let for every $i \in I$ elements $a_i \in A, b_i \in B$ such that $p_i = a_i b_i$. By Proposition 1.1.1 we find a sub-sequence $(i_n)_{n \in \mathbb{N}}$ of I such that $(a_{i_n})_{n \in \mathbb{N}}$ is increasing. Again by Proposition 1.1.1, we find a sub-sequence $(j_n)_{n \in \mathbb{N}}$ of $(i_n)_{n \in \mathbb{N}}$ such that $(b_{j_n})_{n \in \mathbb{N}}$ is increasing. But then so is $(p_{j_n})_{n \in \mathbb{N}}$. From Proposition 1.1.1 it follows that AB is Noetherian.

(2) The set $P(q) \subseteq AB$ is Noetherian. If it was infinite, then we could choose the sequences from (1) such that at least one of $(a_{j_n})_{n \in \mathbb{N}}$ and $(b_{j_n})_{n \in \mathbb{N}}$ is strictly increasing. But then so is the product, contradiction. □

1.2 Ordered structures

Let K be a ring. We let $K^* := K \setminus \{0\}$ and $K^+ := \{k \in K \mid 0 < k\}$. If $K = (K, \leqslant)$ is an order such that the multiplicative and additive groups on K are ordered groups, then we say that $(K, \leqslant)$ is an ordered ring. The **absolute value** $|a|$ of an element $a \in K$ is a, if $0 \leqslant a$, otherwise $-a$. Note that $|a| = 0$ if and only if $a = 0$ and that for all $a, b \in K$

$$\begin{aligned} |a + b| &\leqslant |a| + |b| \\ |a \cdot b| &= |a| \cdot |b|. \end{aligned}$$

We call a totally ordered ring K **archimedean** iff for all $a, b \in K^*$ there exist integers n, m such that $|a| < |nb|$ and $|b| < |ma|$.

Note that every ring R is a $\mathbb{Z}$-module. We extend the notion of archimedean rings. Let R and K be totally ordered rings such that K is an R-module. Then we define relations $\prec_R$, $\preccurlyeq_R$ and $\asymp_R$ on K as follows. Let $a, b \in K$. Then $a \prec_R b$ iff $\forall r, p \in R : |ra| < |pb|$ and $a \preccurlyeq_R b$ iff $\exists r \in R : |a| < |rb|$. We let $a \asymp_R b$ iff $\exists r, p \in R : |a| < |rb| \wedge |b| < |pa|$. We say that K is **archimedean over** R iff for all $a \in K^* : 1 \asymp_R a$. Note that with this notation archimedean means archimedean over $\mathbb{Z}$.

Lemma 1.2.1 *Let K be an R-module. For all $a, b, c \in K$ we have*

(1) $a \preccurlyeq_R a$, and if $a \prec_R b$, then $a \preccurlyeq_R b$,
(2) if $a \preccurlyeq_R b$ and $b \preccurlyeq_R c$, then $a \preccurlyeq_R c$,
(3) if $a \preccurlyeq_R b$ and $b \preccurlyeq_R a$, then $a \asymp_R b$,

Moreover, if R is an A-module, then K is archimedean over A, if K is archimedean over R and R is archimedean over A. In particular, if R is archimedean and K archimedean over R, then K is archimedean.

Proof: Most of the lemma follows directly from the definitions. If for instance $a \prec_R b$, then take $c = 1$ and d arbitrary to show $|a| < |db|$, hence $a \preccurlyeq_R b$. □

Let G be a multiplicative group. For all $g \in G$ and each integer n there is an element g^n in G. In other words, there is a function $p : \mathbb{Z} \times G \to G$ with $p(n+m, g) = p(n, g) \cdot p(m, g)$ for all integers n, m and all $g \in G$. We generalize this notion to rings R. We say that G is a **group with R-powers** iff there is a function $p : R \times G \to G$ such that

RP1. $\forall g \in G : \forall n \in \mathbb{Z} : p(n, g) = g^n$,
RP2. $\forall g \in G : \forall q, r \in R : p(q + r, g) = p(q, g) \cdot p(r, g)$.

We write $p(r, g) = g^r$ in this case. Now suppose that both G and R are ordered. We say that G is an **ordered group with R-powers** iff in addition to the above for all $r \in R$ and all $g \in G$ we have

$$1 \leqslant g \wedge 0 \leqslant r \Rightarrow 1 \leqslant g^r.$$

Note that in ordered groups with R-powers the function g^r is monotone in r for a fixed g, i.e. for $1 \leqslant g$ and $0 \leqslant r \leqslant s$ we have $g^{-r} \leqslant 1 \leqslant g^r \leqslant g^s$. For every totally ordered field K, the set K^+ of strictly positive elements forms a multiplicative group. We say that K is a totally ordered **field with R-powers** iff K^+ is an ordered group with R-powers.

Let K be a totally ordered field and R a ring such that K is an R-module over R. For all $k \in K^*$ we let $\|k\| = |k|$ iff $1 \preccurlyeq_R k$, otherwise $|k^{-1}|$. Note that for all $k, l \in K^*$ we have $\|k\| \cdot \|l\| \leqslant \|kl\|$. Let K have R-powers. We define the relation $\prec\!\!\prec_R$ as follows. Let $k, l \in K^*$, then we let $k \prec\!\!\prec_R l$ iff there is a $p \in R$ such that for all $r \in R^+$ we have $\|k^r\| < \|l^p\|$. If k are such that for some $r, p \in R^+$ we have $\|k\| < \|l^p\|$ and $\|l\| < \|k^r\|$, then we write $k \asymp\!\!\asymp l$.

1.3 Generalized power series

Notation 1.3.1 Let $f : A \to B$ be a function. We write $f_a = f(a)$ for $a \in A$. If the function is indexed, say $f = f_i$ for some i in an index set I, then we write $f_{i,a} = (f_i)_a$. The set $\operatorname{supp} f = \{a \in A \mid f_a \neq 0\}$ is the **support** of f. We let $\operatorname{term} f = \{f_a a \mid a \in \operatorname{supp} f\}$ be the set of **terms** of f.

Let in what follows $C = (C, +, -, \cdot, 1, 0)$ be a ring. The orders that we will work with are in general multiplicative orders.

Notation 1.3.2 Orders which are groups or semi-groups will henceforth be denoted by fraktur type letters. The ordering will be written as $\succcurlyeq$. Thus, from now on, if we work with $\mathfrak{M}$, then $\mathfrak{M} = (\mathfrak{M}, \succcurlyeq)$ is both an order and a (semi-)group. In view of Warning 1.3.4, we point out that a sequence $(\mathfrak{m}_i)_{0 \leqslant i}$ is well-ordered in $\mathfrak{M}$, if for all $0 \leqslant i < j$ we have $\mathfrak{m}_i \not\succcurlyeq \mathfrak{m}_j$.

Definition 1.3.3 *Let $C \neq \emptyset$ be a ring and $\mathfrak{M}$ an ordered semi-group. Then $f \in C[[(\mathfrak{M}, \succcurlyeq)]] = C[[\mathfrak{M}]]$ is called a* **generalized power series** *over $\mathfrak{M}$ with coefficients in C iff $f : \mathfrak{M} \to C$ is a function such that the support of f is Noetherian in $\mathfrak{M}$, i.e.*

$$C[[\mathfrak{M}]] = \{f : \mathfrak{M} \to C \mid \operatorname{supp} f \text{ is Noetherian}\}.$$

Warning 1.3.4 We will in general write $C[[\mathfrak{M}]]$ instead of the longer $C[[(\mathfrak{M}, \succcurlyeq)]]$ to enhance readability. One should nonetheless keep in mind the ordering of $\mathfrak{M}$. Although we try to make it clear with which ordering we are working, the reader should always be aware of this warning.

Remark 1.3.5 A set S of generalized power series with coefficients in C is **complete** iff

$$S = C[[\bigcup_{f \in S} \operatorname{supp} f]].$$

With this definition, every $C[[\mathfrak{M}]]$ is complete. A set of generalized power series which is not complete will be called **incomplete**. We will mainly work with complete sets of series and only occasionally encounter examples incomplete sets. Therefore, we will in general not mention if a set of series is complete. If a set is incomplete, we will say so.

We embed $\mathfrak{M}$ into $C[[\mathfrak{M}]]$ in a canonical way: let $\mathfrak{m} \in \mathfrak{M}$, then we denote the function $f \in C[[\mathfrak{M}]]$ by $f_\mathfrak{m} = 1$ and $\forall \mathfrak{n} \neq \mathfrak{m} : f_\mathfrak{n} = 0$ also by $\mathfrak{m}$. For $c \in C$ we denote by $c\mathfrak{m}$ the function f with $f_\mathfrak{m} = c$ and $\forall \mathfrak{n} \neq \mathfrak{m} : f_\mathfrak{n} = 0$. For $f \in C[[\mathfrak{M}]]$ we write

$$f = \sum_{\mathfrak{m} \in \mathfrak{M}} f_\mathfrak{m} \mathfrak{m}.$$

Let $\mathfrak{M}$ be an ordered, abelian and multiplicative group with neutral element $1_\mathfrak{M}$ and let $C = (C, \cdot, +, 1_C, 0)$ be a commutative ring. We call $\mathfrak{M}$ the set of **monomials** of the set of generalized power series $C[[\mathfrak{M}]]$. Let

$$\begin{aligned} \mathfrak{M}^\uparrow &:= \{\mathfrak{m} \in \mathfrak{M} \mid \mathfrak{m} \succ 1\} \\ \mathfrak{M}^\downarrow &:= \{\mathfrak{m} \in \mathfrak{M} \mid 1 \succ \mathfrak{m}\}. \end{aligned}$$

Elements from $\mathfrak{M}^{\uparrow}$ are called **infinite monomials**, and elements from $\mathfrak{M}^{\downarrow}$ are **infinitesimal monomials**. Note that $\mathfrak{M}^{\uparrow} \cap \mathfrak{M}^{\downarrow} = \emptyset$ and $\mathfrak{M}^{\uparrow} \cup \mathfrak{M}^{\downarrow} \subseteq \mathfrak{M} \setminus \{1_{\mathfrak{M}}\}$ with equality if and only if $\mathfrak{M}$ is totally ordered. We let

$$\begin{aligned} \mathfrak{M}^{\underline{\uparrow}} &:= \mathfrak{M}^{\uparrow} \cup \{1_{\mathfrak{M}}\} \\ \mathfrak{M}^{\overline{\downarrow}} &:= \mathfrak{M}^{\downarrow} \cup \{1_{\mathfrak{M}}\}. \end{aligned}$$

Then $\mathfrak{M}^{\uparrow}, \mathfrak{M}^{\downarrow}, \mathfrak{M}^{\underline{\uparrow}}$, and $\mathfrak{M}^{\overline{\downarrow}}$ are closed under multiplication. Denote in the following $C[[\mathfrak{M}]]$ by $\mathbb{S}$. Let

$$\begin{aligned} \mathbb{S}^{\uparrow} &:= C[[\mathfrak{M}^{\uparrow}]], \\ \mathbb{S}^{\underline{\uparrow}} &:= C[[\mathfrak{M}^{\underline{\uparrow}}]], \\ \mathbb{S}^{\downarrow} &:= C[[\mathfrak{M}^{\downarrow}]], \\ \mathbb{S}^{\overline{\downarrow}} &:= C[[\mathfrak{M}^{\overline{\downarrow}}]]. \end{aligned}$$

Then $f \in \mathbb{S}^{\uparrow}$ if and only if f has purely infinite support, and f is called a **purely infinite** generalized power series. The elements from $\mathbb{S}^{\downarrow}$, i.e. the generalized power series with purely infinitesimal support, are called **purely infinitesimal** generalized power series.

Example 1.3.6 (1) Let $\mathfrak{M} = (\mathbb{N}, \leqslant_{\mathbb{N}}) = \mathbb{N}$, where $\succcurlyeq\, = \leqslant_{\mathbb{N}}$ denotes the natural ordering on $\mathbb{N}$. Notice that $\mathbb{N}$ is a totally ordered semi-group. Then $f \in C[[\mathbb{N}]]$ if and only if $\operatorname{supp} f$ is $\leqslant_{\mathbb{N}}$-well-ordered. But this is always the case for functions $f : \mathbb{N} \to C$. On the other hand, if we take $\mathfrak{N} = (\mathbb{N}, {}_{\mathbb{N}}\!\geqslant)$, then $f \in C[[\mathfrak{N}]]$ if and only if $\operatorname{supp} f$ is ${}_{\mathbb{N}}\!\geqslant$-well-ordered, i.e. if $\operatorname{supp} f$ is finite which is the case if and only if f is a polynomial in x over C. We will later work with the set of elements f of $\mathbb{Z}[[\mathbb{N}]]$ such that $\operatorname{range} f \subseteq \mathbb{N}$, and we will (abusively) denote this set by $\mathbb{N}[[\mathbb{N}]]$.

(2) Let $x^{\mathbb{Z}} = \{x^n \mid n \in \mathbb{Z}\}$. Let $1 = x^0$ and for integers n, m

$$x^n \cdot x^m = x^{n+m} \qquad \text{and} \qquad (x^n)^{-1} = x^{-n}.$$

Then $(x^{\mathbb{Z}}, \cdot, ^{-1}, 1)$ is an abelian multiplicative group. Note that $\varphi : \mathbb{Z} \to x^{\mathbb{Z}}$ with $\varphi(n) = x^n$ is a group isomorphism between $(\mathbb{Z}, +, -, 0)$ and $(x^{\mathbb{Z}}, \cdot, ^{-1}, 1)$. We define the ordering $\succcurlyeq$ on $x^{\mathbb{Z}}$ by

$$\forall n, m \in \mathbb{Z} : x^n \succcurlyeq x^m \Leftrightarrow n \leqslant m.$$

The ordering $\succcurlyeq$ thus defined is total. Let $x^{\mathbb{Z}} = (x^{\mathbb{Z}}, \succcurlyeq)$ and C a ring. Note that $C[[x^{\mathbb{Z}}]]$ is the set of **Laurent series**. It follows that $f \in C[[x^{\mathbb{Z}}]]$ if and only if there is a $k \in \mathbb{Z}$ and $c_n \in C$ for $n \leqslant k$ such that $f = \sum_{k \geqslant n} c_n x^n$. Purely infinite series in $C[[x^{\mathbb{Z}}]]$ are polynomials in x over C. The purely infinitesimal series in $C[[x^{\mathbb{Z}}]]$ are the formal power series in x^{-1} with constant term 0. Examples:

$$\begin{aligned} f_1 &= 3x^{-5} - x^{-2} + 1 + x + 2^2x^2 + 3^3x^3 + \cdots && \text{(Laurent series)} \\ f_2 &= x^{-12} - 22x^{-3} + 4x^{-1} && \text{(polynomial)} \\ f_3 &= x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \cdots && \text{(formal power series in } x) \end{aligned}$$

(3) Let $x^{\mathbb{N}} = \{x^n \mid n \in \mathbb{N}\} \subseteq x^{\mathbb{Z}}$. Then $x^{\mathbb{N}} = (x^{\mathbb{N}}, \cdot, 1, \succcurlyeq)$ is a totally ordered, multiplicative semi-group. For a ring C the ring of generalized power series $C[[x^{\mathbb{N}}]]$ is the set of formal power series over C. If we consider the totally ordered semi-group $\mathfrak{M} = (x^{\mathbb{N}}, \cdot, 1, \preccurlyeq)$, then $C[[\mathfrak{M}]]$ is the set of polynomials in x over C. This illustrates the need to be aware of Warning 1.3.4.

(4) Let C a ring and $n \in \mathbb{N}$. If $k \in \mathbb{Z}^n$, then we write $k = (k_1, \dots, k_n)$. Let $0 = (0, \dots, 0)$ and for $k, l \in \mathbb{Z}^n$

$$k + l := (k_1 + l_1, \dots, k_n + l_n) \qquad \text{and} \qquad -k := (-k_1, \dots, -k_n).$$

Then $\mathbb{Z}^n = (\mathbb{Z}^n, +, -, 0)$ is an abelian additive group. The set $\mathbb{N}^n = (\mathbb{N}^n, +, 0)$ is an abelian semi-group. Let $k, l \in \mathbb{Z}^n$, then we let $k \leqslant l$ iff $\forall i \leqslant n : k_i \leqslant l_i$. Both $\mathbb{Z}^n$ and $\mathbb{N}^n$ are ordered by $\leqslant$. Let $X = (X_1, \dots, X_n)$ and $k \in \mathbb{Z}^n$, then $X^k = X_1^{k_1} \cdots X_n^{k_n}$. Let $X^{\mathbb{Z}} = \{X^k \mid k \in \mathbb{Z}^n\}$ and $X^{\mathbb{N}} = \{X^k \mid k \in \mathbb{N}^n\}$. Let for all $k, l \in \mathbb{Z}^n$

$$\begin{aligned} 1 &= X^0 \\ X^k \cdot X^l &= X^{k+l} \\ (X^k)^{-1} &= X^{-k}. \end{aligned}$$

Again, $X^{\mathbb{Z}} = (X^{\mathbb{Z}}, \cdot, ^{-1}, 1)$ and $X^{\mathbb{N}} = (X^{\mathbb{N}}, \cdot, ^{-1}, 1)$ are multiplicative, abelian groups and semi-groups respectively. The ordering $\succcurlyeq$ on $X^{\mathbb{Z}}$ is defined by $X^k \succcurlyeq X^l \Leftrightarrow k \leqslant l$. The mapping $\varphi : \mathbb{Z}^n \to X^{\mathbb{Z}}$ with $\varphi(k) = X^k$ is a (semi-)group isomorphism between $\mathbb{Z}^n$ and $X^{\mathbb{Z}}$ and between $\mathbb{N}^n$ and $X^{\mathbb{N}}$ respectively. With $X^{\mathbb{N}} = (X^{\mathbb{N}}, \succcurlyeq)$, the ring $C[[X^{\mathbb{N}}]]$ is called the ring of **formal power series** in n indeterminates over the ring C. The series

$$3 + X + Y + 5XY - 2X^2Y + 3XY^3 + \cdots$$

is therefore a formal power series in two indeterminates X and Y over $\mathbb{Z}$.

1.4 Rings of generalized power series

Let us show how to define a ring structure on $\mathbb{S}$. Let $g, h \in \mathbb{S}$. Then $g + h$ is the function $f : \mathfrak{M} \to C$ such that for all $\mathfrak{m} \in \mathfrak{M} : f_{\mathfrak{m}} = g_{\mathfrak{m}} + h_{\mathfrak{m}}$, i.e.

$$g + h = \sum_{\mathfrak{m} \in \mathfrak{M}} (g_{\mathfrak{m}} + h_{\mathfrak{m}})\mathfrak{m}.$$

Notice that $\operatorname{supp} f \subseteq \operatorname{supp} g \cup \operatorname{supp} h$ is indeed Noetherian. The unique element with empty support is denoted 0; for all $f \in \mathbb{S}$ we obtain $f + 0 = 0 + f = f$. We define for $f \in \mathbb{S}$ the function $-f : \mathfrak{M} \to C$ such that for all $\mathfrak{m} \in \mathfrak{M} : (-f)_{\mathfrak{m}} = -(f_{\mathfrak{m}})$, i.e.

$$-f = \sum_{\mathfrak{m} \in \mathfrak{M}} (-f_{\mathfrak{m}})\mathfrak{m}.$$

Notice that $\operatorname{supp} f = \operatorname{supp}(-f)$, $-f \in \mathbb{S}$ and $-f + f = 0$. Since $(C, +)$ is abelian, so is $(\mathbb{S}, +)$.

For our purposes it will be sufficient to consider commutative rings C. Recall that $(\mathfrak{v},\mathfrak{w}) \in P(\mathfrak{m})$ iff $\mathfrak{v}\mathfrak{w} = \mathfrak{m}$ for all $\mathfrak{v},\mathfrak{w},\mathfrak{m} \in \mathfrak{M}$. We define the function $p = p_{f,g} : \mathfrak{M} \to C$ by

$$p = \sum_{\mathfrak{m}\in\mathfrak{M}} \Big(\sum_{(\mathfrak{v},\mathfrak{w})\in P(\mathfrak{m})} f_{\mathfrak{v}} g_{\mathfrak{w}} \Big) \mathfrak{m}.$$

This function is well defined by (2) of Lemma 1.1.6. By (1) of the same lemma, $\operatorname{supp} p$ is Noetherian. Hence $p_{f,g} \in \mathbb{S}$. Note that $p_{f,g} = p_{g,f}$. We write $p_{f,g} = f \cdot g$ and call $f \cdot g$ the **product** of f and g. The multiplication thus defined is commutative and associative. Let $1 : \mathfrak{M} \to C$ be the function with $\operatorname{supp} 1 = \{1_{\mathfrak{M}}\}$ and $1(1_{\mathfrak{M}}) = 1_C$, then $1 \in \mathbb{S}$ and for all $f \in \mathbb{S} : 1 \cdot f = f \cdot 1 = f$.

Lemma 1.4.1 $(\mathbb{S}, \cdot, +, 1, 0)$ *is a ring.*

Proof: It remains to show that for all $f, g, h \in \mathbb{S}$ we have $f(g+h) = fg + fh$. Let $\mathfrak{m} \in \mathfrak{M}$, then

$$\begin{aligned} f(g+h)_{\mathfrak{m}} &= \sum_{\mathfrak{v}\mathfrak{w}=\mathfrak{m}} f_{\mathfrak{v}}(g+h)_{\mathfrak{w}} = \sum_{\mathfrak{v}\mathfrak{w}=\mathfrak{m}} f_{\mathfrak{v}}(g_{\mathfrak{w}} + h_{\mathfrak{w}}) = \sum_{\mathfrak{v}\mathfrak{w}=\mathfrak{m}} (f_{\mathfrak{v}} g_{\mathfrak{w}} + f_{\mathfrak{v}} h_{\mathfrak{w}}) \\ &= \sum_{\mathfrak{v}\mathfrak{w}=\mathfrak{m}} f_{\mathfrak{v}} g_{\mathfrak{w}} + \sum_{\mathfrak{v}\mathfrak{w}=\mathfrak{m}} f_{\mathfrak{v}} h_{\mathfrak{w}} = (fg)_{\mathfrak{m}} + (fh)_{\mathfrak{m}} \\ &= (fg + fh)_{\mathfrak{m}}. \end{aligned}$$

Hence the lemma. □

Note that for $\mathbb{S} = C[[\mathfrak{M}]]$ the subsets $\mathbb{S}^{\uparrow}, \mathbb{S}^{\downarrow}, \mathbb{S}^{\updownarrow}, \mathbb{S}^{\underline{\updownarrow}}$ are closed under addition and multiplication. If C is a ring, then $\mathbb{S}^{\updownarrow}$ and $\mathbb{S}^{\underline{\updownarrow}}$ are subrings of $\mathbb{S}$.

Remark 1.4.2 Some of the structural properties can be introduced for more general C and $\mathfrak{M}$. Let $P \neq \emptyset$ be an ordered set. We let $C[[P]]$ be set set of functions $f : P \to C$ with Noetherian support. Then $+$ can be defined as above, and $(C[[P]], +, 0)$ is an additive group if and only if $(C, +, 0)$ is an additive group. The equivalence remains true, if we consider abelian groups. Suppose that $\mathfrak{M}$ is multiplicative. In this case, if $(C, \cdot, 1)$ is a commutative, associative semi-group, then so is $(C[[\mathfrak{M}]], \cdot, 1)$.

1.5 Noetherian families

From now on, we will work with generalized power series over rings C. For a finite number of generalized power series $f_1, \ldots, f_n \in \mathbb{S} = C[[\mathfrak{M}]]$ we have defined the sum $f_1 + \cdots + f_n \in \mathbb{S}$. We will extend this notion of addition to larger subsets of $\mathbb{S}$.

Notation 1.5.1 Let $F \subseteq \mathbb{S}$, then we will write $F = (f_i)_{i\in I}$ for an index set I. Let $(X_i)_{i\in I}$ be a family of subsets of a set X. Then we write $\coprod_{i\in I} X_i$ for $\bigcup_{i\in I} X_i$, if the sets X_i are pairwise disjoint. If no confusion can arise, we simply write $\coprod X_i$.

Definition 1.5.2 *$F = (f_i)_{i\in I} \in \mathbb{S}^I$ is a* **Noetherian family** *iff*

(a) $\bigcup_{i\in I} \operatorname{supp} f_i$ is Noetherian in $\mathfrak{M}$.

(b) $\forall \mathfrak{m} \in \mathfrak{M} : |\{i \in I \mid \mathfrak{m} \in \operatorname{supp} f_i\}| < \infty$.

For a Noetherian family $F = (f_i)_{i\in I}$ let $\sum F = \sum_{i\in I} f_i = \sum_{\mathfrak{m}\in\mathfrak{M}}(\sum_{i\in I} f_{i,\mathfrak{m}})\mathfrak{m}$. Note that $\sum F \in \mathbb{S}$.

Every generalized power series $f \in \mathbb{S}$ gives rise to a Noetherian family

$$F_f = (f_{\mathfrak{m}}\mathfrak{m})_{\mathfrak{m}\in\mathfrak{M}}.$$

In this case, we have $\operatorname{supp} \sum F_f = \operatorname{supp} f$. Note that in general $\operatorname{supp} \sum F \subseteq \bigcup_{i\in I} \operatorname{supp} f_i$ for a Noetherian family $F = (f_i)_{i\in I}$. For two families $F = (f_i)_{i\in I}$ and $G = (g_j)_{j\in J}$, the product $F \cdot G$ is the set $\{f_i \cdot g_j \mid i \in I, j \in J\}$. Letting $D = I \times J$ and $h_d = f_i g_j$ for $d = (i, j) \in D$, we can write $F \cdot G = (h_d)_{d\in D}$.

Proposition 1.5.3 *Let $F = (f_i)_{i\in I}$ be Noetherian family in $\mathbb{S}$. Then*

(1) If $J \subseteq I$, then $(f_i)_{i\in J}$ is a Noetherian family.

(2) If $I = \coprod_{j\in J} I_j$ and $g_j = \sum_{i\in I_j} f_i$, then $G = (g_j)_{j\in J}$ is a Noetherian family and $\sum F = \sum G$.

(3) Let F and G be two Noetherian families, then $F \cdot G$ is a Noetherian family and $\sum F \cdot \sum G = \sum(F \cdot G)$.

(4) Multiplication and addition with Noetherian families are commutative, distributive and associative, i.e. for Noetherian families $F, G, H \in \mathbb{S}$ we have

(a) $\sum F \cdot \sum G = \sum G \cdot \sum F$,

(b) $\sum F(\sum G + \sum H) = \sum(FG) + \sum(FH)$,

(c) $\sum F \cdot \sum(GH) = \sum(FG) \cdot \sum H = \sum(FGH)$.

Proof: (1) follows from $\bigcup_{i\in J} \operatorname{supp} f_i \subseteq \bigcup_{i\in I} \operatorname{supp} f_i$. We remark that

$$\operatorname{supp} \sum G \subseteq \bigcup_{j\in J} \operatorname{supp} g_j = \bigcup_{j\in J} \bigcup_{i\in I_j} \operatorname{supp} f_i = \bigcup_{i\in I} \operatorname{supp} f_i$$

shows that $\operatorname{supp} \sum G$ is Noetherian. Let $\mathfrak{m} \in \mathfrak{M}$ and $j \in J$ with $\mathfrak{m} \in \operatorname{supp} g_j$, then there is an $i_j \in I$ with $\mathfrak{m} \in \operatorname{supp} f_{i_j}$. Therefore there can only be finitely many $j \in J$ with $\mathfrak{m} \in \operatorname{supp} g_j$. Hence G is a Noetherian family. To show the equality, let $\mathfrak{m} \in \mathfrak{M}$. Let $\{j_1, \ldots , j_n\} = \{j \in J \mid \mathfrak{m} \in \operatorname{supp} g_j\}$ and for every $1 \leqslant k \leqslant n$, $B_k = \{i \in I_{j_k} \mid \mathfrak{m} \in \operatorname{supp} f_i\}$. Then $\mathfrak{m} \in \operatorname{supp} \sum F$ if and only if $\mathfrak{m} \in \bigcup_{1\leqslant k\leqslant n} B_j$, thus

$$\left(\sum F\right)_{\mathfrak{m}} = \sum_{k=1}^{n} \sum_{i\in B_k} f_{i,\mathfrak{m}} = \sum_{k=1}^{n} g_{j_k,\mathfrak{m}} = \left(\sum G\right)_{\mathfrak{m}}.$$

Hence (2).

Let with the above notations $F \cdot G = H = (h_d)_{d\in D}$, then

$$\operatorname{supp} H \subseteq \bigcup \operatorname{supp} h_d \subseteq (\bigcup \operatorname{supp} f_i)(\bigcup \operatorname{supp} g_j),$$

hence by Lemma 1.1.6, the set $\operatorname{supp} H$ is Noetherian. Fix $\mathfrak{m} \in \mathfrak{M}$. By (2) of Lemma 1.1.6 applied to $S = \bigcup_a \operatorname{supp} f_i$ and $T = \bigcup_b \operatorname{supp} g_j$, there are only finitely many $d \in D$ with $\mathfrak{m} \in \operatorname{supp} h_d$. Thus the first part of (3). Let $F = (f_i)_{i \in I}$ be a Noetherian family and g a generalized power series. Then for $\mathfrak{m} \in \mathfrak{M}$

$$(g \sum F)_\mathfrak{m} = \sum_{\mathfrak{v}\mathfrak{w}=\mathfrak{m}} g_\mathfrak{v} \sum F_\mathfrak{w}.$$

For every $\mathfrak{w}$ the sum $\sum F_\mathfrak{w}$ is finite and we have

$$\sum_{\mathfrak{v}\mathfrak{w}=\mathfrak{m}} g_\mathfrak{v} \sum F_\mathfrak{w} = \sum_{i \in I} \sum_{\mathfrak{v}\mathfrak{w}=\mathfrak{m}} g_\mathfrak{v} f_{i,\mathfrak{w}} = \sum_{i \in I} (g f_i)_\mathfrak{m} = \sum (gF)_\mathfrak{m}.$$

Now for $G = (g_j)_{j \in J}$ we get

$$\sum G \cdot \sum F = \sum_{j \in J} g_j \sum F = \sum_{j \in J} \sum g_j F = \sum_{i \in I} \sum_{j \in J} g_j f_i = \sum_{d \in D} h_d = \sum (FG).$$

Hence (3). (4) follows from (3). □

Citerion 1.5.4 Let $F = (f_i)_{i \in I}$ be a family of series in $\mathbb{S}$. We let

$$S_F := \{(i, \mathfrak{m}) \mid i \in I \ \wedge \ \mathfrak{m} \in \operatorname{supp} f_i\}.$$

We define the strict ordering $\succ$ on S_f by $(i, \mathfrak{m}) \succ (j, \mathfrak{n})$ iff $\mathfrak{m} \succ \mathfrak{n}$. Then F is a Noetherian family if and only if $(S_f, \succcurlyeq)$ is Noetherian.

Proposition 1.5.5 *Let $\mathcal{E} = (\varepsilon_i)_{i \in I}$ be a Noetherian family in $\mathbb{S}$ such that $\varepsilon_i \prec 1$ for all $i \in I$. Then the sequence*

$$(\varepsilon_{i_1} \cdots \varepsilon_{i_n})_{(i_1, \ldots, i_n) \in I^\star}$$

is also a Noetherian family in $\mathbb{S}$.

Proof: Let $S_\mathcal{E}$ and $\succcurlyeq$ be defined as in Criterion 1.5.4. Then $(S_\mathcal{E}, \succcurlyeq)$ is Noetherian. The ordering $\succcurlyeq$ induces an ordering $\succcurlyeq_{S_\mathcal{E}^\star}$ on $S_\mathcal{E}^\star$, which is Noetherian as well by Higman's Theorem. Criterion 1.5.4 then shows the proposition. □

Corollary 1.5.6 *Let $f_1, \ldots, f_n \in \mathbb{S}^\downarrow$ and let $f^k = f_1^{k_1} \cdots f_n^{k_n}$ for $k \in \mathbb{N}^n$. Then $(f^k)_{k \in \mathbb{N}^n}$ is a Noetherian family.*

Proposition 1.5.7 *Let $F = (f_n)_{0 \leqslant n}$ be a Noetherian family in $\mathbb{S}$. Then for all $g \in \mathbb{S}^{\updownarrow}$ the sequence $(f_n \cdot g^n)_{0 \leqslant n}$ is again a Noetherian family.*

Proof: By Criterion 1.5.4, the ordering $(S_f, \succcurlyeq)$ is Noetherian. Since $\operatorname{supp} g \preccurlyeq 1$, the set $(\operatorname{supp} g)^\star$ is Noetherian, too, by Higman's Theorem. Consider the mapping

$$\begin{aligned} \varphi : S \times (\operatorname{supp} g)^\star &\longrightarrow \bigcup_{0 \leqslant n} \{(n, \mathfrak{a}) \mid \mathfrak{a} \in \operatorname{supp} f_n \cdot \prod (\operatorname{supp} g)^\star\} \\ ((n, \mathfrak{m}), (\mathfrak{n}_1, \ldots, \mathfrak{n}_k)) &\longmapsto (n, \mathfrak{m}\mathfrak{n}_1 \cdots \mathfrak{n}_k). \end{aligned}$$

Then φ is strictly increasing and surjective. Hence range φ is a Noetherian set. But then again by Criterion 1.5.4, the sequence $(f_n \cdot g^n)_{0 \leqslant n}$ is a Noetherian family. □

Corollary 1.5.8 *Let $(f_n)_{0\leqslant n}$ be a sequence in $\mathbb{S}$ and $\varepsilon \in \mathbb{S}$ such that $(f_n \cdot \varepsilon^n)_{0\leqslant n}$ is a Noetherian family. Then for every $\delta \preccurlyeq \varepsilon$ in $\mathbb{S}$, the sequence $(f_n \cdot \delta^n)_{0\leqslant n}$ is a Noetherian family.*

Proof: Since $\delta/\varepsilon \preccurlyeq 1$ and

$$\left(f_n \cdot \varepsilon^n \cdot \left(\frac{\delta}{\varepsilon}\right)^n\right)_{0\leqslant n} = (f_n \cdot \delta^n)_{0\leqslant n},$$

the corollary follows from Lemma 1.5.7. □

1.6 Strongly linear algebra

Let $F = (f_i)_{i\in I}$ be a Noetherian family in the ring of generalized power series $\mathbb{S}$. Then we have defined a series $\sum F = \sum_I f_i$ in $\mathbb{S}$. Hence for an index set I, we have a summation operator $\sum_I$ which assigns a series from $\mathbb{S}$ to every Noetherian family which is indexed with I. This section will treat such summation operators in an abstract way.

Let G be an abelian, additive group. Assume that for every index set I we have a partially defined summation operator $\sum_I$ such that if $(x_i)_{i\in I} \in G^I \cap \operatorname{dom} \sum_I$, then $\sum_I (x_i)_{i\in I}$ is an element from G. We say that G is a **strong abelian group** iff $\sum_I$ is totally defined for all finite I and if for all I and $(x_i)_{i\in I} \in G^I$ we have

SA1. if $I = \{i_1, \ldots, i_n\}$ is finite, then $\sum_I (x_i)_{i\in I} = x_{i_1} + \cdots + x_{i_n}$;
SA2. $\sum_I (0)_{i\in I} = 0$ for all I;
SA3. if $(x_i)_{i\in I} \in \operatorname{dom} \sum_I$ and φ is a permutation of I, then $(x_{\varphi(i)})_{i\in I} \in \operatorname{dom} \sum_I$ and

$$\textstyle\sum_I (x_i)_{i\in I} = \sum_I (x_{\varphi(i)})_{i\in I};$$

SA4. if $(x_i)_{i\in I} \in \operatorname{dom} \sum_I$ and $I = \coprod_{j\in J} I_j$, then
- for all $j \in J$: $(x_i)_{i\in I_j} \in \operatorname{dom} \sum_{I_j}$,
- $(\sum_{I_j} (x_i)_{i\in I_j})_{j\in J} \in \operatorname{dom} \sum_J$
- $\sum_J (\sum_{I_j} (x_i)_{i\in I_j})_{j\in J} = \sum_I (x_i)_{i\in I}$.

Remark 1.6.1 We will also write $\sum_I x_i$, if no confusion can arise. Hence the last condition in **SA4** can be written as $\sum_J \sum_{I_j} x_i = \sum_I x_i$. Also, if I is clear from the context, we will use $\sum$ instead of $\sum_I$. Notice that we allow the implication in **SA4** only in one direction. One might think of series $\sum_{I_j} x_i = 1 - 1$. Then for $J = \mathbb{N}$ we have $\sum_J \sum_{I_j} x_i = 0$, but $\sum_I x_i = (1-1) + (1-1) + \cdots$.

A **strong ring** is a ring R which is a strong additive group such that for all index sets I, J and all $(x_i)_{i\in I} \in R^I, (y_j)_{j\in J}$ we have

SR. if $(x_i)_I \in \operatorname{dom} \sum_I$ and $(y_j)_J \in \operatorname{dom} \sum_J$, then $(x_i \cdot y_j)_{I\times J} \in \operatorname{dom} \sum_{I\times J}$ and

$$\sum_{I\times J} x_i \cdot y_j = (\sum_I x_i) \cdot (\sum_J y_j).$$

If R is a strong ring and $(x_i)_{i\in I} \in R^I \cap \operatorname{dom} \sum_I$, then for all $r \in R$ we have $(r \cdot x_i)_{i\in I} \in \operatorname{dom} \sum_I$ and $r \cdot \sum_I x_i = \sum_i r \cdot x_i$. Notice that this implies that the converse direction of condition **SR** is true: let I, J be index sets such that $(x_i \cdot y_i)_{I\times J} \in \operatorname{dom} \sum_{I\times J}$. By **SA4**, for all $j \in J$ the set $(x_i \cdot y_j)_{i\in I}$ is in the domain of $\sum_I$. From **SR** follows $\sum_I x_i \cdot y_j = y_j \cdot \sum_I x_i$. We have $\sum x_i \in R$, thus by **SR**

Let R be a ring and M be an R-module. Summation operators in R and M are denoted by $\sum_{I,R}$ and $\sum_{I,M}$ respectively. We say that M is a **strong R-module** iff R is a strong ring, if the additive group of M is strong and if

SM1. for all $(x_i)_{i\in I} \in \operatorname{dom} \sum_{I,M}$ and all $r \in R$ we have $(r \cdot x_i)_{i\in I} \in \operatorname{dom} \sum_{I,M}$ and

$$\sum_{I,M} r \cdot x_i = r \cdot \sum_{I,M} x_i.$$

SM2. for all $(r_i)_{i\in I} \in \operatorname{dom} \sum_{I,R}$ and all $x \in M$ we have $(r_i \cdot x)_{i\in I} \in \operatorname{dom} \sum_{I,M}$ and

$$(\sum_{I,R} r_i) \cdot x = \sum_{I,M} r_i \cdot x.$$

A **strong algebra** is an R-algebra A together with infinite summation symbols, such that A is a strong ring and a strong R-module. Let us apply these definitions to rings of generalized power series. Let C be a ring. We define $\sum_{I,C}$ on C if and only if I is finite. We can do this for each ring R, and call this the **trivial strong ring structure** of R. Note that $\mathbb{S} = C[[\mathfrak{M}]]$ is a C-algebra. For families $F = (f_i)_{i\in I}$ we define $\sum_{I,\mathbb{S}}$ if and only if F is Noetherian, and in this case we let $\sum_{I,\mathbb{S}} F$ be defined as in Section 1.5. We now prove

Proposition 1.6.2 *With the above definitions of $\sum_I$ in C and $\mathbb{S}$ the field $C[[\mathfrak{M}]]$ is a strong C-algebra.*

Proof: Since $\sum_{I,C}$ is only defined for finite I, the ring C is strong. We have to show that $\mathbb{S}$ is both a strong ring and a strong C-module.

The conditions **SA1**, **SA2** and **SA3** need no comment. The condition **SA4** follows from (2) of Proposition 1.5.3. Thus $C[[\mathfrak{M}]]$ is a strong abelian group. The condition **SR** follows from (3) of the same proposition. Hence $\mathbb{S}$ is a strong ring.

Finally, we show that $\mathbb{S}$ is a strong C-module. The condition **SM1** is a special case of **SR**. So is condition **SM2**, since C has the trivial strong ring structure. $\square$.

Let M and N be two strong R-modules. A mapping $\varphi : M \to N$ is a **strongly linear** mapping iff it preserves the infinite summation symbols, i.e. for all I and all $(x_i)_{i\in I} \in \operatorname{dom} \sum_{I,M}$ we have

SL1. $(\varphi(x_i))_{i\in I} \in \operatorname{dom} \sum_{I,N}$ and
SL2. $\varphi(\sum_I x_i) = \sum_I \varphi(x_i)$.

We remark that strongly linear mappings are linear and that $\varphi(\sum_I r_i \cdot x_i) = \sum_I r_i\varphi(x_i)$ for all $(r_ix_i)_{i\in I}$ in the domain of $\sum_{I,M}$. We will consider strong linear mappings between rings of generalized power series.

Proposition 1.6.3 *Let $C[[\mathfrak{M}]]$ and $C[[\mathfrak{N}]]$ be two rings of generalized power series. Let*

$$\varphi : \mathfrak{M} \longrightarrow C[[\mathfrak{N}]]$$

be a **Noetherian mapping***, i.e. a mapping such that the image of a Noetherian set in $\mathfrak{M}$ is a Noetherian family in $C[[\mathfrak{N}]]$. Then φ extends uniquely to a strongly linear mapping $\hat{\varphi} : C[[\mathfrak{M}]] \to C[[\mathfrak{N}]]$.*

Proof: Let $f \in C[[\mathfrak{M}]]$. Then $\operatorname{supp} f$ is Noetherian, thus $(\varphi(\mathfrak{m}))_{\mathfrak{m}\in\operatorname{supp} f}$ is a Noetherian family. So is $F = (f_{\mathfrak{m}}\varphi(\mathfrak{m}))_{\mathfrak{m}\in\operatorname{supp} f}$. We will prove that the mapping

$$\hat{\varphi} : f \longmapsto \sum_{\mathfrak{m}\in\operatorname{supp} f} f_{\mathfrak{m}}\varphi(\mathfrak{m})$$

is the only strong linear mapping which coincides with φ on $\mathfrak{M}$.

We first show that $\hat{\varphi}$ is a strong linear mapping. Let $F = (f_i)_{i\in I}$ be in $\operatorname{dom} \sum_I$. In other words, F is a Noetherian family. Note that $\operatorname{supp} \sum F$ is contained in $S = \bigcup_{i\in I} \operatorname{supp} f_i$ which is a Noetherian set. We claim that $(f_{i,\mathfrak{m}}\varphi(\mathfrak{m}))_{(i,\mathfrak{m})\in I\times S}$ is a Noetherian family in $C[[\mathfrak{N}]]$. Since F is a Noetherian family, we have

$$\bigcup_{(i,\mathfrak{m})\in I\times S} \operatorname{supp} f_{i,\mathfrak{m}}\varphi(\mathfrak{m}) \subseteq \varphi(S),$$

which is Noetherian by our hypothesis about φ. Furthermore, given $\mathfrak{n} \in \mathfrak{N}$, the set $\{\mathfrak{m} \in S \,|\, \varphi(\mathfrak{m})_{\mathfrak{n}} \neq 0\}$ is finite, since $(\varphi(\mathfrak{m}))_{\mathfrak{m}\in S}$ is a Noetherian family. Finally, for each $\mathfrak{m} \in S$ with $\varphi(\mathfrak{m})_{\mathfrak{n}} \neq 0$ the set $\{i \in I \,|\, f_{i,\mathfrak{m}} \neq 0\}$ is also finite, since F is a Noetherian family. Hence the set

$$\{(i, m) \in I \times S \,|\, f_{i,\mathfrak{m}}\varphi(\mathfrak{m}) \neq 0\}$$

is finite. This shows our claim.

Now the claim together with **SA4** proves that $(\hat{\varphi}(f_i))_{i\in I} = (\sum_{\mathfrak{m}\in S} f_{i,\mathfrak{m}}\varphi(\mathfrak{m}))_{i\in I}$ is a Noetherian family and that

$$\textstyle\sum_I \hat{\varphi}(f_i) = \sum_I \displaystyle\sum_{\mathfrak{m}\in S} f_{i,\mathfrak{m}}\varphi(\mathfrak{m}) = \sum_{(i,\mathfrak{m})\in I\times S} f_{i,\mathfrak{m}}\varphi(\mathfrak{m}) = \sum_{m\in S}\sum_{i\in I} f_{i,\mathfrak{m}}\varphi(\mathfrak{m}) = \hat{\varphi}\left(\textstyle\sum_I f_i\right).$$

This shows the strong linearity of $\hat{\varphi}$.

In order to show that $\hat{\varphi}$ is unique, it suffices to remark that for each $f \in C[[\mathfrak{M}]]$ we must have $\hat{\varphi}(f_{\mathfrak{m}}\mathfrak{m}) = f_{\mathfrak{m}}\varphi(\mathfrak{m})$ by linearity and $\hat{\varphi}(f) = \sum_{\operatorname{supp} f} f_{\mathfrak{m}}\varphi(\mathfrak{m})$ by strong linearity. □

Corollary 1.6.4 *Let $\varphi : \mathfrak{M} \to C[[\mathfrak{N}]]$ and $\psi : \mathfrak{N} \to C[[\mathfrak{V}]]$ be two mappings as in Proposition 1.6.3 and $\hat{\varphi}$ and $\hat{\psi}$ their unique extensions to $C[[\mathfrak{M}]]$ and $C[[\mathfrak{N}]]$ as strong linear mappings. Note that $\hat{\psi} \circ \varphi : \mathfrak{M} \to C[[\mathfrak{V}]]$ is a mapping such that the image of Noetherian sets in $\mathfrak{M}$ are Noetherian families in $C[[\mathfrak{V}]]$. Then*

$$\widehat{\hat{\psi} \circ \varphi} = \hat{\psi} \circ \hat{\varphi}.$$

Proof: Note that $\hat{\psi}\circ\hat{\varphi}$ is a strong linear mapping extending $\hat{\psi}\circ\varphi$. Then the corollary follows from the uniqueness. $\square$

Let us finally give an application of strong linearity as a tool. Let $\mathbb{S} = C[[\mathfrak{M}]]$ be a ring of generalized power series and $f_1, \dots, f_n \in \mathbb{S}^{\downarrow}$. For $k \in \mathbb{N}^n$ we let $f^k = f_1^{k_1} \cdots f_n^{k_n}$. Corollary 1.5.6 implies that $(f^k)_{k\in\mathbb{N}^n}$ is a Noetherian family. Hence for every formal power series $g \in C[[X^{\mathbb{N}}]]$ in n indeterminates, the family $(g_k f^k)_{k\in\mathbb{N}^n}$ is Noetherian. We denote its sum by $g \circ \bar{f}$. If $g_0 = 0$, then $g \circ \bar{f} \in \mathbb{S}^{\downarrow}$.

Now let $g_1, \dots, g_m \in C[[X^{\mathbb{N}}]]^{\downarrow}$ be formal power series without absolute term. Then $g_i \circ \bar{f} \in \mathbb{S}^{\downarrow}$ for $1 \leqslant i \leqslant m$. On the other hand, for every formal power series h in m indeterminates over C the above implies that $h \circ (g_1, \dots g_m) = h \circ \bar{g}$ is a formal power series in n indeterminates with coefficients in C.

This way we get two series $h \circ (g_1 \circ \bar{f}, \dots, g_m \circ \bar{f}) = h \circ (\bar{g} \circ \bar{f})$ and $(h \circ \bar{g}) \circ \bar{f}$ in $\mathbb{S}$. We want to show that they are identical. Instead of writing down each series g_i and h and evaluating every term in the development, we use the strong linearity.

Lemma 1.6.5 *Let $C[[\mathfrak{M}]]$ and $C[[\mathfrak{N}]]$ be two rings of generalized power series and*

$$\varphi : \mathfrak{M} \longrightarrow C[[\mathfrak{N}]]$$

be a mapping such that the image of a Noetherian set in $\mathfrak{M}$ is a Noetherian family in $C[[\mathfrak{N}]]$.
(1) If φ preserves multiplication, then so does its unique strong linear extension $\hat{\varphi}$.
(2) If $\varphi(\mathfrak{m}\,\mathfrak{n}) = \varphi(\mathfrak{m}) \cdot \mathfrak{n} + \mathfrak{m} \cdot \varphi(\mathfrak{n})$ for all $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$, then $\hat{\varphi}(f\,g) = \hat{\varphi}(f) \cdot g + f \cdot \hat{\varphi}(g)$ for all $f, g \in C[[\mathfrak{M}]]$.

Proof: (1) We have for all $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$ that $\varphi(\mathfrak{m}\mathfrak{n}) = \varphi(\mathfrak{m}) \cdot \varphi(\mathfrak{n})$. Let $\hat{\varphi}$ be the unique strong linear extension of φ to $C[[\mathfrak{M}]]$. Fix $f, g \in C[[\mathfrak{M}]]$. We have to show $\hat{\varphi}(fg) = \hat{\varphi}(f) \cdot \hat{\varphi}(g)$.

For $\mathfrak{m} \in \mathfrak{M}$ we have by strong linearity

$$\hat{\varphi}(\mathfrak{m}g) = \hat{\varphi}(\mathfrak{m} \sum_{\mathfrak{n}\in\mathfrak{M}} g_{\mathfrak{n}}\mathfrak{n}) = \sum_{\mathfrak{n}\in\mathfrak{M}} g_{\mathfrak{n}} \cdot \varphi(\mathfrak{m}\mathfrak{n}) = \sum_{\mathfrak{n}\in\mathfrak{M}} g_{\mathfrak{n}} \cdot \varphi(\mathfrak{m}) \cdot \varphi(\mathfrak{n}).$$

From the multiplication in $\mathbb{S}$ follows

$$\sum_{\mathfrak{M}} g_{\mathfrak{n}} \cdot \varphi(\mathfrak{m}) \cdot \varphi(\mathfrak{n}) = \varphi(\mathfrak{m}) \cdot \sum_{\mathfrak{m}\in\mathfrak{M}} g_{\mathfrak{n}} \cdot \varphi(\mathfrak{n}) = \varphi(\mathfrak{m}) \cdot \hat{\varphi}(g).$$

Thus $\hat{\varphi}(\mathfrak{m}g) = \varphi(\mathfrak{m}) \cdot \hat{\varphi}(g)$. From the strong linearity we now obtain

$$\hat{\varphi}(fg) = \hat{\varphi}\left(\sum_{\mathfrak{m}\in\mathfrak{M}} f_{\mathfrak{m}}\mathfrak{m} \cdot g\right) = \sum_{\mathfrak{m}\in\mathfrak{M}} f_{\mathfrak{m}}\varphi(\mathfrak{m}) \cdot \hat{\varphi}(g) = \hat{\varphi}\left(\sum_{\mathfrak{m}\in\mathfrak{M}} f_{\mathfrak{m}}\mathfrak{m}\right) \cdot \hat{\varphi}(g).$$

This shows $\hat{\varphi}(fg) = \hat{\varphi}(f) \cdot \hat{\varphi}(g)$.

(2) For $\mathfrak{m} \in \mathfrak{M}$ and $g \in C[[\mathfrak{M}]]$ we have by strong linearity

$$\begin{aligned} \hat{\varphi}(\mathfrak{m}\,g) &= \hat{\varphi}(\mathfrak{m} \sum_{\mathfrak{n}} g_{\mathfrak{n}}\,\mathfrak{n}) = \sum_{\mathfrak{n}} g_{\mathfrak{n}}\,\varphi(\mathfrak{m}\,\mathfrak{n}) = \sum_{\mathfrak{n}} g_{\mathfrak{n}}\varphi(\mathfrak{m}) \cdot \mathfrak{n} + \sum_{n} g_{\mathfrak{n}}\,\mathfrak{m} \cdot \varphi(\mathfrak{n}) \\ &= \varphi(\mathfrak{m}) \cdot g + \mathfrak{m} \cdot \hat{\varphi}(g). \end{aligned}$$

Then

$$\begin{aligned}\hat{\varphi}(f\,g) &= \hat{\varphi}(\sum_{\mathfrak{m}} f_{\mathfrak{m}}\mathfrak{m}\,g) = \sum_{\mathfrak{m}} f_{\mathfrak{m}}\hat{\varphi}(\mathfrak{m}\,g) = \sum_{\mathfrak{m}} f_{\mathfrak{m}}(\varphi(\mathfrak{m})\cdot g + \mathfrak{m}\cdot\hat{\varphi}(g)) \\ &= (\sum_{\mathfrak{m}} f_{\mathfrak{m}}\varphi(\mathfrak{m}))\cdot g + (\sum_{\mathfrak{m}} f_{\mathfrak{m}}\mathfrak{m})\cdot\hat{\varphi}(g) = \hat{\varphi}(f)\cdot g + f\cdot\hat{\varphi}(g).\end{aligned}$$

□

Proposition 1.6.6 *Let $n, m \in \mathbb{N}$ and $\mathbb{S} = C[[\mathfrak{M}]]$ be a ring of generalized power series. Let $f_1, \dots, f_n \in \mathbb{S}^{\downarrow}$ and $g_1, \dots, g_m \in C[[X_1^{\mathbb{N}} \cdots X_n^{\mathbb{N}}]]$ be formal power series without absolute term. Let $h \in C[[X_1^{\mathbb{N}} \cdots X_m^{\mathbb{N}}]]$ and $g_i \circ \bar{f} := g_i(f_1, \dots, f_n)$ and $h \circ \bar{g} := h(g_1, \dots, g_m)$. Then $(h \circ \bar{g}) \circ \bar{f} = h \circ (\bar{g} \circ \bar{f})$.*

Proof: Fix $f_1, \dots, f_n$ and $g_1, \dots, g_m$ as above. We define mappings

$$\begin{aligned}\varphi : X_1^{\mathbb{N}} \cdots X_m^{\mathbb{N}} &\longrightarrow C[[X_1^{\mathbb{N}} \cdots X_n^{\mathbb{N}}]] \\ \psi : X_1^{\mathbb{N}} \cdots X_n^{\mathbb{N}} &\longrightarrow C[[\mathfrak{M}]] \\ \vartheta : X_1^{\mathbb{N}} \cdots X_m^{\mathbb{N}} &\longrightarrow C[[\mathfrak{M}]]\end{aligned}$$

by

$$\begin{aligned}\varphi(X^k) &:= g_1^{k_1} \cdots g_m^{k_m} = X^k \circ \bar{g} \\ \psi(X^l) &:= f_1^{l_1} \cdots f_n^{l_n} = X^l \circ \bar{f} \\ \vartheta(X^k) &:= \varphi(\bar{g}) \circ \bar{f}.\end{aligned}$$

We remark that φ, ψ and ϑ are multiplicative, and by Corollary 1.5.6 they extent uniquely to strongly linear operators

$$\begin{aligned}\hat{\varphi} : C[[X_1^{\mathbb{N}} \cdots X_m^{\mathbb{N}}]] &\longrightarrow C[[X_1^{\mathbb{N}} \cdots X_n^{\mathbb{N}}]] \\ \hat{\psi} : C[[X_1^{\mathbb{N}} \cdots X_n^{\mathbb{N}}]] &\longrightarrow C[[\mathfrak{M}]],\end{aligned}$$

which are also multiplicative. Then $\hat{\psi} \circ \hat{\varphi} : C[[X_1^{\mathbb{N}} \cdots X_m^{\mathbb{N}}]] \to C[[\mathfrak{M}]]$ is strongly linear and multiplicative. Thus $\hat{\vartheta} = \hat{\psi} \circ \hat{\varphi}$. Now let h as above, then

$$h \circ (\bar{g} \circ \bar{f}) = \hat{\vartheta}(h) = \sum_{k \in \mathbb{N}^m} h_k \cdot \hat{\psi} \circ \hat{\varphi}(X^k).$$

By strong linearity, this means

$$h \circ (\bar{g} \circ \bar{f}) = \hat{\psi}\left(\sum_{k \in \mathbb{N}^m} h_k \cdot \hat{\varphi}(X^k)\right) = \left(\sum_{k \in \mathbb{N}^m} h_k \cdot \hat{\varphi}(X^k)\right) \circ \bar{f}.$$

From $\sum_{k \in \mathbb{N}^m} h_k \cdot \hat{\varphi}(X^k) = \hat{\varphi}(h) = h \circ \bar{g}$ we then obtain

$$h \circ (\bar{g} \circ \bar{f}) = (h \circ \bar{g}) \circ \bar{f}.$$

This shows the proposition. □

1.7 Totally ordered supports

In this section, we will work with sets of generalized power series $\mathbb{S} = C[[\mathfrak{M}]]$ such that both C and $\mathfrak{M}$ are totally ordered. In this case, we introduce some canonical representations of generalized power series from $\mathbb{S}$ as well as two orderings on this set.

1.7.1 Representations of generalized power series

Since $\mathfrak{M}$ is totally ordered, we obtain $\mathfrak{M} = \mathfrak{M}^{\uparrow} \cup \{1_{\mathfrak{M}}\} \cup \mathfrak{M}^{\downarrow}$. For every subset $S \subseteq \mathfrak{M}$ there are uniquely determined sets $S^{\uparrow} \subseteq \mathfrak{M}^{\uparrow}$, $S^{\downarrow} \subseteq \mathfrak{M}^{\downarrow}$, and $S^{C} \subseteq \{1_{\mathfrak{M}}\}$ such that $S = S^{\uparrow} \cup S^{C} \cup S^{\downarrow}$. Let $f \in \mathbb{S}$, then $\operatorname{supp} f = (\operatorname{supp} f)^{\uparrow} \cup (\operatorname{supp} f)^{C} \cup (\operatorname{supp} f)^{\downarrow}$. We let $f^{\uparrow}$ be the generalized power series with support $(\operatorname{supp} f)^{\uparrow}$ and $\forall \mathfrak{m} \in (\operatorname{supp} f)^{\uparrow} : f_{\mathfrak{m}} = f^{\uparrow}_{\mathfrak{m}}$, i.e.

$$f^{\uparrow} = \sum_{\mathfrak{m} \in \mathfrak{M}^{\uparrow}} f_{\mathfrak{m}} \mathfrak{m}.$$

Similarly, we define f^{C} and $f^{\downarrow}$. We let $f^{↥} = f^{\uparrow} + f^{=}$ and $f^{↧} = f^{\downarrow} + f^{=}$. Then

$$\begin{aligned} f &= f^{\uparrow} + f^{=} + f^{\downarrow} \\ &= f^{↥} + f^{\downarrow} \\ &= f^{\uparrow} + f^{↧}. \end{aligned}$$

We remark that $f^{\uparrow}, f^{↥}, f^{=}, f^{\downarrow}$ and $f^{↧}$ are uniquely determined. We call $f^{\uparrow}, f^{=}$ and $f^{\downarrow}$ the **purely infinite, constant** and **purely infinitesimal part** of f. Note that $f^{\uparrow} \in \mathbb{S}^{\uparrow}$ and $f^{\downarrow} \in \mathbb{S}^{\downarrow}$.

Example 1.7.1 We illustrate the above notations with an example from $C[[x^{\mathbb{Z}}]]$. Let $f = 3x^{-5} - x^{-2} + 1 + x + 2^2x^2 + 3^3x^3 + \cdots$. Then

$$\begin{aligned} f^{\uparrow} &= 3x^{-5} - x^{-2} \\ f^{↥} &= 3x^{-5} - x^{-2} + 1 \\ f^{=} &= 1 \\ f^{↧} &= 1 + x + 2^2x^2 + 3^3x^3 + \cdots \\ f^{\downarrow} &= x + 2^2x^2 + 3^3x^3 + \cdots . \end{aligned}$$

There is a second canonical representation of a generalized power series $f \in \mathbb{S}$. We call $\min \operatorname{supp} f$ the **leading monomial** of f and denote it by $\mathfrak{d}_f$. We let $c_f = f_{\mathfrak{d}_f}$ and call c_f the **leading coefficient** of f. The series $\tau_f = c_f \mathfrak{d}_f$ is called the **leading term** of f. Let the **remainder** be the series R_f with $\operatorname{supp} R_f = \operatorname{supp} f \setminus \{\mathfrak{d}_f\}$ and $\forall \mathfrak{m} \in \operatorname{supp} R_f : R_{f,\mathfrak{m}} = f_{\mathfrak{m}}$, i.e.

$$R_f = \sum_{\mathfrak{m} \in \operatorname{supp} f \setminus \{\mathfrak{d}_f\}} f_{\mathfrak{m}} \mathfrak{m}.$$

Then $f = \tau_f + R_f$ and $\mathfrak{d}_f \succ \operatorname{supp} R_f$. The function $\delta_f : \mathfrak{M} \to C$ which is defined by

$$\delta_f = \sum_{\mathfrak{m} \in \operatorname{supp} f \setminus \{\mathfrak{d}_f\}} (f_{\mathfrak{m}} / c_f) \cdot \mathfrak{m} / \mathfrak{d}_f$$

is in $\mathbb{S}$, and we have $f = c_f \mathfrak{d}_f (1+\delta_f)$ and $c_f \mathfrak{d}_f \delta_f = R_f$. Note that $\operatorname{supp} \delta_f = \{\mathfrak{m}/\mathfrak{d}_f \mid \mathfrak{m} \in \operatorname{supp} R_f\}$ contains only infinitesimal elements. In later chapters, we will frequently use this product representation for elements $f \in \mathbb{S}$, and we will write $f = c\mathfrak{d}(1+\delta)$ instead of $f = c_f \mathfrak{d}_f (1+\delta_f)$, if no confusion can arise. We summarize:

$$\begin{aligned} f &= c_f \mathfrak{d}_f (1+\delta_f) \\ &= c_f \mathfrak{d}_f + R_f \\ &= \tau_f (1+\delta_f) \\ &= \tau_f + R_f. \end{aligned}$$

Example 1.7.2 We use the series f from Example 1.7.1. Then

$$\begin{aligned} \mathfrak{d}_f &= x^{-5} \\ c_f &= 3 \\ R_f &= -x^{-2} + 1 + x + 2^2 x^2 + 3^3 x^3 + \cdots \\ \delta_f &= -\frac{1}{3} x^3 + \frac{1}{3} x^5 - \frac{1}{3} x^6 + \frac{2^2}{3} x^7 + \frac{3^3}{3} x^8 + \cdots \end{aligned}$$

Example 1.7.3 We will use Proposition 1.6.6 mainly to show equalities between formal power series that will be defined in different ways. Let us give an example of what we mean by this. Assume that C is a field and that $\mathbb{S} = C[[\mathfrak{M}]]$. Let $f \in \mathbb{S}$ and $f = c\mathfrak{d}(1+\delta)$ as above. Then $(\delta^i)_{i \geqslant 0}$ is a Noetherian family, and for $F(x) = \sum_{0 \leqslant i} X^i$ we know that $1 = (1+X)F(-X)$. From Lemma 1.6.6 we now get $1 = (1+\delta)F(-\delta)$ and $F(-\delta) \in \mathbb{S}^{\downarrow}$. Since $c^{-1} \in C$ and $\mathfrak{d}^{-1} \in \mathfrak{M}$, we obtain $(c\mathfrak{d})^{-1} F(-\delta) \cdot f = 1$. Therefore, there exists a multiplicative inverse in $\mathbb{S}$. We have proved:

Corollary 1.7.4 $(C[[\mathfrak{M}]], +, \cdot, 1, 0)$ *is a field if and only if* C *is a field.*

Let $\mathbb{S}^+ := \{f \in \mathbb{S} \mid 0 < f\}$ and $\mathbb{S}_\infty := \{f \in \mathbb{S} \mid \mathfrak{d}_f \in \mathfrak{M}^{\uparrow}\}$. Elements from $\mathbb{S}_\infty$ are called **infinite** series. We let $\mathbb{S}_\infty^+ := \mathbb{S}_\infty \cap \mathbb{S}^+$. Note that $\mathbb{S}^{\uparrow} \subseteq \mathbb{S}_\infty$. The series f from Example 1.7.1 is positive, infinite, i.e. $f \in C[[x^{\mathbb{Z}}]]_\infty^+$. The remainder R_f is still infinite, but not positive. The series δ_f is neither positive nor infinite.

1.7.2 Lexicographic orderings

The total orders C and $\mathfrak{M}$ give rise to a total ordering on the set of generalized power series $\mathbb{S}$.

Let $\succcurlyeq$ and $\leqslant$ be the total orderings on the sets $\mathfrak{M}$ and C respectively. For each series $0 \neq f \in \mathbb{S} = C[[\mathfrak{M}]]$ we let

$$0 < f \quad \text{iff} \quad 0 < c_f.$$

If $f, g \in \mathbb{S}$, then we let $g < f$ iff $0 < f - g$. Note that $\mathbb{S}$ is totally ordered by $\leqslant$. We call this ordering the **lexicographic ordering** of $\mathbb{S}$, and we will also use the symbol $\leqslant_{\text{lex}}$ to denote this ordering. For $f, g \in \mathbb{S}$ with $f <_{\text{lex}} g$ we also say that f is **lexicographically smaller** than g.

Example 1.7.5 Take the ring of generalized power series $C[[x^{\mathbb{Z}}]]$, where C is a totally ordered ring and $x^{\mathbb{Z}}$ totally ordered by $\succcurlyeq$ as in Example 1.3.6. For the series

$$\begin{aligned} f_1 &= 2x^{-1}+5+x+x^2+\cdots \\ f_2 &= 2x^{-1}+4+x+x^2+\cdots \\ f_3 &= x^{-1}+4+x+x^2+\cdots \\ f_4 &= -x^{-2} \\ f_5 &= x^{-2} \end{aligned}$$

we obtain $f_4 <_{\text{lex}} f_3 <_{\text{lex}} f_2 <_{\text{lex}} f_1 <_{\text{lex}} f_5$.

Remark 1.7.6 The ring $(\mathbb{S}, \leqslant_{\text{lex}})$ is a C-module and we have relations $\prec_C$, $\preccurlyeq_C$ and $\asymp_C$ on $\mathbb{S}$ as above. For two $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$ with $\mathfrak{m} \succ \mathfrak{n}$ we obtain $|c\mathfrak{n}| <_{\text{lex}} |d\mathfrak{m}|$ for all $0 \neq c, d \in C$. It follows that $\mathfrak{n} \prec_C \mathfrak{m}$. Hence, we remark that $\succ_C|_{\mathfrak{M}} = \succ$, $\succcurlyeq_C|_{\mathfrak{M}} = \succcurlyeq$ and $\asymp_C|_{\mathfrak{M}} = \mathrm{id}_{\mathfrak{M}}$. Since $\succ_C, \succcurlyeq_C$ are extensions of $\succ, \succcurlyeq$ on $\mathfrak{M}$, we denote them by $\succ$ and $\succcurlyeq$ as well. Note, however, that $\preccurlyeq$ is not a total ordering on $\mathbb{S}$. Take for instance $f = x + x^2$ and $g = x$ in $C[[x^{\mathbb{Z}}]]$, then $f \preccurlyeq g$ and $g \preccurlyeq f$, but $f \neq g$. In fact, we have

$$f \succ g \Leftrightarrow \mathfrak{d}_f \succ \mathfrak{d}_g \qquad \text{and} \qquad f \asymp g \Leftrightarrow \mathfrak{d}_f = \mathfrak{d}_g.$$

To show this, we may assume that f, g are positive. Let $f \prec g$. Then let $c, d \in C$ such that $d > 0$ and $c \cdot c_g - d \cdot c_f > 0$. If $\mathfrak{d}_f \preccurlyeq \mathfrak{d}_g$, then this implies $0 < df < cg$. We remark that the second equivalence can be shown similarly.

An element $f \in \mathbb{S} = C[[\mathfrak{M}]]$ is said to be **weakly decreasing** iff $\forall \mathfrak{m}, \mathfrak{n} \in \operatorname{supp} f : \mathfrak{m} \succcurlyeq \mathfrak{n} \Leftrightarrow f_{\mathfrak{m}} \geqslant f_{\mathfrak{n}} \geqslant 0$ (in C). The set of weakly decreasing generalized power series will be denoted by $\mathbb{S}^{wd}$. For sets $S \subseteq \mathbb{S}$ we let $S^{wd} := S \cap \mathbb{S}^{wd}$. For instance, from the series in Example 1.7.5 only the series $f_5 = x^{-2}$ is a weakly decreasing series. Other examples in the ring $C[[x^{\mathbb{Z}}]]$ are $x + x^2 + x^3 + \cdots$ or $5x^{-4} + 4x^{-3} + 3x^{-2} + 2x^{-1} + 1$.

Lemma 1.7.7 *Let C and $\mathfrak{M}$ be well-ordered. Then $(C[[\mathfrak{M}]]^{wd}, \leqslant_{lex})$ is well-ordered.*

Proof: Suppose that $(f_i)_{0 \leqslant i}$ is a strictly decreasing sequence in $(C[[\mathfrak{M}]]^{wd}, \leqslant_{\text{lex}})$. The set $\bigcup \operatorname{supp} f_i \subseteq \mathfrak{M}$ is well-ordered. Then the set of leading monomials $\{\mathfrak{d}_{f_i} \,|\, i \geqslant 0\}$ cannot be infinitely decreasing; but on the other hand, it cannot be infinitely increasing either, since if it was, we could extract an strictly increasing sub-sequence from $(f_i)_{0 \leqslant i}$. We therefore find an element $\mathfrak{m}_0 \in \mathfrak{M}$ such that $\mathfrak{m}_0 = \mathfrak{d}_{f_i}$ for infinitely many $i \geqslant 0$. We can hence without loss of generality assume that $(f_i)_{0 \leqslant i}$ has this property. For the same reason, the set $\{c_{f_i} \,|\, 0 \leqslant i\} \in C$ of leading coefficients cannot have either infinitely increasing or decreasing sequences. We therefore may assume that we have $\mathfrak{m}_0 \in \mathfrak{M}$ and $c_0 \in C$ such that for all $i \geqslant 0 : c_0 \mathfrak{m}_0 = c_{f_i} \mathfrak{d}_{f_i}$.

Assume now that for an ordinal α, we have constructed a subset $\{\mathfrak{m}_\beta \,|\, \beta < \alpha\} \subseteq \bigcup \operatorname{supp} f_i$ such that

$(1)_\alpha$ $\forall k < l < \alpha : \mathfrak{m}_k \succ \mathfrak{m}_l$,

$(2)_\alpha$ $\forall i \geqslant 0 : \{\mathfrak{m}_\beta \,|\, \beta < \alpha\} \subseteq \operatorname{supp} f_i$ and $\min(\operatorname{supp} f_i \setminus \{\mathfrak{m}_\beta \,|\, \beta < \alpha\}) \prec \mathfrak{m}_\beta$ for all $\beta < \alpha$, and
$(3)_\alpha$ $\forall \beta < \alpha : \exists c_\beta \in C : \forall i \geqslant 0 : f_{i,\mathfrak{m}_\beta} = c_\beta$.
Then let

$$f_i^\alpha := f_i - \sum_{\beta<\alpha} c_\beta \mathfrak{m}_\beta,$$

i.e. $f_i^\alpha \in C[[\mathfrak{M}]]^{wd}$ with $\operatorname{supp} f_i^\alpha = \operatorname{supp} f_i \setminus \{\mathfrak{m}_\beta \,|\, \beta < \alpha\}$ and $\forall \mathfrak{m} \in \operatorname{supp} f_i^\alpha : f_{i,\mathfrak{m}}^\alpha = f_{i,\mathfrak{m}}$. Repeating the above argument, we can find $c_\alpha \in C, \mathfrak{m}_\alpha \in \mathfrak{M}$ such that for an infinite sub-sequence of $(f_i^\alpha)_{0\leqslant i}$ we have $\mathfrak{m}_\alpha = \mathfrak{d}_{f_i^\alpha}$ and $c_\alpha = c_{f_i^\alpha}$. Hence, for successor ordinals, we can always maintain the hypotheses.

Let λ be a limit ordinal such that for all $\alpha < \lambda$ we have $(1)_\alpha$, $(2)_\alpha$, $(3)_\alpha$. Since f_i is weakly decreasing and $\{f_{i,\mathfrak{m}_\beta} \,|\, \beta < \lambda\} \subseteq C$, this set cannot be either strictly increasing nor decreasing. Hence there must be a $c_i \in C$ and a $\beta_i < \lambda$ such that for all β with $\beta_i \leqslant \beta < \lambda$ we have $c_i = f_{i,\mathfrak{m}_\beta}$. But this is true for all i. From $(1)_{\beta_i}$, $(2)_{\beta_i}$, $(3)_{\beta_i}$ it follows now that for $\beta \geqslant \beta_i$ all functions in the sequence take the same value, c_β. Hence, the result of the extraction process is always the same sequence from β_i on. We can therefore define f_i^λ as the element from $C[[\mathfrak{M}]]^{wd}$ with $\operatorname{supp} f_i^\lambda = \operatorname{supp} f_i \setminus \{\mathfrak{m}_\beta \,|\, \beta < \lambda\}$ and $\forall \mathfrak{m} \in \operatorname{supp} f_i^\lambda : f_{i,\mathfrak{m}}^\lambda = f_{i,\mathfrak{m}}$. We continue this process until $\operatorname{supp} f_i$ is empty. But then we have a constant sub-sequence, contradiction. □

Lemma 1.7.7 is not true if we replace $C[[\mathfrak{M}]]^{wd}$ by $C[[\mathfrak{M}]]^+$. Let for instance $C = \mathbb{N}, \mathfrak{M} = \mathbb{N}$ and $f_i = (1, \dots, 1, 2, 0, \dots)$, i.e. the function $f_i : \mathbb{N} \to \mathbb{N}$ with $f_i(j) = 1$ for $1, \dots, i-1$, $f_i(i) = 2$ and $f_i(j) = 0$ else. Then $f_0 \;{}_{\mathrm{lex}}\!\!> f_1 \;{}_{\mathrm{lex}}\!\!> f_2 \cdots$. In fact, after ω extractions in the proof of the lemma, there are no series f_i left. The set $\mathbb{N}[[\mathbb{N}]]^{wd}$, which is by Lemma 1.7.7 well-ordered, will be useful in later chapters.

1.8 On truncations and supports

Let $f, g \in \mathbb{S} = C[[\mathfrak{M}]]$, then g is a **truncation** of f, in symbols $g \trianglelefteq f$, iff there is a monomial $\mathfrak{m}_g \in \mathfrak{M}$ such that for all $\mathfrak{m} \succ \mathfrak{m}_g : f_\mathfrak{m} = g_\mathfrak{m}$ and for all $\mathfrak{m}_g \not\succcurlyeq \mathfrak{m} : g_\mathfrak{m} = 0$, i.e.

$$g = \sum_{\mathfrak{m} \,\succ\, \mathfrak{m}_g} f_\mathfrak{m} \mathfrak{m}.$$

The truncation g of f is **proper** iff $g \neq f$, and we write $g \vartriangleleft f$ in this case. Suppose $0 \neq f$. For every proper truncation $0 \neq g$ of f we have $\operatorname{supp} g \succ \operatorname{supp}(f-g)$ and in particular $\operatorname{supp} g \succ f-g$.

Let $f, g \in \mathbb{S}$, then $h = f \;\vartriangle\; g \in \mathbb{S}$ is called the **maximal common truncation** of g and f iff $h \trianglelefteq g$, $h \trianglelefteq f$ and if for all $h^* \trianglelefteq f$ with $h \vartriangleleft h^*$ we have $\neg(h^* \trianglelefteq g)$.

Remark 1.8.1 To all given $f, g \in \mathbb{S}$ a maximal common truncation exists and it is unique. To see this suppose that h_1 and h_2 are distinct maximal truncations of g and f, but then either $h_1 \vartriangleleft h_2$ or $h_2 \vartriangleleft h_1$, which contradicts the definition.

Let $f_1 > f_2 > f_3$ be generalized power series, then we have

$$\begin{aligned} f_1 \;\vartriangle\; f_3 \;&\trianglelefteq\; f_2 \;\vartriangle\; f_3, \\ f_1 \;\vartriangle\; f_3 \;&\trianglelefteq\; f_1 \;\vartriangle\; f_2. \end{aligned}$$

To see this, we remark that we have $0 < f_2 - f_3 < f_1 - f_3$ and thus $f_2 - f_3 \preccurlyeq f_1 - f_3$. Then for every $t \in \operatorname{term} f_3$ with $t \succ f_1 - f_3$ we have $t \succ f_2 - f_3$. Hence

$$f_1 \vartriangle f_3 = \sum_{\substack{t \in \operatorname{term} f_3: \\ t \succ f_1 - f_3}} t \trianglelefteq \sum_{\substack{t \in \operatorname{term} f_3: \\ t \succ f_2 - f_3}} t = f_2 \vartriangle f_3.$$

Similarly, from $0 < f_1 - f_2 < f_1 - f_3$ we obtain $f_1 - f_2 \preccurlyeq f_1 - f_3$ and with the same argument as above

$$f_1 \vartriangle f_3 = \sum_{\substack{t \in \operatorname{term} f_1: \\ t \succ f_1 - f_3}} t \trianglelefteq \sum_{\substack{t \in \operatorname{term} f_1: \\ t \succ f_1 - f_2}} t = f_1 \vartriangle f_2.$$

It is in general not true that $g \lhd f$ implies $g + h \lhd f + h$. A simple example in $C[[x^{\mathbb{Z}}]]$ is $x \lhd x + x^2$ and $h = x^2$. In the following lemma, we give a condition on the support of the series h under which $g+h$ remains a truncation of $f+h$. (Note that the truncations here are proper.)

Lemma 1.8.2 *Let $f, g, h \in \mathbb{S}$ with $g \lhd f$. Then $g+h \lhd f+h$ if and only if $\operatorname{supp} h \succ \operatorname{supp}(f-g)$.*

Proof: Note that $g \lhd f$ implies that $\emptyset \neq \operatorname{supp}(f-g) = \operatorname{supp} f \setminus \operatorname{supp} g$. First, suppose that $g+h \trianglelefteq f+h$. If there was some $\mathfrak{m} \in \operatorname{supp} h$ with $\mathfrak{m} \not\succ \operatorname{supp}(f-g)$, then we would find a monomial $\mathfrak{n} \in \operatorname{supp}(f-g)$ with $\mathfrak{n} \succcurlyeq \mathfrak{m}$. From $f_{\mathfrak{n}} \neq 0$ we obtain $(h+g)_{\mathfrak{n}} = h_{\mathfrak{n}} \neq f_{\mathfrak{n}} + h_{\mathfrak{n}} = (f+h)_{\mathfrak{n}}$. Since $\mathfrak{n} \succcurlyeq \mathfrak{m} \in \operatorname{supp}(g+h)$ this contradicts $g+h \trianglelefteq f+h$.

Now suppose that $\operatorname{supp} h \succ \operatorname{supp}(f-g)$. For all $\mathfrak{n}$ with $\mathfrak{n} \not\succ \operatorname{supp}(f-g)$ we obtain $(g+h)_{\mathfrak{n}} = 0$. Let $\mathfrak{n} \succ \operatorname{supp}(f-g)$, then $g_{\mathfrak{n}} = f_{\mathfrak{n}}$ implies $(g+h)_{\mathfrak{n}} = (f+h)_{\mathfrak{n}}$. $\square$.

Recall that elements from $\mathbb{S}$ have well-ordered support in $\mathfrak{M}$. The support is then isomorphic to an ordinal. In this section, we will will consider cardinalities of supports. The lemmas shown here will have applications to the fields which we will construct in the next chapters. In the proofs, we will use two general facts about ordinals.

Proposition 1.8.3 *Let $(\mathbb{R}, <)$ be the set of real numbers with its natural order. Then:*

(1) If $X \subset \mathbb{R}$ is such that $(X, <)$ is well-ordered, then X is countable. The same holds for anti-well-ordered sets X.

(2) Let $(X, <)$ be a countable and well-ordered set. Then there is a monomorphism from $(X, <)$ to $(\mathbb{R}, <)$.

Proof: Let β be an ordinal number such that $X = (x_\alpha)_{\alpha<\beta}$ and such that for all $\alpha < \tau < \beta$ we have $x_\alpha < x_\tau$ in $\mathbb{R}$. Let $\alpha < \beta$. Then let $y_\alpha \in \mathbb{Q}$ with $x_\alpha < y_\alpha < x_{\alpha+1}$. Hence for $\alpha < \tau < \beta$ we have $y_\alpha < y_\tau$ and $(y_\alpha)_{\alpha<\beta} \subseteq \mathbb{Q}$. This proves (1).

For (2), we can assume that X is an ordinal. Let δ the smallest ordinal that cannot be embedded into $\mathbb{R}$. Then δ must be a limit ordinal. If δ was countable, then there would be a strictly increasing sequence of countable ordinals $(\delta_\alpha)_{\alpha<\omega}$ with $\delta = \sup_{\alpha<\omega}(\delta_\alpha)$. Let $\delta_{-1} = \emptyset$. For every $n \in \mathbb{N}$ there is an embedding f_n of $\delta_n \setminus \delta_{n-1}$ into $[1 - \frac{1}{n+1}, 1 - \frac{1}{n+2})$. Let $x \in \delta$ and

$n = n_x \in \mathbb{N}$ minimal with $x \in \delta_n$, then we let $f(x) := f_{n_x}(x)$. We have therefore constructed an embedding of δ into $[0, 1)$. Contradiction. □

Remark 1.8.4 (1) Note that the proposition remains true if we replace $\mathbb{R}$ by any archimedean field C. This field must contain $\mathbb{Q}$, and if there were elements $c < d$ such that there are no rational numbers between c and d, then $1 \not\preccurlyeq d - c$.

(2) An alternative proof for part (2) of Lemma 1.8.3 is the following: X is countable, hence we can write $X = (x_n)_{n<\omega}$ such that the x_n are pairwise distinct. We define the embedding $\phi : X \to \mathbb{R}$ with $x_i < x_j \Rightarrow \phi(x_i) < \phi(x_j)$ inductively as follows. Take $\phi(x_0) \in \mathbb{R}$ arbitrary. Suppose that we have defined the embedding ϕ for $x_0, \ldots, x_n$. The element x_{n+1} divides the set $X_n = \{x_0, \ldots, x_n\}$. The set $\phi(X_n) = \{\phi(x_i) \,|\, i \leqslant n\}$ is finite. Since $\mathbb{R}$ is dense, there is a $y \in \mathbb{R}$ which realizes the same cut over $\phi(X_n)$ as x_{n+1} over X_n. We let $\phi(x_{n+1}) = y$. Notice that we could have taken $\mathbb{Q}$ instead of $\mathbb{R}$.

(3) We can modify the morphism f such that the following holds. Let $\lambda < \delta$ a limit ordinal and $\lambda = \sup_{i<\omega}(\lambda_i)$ with $\lambda_i < \lambda$, then $\sup_{i<\omega} f(\lambda_i) < f(\lambda)$. To see this this let $\lambda < \delta$ with $\sup_{i<\omega} f(\lambda_i) = f(\lambda)$, then we replace $f(\lambda)$ by $\frac{1}{2}(f(\lambda) + f(\lambda + 1))$.

Let $P = (P, \leqslant)$ be a total order and κ a cardinal number. We say that P has **cofinal cardinality** $<\kappa$ iff every well-ordered $D \subseteq P$ has cardinality less than κ. From Proposition 1.8.3 it follows that $(\mathbb{R}, \leqslant)$ has cofinal cardinality $< \aleph_1$. If an order has cofinal cardinality $< \kappa$, then so has the inverse order.

Lemma 1.8.5 *Let κ_1 and κ_2 be cardinal numbers such that C and $\mathfrak{M}$ have cofinal cardinality $< \kappa_1$ and $< \kappa_2$, respectively. Let $(f_\alpha)_{\alpha<\tau}$ be a strictly increasing sequence of series in $\mathbb{S} = C[[\mathfrak{M}]]$. Then $|\tau| < \max(\kappa_1, \kappa_2)$.*

Proof: Let $\kappa = \max(\kappa_1, \kappa_2)$. The set $C\mathfrak{M}$ is totally ordered, and we first show that it is of cofinal cardinality $< \kappa$.

Let $(\tau_\beta)_{\beta<\alpha}$ be a strictly increasing sequence in $C\mathfrak{M}$. For every β we have $\tau_\beta = c_\beta\, \mathfrak{m}_\beta$. Suppose that $0 < \tau_\beta$ for all $\beta < \alpha$. Then we have $\mathfrak{m}_\beta \preccurlyeq \mathfrak{m}_\gamma$ for all $\beta < \gamma < \alpha$. Hence the set $\mathfrak{N}_\tau = \{\mathfrak{m}_\beta \mid \beta < \alpha\}$ has cardinality $< \kappa_2$. For every $\mathfrak{m} \in \mathfrak{N}_\tau$, the sequence

$$(c_\alpha)_{\substack{\alpha<\tau:\\ \mathfrak{m}=\mathfrak{m}_\alpha}}$$

is strictly increasing in C and has thus cardinality $< \kappa_1$. This shows our claim if all terms are positive. The general case follows immediately.

Let $f \in C[[\mathfrak{M}]]$ and γ be an ordinal number. Recall that every sequence of $C[[\mathfrak{M}]]$ has an order type, i.e. there is a unique ordinal number such that the support of f is isomorphic to this ordinal. Then for γ the series f either admits a unique truncation of order type γ or there is no such truncation at all. The latter is the case is γ is larger than the order type of f. We denote this truncation by $\mathrm{tr}_\gamma(f)$. Note that $\mathrm{tr}_0(f) = 0$ and $\mathrm{tr}_1(f) = \tau_f$.

We construct a sequence $A_0, A_1, \ldots$ of sets of series from the given sequence with $|A_\gamma| = |\tau|$ for all $\gamma < \kappa$. Let let $A_0 := \{f_i \mid i < \tau\}$. Moreover, once we have defined the set A_γ, we define sets

$$\begin{aligned} B_\gamma &:= \{\mathrm{tr}_\gamma(f) \mid f \in A_\gamma\}, \\ C_\gamma &:= \{f \in A_\gamma \mid \quad |\{i < \tau \,|\, \mathrm{tr}_\gamma(f) = \mathrm{tr}_\gamma(f_i)\}| < |\tau| \quad \}. \end{aligned}$$

Note that $C_0 = \emptyset$. Furthermore, we suppose that

$$A_\gamma = A_0 \setminus \coprod_{\beta<\gamma} C_\beta.$$

We show that for all $\gamma < \kappa$ we have

- $|A_\gamma| = |\tau|$,
- $|C_\gamma| < \kappa$,
- $\forall \nu < \gamma : C_\nu \ \cap \ C_\gamma = \emptyset$.

First we consider the case where $\gamma = \delta + 1$ is a successor ordinal. We claim that $|C_\gamma| < |\tau|$. By definition, for every series $f \in A_\gamma$ the set

$$D_f = \{i < \tau \mid \mathrm{tr}_\delta(f) = \mathrm{tr}_\delta(f_i)\}$$

has cardinality $|\tau|$. Moreover, the set admits a minimal element j. Hence $\{f_i - f_j \mid i \in D_f\}$ is a strictly increasing sequence of series. Moreover, we have

$$\forall i \in D_f : \mathrm{tr}_{\delta+1}(f_i) = \mathrm{tr}_\delta(f_i) + \mathrm{tr}_1(f_i - f_j).$$

Now $|C_\gamma| \geqslant \kappa$ would imply that for at least one f the set D_f gives rise to an increasing sequence of length κ of terms, which on the other hand contradicts the cofinal cardinality $< \kappa$ of $C\mathfrak{M}$.

Now that $|C_\gamma| < \kappa$, there is at least one $f \in A_\delta$ such that the set

$$\{f_i \in A_\delta \mid \mathrm{tr}_\delta(f) = \mathrm{tr}_\delta(f_i)\}$$

has cardinality $|\tau|$. Since this set is contained in A_γ, our claim holds. Clearly, the set C_γ has no common element with any of the sets B_α for $\alpha < \gamma$. We notice that $|B_1| < \kappa$. Furthermore $|B_\delta| < \kappa$ implies $|B_{\delta+1}| < \kappa$. This finishes the case of successor ordinals.

Let γ be a limit ordinal. From the inductive assumptions it follows that $|\coprod_{\delta<\gamma} C_\delta| < \kappa{\cdot}\gamma = \kappa$. Hence the set A_γ has cardinality $|\tau|$. On the other hand, we have

$$|B_\gamma| \leqslant |\bigcup_{\delta<\gamma} B_\delta| = \kappa.$$

But then $|C_\gamma| < |B_\gamma| \cdot |\tau| = |\tau|$.

We can thus continue the construction of A_γ for all $\gamma < \kappa$ and therefore construct a decreasing sequence of monomials of length κ. This contradiction shows the lemma. □

Lemma 1.8.6 *Let C be archimedean and $\mathfrak{M} = (\mathfrak{M}, \succcurlyeq)$ be such that $(f_\alpha)_{\alpha<\beta}$ is a strictly decreasing sequence in $\mathbb{S} = C[[\mathfrak{M}]]$, i.e. for all $\alpha < \tau < \beta$: $f_\alpha > f_\tau$. Suppose that the support of f_α is countable for all $\alpha < \beta$, and that all well-ordered subsets $S \subseteq \mathfrak{M}$ are countable. Then β is countable. (The same holds for strictly increasing sequences.)*

Proof: Apply Lemma 1.8.5 with $\kappa_1 = \kappa_2 = \aleph_1$. $\square$

Some monomial sets are countable. Take for instance $x^{\mathbb{Z}}$. Then we obtain from the above that every decreasing sequence in $\mathbb{R}[[x^{\mathbb{Z}}]]$ is countable. If C has cofinal cardinality $< \kappa$, then there are no decreasing sequences of cardinality κ in $C[[x^{\mathbb{Z}}]]$. The Lemmas 1.8.5 and 1.8.6 will be applied to larger monomial groups in the next chapters. We finish with a simple consequence of the above.

Lemma 1.8.7 *Let κ be a cardinal, α be an ordinal and $(\mathfrak{M}_i)_{i<\alpha}$ be such that every $\mathfrak{M}_i$ has cofinal cardinality $< \kappa$ and $|\alpha| < \kappa$. Let $\mathfrak{M}_\alpha = \bigcup_{i<\alpha} \mathfrak{M}_i$. Then for every $f \in C[[\mathfrak{M}_\alpha]]$ we have $|\operatorname{supp} f| < \kappa$.*

Proof: Let $\mathfrak{M}_{-1} = \emptyset$ and for $i < \alpha$

$$f_i = \sum_{\mathfrak{m} \in \mathfrak{M}_i \setminus \mathfrak{M}_{i-1}} f_{\mathfrak{m}} \mathfrak{m}.$$

Then $f = \sum_{i<\alpha} f_i$ and $\operatorname{supp} f = \coprod_{i<\alpha} \operatorname{supp} f_i$. Then $|\operatorname{supp} f| < |\alpha| \cdot \kappa = \kappa$. $\square$

Chapter 2

Fields of transseries

In this section, we will introduce the notion of a field of transseries. Such a field will be a field $\mathbb{T} = C[[\mathfrak{M}]]$ of generalized power series, on which we have an additional partial exponentiation function. This exponential function both satisfies a certain number of algebraic requirements (see Section 2.1) and several compatibility conditions with the serial structure of $\mathbb{T}$ (see Section 2.2).

The introduction of the notion of transseries fields provides a great flexibility in the study of transseries. First, it forces one to clearly state the essential properties of such fields. Next, it provides a framework for the construction of complicated fields of transseries, such as fields which contain iterators of the exponential function. But most importantly, it enables us to think of constructions of specific fields of transseries as repeated extensions of algebraic structures.

For example, in Section 2.3.1, we show how to construct the "simplest", non trivial transseries field. In Sections 2.3.2 and 2.3.3, we respectively show how to adjoin exponentials to transseries fields, and how to take inductive limits. Finally, we show in Section 2.5 how to extend transseries fields by nested expressions.

2.1 exp-log fields

Definition 2.1.1 *An ordered field C with ordering $\leqslant$ is called an* **exp-log field** *(or simply an* **exponential field***) iff there is a partial function* $\exp : B \subseteq C \to C$ *such that*

e1. If two among f, g, $f+g \in C$ are in dom exp, then so is the third and

$$\exp(f+g) = (\exp f)(\exp g),$$

e2. For all $f, g \in \operatorname{dom} \exp$:

$$f < g \Leftrightarrow \exp f < \exp g,$$

e3. For all $f \in \operatorname{dom} \exp : f + 1 \leqslant \exp f$.

Remark 2.1.2 If C is an exp-log field, then the function exp is called **exponential function**. If $g \in C$ is such that there is an $f \in C$ with $g = \exp f$, then it follows from **e2** that f is unique and we write $f = \log g$. We call f the **logarithm** of g. It follows from **e1** that $\log gh = \log g + \log h$, whenever two among $\log g$, $\log h$ and $\log gh$ are defined.

In what follows, we will denote $e = \exp 1$, if $1 \in \operatorname{dom} \exp$. We also denote $a^x = \exp(a \log x)$, for $x \in \operatorname{dom} \log$ and $a \log x \in \operatorname{dom} \exp$.

Example 2.1.3 Let C be a totally ordered exp-log field with $\operatorname{dom} \exp = C$ and $\operatorname{dom} \log = C^+$ such as $C = \mathbb{R}$. Let $\mathfrak{M}$ be a monomial group. In view of Lemma 1.5.6 we may extend the exponentiation on C to a partial exponentiation on $\mathbb{S} = C[[\mathfrak{M}]]$ with domain $C[[\mathfrak{M}]]^{\updownarrow}$. Let $e(X)$ and $l(X)$ be the formal power series

$$e(X) = \sum_{0 \leqslant i} \frac{1}{i!} X^i \qquad \text{and} \qquad l(X) = \sum_{1 \leqslant i} \frac{(-1)^{i+1}}{i} X^i.$$

For $f \in C[[\mathfrak{M}]]^{\updownarrow}$ we let

$$\exp f := (\exp f^{=}) \cdot e(f^{\downarrow}).$$

We claim that $C[[\mathfrak{M}]]$ with this function is an exp-log field.

We have to show **e1**, i.e. for $f, g \in C[[\mathfrak{M}]]$ we have to show $\exp(fg) = (\exp f)(\exp g)$. Since C is an exp-log field, we have to show that the equation yields for purely infinitesimal series f, g. By Property 1.6.6 it suffices to show that $e(X)e(Y) = e(X+Y)$ for formal power series. But

$$\begin{aligned} e(X)e(Y) &= \sum_{i \geqslant 0} \frac{1}{i!} X^i \sum_{j \geqslant 0} \frac{1}{j!} Y^j = \sum_{i \geqslant 0} \sum_{j \geqslant 0} \frac{1}{i!j!} X^i Y^j = \sum_{k \geqslant 0} \frac{1}{k!} \sum_{i+j=k} \frac{k!}{i!j!} X^i Y^j \\ &= \sum_{k \geqslant 0} \frac{1}{k!} (X+Y)^k = e(X+Y) \end{aligned}$$

This shows **e1**. Now let $f < g$, both in $\mathbb{S}^{\updownarrow}$. Then $0 < g - f \in \mathbb{S}^{\updownarrow}$, and from the definition of exp we obtain $1 < \exp(g - f)$. Multiplying both sides with $\exp f$ yields **e2**. As to the last property, note that in the case $f^{=} = 0$ we have

$$1 + f^{\downarrow} < 1 + f^{\downarrow} + (f^{\downarrow})^2 (1/2 + f^{\downarrow}/3! + \cdots) = e(f^{\downarrow}).$$

In the case $f^{=} \neq 0$ we have $1 + f^{=} < \exp f^{=}$ in C. Hence for all infinitesimal series g, h we have $1 + f^{=} + g < \exp f^{=} + h$. Taking $g = f^{\downarrow}$ and $h = (\exp f^{=}) \cdot (e(f^{\downarrow}) - 1)$ shows **e3**. This proves our claim.

In a similar way, given $f \in C[[\mathfrak{M}]]^{\updownarrow}$ with $f^{=} > 0$, we may define

$$\log f := \log f^{=} + l(\delta_f).$$

We claim that log is the inverse function of exp. Since $\log f^{=} \in C$ and $l(\delta) \in C[[\mathfrak{M}]]^{\downarrow}$, we only have to show that $l(e(\delta_f) - 1) = \delta_f$ and $e(l(\delta_f)) - 1 = \delta_f$.

Lemma 2.1.4 *In $C[[X^{\mathbb{N}}]]$ the equations $l(e(X) - 1) = X$ and $e(l(X)) - 1 = X$ hold.*

Proof: The series e and l are the Taylor-series developments of analtyic functions, which are inverse one to another. The lemma then follows from the fact that 1 is the development of the identity. □

Remark 2.1.5 Let us give an alternative proof of Lemma 2.1.4 which relies entierly on properties of formal power series. To this end, we start by recalling that for two integers $j \leqslant n$ we have $\sum_{i=j}^{n} \binom{i-1}{j-1} = \binom{n}{j}$. Furthermore, we use the following notations. For $k \in \mathbb{N}^i$ with $k = (k_1, \ldots, k_i)$ we let $|k| = k_1 + \cdots + k_i$ and $k! = k_1! \cdots k_i!$. Let $T(i,n) = \{k \in \mathbb{N}^i \,|\, |k| = n\}$ and $T^*(i,n) = \{k \in T(i,n) \,|\, k \in (\mathbb{N}^+)^i\}$.

Then $l(e(X)-1) = l(\sum_{1 \leqslant n} \frac{1}{n!} X^n) = \sum_{1 \leqslant k} c_k X^k$ where

$$c_k = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} \sum_{n \in T^*(i,k)} \frac{1}{n!} = \sum_{i=1}^{k} \frac{(-1)^{i+1}}{i} \sum_{n \in T^*(i,k)} \frac{1}{n!}.$$

From

$$\sum_{n \in T^*(i,k)} \frac{1}{n!} = \frac{1}{k!} \sum_{j=0}^{i-1} (-1)^j \binom{i}{j} \sum_{n \in T(i-j,k)} \frac{k!}{n!} = \frac{1}{k!} \sum_{j=0}^{i-1} (-1)^j \binom{i}{j} (i-j)^k$$

we obtain

$$\begin{aligned} k! c_k &= \sum_{i=1}^{k} \frac{(-1)^{i+1}}{i} \sum_{j=1}^{i} (-1)^{i-j} \binom{i}{j} j^k = \sum_{i=1}^{k} \sum_{j=1}^{i} (-1)^{j-1} \binom{i-1}{j-1} j^{k-1} \\ &= \sum_{j=1}^{k} (-1)^{j-1} j^{k-1} \sum_{i=j}^{k} \binom{i-1}{j-1} = \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} j^{k-1}. \end{aligned}$$

Note that for $b_k = k! c_k$ we have $b_1 = 1$ and $b_2 = 0$. We want to show that $b_k = 0$ holds for every $k \geqslant 2$.

Let for $1 \leqslant i \leqslant k-3$, $d_i^k = \sum_{j=1}^{k-1} (-1)^j \binom{k-1}{j} j^i$. Note that for all $m \geqslant 2$ we have $d_0^m = -1$. One also can verify that $d_{i+1}^k = (1-k)(1 + d_0^{k-1} + \sum_{l=0}^{i} \binom{i}{l} d_l^{k-1}) = (1-k) \sum_{l=0}^{i} \binom{i}{l} d_l^{k-1}$. We also remark that for $k = 4, i = 1$ we have $d_i^k = 0$. Now suppose that we have already shown that if $d_l^{k-1} = 0$ for all $l \leqslant k-4$, then $d_{i+1}^k = 0$ for all $i \leqslant k-2$. From the definition of b_k we obtain $b_k/k = \sum_{i=1}^{k-3} \binom{k-2}{i} d_i^k - b_{k-1}$ for all $k \geqslant 2$. Hence $b_k = 0$ for all $k \geqslant 2$ and thus $c_k = 0$ for $k \geqslant 2$. We finally obtain $l(e(X)-1) = X$. The second equation follows similarly. From Proposition 1.6.6 the claim now follows.

This proof of Lemma 2.1.4 has its advantages and disadvantages. Due to its technical character, one would certainly prefer the first proof, which also has links to other fields of mathematics. On the other hand, the second proof does not need any extra knowledge.

Note that we have

$$\log(fg) = \log f + \log g$$

for all $f, g \in C[[\mathfrak{M}]]$ with $0 < f^=, g^=$. The field $C[[\mathfrak{M}]]$ together with the partial exponential function exp on $C[[\mathfrak{M}]]^{\downarrow}$ will be called the **basic exp-log field**.

Remark 2.1.6 Let $\mathcal{L}$ be a first-order language. Recall that for two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ we say that $\mathcal{M}$ is a $\mathcal{L}$-substructure of $\mathcal{N}$ iff $|\mathcal{M}| \subseteq |\mathcal{N}|$ and if for every function symbol and every relation symbol of the language $\mathcal{L}$ the restriction of the interpretation of the symbol in the structure $\mathcal{N}$ to elements from $\mathcal{M}$ coincides with the interpretation of the symbol in $\mathcal{M}$. Let $\mathcal{L}_{\exp}$ be the language of ordered rings with a unitary function symbol exp. Let $\mathcal{M}$ and $\mathcal{N}$ be exp-log fields, hence $\mathcal{L}_{\exp}$-structures. If $\mathcal{M}$ is a $\mathcal{L}_{\exp}$-substructure of $\mathcal{N}$, then we also say that $\mathcal{M}$ is an **exp-log-subfield** of $\mathcal{N}$.

2.2 Transseries fields

Let C be a totally ordered exp-log field such that dom exp $= C$ and dom log $= C^+$. Let $\mathfrak{M}$ be a totally ordered monomial group such that $\mathbb{S} = C[[\mathfrak{M}]]$ is an exp-log field. Suppose that the exponentiation restricted to $\mathbb{S}^{\updownarrow}$ is the exponentiation of the basic exp-log field structure from Example 2.1.3. Let $f \in \mathbb{S}^+_\infty$ be such that $f \in \operatorname{dom} \log_n$ for all $n \in \mathbb{N}$. Then we say that f is **log-confluent at order *k*** iff for all $n \geqslant k$ we have $\tau_{\log_{n+1} f} = \log \tau_{\log_n f}$. The series f is **log-confluent** iff it is log-confluent at some order k. Similarly, we say that a set $S \subseteq \mathbb{S}^+_\infty$ is log-confluent (at order k) iff every element of S is log-confluent (at order k). Instead of log-confluent at order 0, we use the expression **log-atomic**.

Note that if f is log-confluent at order k, then for all $i \geqslant 0$ we have

$$\begin{aligned} \log_i \mathfrak{d}_{\log_k f} &= \mathfrak{d}_{\log_{k+i} f} \\ 1 &= c_{\log_{k+i} f}. \end{aligned}$$

To see this, we remark that the logarithm of infinite elements of $\mathbb{S}$ are infinite, since log is monotone and range log $= C$. Then the claim follows from the functional equation.

Definition 2.2.1 *Let C be a totally ordered exp-log field with* dom exp $= C$ *and* dom log $=$ C^+. *Let $\mathbb{T} = C[[\mathfrak{M}]]$ be a complete exponential field of generalized power series, where $\mathfrak{M}$ is a totally ordered monomial group. We say that $\mathbb{T}$ is a* **transseries field** *iff*

- **T1.** dom log $= \mathbb{T}^+$,
- **T2.** $\log \mathfrak{M} \subseteq \mathbb{T}^{\uparrow}$,
- **T3.** $\log(1+f) = l(f)$, *for all* $f \in \mathbb{T}^{\downarrow}$,
- **T4.** *for every sequence* $(\mathfrak{m}_i)_{0 \leqslant i} \subseteq \mathfrak{M}$ *such that* $\mathfrak{m}_{i+1} \in \operatorname{supp} \log \mathfrak{m}_i$ *for all* $0 \leqslant i$, *there is an integer* $n_0 \in \mathbb{N}$ *such that*

$$\forall n_0 \leqslant n : \forall \mathfrak{n} \in \operatorname{supp} \log \mathfrak{m}_n : \mathfrak{n} \succcurlyeq \mathfrak{m}_{n+1} \ \wedge \ (\log \mathfrak{m}_n)_{\mathfrak{m}_{n+1}} = \pm 1.$$

Remark 2.2.2 Note that transseries fields are always complete fields. We will see that there are fields whose elements are transseries which are incomplete. If they are, we will say so. Otherwise, let us remark that we also use the expression fields of transseries for such complete or incomplete fields.

Remark 2.2.3 The condition **T2** gives a characterization of the monomials in $\mathfrak{M}$, in terms of the exponential structure. The condition **T3** states that the exp-log structure on $\mathbb{T}$ extends the basic exp-log structure from Example 2.1.3. Notice that **T3** is equivalent to the condition that $\exp f = e(f)$ for all $f \in \mathbb{T}^{\downarrow}$. Condition **T4** is the most intricate one. Roughly speaking, it ensures that we are able to compose transseries in $\mathbb{T}$ on the left by purely logarithmic transseries. Condition **T4** means that we do not allow monomials of the form

$$e^{x^2+e^{\log_2^2 x+e^{\log_4^2 x+e^{\cdot^{\cdot^{\cdot}}}+\log_5 x}+\log_3 x}+\log x}.$$

We remark that condition **T4** implies that every $\in \mathbb{T}^+_\infty$ is log-confluent. To see this, let $\mathfrak{m}$ be a monomial and let $(\mathfrak{m}_i)_{0 \leqslant i} = (\mathfrak{d}_{\log_i \mathfrak{m}})_{0 \leqslant i}$. Then by **T4** for a certain i_0 we have

$$\forall i_0 \leqslant i : \operatorname{supp} \log \mathfrak{m}_i \succcurlyeq \mathfrak{m}_{i+1},$$

which means that for all $i \geqslant i_0$ we have $\operatorname{supp} \log \mathfrak{m}_i = \{\mathfrak{m}_{i+1}\}$. Then $\mathfrak{m}$ is log-confluent at order i_0.

Let $\mathbb{T}$ be a transseries field. Given $f, g \in \mathbb{T} \setminus C$, the following notations will sometimes be convenient in what follows:

$$\begin{aligned} f \prec\!\!\!\prec g &\Leftrightarrow \log|f| \prec \log|g|; \\ f \preceq\!\!\!\preceq g &\Leftrightarrow \log|f| \preccurlyeq \log|g|; \\ f \asymp\!\!\!\asymp g &\Leftrightarrow \log|f| \asymp \log|g|. \end{aligned}$$

Notice that $f \prec\!\!\!\prec g$ if and only if $|f|^a \prec |g|$ for all $a > 0$ in C, in the case when $|f|^a$ is defined for all such a. For instance, $x \prec\!\!\!\prec e^x$, but $x \not\prec\!\!\!\prec x^{1000}$.

Proposition 2.2.4 *Let $\mathbb{T} = C[[\mathfrak{M}]]$ be a transseries field. Then*

1. *For all $f \in \mathbb{T}^+_\infty$, we have $\log f \prec f$ and even $\log f \prec\!\!\!\prec f$.*

2. *Given $f \in \mathbb{T}^+$, the canonical decomposition of $\log f$ is given by*

$$\begin{array}{ccccccc} \log f & = & (\log f)^{\uparrow} & + & (\log f)^{=} & + & (\log f)^{\downarrow} \\ & & \| & & \| & & \| \\ & & \log \mathfrak{d}_f & & \log c_f & & \log(1+\delta_f). \end{array}$$

3. *For all $f, g \in \mathbb{T}^+$, we have*

$$\begin{aligned} f \prec g &\Leftrightarrow (\log f)^{\uparrow} < (\log g)^{\uparrow}; \\ f \preccurlyeq g &\Leftrightarrow (\log f)^{\uparrow} \leqslant (\log g)^{\uparrow}; \\ f \asymp g &\Leftrightarrow (\log f)^{\uparrow} = (\log g)^{\uparrow}; \\ f \sim g &\Leftrightarrow (\log f)^{\updownarrow} = (\log g)^{\updownarrow}. \end{aligned}$$

Proof: (1) We first claim that for all $n \in \mathbb{N}$ we have $n \log f < f$. Let $g = \log f$, thus $g \in \operatorname{dom} \exp$. Then from **e3** we obtain $g + 1 \leqslant \exp g$, hence $\log f < f$. We have $1 \prec \log f$ for all series $f \in \mathbb{T}^+_\infty$, since $\operatorname{dom} \exp_C = C$. Condition **T1** implies $f/2n \in \operatorname{dom} \log$ for all n. Hence,

$$\log \frac{f}{2n} < \frac{f}{2n} \Rightarrow 2n \log \frac{f}{2n} < f \Rightarrow 2n(\log f - \log 2n) < x.$$

On the other hand, $(2n)^2 < f$ implies $2n \log 2n < n \log f$. Our claim follows by multiplication.

In particular, for $g = \log f$ we obtain $2 \log g < g$. From $1 \prec g$ it follows that $C + \log g < 2 \log g < g$. Since exp is total on C, we obtain $\log(C^+ g) < g$, whence $C^+ \log f < f$ by **e2**. This shows $\log f \prec f$. Finally, $\log f \prec\!\!\prec f \Leftrightarrow \log_2 f \prec \log f$, by definition.

(2) follows from **e1**, **T2** and **T3**.

(3) We have $f \prec g \Leftrightarrow \mathfrak{d}_f \prec \mathfrak{d}_g \Leftrightarrow \mathfrak{d}_f < \mathfrak{d}_g \Leftrightarrow (\log f)^\uparrow = \log \mathfrak{d}_f < \log \mathfrak{d}_g = (\log g)^\uparrow$. The other relations are proved in a similar fashion. □

Remark 2.2.5 We remark that the above properties of log have their exponential counterparts. For instance, for all $f \in \operatorname{dom} \exp$, we have $f \prec \exp f$, $f \prec\!\!\prec \exp f$ and

$$\begin{array}{ccccccc} \exp f & = & (\exp f^\uparrow) & \cdot & (\exp f^=) & \cdot & (\exp f^\downarrow) \\ & & \| & & \| & & \| \\ & & \mathfrak{d}_{\exp f} & & c_{\exp f} & & (1 + \delta_{\exp f}). \end{array}$$

In particular, if $f \in \mathbb{T}^\uparrow$ is in dom log, then $\exp f \in \mathfrak{M}$. Moreover, if $g \in \mathbb{T}$ is such that $\log g \in \mathfrak{M}$, then $g \in \mathfrak{M}$.

We finish this section with some properties concerning the log-confluence.

Lemma 2.2.6 *Let $f, g \in \mathbb{T}^+_\infty$. Then*

(1) If f is log*-confluent at order k and $g \asymp f$, then g is* log*-confluent at order k.*
(2) The set $\mathbb{T}^+_\infty$ is log*-confluent if and only if $\mathfrak{M}^\uparrow$ is* log*-confluent.*
(3) If f is log*-confluent at order k then $R_{\log_k f} \in \mathbb{T}^\updownarrow$ and $R_{\log_{k+1} f} \in \mathbb{T}^\downarrow$.*

Proof: (1) follows then from $\mathfrak{d}_{\log_i f} = \mathfrak{d}_{\log_i \mathfrak{d}_f}$ for all $i \geqslant 0$. Then (2) follows from (3) and $f \asymp \mathfrak{d}_f \in \mathfrak{M}^\uparrow$. As for (3), conditions **T2** and **T3** imply $\mathfrak{d}_n = (\log_n f)^\uparrow$ and $l(\varepsilon) \in \mathbb{T}^\downarrow$ for $\varepsilon \in \mathbb{T}^\downarrow$. This shows the lemma. □

Remark 2.2.7 Define $\mathfrak{d}_n f = \mathfrak{d}_{\log_n f}$. If we consider $\mathfrak{d}_n$ as an operator on the set of positive infinite series in $\mathbb{T}$, then $\mathfrak{d}_n \circ \mathfrak{d}_m = \mathfrak{d}_{n+m}$ for all $n, m \in \mathbb{N}$. In other terms, for all $n, m \in \mathbb{N}$ and all $f \in \mathbb{T}^+_\infty$ we have

$$\mathfrak{d}_n \circ \mathfrak{d}_m f = \mathfrak{d}_{\log_n \mathfrak{d}_{\log_m f}} = \mathfrak{d}_{\log_{n+m} f} = \mathfrak{d}_{n+m} f.$$

The same does in general not remain true if we replace $\mathfrak{d}$ by c, δ or R.

Remark 2.2.8 Let us show that the condition **T4** is independent from the conditions **T1, T2, T3** by constructing a field of generalized power series satisfying these conditions but admitting elements which are not log-confluent.

Let $C_y = C[[\log^{\mathbb{Z}^\star} y]]$ and x such that $C_y < x$. We will define a function $\mathfrak{log}$ on $C_y[[\log^{\mathbb{Z}^\star} x]]$. On the set C_y we let $\mathfrak{log} = \log$. We have to define $\mathfrak{log}$ on the set of monomials $\log^{\mathbb{Z}^\star} x$. For $i \in \mathbb{N}$, $a \in \log^{\mathbb{Z}^\star} x$ and $f \in C_y[[\log^{\mathbb{Z}^\star} x]]^+$, we let

$$\begin{aligned} \mathfrak{log}(\log_i x) &:= \log_{i+1} x + y \\ \mathfrak{log}(\log^a x) &:= \sum_{0=i}^{|a|} a_i \mathfrak{log}(\log_i x) \\ \mathfrak{log}(f) &:= \mathfrak{log}\mathfrak{d}_f + \log(c_f) + l(\delta_f). \end{aligned}$$

By definition, the conditions **T1** and **T3** hold. We cannot take C_y as field of constants, since otherwise the definition of $\mathfrak{log}$ on $\log^{\mathbb{Z}^\star} x$ would not satisfy condition **T2**. Instead, we take C as field of constants. We let

$$\begin{array}{ll} y_0 := y & x_0 := x \\ y_1 := \log y & x_1 := \log x \\ y_2 := \log_2 y & x_2 := \log_2 x \\ \vdots & \vdots \end{array}$$

Then we can rewrite the field $C_y[[\log^{\mathbb{Z}^\star} x]]$ as

$$C[[\cdots; y_1; y_0; \cdots; x_1; x_0]],$$

where the element x_i, y_j are ordered lexicographically. The set of monomials of this field is

$$\{x^a\, y^b \mid a, b \in \mathbb{Z}^\star\},$$

and for monomials $\mathfrak{m} = x^a\, y^b$ in this group we have

$$\mathfrak{log}(\mathfrak{m}) = \mathfrak{log}(x^a) + \mathfrak{log}(y^b) = \sum_{0=i}^{|a|} a_i \mathfrak{log}(\log_i x) + \sum_{0=j}^{|b|} b_j \mathfrak{log}(\log_j y)$$

This shows **T3**. Now we remark that x is not $\mathfrak{log}$-confluent, hence that **T4** does not hold in this field. An immediate consequence is that we cannot have a composition with fields $\mathbb{T}$ where we replace x by a positive infinite series. Take for instance the series $f = x + \mathfrak{log} x + \mathfrak{log}_2 x + \cdots$. Then

$$(\mathfrak{log}_i x \circ x_0)_{1 \leqslant i} = \left(x_i + y_0 + l(\frac{y_0}{x_{i-1}}) \right)_{1 \leqslant i}$$

is not a Noetherian family. In this example, we have an element, which is in infinitely many members of the family. By modifying the construction of the field, we might even have constructed a family $(\mathfrak{log}_i x_0)_i$ such that the supports contain a strictly decreasing subset in $\succcurlyeq$ by taking $\mathfrak{log}(x_i) = x_i + (y_0/y_{i+1})$.

2.3 Construction of transseries fields

2.3.1 Fields of purely logarithmic transseries

Let C be a totally ordered exp-log field with dom exp $= C$ and dom log $= C^+$. Recall that $\mathbb{Z}^\star$ is the set of non-commutative words over $\mathbb{Z}$. Now $\mathbb{Z}^\star$ has the structure of an abelian group by taking $a + b = [a_0 + b_0, \ldots, a_l + b_l]$ for words $a = [a_0, \ldots, a_l]$ and $b = [b_0, \ldots, b_l]$ of the same length; for words of different lengths, we complete the shortest word on the right with zeros. We also define an ordering on $\mathbb{Z}^\star$ by setting $a > 0$, if $a \neq []$ and $a_i > 0$ for the smallest i with $a_i \neq 0$.

For each $a \in \mathbb{Z}^\star$, we now define

$$\log^a x = x^{a_0} \log^{a_1} x \cdots \log_n^{a_n} x$$

and $\log^{\mathbb{Z}^\star} x = \{\log^a x \mid a \in \mathbb{Z}^\star\}$. We give $\log^{\mathbb{Z}^\star} x$ the structure of a totally ordered monomial group, isomorphic to $\mathbb{Z}^\star$ by $1 = \log^0 x$, $(\log^a x)(\log^b x) = \log^{a+b} x$ for $a, b \in \mathbb{Z}^\star$, and $\log^a x \succcurlyeq \log^b x$ if $a \geqslant b$. Notice that

$$x \ggcurly \log x \ggcurly \log_2 x \ggcurly \cdots.$$

Now let $\mathbb{L} = C[[\log^{\mathbb{Z}^\star} x]]$. For monomials $\log^a x$, with $a = [a_0, \ldots, a_l]$, we define

$$\log \log^a x = a_0 \log x + \cdots + a_l \log_{l+1} x.$$

Notice that $\log \log^a x \in \mathbb{L}^\uparrow$ and $\log^a x \prec \log^b x \Leftrightarrow a < b \Leftrightarrow \log \log^a x < \log \log^b x$. We extend the logarithm to $\mathbb{L}^+$ via

$$\log f = \log \mathfrak{d}_f + \log c_f + l(\delta_f),$$

where $l(z) = \log(1 + z) \in C[[z]]$.

Remark 2.3.1 The construction of the group $\log^{\mathbb{Z}^\star} x$ can be extended to a group $\log^{C^\star} x$ by systematically replacing $\mathbb{Z}$ by C. One then obtains a group with C-powers. This extension is necessary, if one works with asymptotic scales. For our purposes it suffices to take $\mathbb{Z}$. Note that the following proposition only generalizes to the extended construction if $\mathfrak{M}$ has C-powers.

Proposition 2.3.2 *$\mathbb{L}$ is a transseries field.*

Proof: First, **e1** is equivalent to the condition that $\log fg = \log f + \log g$ for all $f, g \in \mathbb{L}^+$. This follows from the fact that $\log \mathfrak{d}_{fg} = \log(\mathfrak{d}_f \mathfrak{d}_g) = \log \mathfrak{d}_f + \log \mathfrak{d}_g$ and Property 1.6.6.

As to **e2**, let $f, g \in \mathbb{L}^+$ be such that $f < g$. If $\mathfrak{d}_f < \mathfrak{d}_g$, then $(\log f)^\uparrow < (\log g)^\uparrow$, whence $\log f < \log g$. If $\mathfrak{d}_f = \mathfrak{d}_g$, but $c_f < c_g$, then $(\log g - \log f)^\updownarrow = \log c_g - \log c_f > 0$ and again $\log f < \log g$. If $\tau_f = \tau_g$, then

$$\log g - \log f = l(\delta_g) - l(\delta_f) = l\left(\frac{\delta_g - \delta_f}{1 + \delta_f}\right) = \frac{\delta_g - \delta_f}{1 + \delta_f} + O\left(\left(\frac{\delta_g - \delta_f}{1 + \delta_f}\right)^2\right) > 0.$$

If $f \preccurlyeq 1$, then **e3** follows from Example 2.1.3. Now suppose that $1 \prec f$ and $f \in \operatorname{dom}\exp$. If $f < 0$, then $1 \prec f$ implies $f + 1 < 0 \leqslant e^f$. Otherwise **e3** is equivalent to $\log(f+1) \leqslant f$. Let $\mathfrak{d}_f = \log_i^{a_i} x \cdots \log_n^{a_n} x$ with $a_i > 0$. Then $(\log(f+1))^{\uparrow} = \log \mathfrak{d}_{f+1} = \log \mathfrak{d}_f \asymp \log_{i+1} x$. Hence $\log(f+1) \asymp \log_{i+1} x \prec \mathfrak{d}_f \asymp f$. In particular, we have **e3**.

By construction, **T1**, **T2** and **T3** are satisfied. As to **T4**, let $\mathfrak{m}_0 \in \log^{\mathbb{Z}^\star} x$, then for every $\mathfrak{m}_1 \in \operatorname{supp} \log \mathfrak{m}_0$ we have $\mathfrak{m}_1 = \log_N x$ for some $N \in \mathbb{N}^+$. Consequently, for every sequence $(\mathfrak{m}_i)_{0 \leqslant i}$ such that $\mathfrak{m}_{i+1} \in \operatorname{supp} \log \mathfrak{m}_i$, we have $\mathfrak{m}_{i+1} = \log_{N+i} x$ for all $i \geqslant 0$. But then $n_0 = 1$, and condition **T4** holds. □

Proposition 2.3.3 *Every non-trivial transseries field $\mathbb{T} = C[[\mathfrak{M}]] \neq C$ contains an isomorphic copy of $\mathbb{L}$.*

Proof: We claim that there exists a monomial $x \in \mathfrak{M}^{\uparrow}$ whose iterated logarithms are all monomials. Indeed, choosing $f \in \mathbb{T}_{\infty}^{+}$ arbitrarily, the series f is log-confluent at some order n_0, and we may take $x = \mathfrak{d}_{\log_{n_0} f}$. Our claim implies that $x, \log x, \log_2 x, \ldots$ are all monomials in $\mathfrak{M}$. Hence, $\log^{\mathbb{Z}^\star} x \subseteq \mathfrak{M}$, since $\mathfrak{M}$ is a group. We conclude that $\mathbb{L} \subseteq \mathbb{T}$. □

Remark 2.3.4 The above construction can be slightly generalized by considering the set $\mathbb{Z}^{\star\star}$ of infinite words $a = [a_0, a_1, \ldots]$ over $\mathbb{Z}$ instead of $\mathbb{Z}^\star$. The analogous construction then yields another transseries field $\hat{\mathbb{L}} = C[[\log^{\mathbb{Z}^{\star\star}} x]]$ which strictly contains $\mathbb{L}$.

2.3.2 Exponential closure

Let $\mathbb{T} = C[[\mathfrak{M}]]$ be a transseries field. In this section, we show how to construct an extension $\mathbb{T}_{\exp} = C[[\mathfrak{M}_{\exp}]]$ of $\mathbb{T}$, which itself is a transseries field such that the exponential function in $\mathbb{T}_{\exp}$ is totally defined on $\mathbb{T}$.

We take $\mathfrak{M}_{\exp} = \exp \mathbb{T}^{\uparrow}$, i.e. $\mathfrak{M}_{\exp}$ is the set of formal expressions $\exp f$ with $f \in \mathbb{T}^{\uparrow}$. We give $\mathfrak{M}_{\exp}$ the structure of a totally ordered monomial group, isomorphic to the additive group of $\mathbb{T}^{\uparrow}$ by setting $(\exp f)(\exp g) = \exp(f+g)$ and $\exp f \succcurlyeq \exp g \Leftrightarrow f \geqslant g$ for $f, g \in \mathbb{T}^{\uparrow}$. Let $f \in (\mathbb{T}_{\exp})_{\infty}^{+}$ with $\mathfrak{d}_f = \exp g$ for some $g \in \mathbb{T}^{\uparrow}$. Then we define

$$\log f = g + \log c_f + l(\delta_f),$$

where $l(z) = \log(1+z) \in C[[z]]$, as above. The field $\mathbb{T}_{\exp}$ together with the function log is called the **exponential extension** or **exp-extension** of $\mathbb{T}$.

Proposition 2.3.5 $\mathbb{T}_{\exp}$ *is a transseries field.*

Proof: We prove **e1**, **e2** and **e3** in the case when $f \notin (\mathbb{T}_{\exp})_{\infty}^{+}$ in a similar way as in Proposition 2.3.2. So assume that $f \in (\mathbb{T}_{\exp})_{\infty}^{+} \cap \operatorname{dom}\exp$. Since $\mathbb{T}$ is a transseries field and $\mathfrak{d}_{\log \mathfrak{d}_f} \in \mathfrak{M}$, we have $\log \mathfrak{d}_{\log \mathfrak{d}_f} \prec \mathfrak{d}_{\log \mathfrak{d}_f} \asymp \log \mathfrak{d}_f$. In particular, $(\log \mathfrak{d}_{\log \mathfrak{d}_f})^{\uparrow} < (\log \mathfrak{d}_f)^{\uparrow}$, so that $\log \mathfrak{d}_f \prec \mathfrak{d}_f$, by construction. We conclude that $\log(f+1) \asymp \log \mathfrak{d}_f \prec \mathfrak{d}_f \asymp f$, which implies **e3**.

By construction, we again have **T1**, **T2** and **T3**. We observe that these conditions imply that $\tau_{\log f} = \tau_{\log \tau_f}$ for all $f \in \mathbb{T}_{\infty}^{+}$. By induction, this yields $\tau_{\log_n f} = \tau_{\log_n \tau_f}$ for all $n > 0$. Now let $f \in \mathbb{T}_\infty^+$. Since $\mathbb{T}$ is a transseries field, the series $\log \tau_f \in \mathbb{T}$ is log-confluent at some order $n_0 \in \mathbb{N}$, i.e. $\tau_{\log_{n+1} \log \tau_f} = \log \tau_{\log_n \log \tau_f}$ for all $n \geqslant n_0$. Then the above observation implies that $\tau_{\log_{n+2} \log f} = \log \tau_{\log_{n+1} f}$ for all $n \geqslant n_0$. In other words, f is log-confluent at order $n_0 + 1$.

We finally remark that for every sequence $(\mathfrak{m}_i)_{0 \leqslant i}$ with $\mathfrak{m}_{i+1} \in \operatorname{supp} \log \mathfrak{m}_i$, the sequence $(\mathfrak{m}_i)_{1 \leqslant i}$ is in $\mathfrak{M}$. Hence since $\mathbb{T}$ is a transseries field, there is an integer $n_0 \geqslant 1$ such that **T4** holds for $(\mathfrak{m}_i)_{1 \leqslant i}$. But then **T4** holds for $(\mathfrak{m}_i)_{0 \leqslant i}$. □

Example 2.3.6 The series

$$\begin{aligned} f &= \frac{1}{x} + \frac{1}{e^{\log^2 x}} + \frac{1}{e^{\log^3 x}} + \cdots ; \\ g &= e^{e^x} + e^{e^x/x} + e^{e^x/x^2} + \cdots \end{aligned}$$

respectively belong to $\mathbb{L}_{\exp}$ and $\mathbb{L}_{\exp,\exp}$.

Remark 2.3.7 First we remark that the exponential function of $\mathbb{T}_{\exp}$ is totally defined on $\mathbb{T}$ for all transseries fields. Secondly, note that $C[[\log^{\mathbb{Z}^{\star\star}} x]]$ is a subset of $\mathbb{L}_{\exp}$, since $\log(\log^a x)$ is an element of $\mathbb{L}^{\uparrow}$. Recall that $\mathbb{L}_\infty^+$ is log-confluent at order 2. That is not true for $\mathbb{L}_{\exp}$ anymore. Take for instance $\exp 5x^2 \log^3 x$, which is log-confluent at order 3, but not at order 2. In general, in the n-th exp-extension of $\mathbb{L}$ the monomial $\exp_n 5x^2 \log^3 x$ is not log-confluent at order $n+1$.

Remark 2.3.8 Using exp-extensions, we can introduce an exponentiation with elements from C on $\mathbb{T}^+$ as follows. Let $f \in \mathbb{T}^+$ and $d \in C$. The series $d \cdot \log f$ is in $\mathbb{T}$, thus

$$f^d := e^{d \cdot \log f} \in \mathbb{T}_{\exp}.$$

We remark that $f^{c+d} = f^c \cdot f^d$ and that f^n defined as above coincides with the n-fold multiplication of f with itself.

Moreover, if $\mathfrak{M}$ is a group with C powers, then the above definition is coherent with the following alternative definition of exponentiation. Let $f = c\mathfrak{d} \cdot (1 + \delta)$. Then $c^d = e^{d \cdot \log c} \in C$, since $c > 0$. We define a formal power series $(1 + X)^d \in C[[X^{\mathbb{N}}]]$ as follows.

$$\begin{aligned} \binom{d}{i} &:= \prod_{j=1}^{i} \frac{d - (j-1)}{j} \\ (1+X)^d &:= \sum_{0 \leqslant i} \binom{d}{i} X^i. \end{aligned}$$

One checks that $(1+X)^{d_1} \cdot (1+X)^{d_2} = (1+X)^{d_1+d_2}$ for $d_1, d_2 \in C$. Applying Proposition 1.6.6 yields $(1+\delta)^d \in \mathbb{T}$ and $(1+\delta)^{d_1} \cdot (1+\delta)^{d_2} = (1+\delta)^{d_1+d_2}$. One verifies

$$f^d = c^d \cdot \mathfrak{d}^d \cdot (1+\delta)^d.$$

In particular, if $\mathfrak{M}$ has C-powers, then $f^d \in \mathbb{T}$ for all $f \in \mathbb{T}^+$ and $d \in C$.

Note that the definition of f^d makes it possible to define a relation $\prec\!\!\prec_C$ as in Section 1.2, i.e. $f \prec\!\!\prec_C g$ iff $\|f^c\| < \|g^d\|$ in $\mathbb{T}_{\exp}$. We remark that $\prec\!\!\prec_C$ coincides with the definition of $\prec\!\!\prec$ in this chapter.

2.3.3 Inductive limits

Let I be a totally ordered index set and $(\mathbb{T}_i)_{i\in I}$ a family of transseries fields $\mathbb{T}_i = C[[\mathfrak{M}_i]]$ such that $\mathbb{T}_i$ is an exp-log subfield of $\mathbb{T}_j$ whenever $i \leqslant j$. In particular, we have $\mathfrak{M}_i \subseteq \mathfrak{M}_j$ and $\log_{\mathbb{T}_i}$ is the restriction of $\log_{\mathbb{T}_j}$ to $\mathbb{T}_i^+$. Consider the fields

$$\begin{aligned}\check{\mathbb{T}} &= \bigcup_{i\in I} \mathbb{T}_i;\\ \mathbb{T} &= C[[\mathfrak{M}]], \text{ with } \mathfrak{M} = \bigcup_{i\in I} \mathfrak{M}_i.\end{aligned}$$

Then $\check{\mathbb{T}}$ naturally has the structure of an exp-log field, such that each $\mathbb{T}_i$ is an exp-log subfield of $\check{\mathbb{T}}$. Given $f \in \mathbb{T}^+$ with $\mathfrak{d}_f \in \mathfrak{M}_i$, we define its logarithm by $\log f = \log_{\mathbb{T}_i} \mathfrak{d}_f + \log c_f + \log \delta_f$.

Clearly, $\check{\mathbb{T}} \subseteq \mathbb{T}$, but this inclusion is usually strict: consider $(\mathbb{T}_i)_{i\in\mathbb{N}}$ with $\mathbb{T}_0 = \mathbb{L}, \mathbb{T}_1 = \mathbb{L}_{\exp}, \mathbb{T}_2 = \mathbb{L}_{\exp,\exp}, \ldots$. Then

$$\frac{1}{x} + \frac{1}{\exp x} + \frac{1}{\exp \exp x} + \cdots$$

is in $\mathbb{T}$, but not in $\check{\mathbb{T}}$. In fact, the field $\check{\mathbb{T}}$ will in general be incomplete.

Proposition 2.3.9 *$C[[\bigcup_{i\in I} \mathfrak{M}_i]]$ is a transseries field.*

Proof: We prove **e1**, **e2** and **e3** in the case when $f \notin \mathbb{T}_\infty^+$ in a similar way as in Proposition 2.3.2. So assume that $f \in \mathbb{T}_\infty^+ \cap \operatorname{dom} \exp$. Then there exists an $i \in I$ with $\mathfrak{d}_f \in \mathfrak{M}_i$. Consequently, we have $\log(f+1) \asymp \log \mathfrak{d}_f \prec \mathfrak{d}_f \asymp f$, since $\log \mathfrak{d}_f$ and $\mathfrak{d}_f$ are both in $\mathbb{T}_i$. In particular, $\log(f+1) < f$, which implies **e3**.

The properties **T1**, **T2** and **T3** are satisfied by construction. As to **T4**, let $\mathfrak{m} \in \bigcup_I \mathfrak{M}_i$, then $\mathfrak{m} \in \mathfrak{M}_i$ for some $i \in I$. Condition **T4** for $\mathfrak{m}$ follows now from the same condition in $\mathbb{T}_i$. □

Proposition 2.3.10 *Let α be an ordinal, and let $\mathfrak{M}_\beta$ be monomial groups for $\beta < \alpha$ such that all $\mathbb{T}_\beta = C[[\mathfrak{M}_\beta]]$ are transseries fields and such that $\mathbb{T}_\beta$ is an* exp-log *sub-field of $\mathbb{T}_\gamma$ for all $\beta \leqslant \gamma$. Suppose that $J \subseteq \alpha$ is cofinal in α and such that $\mathfrak{M}_{j+1} = \mathfrak{M}_{j,\exp}$ for all $j \in J$. Then* exp *is total on $\check{\mathbb{T}}$.*

Proof: Let $f \in \check{\mathbb{T}}$. Then $f \in \mathbb{T}_\beta$ for some $\beta < \alpha$. Let $\beta < j \in J$. Then $f^\uparrow \in \mathbb{T}_j^\uparrow$ and thus $\exp f^\uparrow \in \mathfrak{M}_{j+1}$. But then $\exp f = \exp f^\uparrow \cdot \exp f^= \cdot e(f^\downarrow) \in \mathbb{T}_{j+1}$. □

2.3.4 Inductive exponential closure

Let $\mathbb{T}$ be a transseries field. As an application of Propositions 2.3.5 and 2.3.9, we construct for each ordinal α a transseries field $\mathbb{T}_\alpha = C[[\mathfrak{M}_\alpha]]$ as follows:

$$\begin{aligned}\mathbb{T}_0 &= \mathbb{T};\\ \mathbb{T}_{\alpha+1} &= \mathbb{T}_{\alpha,\exp}, \text{ for successor ordinals } \alpha+1;\\ \mathbb{T}_\lambda &= C[[\bigcup_{\alpha<\lambda} \mathfrak{M}_\alpha]], \text{ for limit ordinals } \lambda.\end{aligned}$$

For limit ordinals λ, we also define

$$\mathbb{T}_{<\lambda} = \bigcup_{\alpha<\lambda} \mathbb{T}_\alpha.$$

Note that exp is total on $\mathbb{T}_{<\omega}$. We call $\mathbb{T}_{<\omega}$ the **exponential closure** of $\mathbb{T}$ and denote it also by $C_{<\omega}[[\mathfrak{M}]]$. We remark that the exponential closures of $C[[\log^{\mathbb{Z}^\star} x]]$ and $C[[\log^{\mathbb{Z}^{\star\star}} x]]$ are the same fields, i.e.

$$C_{<\omega}[[\log^{\mathbb{Z}^\star} x]] = C_{<\omega}[[\log^{\mathbb{Z}^{\star\star}} x]].$$

Remark 2.3.11 We will later show that if one puts a restriction on the cardinality of the support, then the transfinite sequence $\mathbb{T}_\alpha$ stabilizes from a certain limit ordinal λ on. For instance, if one adds to the definition of a transseries that the support should not have a cardinality larger then some fixed cardinal κ, then we will find such a λ which depends on κ. At this stage, we have $\mathbb{T}_\lambda = \mathbb{T}_{<\lambda}$.

This property does no longer hold if one does not put a restriction on the cardinality of the support, which means that $\mathbb{T}_\lambda$ is incomplete and thus no transseries field.

Example 2.3.12 Let $\mathbb{T} = \mathbb{L} = C[[\log^{\mathbb{Z}^\star} x]]$. Then we also denote $\mathbb{T}_\alpha$ by $C_\alpha[[[x]]]$. Accordingly, we write $\mathbb{T}_{<\lambda} = C_{<\lambda}[[[x]]]$. Note that

$$e^{-x} + e^{-e^x} + e^{-e^{e^x}} + \cdots \in C_\omega[[[x]]] \setminus C_{<\omega}[[[x]]].$$

2.4 More on the supports

In this section we apply the results from Section 1.8 to transseries fields. Throughout this section we assume that C has cofinal cardinality $< \kappa_1$. We start with a direct consequence of Lemma 1.8.6.

Lemma 2.4.1 *Let $\mathbb{T} = C[[\mathfrak{M}]]$ and $\widetilde{\mathbb{T}} = C[[\mathfrak{N}]]$ such that $\mathfrak{N}$ has cofinal cardinality $< \kappa_2$. Let $f \in \mathbb{T}$ such that for every $\mathfrak{m} \in \mathfrak{M}$ we have $\log \mathfrak{m} \in \widetilde{\mathbb{T}}$. Then the support of f has cardinality $< \max(\kappa_1, \kappa_2)$.*

Proof: The sequence $(\log \mathfrak{m})_{\mathfrak{m} \in \operatorname{supp} f}$ is a strictly decreasing sequence in $\widetilde{\mathbb{T}}$. Now the lemma follows from Lemma 1.8.5. □

Example 2.4.2 Let $\mathfrak{M} = \log^{\mathbb{Z}^\star} x$ and $C = \mathbb{R}$. For every $\mathfrak{m} = \log^k x \in \mathfrak{M}^\uparrow$ with $k \in \mathbb{Z}^n$ the support of $\log \mathfrak{m} = \sum k_i \log_{i+1} x$ is finite, hence countable. Moreover, in order to apply Lemma 2.4.1 we can take $\widetilde{\mathbb{T}} = \mathbb{T}$. Hence every element in $\mathbb{L} = \mathbb{R}[[\log^{\mathbb{Z}^\star} x]]$ has countable support. In general, the lemma shows that no element of $C_1[[[x]]]$ has support κ_1.

Lemma 2.4.3 *Let $f \in \mathbb{T}_{\exp}$. Let κ_2 be a cardinal such that $\mathfrak{M}$ has cofinal cardinality $< \kappa_2$. Then f has support of cardinality $< \max(\kappa_1, \kappa_2)$.*

Proof: Apply Lemma 2.4.1 with $\widetilde{\mathbb{T}} = \mathbb{T}$. . □

Remark 2.4.4 It should be noticed that the assumption on C is essential here, if $\kappa_2 < \kappa_1$. Choose an ordinal α with $|\alpha| = \kappa_1$. Suppose that $B = (b_i)_{i<\alpha}$ is a well-ordered subset of C. Let $\mathfrak{m} \in \mathfrak{M}^{\uparrow}$ and $\varphi_i = e^{-b_i \mathfrak{m}}$ for all $i < \alpha$. Then

$$\varphi_0 \succ \varphi_1 \succ \varphi_2 \succ \cdots$$

and $\sum_{i<\alpha} \varphi_i$ is a series in $C[[\mathfrak{M}_{\exp}]]$ which has a support of cardinality α.

Let again $\mathfrak{M}$ have cofinal cardinality $< \kappa_2$. From Lemma 2.4.3 it follows that series from $\mathbb{T}_{\exp}$ cannot have a support of cardinality $\max(\kappa_1, \kappa_2)$. Replacing $\mathbb{T}$ and $\mathbb{T}_{\exp}$ by $\mathbb{T}_{\exp}$ and its exp-extension, we find that the same holds for $\mathbb{T}_{\exp,\exp}$. We can continue this process and ask how long we can keep control over the support. To this end let $\mathfrak{M}_0 = \mathfrak{M}$, $\mathbb{T}_\alpha = C[[\mathfrak{M}_\alpha]]$ and $\mathfrak{M}_{\alpha+1} = \exp \mathbb{T}_\alpha^{\uparrow}$. For limit ordinals λ we let $\mathfrak{M}_\lambda = \bigcup_{\alpha<\lambda} \mathfrak{M}_\alpha$. Moreover, we let $\mathfrak{M}_{-1} = \emptyset$.

Corollary 2.4.5 *Let α be an ordinal with $|\alpha| < \max(\kappa_1, \kappa_2)$. Then the support of $f \in \mathbb{T}_\alpha$ has cardinality $< \max(\kappa_1, \kappa_2)$. In particular, if α is a countable ordinal, then for $f \in \mathbb{R}_\alpha[[[x]]]$ the support* $\operatorname{supp} f$ *is countable.*

Proof: Let α be the smallest ordinal such that the corollary is wrong. From Example 2.4.2 it follows that $0 < \alpha$. If α was a successor ordinal, then this would contradict Lemma 2.4.3. Let α be a limit ordinal and $(\mathfrak{m}_i)_{i\in I}$ a well-ordered set in $\mathfrak{M}_\alpha$. Let for all successor ordinals $\beta < \alpha$

$$I_\beta = \{i \in I \,|\, \mathfrak{m}_i \in \mathfrak{M}_\beta \setminus \mathfrak{M}_{\beta-1}\}.$$

Every $(\mathfrak{m}_i)_{i\in I_\beta}$ has cardinality $< \max(\kappa_1, \kappa_2)$. The first part of the lemma follows from $|I| \leqslant |\alpha|$. For the second part, apply Lemma 1.8.6. □

Remark 2.4.6 Even if we replace $\log^{\mathbb{Z}^\star} x$ by the larger group $\log^{\mathbb{Z}^{\star\star}} x$, the support of every element of the resulting field $C[[\log^{\mathbb{Z}^{\star\star}} x]]$ is countable. To see this, we apply Lemma 2.4.1 with $\widetilde{\mathbb{T}} = C[[\log^{\mathbb{Z}^\star} x]]$. It should be noticed that we do not demand C to be archimedean. This hypothesis is, however, essential in the study of supports of series in the field of logarithmic-exponential series in [vdDMM97]. From the construction of the field $\mathbb{R}((t))^{LE}$ and Lemma 2.4.1 it follows that every element of this field has countable support if and only if every element of $\mathbb{R}((t^{\mathbb{R}}))$ has countable support. But this is ensured by Proposition 1.8.3.

2.5 Nested transmonomials and transseries

Given a transseries field $\mathbb{T}$, we have seen how to construct an extension $\mathbb{T}_{\exp}$ such that the domain of exp contains $\mathbb{T}$. Taking inductive limits, we have shown how to extend $\mathbb{T}$ into fields of generalized power series which are closed under exponentiation. However, a nested transseries like

$$e^{x^2 + e^{\log_2^2 x + e^{\log_4^2 x + e^{\cdot^{\cdot^{\cdot}}}}}} \tag{2.1}$$

has no reason to belong to $\mathbb{T}$. This transseries occurs for instance as natural solution to the functional equation

$$f(x) = e^{x^2 + f(\log_2 x)}.$$

In this section, we show how to construct extensions of $\mathbb{T}$ which contain expressions like (2.1).

More precisely, we want to add expressions of the form

$$e^{\varphi_0 \pm e^{\varphi_1 \pm e^{\cdot^{\cdot^{\cdot}}}}},$$

where $\varphi_0, \varphi_1, \ldots \in \mathbb{T}$. The series φ_i will have to satisfy a certain condition imposed by condition **T4** in order to avoid expressions like

$$e^{x^2 + e^{\log_2^2 x + e^{\log_4^2 x + e^{\cdot^{\cdot^{\cdot}}}} + \log_5 x} + \log_3 x} + \log x}. \tag{2.2}$$

Next we introduce an ordering on the multiplicative group generated by the new expressions. We define a logarithm and show that the field over the new ordered multiplicative group is a transseries field.

Remark 2.5.1 Nested expressions like (2.1) also occur naturally in the characterization of intervals of transseries. For more on this see [vdH97].

2.5.1 Determining sequences

Let $\varphi = (\varphi_0, \varphi_1, \ldots)$ and $\sigma = (\sigma_0, \sigma_1, \ldots) \in \{-1, +1\}^{\mathbb{N}}$ be sequences such that

NM1. $\forall i \geqslant 0 : \varphi_i \in \mathbb{T}^{\uparrow} \wedge 0 < \varphi_{i+1}$,

NM2. $\forall i \geqslant 0 : \forall \mathfrak{m} \in \operatorname{supp} \varphi_i : \exists j > i : \forall \psi \in \mathbb{T}^{\uparrow}$:

$$\operatorname{supp} \varphi_j \succ \psi \Rightarrow \mathfrak{m} \succ \sigma_i \, e^{\varphi_{i+1} + \sigma_{i+1} \, e^{\cdot^{\cdot^{\cdot \sigma_{j-1} \, e^{\varphi_j + \psi}}}}}.$$

We say that the pair (σ, φ) **determines** the **nested transmonomial**

$$\mathfrak{n}_{\sigma,\varphi} = e^{\varphi_0 + \sigma_0 \, e^{\varphi_1 + \sigma_1 \, e^{\cdot^{\cdot^{\cdot}}}}}.$$

Remark 2.5.2 Condition **NM1** makes sure that nested transmonomials cannot be developed as series, which corresponds to condition **T2**. Moreover, it will ensure that the logarithm of a nested transmonomial will be a series with purely infinite support.

Similarly, condition **NM2** corresponds to condition **T4**. Actually, if one thinks of the nested transmonomial determined by a pair (σ, φ), then we should have

$$\operatorname{supp} \varphi_i \succ e^{\varphi_{i+1} + \sigma_{i+1} \, e^{\cdot^{\cdot^{\cdot}}}}$$

for each i. This, however, presumes an ordering on the set of nested transmonomials, which is yet to be defined. We have thus to find a condition in $\mathbb{T}$ expressing this property. Actually, if we weaken condition **MN2** to

$$\forall i \geqslant 0 : \exists j > i : \forall \psi \in \mathbb{T}^{\uparrow} : \operatorname{supp} \varphi_j \succ \psi \Rightarrow \operatorname{supp} \varphi_i \succ \sigma_i \, e^{\varphi_{i+1}+\sigma_{i+1} \, e^{\cdot^{\cdot^{\cdot \sigma_{j-1} \, e^{\varphi_j+\psi}}}}}, \tag{2.3}$$

then we lose transmonomials. Let for instance

$$\mathfrak{a}_0 = e^{x^2}, \ \mathfrak{a}_1 = e^{x^2 - e^{\log_2^2 x}}, \ \mathfrak{a}_2 = e^{x^2 - e^{\log_2^2 x + e^{\log_4^2 x}}}, \ \ldots, \mathfrak{a}_i = e^{x^2 - e^{\log_2^2 x + e^{\cdot^{\cdot^{\cdot e^{\log_{2i}^2 x}}}}}}, \ldots .$$

Then $\mathfrak{a}_0 \succ \mathfrak{a}_1 \succ \cdots$. The series $\varphi_0 = \sum_i \mathfrak{a}_i$ exists in every transseries field containing the exponential closure of $\mathbb{L}$. We let $\sigma = (1, -1, 1, 1, \ldots)$ and $\varphi_i = \log_{2i-2}^2 x$ for $i \geqslant 1$. The couple (σ, φ) satisfies the conditions **NM1** and **NM2**. Hence it determines a nested transmonomial. Condition (2.3) fails. For j we may choose $\psi = \log_{2j+1} x$ to obtain a counter-example.

Remark 2.5.3 Instead of restricting the values of σ_i to ± 1, one might want to let these coefficients range over all non-zero elements from C. In fact, it would be possible to modify our definition of nested monomials by allowing sequences $\sigma = (\sigma_0, \sigma_1, \ldots)$ with $\sigma_i \in C^*$ such that $\sigma_i = \pm 1$ for all i greater than some $I \in \mathbb{N}$. We remark, though, that these monomials can be obtained by using the more restrictive definition and exponential extensions. Since we eventually construct exponential closures, we will obtain all monomials with σ above even with our definition.

It should be noticed, however, that one cannot let σ be an arbitrary sequence in C^*. Allowing σ that general can lead to incoherences, for instance concerning the definition of a total ordering on the extended set of monomials. To illustrate the last point, let

$$f(x) = 2e^{\sqrt{x}+e^{\sqrt{\log x}+2e^{\sqrt{\log_2 x}+e^{\cdot^{\cdot^{\cdot}}}}}} \quad \text{and} \quad f(x) = e^{\sqrt{x}+2e^{\sqrt{\log x}+e^{\sqrt{\log_2 x}+2e^{\cdot^{\cdot^{\cdot}}}}}}.$$

Then $f(x)$ and $g(x)$ are formal solutions of the system of equations

$$\begin{aligned} f(x) &= 2e^{\sqrt{x}+g(\log x)} \\ g(x) &= e^{\sqrt{x}+f(\log x)}. \end{aligned}$$

Assume that $f(x) \prec g(x)$. Replacing x by $\log x$ should preserve the ordering, hence

$$f(\log x) \prec g(\log x).$$

This implies $\sqrt{x} + f(\log x) < \sqrt{x} + g(\log x)$. Whence

$$g(x) = e^{\sqrt{x}+f(\log x)} \prec e^{\sqrt{x}+g(\log x)} = \frac{1}{2} f(x).$$

Thus $f(x) \prec g(x)$ implies $g(x) \prec f(x)$. A similar contradiction can be obtained from the assumption $g(x) \prec f(x)$.

Remark 2.5.4 The monomial group $\mathfrak{M}$ can already contain nested transmonomials $\mathfrak{n}$. By this we mean monomials also in the broader sense where σ_i can take any value from C^* for a finite number of integers i. The sequences σ and φ corresponding to $\mathfrak{n}$ are uniquely determined as follows. By condotion **T2**, the series $\log \mathfrak{n}$ is purely infinite. Since $\mathfrak{n}$ is nested, the support of $\log \mathfrak{n}$ has a least element in the ordering $\succcurlyeq$. More generally, if we let $\mathfrak{m}_{-1} = \mathfrak{n}$, then for all $i \geqslant 0$ we have

$$\exists \mathfrak{m}_i \in \operatorname{supp} \log \mathfrak{m}_{i-1} : \forall \mathfrak{m} \in \operatorname{supp} \log \mathfrak{m}_{i-1} : \mathfrak{m} \succcurlyeq \mathfrak{m}_i.$$

Then we let

$$\begin{aligned} \varphi_i &:= \log \mathfrak{m}_{i-1} - \mathfrak{m}_i \\ \sigma_i &:= (\log \mathfrak{m}_{i-1})_{\mathfrak{m}_i}. \end{aligned}$$

One shows inductively that σ_i and φ_i exist for all i. By **T4**, there is some i_0 such that $\sigma_i \in \{-1, +1\}$ for all $i \geqslant i_0$. If $\mathfrak{n}$ was not a nested monomial, then the above process terminates since for some i the support of $\log \mathfrak{m}_{i-1}$ has not a least element.

Inversely, let $\sigma \in C^{\mathbb{N}}$ such that eventually $\sigma_i \in \{-1, 0, +1\}$, and $\varphi \in \mathbb{T}^{\mathbb{N}}$ be arbitrary sequences with properties **NM1** and **NM2**. Then we remark that there are three different cases to distinguish. First consider sequences with $\exists n : \sigma_n = 0$. The monomials $\mathfrak{n}_{\sigma,\varphi}$ are said to be **non-nested**. Note that since $\varphi_n \in \mathbb{T}^{\uparrow}$, the monomial $\mathfrak{n}_{\sigma,\varphi}$ is the element of some transfinite exponential extension $\mathbb{T}_\alpha$ (where $0 \leqslant \alpha$).

The remaining two cases suppose $\forall n : \sigma_n \neq 0$. For every $i \geqslant 0$ we let $\mathfrak{n}_i$ be the nested transmonomial determined by the sequences $(\sigma_i, \sigma_{i+1}, \dots)$ and $(\varphi_i, \varphi_{i+1}, \dots)$. If for some $i \in \mathbb{N}$, the monomial $\mathfrak{n}_i$ is in some transfinite exponential extension of $\mathbb{T}$, then so is $\mathfrak{n}_{\sigma,\varphi}$. In other words, we will capture $\mathfrak{n}_{\sigma,\varphi}$ using the exponential extension process.

The last case concerns sequences σ, φ such that no $\mathfrak{n}_i$ is in some transfinite exponential extension, which is to say that $\mathfrak{n}_{\sigma,\varphi}$ does not result from the exponential extension process. It is this kind of monomials we want to add in this section.

The set of all sequences (σ, φ) with $\sigma \in \{-1, +1\}^{\mathbb{N}}$ and properties **NM1, NM2** such that $\mathfrak{n}_{\sigma,\varphi}$ is not in some transfinite exponential extension of $\mathbb{T}$ will be denoted by $\mathcal{N}_{\mathbb{T}}$. We let

$$\mathfrak{N}_{\mathbb{T}} := \{\mathfrak{n}_{\sigma,\varphi} \mid (\sigma, \varphi) \in \mathcal{N}_{\mathbb{T}}\}.$$

2.5.2 Nested extensions: One-by-one vs. All-at-once

Before describing the extension by nested transmonomials, let us discuss two options. Either we extend $\mathfrak{M}$ by all of $\mathfrak{N}_{\mathbb{T}}$, or we take one couple $(\sigma, \varphi) \in \mathcal{N}_{\mathbb{T}}$ and extend $\mathfrak{M}$ by $\mathfrak{n}_{\sigma,\varphi}$. Let us informally consider the pros and cons of both possibilities.

The first option appeals for its strength; one adds all possible monomials. In opposition to the second approach, one will never have to be concerned about sequences from $\mathcal{N}_{\mathbb{T}}$ anymore, once that extension step is done. There are, however, disadvantages when using this extension process. Formulating the conditions about the monomials becomes rather cumbersome, as many indices are involved. Especially, defining the ordering on the new set of monomials is very tedious.

One has to distinguish a number of cases, which does not really contribute to the understanding of the nature of nested transmonomials. We remark that this is nonetheless possible.

Generally speaking, the disadvantages of one method are the advantages of the other. In this sense the description of the extended monomial group and its ordering become easier and more transparent for the second option. This also enhances working with the extended set. On the other hand, we lose the advantage of having added all possible nested monomials. Also, one might ask whether inductively applying this extension process actually depends on the order in which couples from $\mathcal{N}_{\mathbb{T}}$ are chosen.

Keeping in mind that one will in general not be interested to extend just one transseries field by nested monomials, but that one will rather aim at constructing fields which are closed under nested monomials, we state that this virtual disadvantage is actually none. Using the first option, i.e. adding all possible nested transmonomials determined by a field $\mathbb{T}$, we obtain a field $\mathbb{T}_{\text{nest}}$ which in general will not be closed under nested monomials. Hence in this case, we would have to continue the extension process in a similar way as we had to do for the exponential closure. We will thus not escape from taking inductive limits in pursuing this aim.

Another similarity to the exponential case is that taking inductive limits only stabilizes the field when we work under the assumption of a support-constraint. But then both methods will lead to the same stable field.

For these reasons we have opted to continue with the extension process in the case where we only add one single new nested transmonomial.

2.5.3 Extending by nested monomials

Throughout this section we fix a couple $(\sigma, \varphi) \in \mathcal{N}_{\mathbb{T}}$. Recall that for every $i \geqslant 0$ we let $\mathfrak{n}_i$ be the nested transmonomial determined by the sequences $(\sigma_i, \sigma_{i+1}, \dots)$ and $(\varphi_i, \varphi_{i+1}, \dots)$. In particular, $\mathfrak{n}_i \notin \mathfrak{M}$.

If $\hat{\mathbb{T}}$ is a transseries field extension of $\mathbb{T}$ with monomial group $\hat{\mathfrak{M}}$ and $\mathfrak{n}_i \in \hat{\mathfrak{M}}$, then $\mathfrak{n}_{i+1} \in \hat{\mathfrak{M}}$ since $\log \mathfrak{n}_i \in \hat{\mathbb{T}}^{\uparrow}$ by **T2** and thus, since $\mathfrak{n}_{i+1}$ is a monomial,

$$\mathfrak{n}_{i+1} = \frac{1}{\sigma_i}(\log \mathfrak{n}_i - \varphi_i) \in \hat{\mathfrak{M}}^{\uparrow}.$$

Hence, if we want to add $\mathfrak{n}_0$, we have to add $\mathfrak{n}_1, \mathfrak{n}_2, \dots$ as well. The smallest group containing $\mathfrak{M}$ and $\mathfrak{n}_0$ is thus the multiplicative group generated by

$$\mathfrak{M} \cup \{\mathfrak{n}_0, \mathfrak{n}_1, \dots\}.$$

Recall that $\mathfrak{n}^a = \mathfrak{n}_0^{a_0} \cdots \mathfrak{n}_n^{a_n}$ for $a \in \mathbb{Z}^\star$ with $n = |a|$ and that $\mathfrak{n}^0 = 1$. Let

$$\mathfrak{M}_{\sigma,\varphi} := \{\mathfrak{a} \cdot \mathfrak{n}^a \mid \mathfrak{a} \in \mathfrak{M} \wedge a \in \mathbb{Z}^\star\}.$$

We define a multiplication on $\mathfrak{M}_{\sigma,\varphi}$ by

$$(\mathfrak{a} \cdot \mathfrak{n}^a) \cdot (\mathfrak{b} \cdot \mathfrak{n}^b) := \mathfrak{a}\mathfrak{b} \cdot \mathfrak{n}^{a+b}.$$

Then $\mathfrak{M}_{\sigma,\varphi}$ is a multiplicative group extending $\mathfrak{M}$. In order to extend $\succcurlyeq$ to $\mathfrak{M}_{\sigma,\varphi}$, we will first characterize when $\mathfrak{n}^a \succ \mathfrak{a}$ for $\mathfrak{a} \in \mathfrak{M}$ and $a \in \mathbb{Z}^\star$. We distinguish three cases:

Case 1: $\mathfrak{n}^a = \mathfrak{n}_0$
Case 2: $\mathfrak{n}^a = \mathfrak{n}_i$ with $i \geqslant 0$
Case 3: $\mathfrak{n}^a$ for general $a \in \mathbb{Z}^\star$

Remark 2.5.5 The cases are ordered by their generality, i.e. case 1 is a sub-case of case 2, which in turn is a sub-case of case 3. We point out that a definition for case 3 alone is possible, but we have decided to differentiate for better readability.

Case 1: Let ϑ_0 be the maximal truncation of $\log \mathfrak{a}$ such that

$$\forall \mathfrak{v} \in \operatorname{supp} \vartheta_0 : \exists j > 0 : \forall \psi \in \mathbb{T}^\uparrow : \operatorname{supp} \varphi_j \succ \psi \Rightarrow \mathfrak{v} \succ e^{\varphi_1 + \sigma_1 e^{\cdot^{\cdot^{\cdot \varphi_j + \psi}}}}.$$

Note that ϑ_0 is uniquely defined. Let $\mathfrak{m}_1$ and d_0 be the leading monomial and coefficient of $\log \mathfrak{a} - \vartheta_0$, i.e.

$$d_0 \mathfrak{m}_1 = \tau(\log \mathfrak{a} - \vartheta_0).$$

Moreover, let $\rho_0 \in \mathbb{T}^\uparrow$ such that $\log \mathfrak{a} = \vartheta_0 + d_0 \mathfrak{m}_1 + \rho_0$. Either of d_0 and ρ_0 can be 0. We recursively define

$$\mathfrak{n}_0 \succ \mathfrak{a} \quad :\Leftrightarrow \quad \begin{cases} \varphi_0 > \vartheta_0 & & & \text{or} \\ \varphi_0 = \vartheta_0 & \wedge\, \mathfrak{m}_1 \prec \mathfrak{n}_1 & \wedge\, 0 < \sigma_0 & \text{or} \\ \varphi_0 = \vartheta_0 & \wedge\, \mathfrak{m}_1 \succ \mathfrak{n}_1 & \wedge\, 0 > d_0 & \text{or} \\ \varphi_0 = \vartheta_0 & \wedge\, d_0 = 0 & \wedge\, 0 < \sigma_0. & \end{cases}$$

Indeed, in order to decide whether $\mathfrak{n}_1 \succ \mathfrak{m}_1$, we use the same procedure with $\mathfrak{m}_1$ and $\mathfrak{n}_1$ in place of $\mathfrak{a}$ and $\mathfrak{n}_0$ respectively. We have to show, though, that this procedure terminates. To do this, we construct sequences

$$\begin{gathered} (\mathfrak{m}_0, \mathfrak{m}_1, \dots) \\ (\vartheta_0, \vartheta_1, \dots) \\ (d_0, d_1, \dots) \\ (\rho_0, \rho_1, \dots) \end{gathered}$$

as follows. Let $\mathfrak{m}_0 := \mathfrak{a}$ and $\mathfrak{m}_1, \vartheta_0, d_0$ and ρ_0 as above. For $i > 0$, we suppose that $\mathfrak{m}_i, \vartheta_{i-1}, d_{i-1}$ and ρ_{i-1} are already defined. Then let ϑ_i be the maximal truncation of $\log \mathfrak{m}_i$ such that

$$\forall \mathfrak{v} \in \operatorname{supp} \vartheta_i : \exists j > i : \forall \psi \in \mathbb{T}^\uparrow : \operatorname{supp} \varphi_j \succ \psi \Rightarrow \mathfrak{v} \succ e^{\varphi_{i+1} + \sigma_{i+1} e^{\cdot^{\cdot^{\cdot \varphi_j + \psi}}}}.$$

Again, ϑ_i is (uniquely) defined, if $d_{i-1} \neq 0$. We denote the leading monomial and coefficient of $\log \mathfrak{m}_i - \vartheta_i$ by $\mathfrak{m}_{i+1}$ and d_i respectively, i.e.

$$d_i \mathfrak{m}_{i+1} = \tau(\log \mathfrak{m}_i - \vartheta_i).$$

The process terminates if $d_i = 0$ for some i. If this is the case, then $\mathfrak{n}_i \succ \mathfrak{m}_i$ can be decided. We may thus assume that $d_i \neq 0$ for all i. Note that by condition **T4** for the field $\mathbb{T}$, there is some $i_0 \in \mathfrak{n}$ such that for all $i \geqslant i_0$:

$$\begin{array}{rcl} \rho_i & = & 0 \\ d_i & \in & \{-1,+1\}. \end{array}$$

Let p be minimal such that $\vartheta_p \neq \varphi_p$. If p exists, then $\mathfrak{m}_p \succ \mathfrak{n}_p$ is decidable, and we have thus recursively defined $\mathfrak{n}_0 \succ \mathfrak{a}$. It remains the situation where p does not exist, i.e. where $\vartheta_i = \varphi_i$ for all i. Let i_0 be large enough such that $\sigma_i, d_i \in \{-1,+1\}$ for all $i \geqslant i_0$. Since $\mathfrak{n}_0 \in \mathfrak{N}_{\mathbb{T}}$, there must be some minimal $q \geqslant i_0$ with $\sigma_i \neq d_i$. We let

$$\mathfrak{m}_q \succ \mathfrak{n}_q \quad :\Leftrightarrow \quad \sigma_q = -1 < 1 = d_q.$$

Again, we can now recursively decide $\mathfrak{n}_{q-1} \succ \mathfrak{m}_{q-1}, \ldots, \mathfrak{n}_0 \succ \mathfrak{m}_0$. This finishes the first case.

Case 2: Replace in case 1 systematically $\mathfrak{n}_j$ by $\mathfrak{n}_{i+j}$ for all $j \geqslant 0$.

Case 3: Let $a \in \mathbb{Z}^\star$ and i be minimal with $a_i \neq 0$. Note that

$$t = \log \mathfrak{a} - (a_i \varphi_i + \cdots + a_n \varphi_n) \in \mathbb{T}^{\uparrow}.$$

Since $\mathfrak{n}_i \notin \mathfrak{M}$, we have $\mathfrak{n}_i \neq \mathfrak{d}_t$, and we let

$$\mathfrak{n}^a \succ \mathfrak{a} :\Leftrightarrow \sigma_i a_i \, \mathfrak{n}_{i+1} > \tau_t \;\Leftrightarrow\; \begin{cases} \mathfrak{d}_t \prec \mathfrak{n}_{i+1} & \wedge\ 0 < \sigma_i a_i \\ \\ \mathfrak{d}_t \succ \mathfrak{n}_{i+1} & \wedge\ 0 < \sigma_i c_t. \end{cases}$$

This finishes the third case of the definition of the ordering on $\mathfrak{M}_{\sigma,\varphi}$.

Remark 2.5.6 In order to show that the definition is correct, we claim that one of p and q in the first case exists. Suppose that p and q do not exist. Then we have in particular $d_i \neq 0$ for all i. Condition **T4** implies then that there is an integer i such that $\mathfrak{m}_j = \mathfrak{n}_{j+i}$ for all $j \geqslant i$. But this contradicts $(\sigma, \varphi) \in \mathcal{N}_{\mathbb{T}}$.

Remark 2.5.7 Let us motivate the definition of the ordering in case 3. In order to extend $\mathbb{T}$ into a transseries field, we will define a logarithm. This function log has to extend the logarithm of $\mathbb{T}$, it has to satisfy the functional equation $\log xy = \log x + \log y$ and it has to satisfy $1 < x \Rightarrow 0 < \log x$. In particular,

$$\begin{array}{rcl} \mathfrak{a} \prec \mathfrak{n}^a & \Leftrightarrow & \log \mathfrak{a} < a_i \mathfrak{n}_i + \cdots + a_n \mathfrak{n}_n \\ & \Leftrightarrow & \log \mathfrak{a} < (a_i \varphi_i + \cdots + a_n \varphi_n) + (\sigma_i a_i \, \mathfrak{n}_{i+1} + \cdots + \sigma_n a_n \, \mathfrak{n}_{n+1}). \end{array}$$

Also, we will show that $\mathfrak{n}_i \succ \mathfrak{n}_{i+1}, \mathfrak{n}_{i+2}, \ldots$. Since $t \in \mathbb{T}$, we only need to compare the leading terms τ_t and $\sigma_i a_i \mathfrak{n}_{i+1}$.

Let us show some consequences of the definition of $\mathfrak{n}^a \succ \mathfrak{a}$. In particular, we prepare the ground for the definition of a compatible total ordering on $\mathfrak{M}_{\sigma,\varphi}$.

Lemma 2.5.8 *Let $(\sigma,\varphi)\in\mathcal{N}_{\mathbb{T}}$. Then $\forall i\geqslant 0:\operatorname{supp}\varphi_i \succ\!\!\succ \operatorname{supp}\varphi_{i+1}$, i.e. we have*

$$\operatorname{supp}\varphi_0 \succ\!\!\succ \operatorname{supp}\varphi_1 \succ\!\!\succ \operatorname{supp}\varphi_2 \succ\!\!\succ \cdots.$$

Proof: Let $\mathfrak{m}\in\operatorname{supp}\varphi_i$. By **NM2**, there is some $j>i$ such that for all $\psi\in\mathbb{T}^{\uparrow}$ with $\operatorname{supp}\varphi_j\succ\psi$ we have

$$\mathfrak{m}\succ e^{\varphi_{i+1}+\sigma_{i+1}e^{\cdot^{\cdot^{\cdot \sigma_{j-1}e^{\varphi_j+\psi}}}}} = e^{\varphi_{i+1}+\sigma_{i+1}\cdot\mathfrak{a}(\psi)}.$$

Hence $\mathfrak{m}\succ\!\!\succ \log\mathfrak{m}>\varphi_{i+1}+\sigma_{i+1}\cdot\mathfrak{a}(\psi)$. But this holds for all $\psi\in\mathbb{T}^{\uparrow}$ with $\psi\prec\operatorname{supp}\varphi_j$, thus there are ψ with $\varphi_{i+1}\not\asymp\mathfrak{a}(\psi)$. Then for such series ψ we have

$$\varphi_{i+1}\preccurlyeq\varphi_{i+1}+\sigma_{i+1}\cdot\mathfrak{a}(\psi)<\log\mathfrak{m},$$

which implies $\mathfrak{m}\succ\!\!\succ\varphi_{i+1}$. □

Remark 2.5.9 Similarly, one shows $\operatorname{supp}\varphi_i\succ\mathfrak{n}_j$ and $\forall k\in\mathbb{Z}:1\prec\mathfrak{n}_i\mathfrak{n}_j^k$ for all $0<i<j$. The former property corresponds to the condition that $\mathfrak{n}_{i+1}$ is the smallest element in the support of the series $\varphi_i+\sigma_i\mathfrak{n}_{i+1}$, whereas the latter one will imply $\mathfrak{n}_1\succ\mathfrak{n}_2\succ\cdots$.

Lemma 2.5.10 *Let $\mathfrak{a}\in\mathfrak{M}$ and $i\in\mathbb{N}$ be such that $\mathfrak{n}_i\succ\mathfrak{a}$. Then*

$$\forall\mathfrak{b}\in\mathfrak{M}:\mathfrak{a}\succ\mathfrak{b}\Rightarrow\mathfrak{n}_i\succ\mathfrak{b}.$$

Proof: Since $\succ$ is recursively defined, we will show the lemma recursively as well. We describe a procedure that eventually terminates and thus proves the lemma. Also, it suffices to show the lemma for $\mathfrak{n}_i=\mathfrak{n}_0$. The general statement follows from re-indexing.

Recall from the definition of $\mathfrak{n}_0\succ\mathfrak{a}$ that we have constructed sequences $(\vartheta_i)_{0\leqslant i}$, $(d_i)_{0\leqslant i}$ and $(\mathfrak{m}_i)_{0\leqslant i}$, where $\mathfrak{m}_0=\mathfrak{a}$. We let $(\hat\vartheta_i)_{0\leqslant i}$, $(\hat d_i)_{0\leqslant i}$ and $(\hat{\mathfrak{m}}_i)_{0\leqslant i}$ be the corresponding sequences for $\mathfrak{b}$ (in particular, we have $\hat{\mathfrak{m}}_0=\mathfrak{b}$).

From $\mathfrak{n}_0\succ\mathfrak{m}_0$ we obtain $\vartheta_0\leqslant\varphi_0$, and equality implies $d_0\mathfrak{m}_0<\sigma_0\mathfrak{n}_1$. Similarly, $\mathfrak{m}_0\succ\hat{\mathfrak{m}}_0$ leads to $\hat\vartheta_0\leqslant\vartheta_0$, and $\hat\vartheta_0=\vartheta_0$ implies $\hat d_0\hat{\mathfrak{m}}_1<d_0\mathfrak{m}_0$. If one of the inequalities is strict, then $\hat\vartheta_0<\varphi_0$, which immediately yields $\mathfrak{n}_0\succ\hat{\mathfrak{m}}_0=\mathfrak{b}$.

It remains to consider the case $\varphi_0=\vartheta_0=\hat\vartheta_0$ and $\hat d_0\hat{\mathfrak{m}}_1<d_0\mathfrak{m}_0<\sigma_0\mathfrak{n}_1$. We are done if $\hat d_0\leqslant 0<\sigma_0$, for then $\hat d_0\hat m_1<\sigma_0\mathfrak{n}_1$ and thus $\mathfrak{n}_0\succ\hat{\mathfrak{m}}_1$. Hence we have to consider the case where $0<\sigma_0\cdot\hat d_0$. In particular, we remark that σ_0 and d_0 have the same sign. In other words, if σ_0 and d_0 have different signs, then we are in one of the preceding cases, the procedure stops, and the lemma is proven. If both σ_0 and $\hat d_0$ are positive, then we have to show

$$\mathfrak{m}_1\succcurlyeq\hat{\mathfrak{m}}_1\ \wedge\ \mathfrak{n}_1\succ\mathfrak{m}_1\ \Rightarrow\ \mathfrak{n}_1\succ\hat{\mathfrak{m}}_1.$$

Otherwise, if both are negative, then it remains to prove

$$\hat{\mathfrak{m}}_1\succcurlyeq\mathfrak{m}_1\ \wedge\ \mathfrak{m}_1\succ\mathfrak{n}_1\ \Rightarrow\ \hat{\mathfrak{m}}_1\succ\mathfrak{n}_1.$$

Taking this process further, we see that the lemma holds if there is an $i\in\mathbb{N}$ such that at least one of $\vartheta_i\neq\hat\vartheta$ and $\varphi_i\neq\vartheta_i$ holds. Otherwise, if $\varphi_i=\vartheta_i=\hat\vartheta_i$ for all i, then the process terminates if σ_i and d_i have different signs for some i. But this will be the case by condition **T4** and $(\sigma,\varphi)\in\mathcal{N}_{\mathbb{T}}$. □

Lemma 2.5.11 *Let $\mathfrak{a} \in \mathfrak{M}$ and $i \in \mathbb{N}^+$ such that $\mathfrak{a} \succ \mathfrak{n}_i$. Then $\mathfrak{a} \succ \mathfrak{n}_j$ for all $j > i$. Moreover, if $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}$ and $i < j$ are such that $\mathfrak{n}_i \succ \mathfrak{a}$ and $\mathfrak{n}_j \succ \mathfrak{b}$, then for all $c, d \in C : \mathfrak{n}_i \succ c\mathfrak{a} + d\mathfrak{b}$.*

Proof: We only need to show the first part for $j = i + 1$. The full statement follows by induction. Let ϑ_0 be the maximal truncation of $\log \mathfrak{a}$ with

$$\forall \mathfrak{a} \in \operatorname{supp} \vartheta_0 : \exists j > i : \forall \psi \in \mathbb{T}^{\uparrow} : \operatorname{supp} \varphi_j \succ \psi \Rightarrow \mathfrak{a} \succ e^{\varphi_{i+1} + \sigma_{i+1} e^{\cdot^{\cdot^{\cdot \varphi_j + \psi}}}}.$$

Then from the definition of $\mathfrak{a} \succ \mathfrak{n}_i$ it follows that $\vartheta_0 \geqslant \varphi_i$.We have $\varphi_i > \varphi_{i+1}$, since $\varphi_i, \varphi_{i+1} > 0$ and $\operatorname{supp} \varphi_i \succ \operatorname{supp} \varphi_{i+1}$ (by Lemma 2.5.8). Thus $\vartheta_0 > \varphi_{i+1}$, and therefore $\mathfrak{a} \succ \mathfrak{n}_{i+1}$.

To show the second assertion, we remark that $\mathfrak{b} \prec \mathfrak{n}_j$ and $i < j$ imply $\mathfrak{b} \prec \mathfrak{n}_i$. Thus $\tau_{c\mathfrak{a}+d\mathfrak{b}} \preccurlyeq \mathfrak{a}, \mathfrak{b} \prec \mathfrak{n}_i$. This finishes the proof. □

Proposition 2.5.12 *Let $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}$ and $a, b \in \mathbb{Z}^{\star}$ be such that $\mathfrak{n}^a \succ \mathfrak{a}$ and $\mathfrak{n}^b \succ \mathfrak{b}$. Then $\mathfrak{n}^{a+b} \succ \mathfrak{a}\mathfrak{b}$.*

Proof: We define I, J, M, and N by

$$I := \min\{\, i \mid a_i \neq 0\} \qquad M := \max\{\, i \mid a_i \neq 0\}$$
$$J := \min\{\, i \mid b_i \neq 0\} \qquad N := \max\{\, i \mid b_i \neq 0\}.$$

If I does not exist, then neither does M, and in this case we have $a = 0$. The same holds for J. Hence, if neither of I and J exists, then $a = b = 0$, and the proposition follows from the compatibility of the ordering with the group structure of $\mathfrak{M}$. Therefore we will in the following assume that at least one of I and J exists.

We let

$$\begin{aligned} t &:= \log \mathfrak{a} - (a_I \varphi_I + \cdots + a_M \varphi_M) \quad \text{if } I \text{ exists,} \\ s &:= \log \mathfrak{b} - (b_J \varphi_J + \cdots + b_N \varphi_N) \quad \text{if } J \text{ exists.} \end{aligned}$$

We first treat the case **I** where exactly one of I and J exists, say $I \in \mathbb{N}$. Then we have $\mathfrak{b} \prec 1$ and thus $\log \mathfrak{b} < 0$. We distinguish two subcases.

Sub-case **I.1***:* $\mathfrak{d}_t \succ \mathfrak{n}_{i+1}$. Then from $\mathfrak{a} \prec \mathfrak{n}^a$ we obtain $c_t < 0$. If $\log \mathfrak{b} \prec t$, then $t + \log \mathfrak{b} \asymp t$ and $c_{t+\log \mathfrak{b}} = c_t < 0$. Hence the inequality

$$\log(\mathfrak{a}\mathfrak{b}) - (a_I \varphi_I + \cdots + a_M \varphi_M) < \sigma_I a_I \mathfrak{n}_{I+1} \tag{2.4}$$

holds. On the other hand, if $\log \mathfrak{b} \succ t$, then $t + \log \mathfrak{b} \asymp \log \mathfrak{b}$ and $c_{t+\log \mathfrak{b}} = c_{\log \mathfrak{b}} < 0$, which implies inequality (2.4) again. Finally, if $t \asymp \log \mathfrak{b}$, then $t + \log \mathfrak{b} \asymp t \asymp \log \mathfrak{b}$ and $c_{t+\log \mathfrak{b}} = c_t + c_{\log \mathfrak{b}} < 0$. Hence in all cases we obtain $\mathfrak{a}\mathfrak{b} \prec \mathfrak{n}^a$.

Sub-case **I.2***:* $\mathfrak{d}_t \prec \mathfrak{n}_{i+1}$. Then $\sigma_I a_I > 0$. Again, we distinguish three cases. First, if $\log \mathfrak{b} \prec t$, then $t \asymp t + \log \mathfrak{b} \prec \mathfrak{n}_{I+1}$ and thus $t + \log \mathfrak{b} < \sigma_i a_i \, \mathfrak{n}_{I+1}$. Next, if $t \prec \log \mathfrak{b}$, then $\log \mathfrak{b} \asymp t + \log \mathfrak{b}$ which implies $c_{t+\log \mathfrak{b}} = c_{\log \mathfrak{b}} < 0$. Thus again $t + \log \mathfrak{b} < \sigma_i a_i \, \mathfrak{n}_{I+1}$. And finally, if $t \asymp \log \mathfrak{b}$, then $t + \log \mathfrak{b} \preccurlyeq t \prec \mathfrak{n}_{I+1}$. Again, $\sigma_I a_I > 0$ implies $t + \log \mathfrak{b} < \sigma_i a_i \, \mathfrak{n}_{I+1}$. This finishes the case **I**.

Next, we treat the case **II** where both I and J exist. Let $L := \min\{ i \mid a_i + b_i \neq 0\}$. Then $L \geqslant \min(I, J)$. We consider four sub-cases.

Sub-case **II.1***:* $\mathfrak{d}_t \succ \mathfrak{n}_{I+1}$ and $\mathfrak{d}_s \succ \mathfrak{n}_{J+1}$. Then $c_t, c_s < 0$. Hence

$$\mathfrak{d}_{t+s} = \max(\mathfrak{d}_t, \mathfrak{d}_s) \succ \mathfrak{n}_{I+1}, \mathfrak{n}_{J+1}$$

and $c_{t+s} < 0$. Since $L \geqslant \min(I, J)$, we obtain by Lemma 2.5.11 that $\mathfrak{d}_{t+s} \succ \mathfrak{n}_{L+1}$. Thus $t + s < \sigma_L(a_L + b_L)\, \mathfrak{n}_{L+1}$.

Sub-case **II.2***:* $\mathfrak{d}_t \prec \mathfrak{n}_{I+1}$ and $\mathfrak{d}_s \prec \mathfrak{n}_{J+1}$. Then $\sigma_I a_I, \sigma_J b_J > 0$. Hence $\sigma_L(a_L + b_L) > 0$. Moreover, by Lemma 2.5.11 we have

$$\mathfrak{d}_{t+s} \preccurlyeq \max(\mathfrak{d}_t, \mathfrak{d}_s) \prec \mathfrak{n}_{L+1}.$$

Hence $t + s < \sigma_L(a_L + b_L)\, \mathfrak{n}_{L+1}$.

Sub-case **II.3***:* $\mathfrak{d}_t \prec \mathfrak{n}_{I+1}$ and $\mathfrak{d}_s \succ \mathfrak{n}_{J+1}$. Then $\sigma_I a_I > 0$ and $c_s < 0$. If $J \leqslant I$, then by Lemma 2.5.10 we have $\mathfrak{d}_t \prec \mathfrak{d}_s$. Thus $t + s \asymp s \succ \mathfrak{n}_{J+1}$ and $c_{t+s} = c_s < 0$. From $L \geqslant J$, $s \succ \mathfrak{n}_{J+1}$ and Lemma 2.5.11 it then follows that

$$\mathfrak{d}_{t+s} \succ \mathfrak{n}_{L+1}.$$

Hence $t + s < \sigma_L(a_L + b_L)\, \mathfrak{n}_{L+1}$. If $J > I$, then $L = I$ and $\sigma_L(a_L + b_L)\, \mathfrak{n}_{L+1} = \sigma_I a_I\, \mathfrak{n}_{I+1}$. Furthermore, $t + s \preccurlyeq \max(\mathfrak{d}_t, \mathfrak{d}_s)$. If $\mathfrak{d}_t \succ \mathfrak{d}_s$, then $t + s \asymp t \prec \mathfrak{n}_{L+1}$ and $c_{t+s} = c_t$. From $\sigma_I a_I > 0$ it now follows that

$$t + s < \sigma_L(a_L + b_L)\, \mathfrak{n}_{L+1}. \tag{2.5}$$

On the other hand, if $\mathfrak{d}_t \prec \mathfrak{d}_s$, then $t + s \asymp s$ and $c_{t+s} = c_s < 0$. This also shows the inequality (2.5). Finally, if $t \asymp s$, then $t + s \preccurlyeq t \prec \mathfrak{n}_{L+1}$. Again, $\sigma_I a_I > 0$ shows inequality (2.5).

Sub-case **II.4**: $\mathfrak{d}_t \succ \mathfrak{n}_{I+1}$ and $\mathfrak{d}_s \prec \mathfrak{n}_{J+1}$. This case is similar to the case **II.3**.

Thus, we have shown $t + s < \sigma_L(a_L + b_L)\, \mathfrak{n}_{L+1}$ in the case **II**, from which the proposition follows. □

Remark that for all $\mathfrak{a} \in \mathfrak{M}$ and all $a \in \mathbb{Z}^\star$ with $0 \neq a$ we have either $\mathfrak{a} \succ \mathfrak{n}^a$ or $\mathfrak{n}^a \succ \mathfrak{a}$. We extend the relation $\succ$ to $\succcurlyeq$ by

$$\mathfrak{n}^a \succcurlyeq \mathfrak{a} \quad :\Leftrightarrow \quad \mathfrak{n}^a \succ \mathfrak{a} \ \vee \ (\mathfrak{a} = 1 \ \wedge \ a = 0).$$

We define the binary relation $\succcurlyeq$ on $\mathfrak{M}_{\sigma,\varphi}$ as follows. Let $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}$ and $a, b \in \mathbb{Z}^\star$. Then we let

$$\mathfrak{a}\, \mathfrak{n}^a \succcurlyeq \mathfrak{b}\, \mathfrak{n}^b \quad :\Leftrightarrow \quad \mathfrak{a}\mathfrak{b}^{-1} \succcurlyeq \mathfrak{n}^{b-a}.$$

Proposition 2.5.13 *The relation $\succcurlyeq$ is a total ordering on $\mathfrak{M}_{\sigma,\varphi}$. It extends the ordering of $\mathfrak{M}$ and is compatible with the group structure of $\mathfrak{M}_{\sigma,\varphi}$.*

Proof: Throughout this proof, let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathfrak{M}$ and $a, b, c \in \mathbb{Z}^\star$. If one of $\mathfrak{a} \neq \mathfrak{b}$ and $a \neq b$ holds, then either $\mathfrak{a}\,\mathfrak{n}^a \succ \mathfrak{b}\,\mathfrak{n}^b$ or $\mathfrak{b}\,\mathfrak{n}^b \succ \mathfrak{a}\,\mathfrak{n}^a$. Hence the relation $\succcurlyeq$ is total. Next we show **PO1** – **PO3**. Suppose

$$\mathfrak{a}\,\mathfrak{n}^a \succcurlyeq \mathfrak{b}\,\mathfrak{n}^b \quad \text{and} \quad \mathfrak{b}\,\mathfrak{n}^b \succcurlyeq \mathfrak{a}\,\mathfrak{n}^a.$$

Applying the definition, this is equivalent to

$$\mathfrak{a}\,\mathfrak{b}^{-1} \succcurlyeq \mathfrak{n}^{b-a} \quad \text{and} \quad \mathfrak{n}^{b-a} \succcurlyeq \mathfrak{a}\,\mathfrak{b}^{-1}.$$

If one of the inequalities were proper, then the other inequality would have to be proper as well. But then we have an immediate contradiction, which shows **PO1**. As for **PO2**, we remark that

$$\mathfrak{a}\,\mathfrak{n}^a \succcurlyeq \mathfrak{a}\,\mathfrak{n}^a \quad \Leftrightarrow \quad \mathfrak{a}\,\mathfrak{a}^{-1} \succcurlyeq \mathfrak{n}^{a-a} \quad \Leftrightarrow \quad 1 \succcurlyeq \mathfrak{n}^0,$$

which holds by the definition of $\succcurlyeq$. Finally, we show **PO3**. Assume that $\mathfrak{a}\,\mathfrak{n}^a \succcurlyeq \mathfrak{b}\,\mathfrak{n}^b$ and $\mathfrak{b}\,\mathfrak{n}^b \succcurlyeq \mathfrak{c}\,\mathfrak{n}^c$. If at least one of these inequalities is an equality, we are done. Let as thus assume that $\mathfrak{a}\,\mathfrak{n}^a \succ \mathfrak{b}\,\mathfrak{n}^b$ and $\mathfrak{b}\,\mathfrak{n}^b \succ \mathfrak{c}\,\mathfrak{n}^c$. Then by Proposition 2.5.12 we have

$$\mathfrak{a}\mathfrak{c}^{-1} = \mathfrak{a}\,\mathfrak{b}^{-1}\,\mathfrak{b}\,\mathfrak{c}^{-1} \succ \mathfrak{n}^{b-a+c-b} = \mathfrak{n}^{c-a},$$

which is equivalent to $\mathfrak{a}\,\mathfrak{n}^a \succ \mathfrak{c}\,\mathfrak{n}^c$. This shows **PO3** and the compatibility with the group structure. □

Let

$$\mathbb{T}_{\sigma,\varphi} := C[[\mathfrak{M}_{\sigma,\varphi}]].$$

We define log on $\mathfrak{M}_{\sigma,\varphi}$ and $\mathbb{T}^+_{\sigma,\varphi}$ as follows. Let $\mathfrak{a} \in \mathfrak{M}$ and $a \in \mathbb{Z}^\star$ with $n = |a|$. Then with $i \in \mathbb{N}$:

$$\begin{aligned} \log \mathfrak{n}_i &:= \varphi_i + \sigma_i\,\mathfrak{n}_{i+1} \\ \log(\mathfrak{a}\,\mathfrak{n}^a) &:= \log \mathfrak{a} + a_0 \log \mathfrak{n}_0 + \cdots + a_n \log \mathfrak{n}_n \\ \log f &:= \log \mathfrak{d}_f + \log c_f + l(\delta_f). \end{aligned}$$

Proposition 2.5.14 *The field $\mathbb{T}_{\sigma,\varphi}$ is a transseries field.*

Proof: We prove **e1**, **e2** and **e3** (the latter for $f \notin (\mathbb{T}_{\sigma,\varphi})^+_\infty$) in a similar way as in Proposition 2.3.2. It remains thus to show **e3** for series $f \in (\mathbb{T}_{\sigma,\varphi})^+_\infty$. First remark that $\log(f+1) \asymp \log \mathfrak{d}_f$ and that $\mathfrak{d}_f = \mathfrak{a}\,\mathfrak{n}^a$ for some $\mathfrak{a} \in \mathfrak{M}$ and $a \in \mathbb{Z}^\star$. If $\log \mathfrak{d}_f \asymp \log \mathfrak{a}$, then **e3** follows from the same property of $\mathbb{T}$. If $\log \mathfrak{d}_f \asymp \log \mathfrak{n}^a$, then let i be minimal with $a_i \neq 0$. From $1 \prec f$ it then follows that $a_i > 0$, hence that $\mathfrak{n}_{i+1} \prec \mathfrak{a}\,\mathfrak{n}^a$. Therefore

$$\log \mathfrak{n}^a \asymp \mathfrak{n}_{i+1} \prec \mathfrak{a}\,\mathfrak{n}^a.$$

In particular, $\log(f+1) < f$, which implies **e3**. Hence, $\mathbb{T}_{\sigma,\varphi}$ is an exp-log field.

Property **T1** follows from the above definition of log. In order to show **T2** we have to show that all $\mathfrak{n}_i$ are infinite monomials. But this follows from the definition of the ordering and from $0 < \varphi_i$ for $i > 0$. By construction, **T3** is true. We show **T4**. Let $\mathfrak{m}_0, \mathfrak{m}_1, \dots$ a sequence of monomials in $\mathfrak{M}_{\sigma,\varphi}$ such that

$$\forall 0 \leqslant i : \mathfrak{m}_{i+1} \in \operatorname{supp} \log \mathfrak{m}_i.$$

Let $n \in \mathbb{N}$ be minimal such that $\mathfrak{m}_n \in \mathfrak{M}$. If n exists, then the property follows from condition **T4** for $\mathfrak{M}$. Otherwise there are $a \in \mathbb{Z}^\star$ and $0 < i$ such that $\mathfrak{m}_0 = \mathfrak{n}^a$ and $\mathfrak{m}_j = \mathfrak{n}_{j+i}$ for all $j \in \mathbb{N}$. But then **T4** holds since $\operatorname{supp} \varphi_i \succ \mathfrak{n}_{i+1}$ for all i. □

Remark 2.5.15 Nested extensions will not play any role in the rest of this thesis. One reason to introduce them was to give another example of a possible extension of a transseries field, thus illustrating the general method – extending the group $\mathfrak{M}$ to a set $\hat{\mathfrak{M}}$, introducing compatible multiplication and ordering on $\hat{\mathfrak{M}}$, defining a function log on $\mathfrak{M}$ such that $C[[\hat{\mathfrak{M}}]]$ is a transseries field – once more.

Another reason was to extend transseries fields by canonical solutions of certain functional equations. In fact, constructing a super-exponential function can be motivated in that way, too. Therefore, introducing nested extensions does not provide a tool for every functional equation. On the other hand, the functional equation

$$f(x) = e^{x^2 + f(\log_2 x) + \log x} \tag{2.6}$$

has natural solutions like (2.2), which causes problems, since this expression does not belong to any transseries field. It should be noticed that there is a solution of (2.6) which can be expressed in terms of nested expressions. (See [vdH97], p. 86 for more on this.)

Chapter 3

Trees

So far, we have defined sets of generalized power series, given them a field structure and additional functions exp and log. This chapter will study the combinatorial aspects of the theory of transseries.

Transseries admit several properties which cause such properties to emerge naturally. Let us mention three of them.

- Transseries can be represented as trees.
- Paths and sub-trees of such tree-representations can be used to define derivations and compositions on transseries fields.
- Noetherian operators and a generalized form of Kruskal's theorem are the combinatorial analog to the concept of strongly linear algebras.

In this chapter, we cover the first and the last point. As for the second point, we use the properties shown in this chapter in Chapters 4 and 5 in order to define derivations and compositions.

3.1 Basic notions

An order $T = (T, \leqslant_T)$ is a **tree** iff

Tr1. $\exists \mathsf{r} \in T : \forall \mathsf{n} \in T : \mathsf{r} \leqslant_T \mathsf{n}$,

Tr2. for all $\mathsf{n} \in T$, the set $i_{\mathsf{n}} = \{\mathsf{s} \in T \mid \mathsf{s} <\!\!\cdot_T \mathsf{n}\}$ is finite.

Note that the element r in condition **Tr1** is unique. We call r the **root** of the tree T, in symbols $\mathsf{r} = \mathsf{r}(T)$. The order-type of i_{n} is called the **height** of n, symbolized by $\mathsf{h}(\mathsf{n})$. The root is the only element of a tree with height 0. If there is an integer N such that the height of each element of the tree T is less than N, then we call T a **uniformly finite** tree.

Elements of a tree will also be called **nodes**. If the height of a node $\mathsf{n} \in T$ is a successor ordinal, then there is a unique node $\mathsf{p} \in T$ with $\mathsf{p} <\!\!\cdot_T \mathsf{n}$ such that for no other $\mathsf{s} \in T$ we have $\mathsf{p} <\!\!\cdot_T \mathsf{s} <\!\!\cdot_T \mathsf{n}$. In this case, we call p the **predecessor** of n, and we write $\mathsf{p} = \mathrm{pred}(\mathsf{n})$. Then the set $\mathrm{succ}(\mathsf{n}) = \{\mathsf{s} \in T \mid \mathsf{n} = \mathrm{pred}(\mathsf{s})\}$ is the set of **successors** of n. A **leaf** is a node without successors. (See Figure 3.1.)

Remark 3.1.1 We remark that trees are often defined more generally. Condition **Tr2** can be modified by replacing "finite" by "well-ordered". This allows trees to have nodes n with

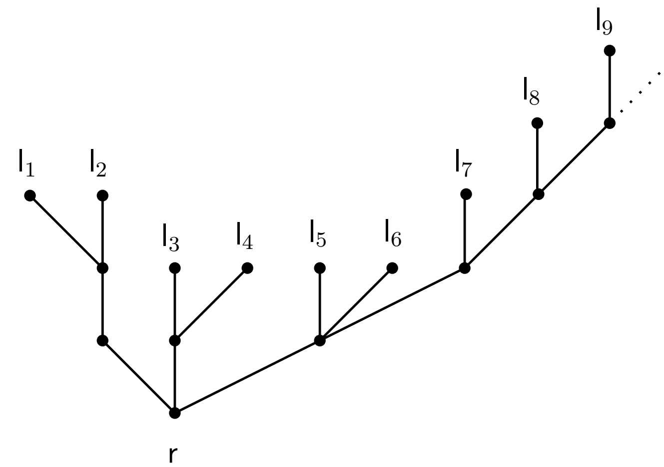


Figure 3.1: A tree of finite height (but not uniformly finite) with root r and leaves $\mathsf{l}_1, \mathsf{l}_2, \dots$.

$\mathsf{h}(\mathsf{n}) > \omega$. In this setting, a tree is said to be of finite height iff every element has finite height. We will later associate trees to transseries. Those trees will be of finite height. Since this is the only type of trees we will encounter, we have decided to define trees more restrictively.

Moreover, in parts of the literature [Jec78],[Kun80], the definition of a tree does not demand the existence of a root, i.e. one does not have condition **Tr1**. With this definition, every subset of a tree is again a tree. This is in general not the case with the present definition of a tree. If, for instance, the root of a given tree T has more than one successor, then the set $T \setminus \{\mathsf{r}(T)\}$ is not a tree anymore. On the other hand, every subset containing the root will be a tree.

A well-ordered subset P of T is a **path** iff $i_{\mathsf{n}} \subset P$ for all $\mathsf{n} \in P$ and $P \not\subseteq i_{\mathsf{n}}$ for all nodes $\mathsf{n} \in T$. For a fixed tree, every path has $\mathsf{r}(T)$ as a minimal element. For a path P and an ordinal α, we denote by $\mathsf{n}_{P,\alpha}$ the element of the path with height α, if there is such an element. If a path is finite, then we write $P = [\mathsf{n}_{P,0}, \dots, \mathsf{n}_{P,|P|}]$ for the path. Note that trees can have infinite paths.

Remark 3.1.2 For every path of a tree T, the least element of P is $\mathsf{n}_{P,0} = \mathsf{r}(T)$. The first characteristic element of P is therefore a successor of $\mathsf{r}(T)$. For this reason, we will sometimes not mention $\mathsf{n}_{P,0}$ and start instead with the successor of the root, i.e. the element of the path with height 1. When we do so, we will mention it, in order to avoid confusion.

For each node n, the set $K_{\mathsf{n}} = \{\mathsf{s} \in T \mid \mathsf{n} \leqslant_T \mathsf{s}\}$ is a tree. For all $\mathsf{n} \in \operatorname{succ}(\mathsf{p})$, we call the trees K_{n} the **children** of the node p, and we call p the **parent** of $\mathsf{n} \in \operatorname{succ}(\mathsf{p})$. (See Figure 3.2.) Note that for all $\mathsf{s}, \mathsf{t} \in T$ the set $\{\mathsf{p} \in T \mid \mathsf{p} \leqslant \mathsf{s}, \mathsf{t}\}$ is non-empty and well-ordered. Moreover, this set has a maximal element. (We remark that this is in general not true, if we allow trees to have nodes n with height $\mathsf{h}(\mathsf{n}) \geqslant \omega$ as in Remark 3.1.1.) We denote this element by $\mathsf{s} \vee \mathsf{t}$.

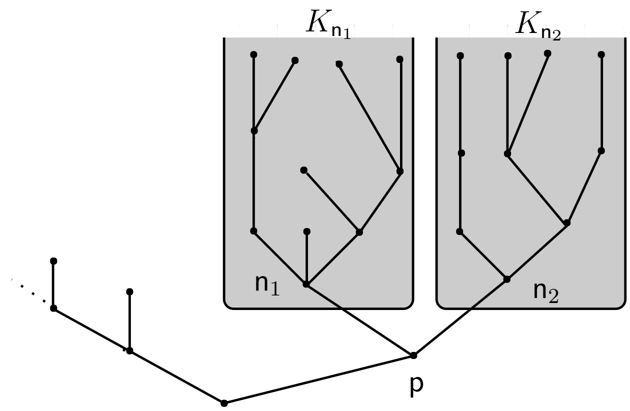


Figure 3.2: Children $K_{\mathsf{n}_1}, K_{\mathsf{n}_2}$ of the parent-node p.

Let $\mathrm{leaf}(T)$ be the set of leaves of a tree T and $(S_\mathsf{l})_{\mathsf{l}\in\mathrm{leaf}(T)}$ be a family of trees. There is a canonical way to substitute the trees S_l into T by replacing $\mathsf{l} \in \mathrm{leaf}(T)$ by the root of S_l. Formally, this is done as follows.

Let $\hat{S}_\mathsf{l} = S_\mathsf{l} \times \{\mathsf{l}\}$ for all $\mathsf{l} \in \mathrm{leaf}(T)$ and $\hat{T} = T \setminus \mathrm{leaf}(T)$. We put $U := \hat{T} \cup \coprod_{\mathsf{l}\in\mathrm{leaf}(T)} \hat{S}_\mathsf{l}$, and we define $\leqslant_U$ on U by

$$\mathsf{n} \leqslant_U \mathsf{m} \qquad \text{iff} \qquad \begin{cases} \mathsf{n},\mathsf{m} \in \hat{T} \quad \text{and} \quad \mathsf{n} \leqslant_T \mathsf{m} \quad \text{or} \\ \\ \mathsf{n},\mathsf{m} \in \hat{S}_\mathsf{l} \quad \text{with} \quad \mathsf{n} = (\mathsf{n}_S, \mathsf{l}), \mathsf{m} = (\mathsf{m}_S, \mathsf{l}) \quad \text{and} \quad \mathsf{n}_S \leqslant_{S_\mathsf{l}} \mathsf{m}_S \quad \text{or} \\ \\ \mathsf{n} \in i_\mathsf{l} \quad \text{and} \quad \mathsf{m} \in \hat{S}_\mathsf{l} \quad \text{for} \quad \mathsf{l} \in \mathrm{leaf}(T). \end{cases}$$

One verifies that every node of U has finite height, hence that $U = (U, \leqslant_U)$ is a tree. We also write $U = T[S_\mathsf{l}]_{\mathsf{l}\in\mathrm{leaf}(T)}$.

Let $T = (T, \leqslant_T)$ be a tree, M a set and $l : T \to M$ a function. We call the tuple (T, l) an ***M*-labeled tree**. We can substitute M-labeled trees into M-labeled trees – given that the labelings of the leaves and the roots are the same – by substituting the underlying trees and adjusting the mapping in the natural way.

Remark 3.1.3 The reason to introduce labeled trees is that trees only provide information about structure. For our purposes, this will not be enough.

One disadvantage of trees is that their nodes are pairwise distinct. For representations of transseries, this limits the use of trees as a tool considerably, as we will see later. Adding a labeling to a tree provides thus an easy way to extend the range of applications of trees.

Example 3.1.4 Let T be an M-labeled tree, where M has a total ordering $\leqslant_M$. For a node $\mathsf{n} \in T$, the set $\{l(\mathsf{s}) \,|\, \mathsf{s} \in \mathrm{succ}(\mathsf{n})\}$ is therefore totally ordered. We can thus totally order the children of a node. Moreover, suppose that if $\mathsf{s} \neq \mathsf{p} \in \mathrm{succ}(\mathsf{n})$, then $l(\mathsf{s}) \neq l(\mathsf{p})$. This will always be the case in our applications.

We define a total order on the set of paths in T. Let $P, Q \in \mathrm{path}(T)$, and let i be minimal with $l(\mathsf{n}_{P,i}) \neq l(\mathsf{n}_{Q,i})$. Remark that if $P \neq Q$, then i exists and $0 < i$. Then we let

$$P < Q \quad :\Leftrightarrow \quad l(\mathsf{n}_{P,i}) < l(\mathsf{n}_{Q,i}).$$

One checks that this defines indeed a total ordering. This ordering is in general not Noetherian. However, if T is a uniformly finite tree and if for every node $\mathsf{n} \in T$ the set

$$\{l(\mathsf{s}) \mid \mathsf{s} \in \mathrm{succ}(\mathsf{n})\}$$

is well-ordered in $(M, \leqslant_M)$, then the set of paths is also well-ordered. For if this was not the case, then there would be a strictly decreasing sequence of paths $(P_0, P_1, \dots)$. Thus the sequence $(l(\mathsf{n}_{P_i,1}))_{0 \leqslant i}$ is decreasing. Hence for some $v \in M$, the set $\{P_i \mid l(\mathsf{n}_{P_i,1}) = v\}$ contains a strictly decreasing sub-sequence. Repeating this argument inductively, we can construct paths of arbitrary lengths, which contradicts the assumption about T.

3.2 Tree-representations of transseries

Throughout this section, let $\mathbb{T} = C[[\mathfrak{M}]]$ be a transseries field. We will associate series $f \in \mathbb{T}$ with labeled trees.

3.2.1 Definition of tree-representations

Definition 3.2.1 *A labeled tree $T = (T, l)$ of finite height is a* **tree-representation** *of $f \in \mathbb{T}$ iff the labeling $l : T \setminus \mathsf{r}(T) \to C\mathfrak{M}$ is such that*

TR1. $l(\mathsf{r}(T)) = f$,

TR2. $l : T \setminus \mathsf{r}(T) \to C\mathfrak{M}$,

TR3. *for each* $\mathsf{n} \in T \setminus (\mathrm{leaf}(T) \cup \mathsf{r}(T))$ *there exists a bijection* $\varphi : \mathrm{supp} \log \mathfrak{d}_{l(\mathsf{n})} \to \mathrm{succ}(\mathsf{n})$ *with*

(i) $\forall \mathfrak{m}, \mathfrak{n} \in \mathrm{supp} \log \mathfrak{d}_{l(\mathsf{n})} : \mathfrak{m} \succ \mathfrak{n} \Leftrightarrow l(\varphi(\mathfrak{m})) \succ l(\varphi(\mathfrak{n}))$ *and*

(ii) $\forall \mathfrak{m} \in \mathrm{supp} \log \mathfrak{d}_{l(\mathsf{n})} : l(\varphi(\mathfrak{m})) = (\log \mathfrak{d}_{l(\mathsf{n})})_{\mathfrak{m}} \mathfrak{m}$.

We say that $T = (T, l)$ **represents** *the term $l(\mathsf{r}(T)) = f$.*

Example 3.2.2 Every $f \in \mathbb{T}$ has a **trivial** tree-representation $T_{f,\mathrm{triv}}$, namely the one-point tree which is labeled with f. Clearly, it satisfies **TR1**. Since $\mathrm{leaf}(T_{f,\mathrm{triv}}) = \emptyset$, there is nothing to show for the remaining conditions. Denote the labeling of the trivial tree-representation by $l_{f,\mathrm{triv}}$.

Let $t \in C\mathfrak{M}$. We let S_t be the tree of height 1 such that there is a bijection

$$l : \mathrm{succ}\, \mathsf{r}(S_t) \to \mathrm{term} \log \mathfrak{d}_t.$$

We extend l to $\mathsf{r}(S_t)$ by $l(\mathsf{r}(S_t)) := t$. We say that (S_t, l) is the **log-tree** of t (for an example, see Figure 3.3).

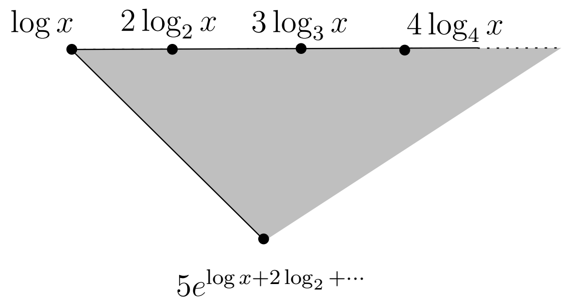


Figure 3.3: The log-tree of $t = 5e^{\log x+2\log_2 x+3\log_3 x+\cdots}$.

3.2.2 Maximal tree-representations

Let $T = (T, f)$ be a tree-representation of some series $f \in \mathbb{T}$. We define a labeled tree $T^+ = (T^+, l^+)$ by replacing all leaves $\mathsf{n} \in \operatorname{leaf}(T)$ by their respective log-trees $S_{l(\mathsf{n})}$:

$$T^+ := T[S_{l(\mathsf{n})}]_{\mathsf{n}\in\operatorname{leaf}(T)}.$$

Note that substituting the log-trees into the series is possible, since the labelings of the leaves and the roots match. One checks that T^+ is a tree-representation of f. The restriction of l^+ to T is l. Note that $T = T^+$ if and only if $\operatorname{leaf}(T) = \emptyset$.

A $C\mathfrak{M}$-labeled tree $T = (T, l)$ is a **maximal** tree-representation of f iff there exists a sequence $(T_i)_{0\leqslant i}$ of tree-representations $T_i = (T_i, l_i)$ of f such that

$\mathbf{T_{max}1}$. $T_0 = T_{f,\mathrm{triv}}$,
$\mathbf{T_{max}2}$. $\forall i \geqslant 0 : T_{i+1} = T_i^+$,
$\mathbf{T_{max}3}$. $T := \bigcup_{i<\omega} T_i$, the inductive limit of $(T_i)_{0\leqslant i}$, and $l := \bigcup_i l_i$ the induced labeling.

Remark 3.2.3 Note that we have to show that $\bigcup_{i<\omega} T_i$ is a tree-representation. Moreover, we have to make sure that $\bigcup_{i<\omega} T_i$ is well defined.

By condition $\mathbf{T_{max}2}$, the ordering on $T_{f,i+1}$ extends the ordering on $T_{f,i}$. Thus the inductive limit in $\mathbf{T_{max}3}$ exists. For $\mathsf{n}, \mathsf{m} \in \bigcup_{i<\omega} T_i$ we let $\mathsf{n} <\!\cdot\ \mathsf{m}$ iff there is an $i \in \mathbb{N}$ with $\mathsf{n} <\!\cdot_{T_i} \mathsf{m}$, where $\leqslant_{T_i}$ is the ordering of T_i. Hence $\bigcup_{i<\omega} T_i$ exists and is a labeled tree of finite height. The conditions **TR1** – **TR3** can be easily verified.

Proposition 3.2.4 *There is exactly one maximal tree-representation $T_{f,\max}$ of f.*

Proof: We define a sequence $(T_i)_{0\leqslant i}$ as in conditions $\mathbf{T_{max}1}$ and $\mathbf{T_{max}2}$. The inductive limit exists. Hence, it remains to show the uniqueness. Let T and T' be two different maximal tree-representations. Then there are sequences $(T_i)_i$ and $(T'_i)_i$ such that T and T' are the inductive limits of these sequences.

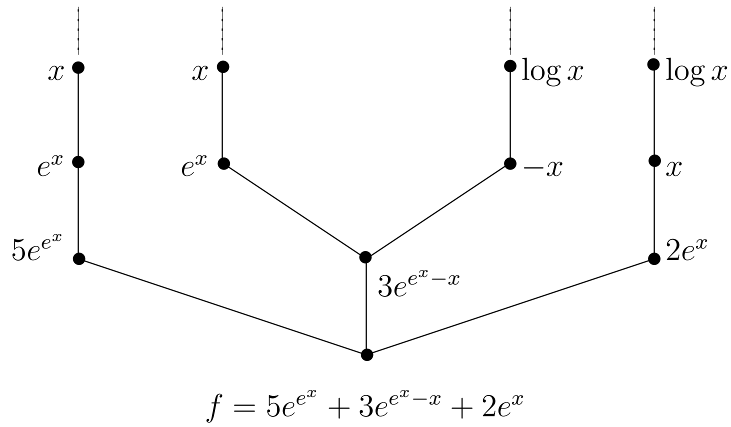


Figure 3.4: The maximal tree-representation of $f = 5e^{e^x} + 3e^{e^x-x} + 2e^x$

If there was a $P \in \text{path}(T) \setminus \text{path}(T')$ (in the labeled tree), then there is a minimal $i \in \mathbb{N}$ such that

$$\begin{array}{rcl}[t_{P,0}, \dots, t_{P,i}] & \in & \text{path}(T'_i) \\ [t_{P,0}, \dots, t_{P,i+1}] & \notin & \text{path}(T'_{i+1}).\end{array}$$

But $t_{P,i+1}$ is a label of a leaf in the log-tree of $t_{P,i}$. Since $t_{P,i}$ is also the label of a leaf of T'_i it follows from $\mathbf{T_{max}2}$ that $t_{P,i+1}$ is a label of a leaf of $T'_{f,i+1}$. This contradiction shows that $\text{path}(T) \subseteq \text{path}(T')$. Similarly, on obtains equality. This shows the proposition. $\square$

Remark 3.2.5 The tree-representation $T_{f,\max}$ is maximal in the sense that every tree-representation $T = (T, l)$ of f is a sub-tree of $T_{f,\max}$. By that we mean that paths of T are truncations of paths in $T_{f,\max}$ and that the restriction of the labeling $l_{f,\max}$ to T is l.

Notation 3.2.6 Let P be a path in a tree-representation. By $t_{P,i}$ we will denote the term which labels the node $\mathsf{n}_{P,i}$. Since for $i > 0$ we have $t_{P,i} \in C\mathfrak{M}$, for every i there are $c_{P,i} \in C$ and $\mathfrak{m}_{P,i} \in \mathfrak{M}$ with

$$t_{P,i} = c_{P,i}\mathfrak{m}_{P,i}.$$

We will henceforth write $\mathfrak{m}_{P,i}$ instead of $\mathfrak{d}t_{P,i}$.

A path P in the maximal tree-representation of f is **convergent** iff there is some i such that $t_{P,i}$ is log-atomic. We say that P is a **right-most** path iff $t_{P,i+1}$ is the least element in term $\log \mathfrak{m}_{P,i}$ for all $i \geqslant 0$. By condition **T4** of the definition of transseries, for all paths P in $T_{f,\max}$ there is some i_0 such that

$$[t_{P,i_0}, t_{P,i_0+1}, \dots]$$

is a right-most path. A path P has **cofinal bifurcations** iff

$$\forall i : \exists j \geqslant i : \exists s \in \operatorname{term} \log \mathfrak{m}_{P,j} : \qquad s \succ t_{j+1}.$$

If P has no cofinal bifurcations, then we say that P is **eventually bifurcation-free**.

Remark 3.2.7 Not every path is necessarily convergent. Take for instance the nested monomial

$$\mathfrak{n} = e^{x^2 + e^{\log_2^2 x + e^{\log_4^2 x + e^{\cdot^{\cdot^{\cdot}}}}}}.$$

Then $T_{\mathfrak{n},\max}$ contains the path $P = [\mathfrak{n}, \mathfrak{n}_1, \mathfrak{n}_2, \ldots]$ where

$$\mathfrak{n}_i = e^{\log_i^2 x + e^{\log_{i+1}^2 x + e^{\log_{i+2}^2 x + e^{\cdot^{\cdot^{\cdot}}}}}}.$$

Note also that P is a right-most path which has cofinal bifurcations. On the other hand, all other paths in $T_{\mathfrak{n},\max}$ are convergent.

Proposition 3.2.8 *Paths are either* log*-convergent or have cofinal bifurcations.*

Proof: Let P be convergent, then let i be such that $t_{P,i}$ is log-atomic. Thus

$$\operatorname{term} \log \mathfrak{m}_{P,i} = \{\log t_{P,i}\} \subseteq \mathfrak{M}.$$

This shows that P is eventually bifurcation-free.

Now let P be a path which is not convergent. Then no $t_{P,i}$ is log-atomic. By condition **T4**, we may assume that for sufficiently large i, the leading coefficient of $t_{P,i}$ is ± 1. Then for every i there is an integer $j \geqslant i$ such that either $\operatorname{term} \log \mathfrak{m}_{P,j}$ has more than one element, and $t_{P,j+1}$ cannot be eventually the leading term of $\log \mathfrak{m}_{P,j}$. But then we have a cofinal sequence of bifurcations, and P is therefore not bifurcation-free. □

3.2.3 Minimal tree-representations

A tree-representation $T = (T, l)$ of f is **minimal** iff

T$_{\min}$1. $T \subseteq T_{f,\max}$ and $l \subseteq l_{f,\max}$,
T$_{\min}$2. $\mathsf{t} \in \operatorname{leaf}(T)$ if and only if $l(\mathsf{t})$ is log-atomic.

Proposition 3.2.9 *There is a unique minimal tree-representation $T_{f,\min}$ of f.*

Proof: We start with the existence of minimal tree-representations. We define a labeled tree T by defining its set of paths. Note that this completely determines T. Let $P \in \operatorname{path}(T)$ iff

- either P is not a convergent path in $T_{f,\max}$,

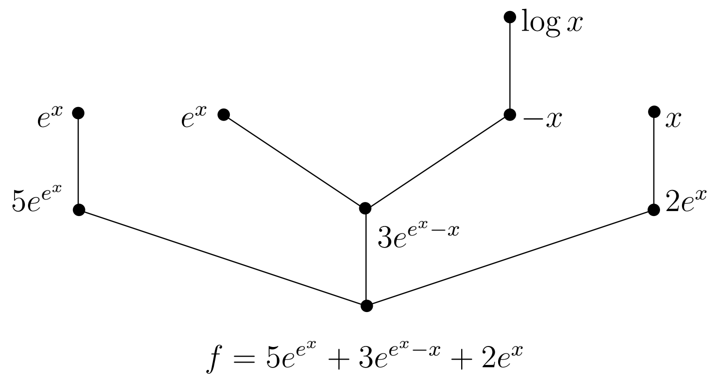


Figure 3.5: The minimal tree-representation of $f = 5e^{e^x} + 3e^{e^x-x} + 2e^x$

- or there is a convergent path $Q \in \text{path}(T_{f,\max})$ such that

$$P = [t_{Q,0}, \dots, t_{Q,i}]$$

  and i is minimal such that $t_{Q,i}$ is log-atomic.

Note that all nodes of T have finite height. We let l be the restriction of $l_{f,\max}$ to T. The conditions **Tr1** – **Tr3** hold by construction. Thus $T = (T, l)$ is a tree-representation. Condition $\mathbf{T_{min}1}$ also holds by construction. A node n of T is a leaf if and only if it is the minimal node on an convergent path such that its label $l(\mathsf{n})$ is log-atomic. This immediately implies $\mathbf{T_{min}2}$. The tree (T, l) is thus minimal.

As for the uniqueness, we assume that T, T' are two different minimal trees. Then we let T_ω, T'_ω be the trees that result from substituting the maximal trees $S_{l(t)}$ into all leaves t of T and T' respectively. Then $T_\omega = T'_\omega$, since both are maximal tree-representations. But then the set of paths in T and T' are identical. Thus $T = T'$. Contradiction. □

Remark 3.2.10 In other words, the minimal tree-representation results from cutting off the branches of the maximal tree-representation where they start to become non-branching trees.

Moreover, by Proposition 3.2.8, the only non-finite paths in $T_{f,\min}$ are the paths in the maximal tree-representation $T_{f,\max}$ which are not convergent. The minimal and maximal tree-representation provide thus the same information about f.

We will show next, that the paths which are not convergent do not play an important role in neither the minimal nor maximal tree-representation of f.

Proposition 3.2.11 *The minimal tree-representation $T_{f,\min}$ is completely determined by its set of finite paths. In other words, the maximal tree-representation is completely described by its set of* log*-convergent paths.*

Proof: Let P be a path which is not convergent. We have to show that for every i there is a convergent path Q in $T_{f,\max}$ such that

$$\forall j \leqslant i : \quad t_{P,j} = t_{Q,j}. \tag{3.1}$$

Suppose not. Then for some $i = i_0$, all paths Q in $T_{f,\max}$ with condition (3.1) are not convergent. There cannot be a finite number of such paths, for P has cofinal bifurcations. Hence, we construct a contradiction as follows. Let $i_1 > i_0$ be such that P bifurcates in t_{P,i_1}. Let $Q_0 := P$, and let Q_1 be a path such that

$$\forall j \leqslant i_1 : \quad t_{P,j} = t_{Q_1,j} \ \wedge \ t_{Q_1,i_1+1} \succ t_{P,i_1+1}.$$

Now suppose that we have already constructed a sequence $i_n > \cdots > i_0$ such that for all $n \geqslant m > 1$ we have

$$\forall j \leqslant i_m : \quad t_{Q_{m-1},j} = t_{Q_m,j} \ \wedge \ t_{Q_{m-1},i_m+1} \prec t_{Q_m,i_m+1}.$$

Then Q_n bifurcates in some $t_{Q_n,i_{n+1}}$ with $i_{n+1} > i_n$ such that for some non-convergent path Q_{n+1}, which coincides with Q_n up to $t_{Q_n,i_{n+1}}$ we have

$$t_{Q_{n+1},i_{n+1}+1} \succ t_{Q_n,i_{n+1}+1}.$$

This finishes the construction. Let $\mathfrak{m}_n = \mathfrak{m}_{Q_n,i_n}$ be the monomial of the term t_{Q_n,i_n}. Then $(\mathfrak{m}_n)_{0 \leqslant n}$ violates condition **T4**. This shows the proposition. □

3.2.4 Relative tree-representations with respect to transseries fields

Minimal and maximal tree-representations exist uniquely for all transseries. We change now the setting. Recall that $\mathbb{T}_{\exp}$ is the exp-extension of $\mathbb{T}$ and that we have defined transseries fields $\mathbb{T}_\alpha = C[[\mathfrak{M}_\alpha]]$ for all ordinals α by letting $\mathbb{T}_0 = \mathbb{T}$, $\mathbb{T}_{\alpha+1} = \mathbb{T}_{\alpha,\exp}$ and $\mathbb{T}_\lambda = C[[\bigcup_{\alpha<\lambda} \mathfrak{M}_\alpha]]$ for limit ordinals λ. The fields $\mathbb{T}_\alpha$ are called **transfinite exponential extensions** of $\mathbb{T}$.

Let in the following $f \in \mathbb{T}_\alpha$. A tree-representation $T = (T, l)$ of f is **relative with respect to** $\mathbb{T}$ iff for all nodes $\mathsf{n} \in T$ we have $l(\mathsf{n}) \in C\mathfrak{M} \Rightarrow \mathsf{n} \in \operatorname{leaf}(T)$. We denote the relative tree-representation of f w.r.t. $\mathbb{T}$ by $T_{f,\mathbb{T}}$, and we will not mention $\mathbb{T}$, if it is clear from the context. (Note that in the definition of this tree-representation, the group $\mathfrak{M}_\alpha$ replaces $\mathfrak{M}$.)

Example 3.2.12 Let $f \in \mathbb{T}_\alpha$. We define a labeled tree $T_f = (T_f, l_f)$ as follows.

First assume that $\alpha = 0$. If $f \in C\mathfrak{M}$, then we let $T_f = \{\bullet\}$ and $l_f(\bullet) = f$. Hence, T_f is the unique tree of height 0 which is labeled with f. Clearly, this is a relative tree-representation of f w.r.t. $\mathbb{T}$.

Otherwise let T_f be the tree of height 1 such that $l_f(\mathsf{r}(T_f)) = f$ and $l_f(\operatorname{leaf}(T_f)) = \operatorname{term} f$. This determines the labeled tree T_f uniquely. Again, the tree T_f is relative w.r.t. $\mathbb{T}$.

Now assume that $\alpha > 0$ and that for all $\beta < \alpha$ and all $g \in \mathbb{T}_\beta$ a relative tree-representation T_g w.r.t. $\mathbb{T}$ has already been defined. If there is an ordinal $\beta < \alpha$ such that $f \in \mathbb{T}_\beta$, then let T_f be the labeled tree defined in $\mathbb{T}_\beta$. If not, then let $\operatorname{pre}(T_f)$ be the labeled tree of height 1 with $\mathsf{r}(\operatorname{pre}(T_f)) = f$ and $\operatorname{leaf}(\operatorname{pre}(T_f)) = \operatorname{term} f$. (See for example Figure 3.6.)

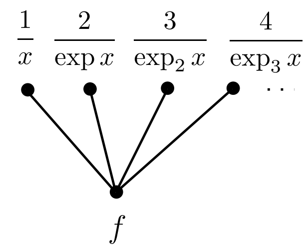


Figure 3.6: The pre-tree of $f = \frac{1}{x} + \frac{2}{\exp x} + \frac{3}{\exp_2 x} + \cdots \in \mathbb{L}_\omega$.

For every $\mathfrak{m} \in \operatorname{supp} f$, there is an ordinal $\beta_\mathfrak{m} < \alpha$ such that $\log \mathfrak{m} \in \mathbb{T}_{\beta_\mathfrak{m}}$. Hence $\log \mathfrak{m}$ admits already a relative tree-representation $T_{\log \mathfrak{m}}$ in $\mathbb{T}_{\beta_\mathfrak{m}}$ w.r.t. $\mathbb{T}$. Let $\hat{T}_\mathfrak{m}$ be the labeled tree which is identical to $T_{\log \mathfrak{m}}$ except that

$$\mathsf{r}(\hat{T}_\mathfrak{m}) = f_\mathfrak{m} \mathfrak{m}.$$

Then we substitute the family $(\hat{T}_\mathfrak{m})_{\mathfrak{m} \in \operatorname{supp} f}$ into the labeled tree $\operatorname{pre}(T_f)$ by replacing $f_\mathfrak{m}\mathfrak{m}$ by $\hat{T}_\mathfrak{m}$:

$$T_f := \operatorname{pre}(T_f)[\hat{T}_\mathfrak{m}]_{\mathfrak{m} \in \operatorname{supp} f}.$$

Now, T_f is a tree-representation of f, and form the relativity of all $T_{\log \mathfrak{m}}$, it follows that T_f is relative.

Proposition 3.2.13 *Each series $f \in \mathbb{T}_\alpha$ admits a unique relative tree-representation $T_{f,\mathbb{T}}$.*

Proof: The existence follows from the construction of T_f in Example 3.2.12. We have to show the uniqueness. Let α be minimal such that there is a series f with two distinct relative tree-representations T and T'. The uniqueness in the case $\alpha = 0$ follows directly from the fact that the root is labeled with f and that the successors of the root are labeled with elements from $C\mathfrak{M}$.

Hence $\alpha > 0$. Note that $\mathsf{r}(T) = \mathsf{r}(T')$. Let $t \in \operatorname{term} f$ and $\tilde{T}'_t$ the child of the root in T' which is labeled with t. Replace the root of $\tilde{T}'_t$ by $\log \mathfrak{d}_t$, then the resulting tree T'_t is a tree-representation of $\log \mathfrak{d}_t$. Since $\operatorname{leaf}(T'_t) \subseteq \operatorname{leaf}(T)$, these tree-representations are relative with respect to the field $\mathbb{T}$, hence $T'_t = T_t$ for all $t \in \operatorname{term} f$. This shows $T' = T$. □

Let $\leqslant$ be the ordering in the underlying tree of the relative tree-representation $T_{f,\mathbb{T}}$ of $f \in \mathbb{T}_\alpha$. Then the next proposition shows that $T_{f,\mathbb{T}}$ has no infinite paths.

Proposition 3.2.14 *The relative tree-representation w.r.t. $\mathbb{T}$ does not contain infinite chains for $\leqslant$.*

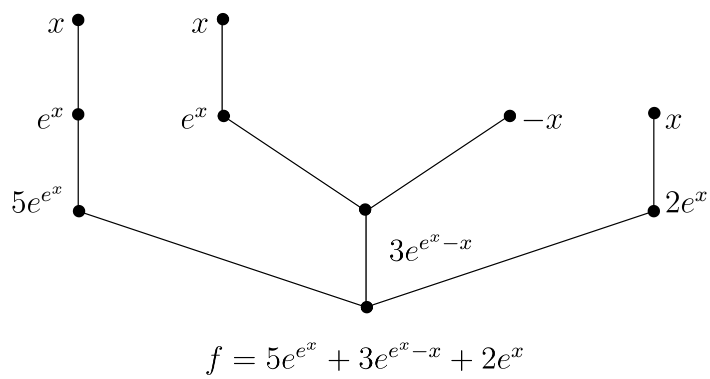


Figure 3.7: The relative tree-representation of $f = 5e^{e^x} + 3e^{e^x-x} + 2e^x$ w.r.t. $\mathbb{L} = C[[\log^{\mathbb{Z}^\star} x]]$

Proof: Suppose not, and let $\mathsf{n}_0 \lessdot \mathsf{n}_1 \lessdot \cdots$ be an infinite chain of nodes in $T_{f,\mathbb{T}}$ with

$$\begin{array}{rcl} \mathsf{n}_0 & = & \mathsf{r}(T_{f,\mathbb{T}}) \\ \mathsf{n}_1 & \in & \operatorname{succ}(\mathsf{n}_0) \\ \mathsf{n}_2 & \in & \operatorname{succ}(\mathsf{n}_1) \\ & \vdots & \end{array}$$

For $i \geqslant 0$, let β_i be the minimal ordinal such that $l(\mathsf{n}_i) \in C\mathfrak{M}_{\beta_i}$, where l is the labeling of $T_{f,\mathbb{T}}$.

Fix $i \geqslant 0$, then $\beta_i > 0$, for otherwise $l(\mathsf{n}_i) \in C\mathfrak{M}$ and the relativity imply that n_i is a leaf of T_f, hence $\operatorname{succ}(\mathsf{n}_i) = \emptyset$. Furthermore, β_i is a successor ordinal: if it was a limit ordinal, then $l(\mathsf{n}_i) \in \bigcup_{\beta<\beta_i} \mathfrak{M}_\beta$ implies $l(\mathsf{n}_i) \in C\mathfrak{M}_\beta$ for some $\beta < \beta_i$, which contradicts the minimality of β_i.

Hence for all $i \geqslant 0$ there is an ordinal α_i with $\beta_i = \alpha_i + 1$. By **TR3** we have $l(\mathsf{n}_{i+1}) \in \operatorname{term} \log \mathfrak{d}_{l(\mathsf{n}_i)}$. Since $\log \mathfrak{d}_{l(\mathsf{n}_i)} \in \mathbb{T}^{\uparrow}_{\alpha_i}$, this means $l(\mathsf{n}_{i+1}) \in C\mathfrak{M}_{\alpha_i}$ which proves $\beta_{i+1} < \beta_i$ for all $i \geqslant 0$.

Therefore, $(\beta_i)_{0 \leqslant 0}$ is a strictly decreasing sequence of ordinals. This contradiction shows the proposition. $\square$

The next proposition shows that one can represent series with less information. Indeed, the inner nodes (that is, nodes which are neither leaves nor the root) only need to be labeled by elements from the field of constants C.

Proposition 3.2.15 *The labeling $l : T_{f,\mathbb{T}} \setminus \{\mathsf{r}(T_{f,\mathbb{T}})\} \to C\mathfrak{M}_\alpha$ of a relative tree-representation $T_{f,\mathbb{T}}$ is uniquely determined by its restriction to* $\operatorname{leaf}(T_{f,\mathbb{T}})$ *and by the mapping $T_{f,\mathbb{T}} \setminus \{\mathsf{r}(T_{f,\mathbb{T}})\} \to C$ which is defined by $c(t) = c_{l(t)}$.*

Proof: We prove the proposition by transfinite induction over the depth α of the tree $T_{f,\mathbb{T}}$. If $\alpha = 0$, then $T_{f,\mathbb{T}}$ is reduced to the root and the leaves, and there is nothing to prove.

Hence assume that $\alpha > 0$ and that we have proved the proposition for all $\beta < \alpha$. Let S be a child of the root of $T_{f,\mathbb{T}}$. We have to show the proposition for S. Each of the children of $\mathsf{r}(S)$ is a tree-representation of depth $< \alpha$. Hence l is uniquely determined on $S \setminus \{\mathsf{r}(S)\}$.

Now assume that $l, l' : S \to C\mathfrak{M}_\alpha$ are two labelings with

$$\begin{aligned} c_{l(\mathsf{r}(S))} &= c_{l'(\mathsf{r}(S))} \\ l|_{S\setminus\{\mathsf{r}(S)\}} &= l'|_{S\setminus\{\mathsf{r}(S)\}}, \end{aligned}$$

such that (S,l) and (S,l') are both relative tree-representations w.r.t. $\mathbb{T}$. Let $s = \mathsf{r}(S)$. By **TR3** we have bijections

$$\begin{aligned} \varphi : \text{supp} \log \mathfrak{d}_{l(s)} &\to \text{succ}(s) \\ \varphi' : \text{supp} \log \mathfrak{d}_{l'(s)} &\to \text{succ}(s) \end{aligned}$$

which satisfy the conditions from Definition 3.2.1. For each $\mathfrak{m} \in \text{supp} \log \mathfrak{d}_{l(s)}$ we then have

$$(\log \mathfrak{d}_{l(s)})_{\mathfrak{m}}\mathfrak{m} = l(\varphi(\mathfrak{m})) = (\log \mathfrak{d}_{l'(s)})_{(\varphi')^{-1}(\varphi(\mathfrak{m}))}(\varphi')^{-1}(\varphi(\mathfrak{m})).$$

Hence $\varphi = \varphi'$ and $(\log \mathfrak{d}_{l(s)})_{\mathfrak{m}} = (\log \mathfrak{d}_{l'(s)})_{\mathfrak{m}}$ for all $\mathfrak{m}$. Hence $\mathfrak{d}_{l(s)} = \mathfrak{d}_{l'(s)}$. By assumption, we have $c_{l(s)} = c_{l'(s)}$. Thus $l(s) = l'(s)$. □

Remark 3.2.16 It should be noticed that the relative tree-representation of some series f with respect to a field $\mathbb{T}$ can always be extended to a tree-representation such that all leaves are log-atomic. This can be done by replacing every leaf of $T_{f,\mathbb{T}}$ by its unique minimal tree-representation. The result is the unique tree-representation of f such that

- all leaves are monomials from $\mathfrak{M}$ which are log-atomic,
- if a node is in $\mathfrak{M}$ and log-atomic, then it is a leaf.

We call this tree the **relative-minimal** tree-representation of f with respect to $\mathbb{T}$. We denote it by $T_{f,\text{rm},\mathbb{T}}$ (See Figure 3.8.)

Notation 3.2.17 Let t be a term, then we let

$$\begin{aligned} \text{path}(t) &:= \text{path}(T_{t,\max}), \\ \text{path}(\mathbb{T}) &:= \bigcup_{\mathfrak{m}\in\mathfrak{M}} \text{path}(\mathfrak{m}). \end{aligned}$$

Similarly, we define in transfinite exponential extensions

$$\begin{aligned} \text{path}_{\mathbb{T}}(t) &:= \text{path}(T_{t,\mathbb{T}}), \\ \text{path}_{\mathbb{T}}(T_\alpha) &:= \bigcup_{\mathfrak{m}\in\mathfrak{M}_\alpha} \text{path}_{\mathbb{T}(\mathfrak{m})}. \end{aligned}$$

Remark 3.2.18 Let us finish this section with a short remark about the connection between the different types of trees we have defined. Let $f \in \mathbb{T}_\alpha$. Then $T_{f,\min}$, $T_{f,\mathbb{T}}$ and $T_{f,\text{rm},\mathbb{T}}$ are sub-trees of $T_{f,\max}$. Moreover, we have that $T_{f,\mathbb{T}}$ and $T_{f,\min}$ are sub-trees of $T_{f,\text{rm},\mathbb{T}}$.

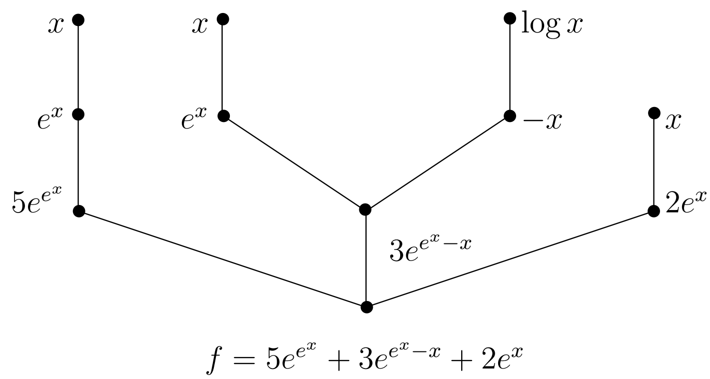


Figure 3.8: $T_{f,\mathrm{rm},\mathbb{L}}$ for $f = 5e^{e^x} + 3e^{e^x-x} + 2e^x$.

Something similar does not hold for the relative tree-representation w.r.t. $\mathbb{T}$ and the minimal tree-representation. Let for instance $\mathbb{T} = \mathbb{L} = C[[\log^{\mathbb{Z}^\star} x]]$ and $f \in \mathbb{L}_{\exp}$ with

$$f = e^{x^2 \log x} + e^x + e^{x - \log x} + e^{x - \log x - \log_2 x} + \cdots .$$

Then the first term $e^{x^2 \log x}$ provides an example where the path in $T_{f,\mathbb{T}}$ is shorter than the paths in $T_{f,\min}$. In fact, $x^2 \log x$ is a monomial in $\log^{\mathbb{Z}^\star} x$, but not log-atomic. The second term e^x is log-atomic, but not in $\log^{\mathbb{Z}^\star} x$. Thus the path in $T_{f,\mathbb{T}}$ is in this case longer.

For every series f, any of the above tree-representations is uniquely determined by the respective tree-representations of the terms of f. The only difference is that paths in a representation of f always start with the label f. Already the successor of the root determines in which term the paths is continued, and there can be only one such term. We can therefore see any of the above tree-representations as the distinct union of the tree-representations for elements of term f. For instance

$$T_{f,\mathbb{T}} = \coprod_{t \in \operatorname{term} f} T_{t,\mathbb{T}}.$$

We call the right-hand side union the **forest** of the series f.

3.3 Closure properties for series with support-constraints

Let us give an application of the properties shown in the last section. We will consider a special type of generalized power series. Indeed, we will assume that for an infinite cardinal $\kappa > \aleph_0$ generalized power series have a support of cardinality $< \kappa$. Our aim is to show that adding this condition to the definition of generalized power series stabilizes the extension process $\mathbb{T} \to \mathbb{T}_{\exp} \to \mathbb{T}_{\exp,\exp} \to \cdots$.

For the rest of this section, let us fix a cardinal $\kappa > \aleph_0$. We say that a series $f \in C[[\mathfrak{M}]]$ has **κ-support** iff $|\operatorname{supp} f| < \kappa$. We only consider series with κ-support. If C and $\mathfrak{M}$ have cofinal cardinality $< \kappa_1$ and $< \kappa_2$ respectively, then $\kappa_2 \leqslant \kappa$. If $\max(\kappa_1, \kappa_2) \leqslant \kappa$, then Lemma 2.4.3 implies that all series in $\mathbb{T}_{\exp} = C[[\mathfrak{M}]]_{\exp}$ have κ-support, and even more, by Corollary 2.4.5, for every ordinal $\alpha < \max(\kappa_1, \kappa_2)$ the set $\mathbb{T}_\alpha$ contains only series with κ-support.

On the other hand, we have seen that if $\kappa_2 < \kappa_1$, then the series from $\mathbb{T}_{\exp}$ can have supports with cardinality $\geqslant \kappa_2$. Thus, if we have $\kappa = \kappa_2$, then for $\mathfrak{m} \in \mathfrak{M}^{\uparrow}$ and a well-ordered sequence $(b_\alpha)_{\alpha<\kappa_1} \subseteq C$, the series $\sum_{\alpha<\kappa_1} \exp(-b_\alpha \mathfrak{m})$ would not be in the exp-extension of $\mathbb{T}$ anymore, since this series violates the κ-support condition.

Let us show that the extension process is stabilizing under the additional assumption that all series have κ-support.

Proposition 3.3.1 *There exists a unique ordinal λ such that*

1. $\forall \alpha < \lambda : \mathbb{T}_\alpha \subsetneq \mathbb{T}_\lambda$,
2. $\forall \alpha \geqslant \lambda : \mathbb{T}_\alpha = \mathbb{T}_\lambda$.

Moreover, this ordinal is either 0 or a limit ordinal.

Proof: Consider the class $\mathcal{T}$ of labeled trees T such that

$\mathcal{T}$1. each node in T has less than κ successors,
$\mathcal{T}$2. T is of finite height,
$\mathcal{T}$3. the inner nodes are labeled by constants from C,
$\mathcal{T}$4. the leaves of T are labeled by terms in $C\mathfrak{M}$.

We claim that $\mathcal{T}$ is a set. To see this we remark that every path in every $T \in \mathcal{T}$ is represented by a tuple in $C^\star \times C\mathfrak{M}$. There are at most $2^{|C|} \times 2^{|C\mathfrak{M}|}$ such tuples. Trees are then subsets of this set, hence

$$|\mathcal{T}| \leqslant 2^{2^{|C|} \times 2^{|C\mathfrak{M}|}}.$$

This shows the claim.

The propositions of the previous section imply that we have an injection of $\mathbb{T}_\alpha$ into $\mathcal{T}$ for each α, hence that $|\mathbb{T}_\alpha| \leqslant |\mathcal{T}|$. Now assume for a contradiction that $\mathbb{T}_\alpha \subsetneq \mathbb{T}_\beta$ for all $\alpha < \beta$. Then let $(f_\gamma)_{\gamma<|\mathcal{T}|}$ with $f_\gamma \in \mathbb{T}_{\gamma+1} \setminus \mathbb{T}_\gamma$. Hence $|\mathbb{T}_\alpha| \geqslant |\alpha|$, contradiction. Consequently, there are ordinals $\alpha < \beta$ with $\mathbb{T}_\alpha = \mathbb{T}_\beta$.

Let λ be minimal such that for some $\alpha > \lambda$ we have $\mathbb{T}_\lambda = \mathbb{T}_\alpha$. Then $\mathbb{T}_\alpha \subsetneq \mathbb{T}_\lambda$ for all $\alpha < \lambda$. On the other hand,

$$\forall \lambda < \beta < \alpha : \mathbb{T}_\lambda \subseteq \mathbb{T}_\beta \subseteq \mathbb{T}_\alpha$$

implies $\mathbb{T}_\lambda = \mathbb{T}_\beta$ for all $\lambda \leqslant \beta \leqslant \alpha$. Next, we show that $\mathbb{T}_\lambda = \mathbb{T}_\beta$ for all $\alpha \leqslant \beta$. Assume that $\beta > \alpha$ and that we have shown the assertion for all smaller ordinals. If $\beta = \gamma + 1$, then

$$\mathbb{T}_\beta = \mathbb{T}_{\gamma,\exp} = \mathbb{T}_{\lambda,\exp} = \mathbb{T}_\alpha,$$

since $\lambda < \lambda + 1 \leqslant \alpha$. If β is a limit ordinal, then we have

$$\mathfrak{M}_\beta = \bigcup_{\gamma<\beta} \mathfrak{M}_\gamma = \bigcup_{\gamma<\lambda} \mathfrak{M}_\gamma \cup \bigcup_{\lambda\leqslant\gamma<\beta} \mathfrak{M}_\gamma = \mathfrak{M}_\lambda.$$

This shows the existence and the uniqueness.

Let us finally show that λ is either zero or a limit ordinal. If not, then $\lambda = \alpha + 1$, and $\mathfrak{M}_\alpha \subsetneq \mathfrak{M}_\lambda$. Choose $\mathfrak{m} \in \mathfrak{M}_\lambda^\uparrow \setminus \mathfrak{M}_\alpha$. Then we claim that $\mathfrak{m} \notin \exp \mathfrak{M}_\alpha^\uparrow$. Otherwise $\exp \mathfrak{m} \in \mathfrak{M}_\lambda$, since $\mathfrak{m} = \mathfrak{m}^\uparrow$. Hence $\mathfrak{m} \in \mathbb{T}_\alpha^\uparrow$, since $\mathfrak{M}_\lambda = \exp \mathfrak{M}_\alpha^\uparrow$. But then $\mathfrak{m} \in \mathfrak{M}_\alpha^\uparrow$, since it is a monomial. This contradiction finishes the proof. □

Remark 3.3.2 We will later show that without the constraint on the support, the exp-extension process is not stabilizing.

3.4 Embeddings in maximal tree-representations

Every subset T' of a tree T is an ordered set with the induced ordering. If for one element of T' condition **Tr1** holds, then T' is again a tree. In particular, this is true, if it contains the root of T. In this sense, a sub-tree can be seen as an injective embedding of a tree into its host-tree.

We will need tree-embeddings in a broader sense. For instance, we will not demand injectivity, but we will always demand that the roots of the sub- and host-tree coincide. What is more, in our applications we will encounter situations where paths in the sub-trees are merely truncations of paths in the host-tree. The purpose of this section is to introduce the right setting for this kind of tree-embeddings.

Let T and U be trees and $\psi : T \to U$ be a mapping. In particular, if P is a path in T, then to every node $\mathsf{n}_{P,i}$ of height i in the path we find an image $\psi(\mathsf{n}_{P,i})$ in U. The pair (U, ψ) is a **tree-embedding** iff for every $P \in \operatorname{path}(T)$ there is a path $Q \in \operatorname{path}(U)$ such that

$$\forall \mathsf{n} \in P : \quad \mathsf{h}(\mathsf{n}) = i \quad \Longrightarrow \quad \psi(\mathsf{n}) = \mathsf{n}_{Q,i}.$$

We denote the sequence $[\psi(\mathsf{n}_{P,0}),\ \psi(\mathsf{n}_{P,1}), \ldots]$ by $\psi(P)$. A tree-embedding is said to be **faithful** iff all $P \in \operatorname{path}(T)$ of length at least 2 are mapped onto paths.

Remark 3.4.1 Note that the definition of tree-embeddings does not suppose that either of the trees is finite. On the other hand, we even allow embeddings where T has only one element, the root. In such cases, of course, the root is mapped to the root of U.

In fact, the condition on ψ makes sure that paths are mapped on truncations of paths in the sense that there are no gaps in $\psi(P)$. Faithfully embedded trees (with the exception of the case where T is a one-point tree and U is not) have the property that $\psi(P)$ streches over all of U. That is, not only do we map the root of T to the root of U, we also map the leaf of P – if there is one – onto a leaf of U.

We include the particular case of trees with only one element, since they will appear naturally in our applications. Hence even though the extra condition for faithfully embedded trees may look unmotivated at this point, it will serve us well in future and render the treatment of

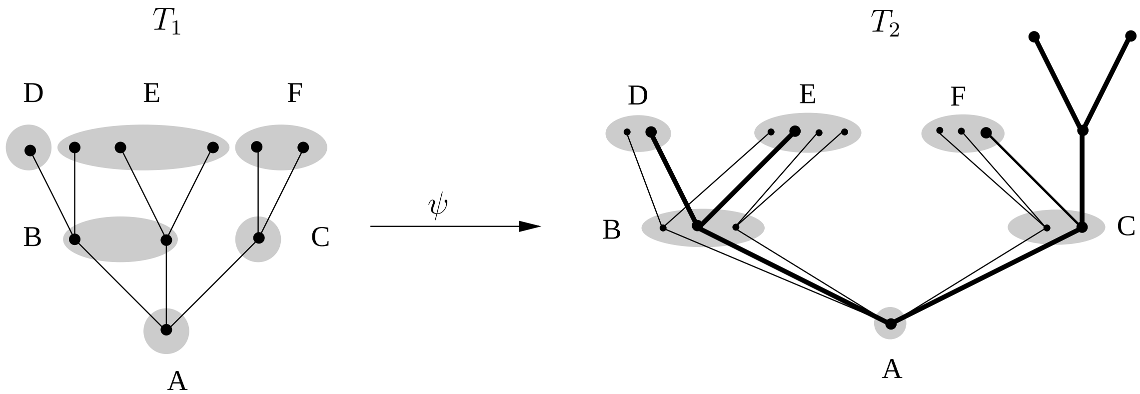


Figure 3.9: A faithful tree-embedding ψ.

sequences of trees easier. Figure 3.9 shows an example of a tree-embedding, where all paths are mapped onto paths and which is therefore faithful.

We extend the notion of faithful tree-embeddings to labeled trees. To start with, we generalize the labeling notion for path to trees. Let T be a labeled tree and $\mathsf{n} \in T$. Then $l_{T,\mathsf{n}}$ denotes the label of n in T. In particular, if the range of the labeling is a set $C\mathfrak{M}$ of terms, then we write

$$t_{T,\mathsf{n}} = c_{T,\mathsf{n}}\,\mathfrak{m}_{T,\mathsf{n}}$$

instead of $l_{T,\mathsf{n}}$. If T and U are both M-labeled trees, then T is a **faithfully M-embedded** sub-tree of U iff there is an embedding $\psi : T \to U$ of the underlying trees such that (T, ψ) is a faithful tree-embedding into U and $l_{T,\mathsf{n}} = l_{U,\psi(\mathsf{n})}$ for all nodes $\mathsf{n} \in T$.

We are particularly interested in faithfully embedded trees of tree-representations of terms and monomials. Indeed, for such settings there is an alternative way to express that a labeled tree is a faithfully embedded labeled tree. Recall that for any term $t \in C\mathfrak{M}$, the maximal tree-representation $T_{t,\max}$ is the inductive limit of the sequence $(T_i, l)_{0 \leqslant i}$.

Proposition 3.4.2 *U is a faithfully $C\mathfrak{M}$-embedded tree of $T_{t,\max}$ if and only if U is either the one-point tree with label t or if it is the inductive limit of a sequence $(U_i)_{0\leqslant i}$ with*

ss1. $\forall i \geqslant 0 : U_i$ *is a faithfully $C\mathfrak{M}$-embedded tree of T_i,*

ss2. $\forall i \geqslant 0 : \forall \mathsf{l} \in \mathrm{leaf}(U_i)$: *there is a faithful tree-embedding $0S'_{\mathsf{l}}$ of the* log*-tree S_{l} such that*

$$U_{i+1} = U_i[S'_{\mathsf{l}}]_{\mathsf{l}\in \mathrm{leaf}(U_i)},$$

(Note that condition **ss2** *implies that U_i is a sub-tree of U_{i+1}, hence that the inductive limit exists.)*

Proof: Suppose that U is not the one-point tree labeled by t. Let $U = (U, v)$ be a faithfully $C\mathfrak{M}$-embedded tree of $T_{t,\max}$ with embedding $\psi : U \to T_{t,\max}$. We define inductively $C\mathfrak{M}$-labeled trees $U_i = (U_i, v_i)$ and embeddings $\psi_i : U_i \to T_{t,i}$ such that **ss1** and **ss2** hold and such that U is the inductive limit of this sequence. Let in the following l be the labeling of $T_{t,\max}$.

We let $U_0 := T_{t,\mathrm{triv}}$ as labeled trees and $\psi_0 := \psi|_{U_0}$. Then U_0 is a faithfully $C\mathfrak{M}$-embedded tree of $T_{t,\mathrm{triv}}$ via ψ_0. This shows **ss1**. For the other **ss2** there is nothing to show.

Now suppose that we have constructed U_i, v_i and ψ_i with properties **ss1** and **ss2**. Let

$$\begin{aligned} U_{i+1} &:= U_i \coprod \{\mathsf{n} \in U \mid \exists \mathsf{p} \in \mathrm{leaf}(U_i) : \mathsf{p} = \mathrm{pred}(\mathsf{n})\}, \\ v_{i+1} &:= v|_{U_{i+1}}, \\ \psi_{i+1} &:= \psi|_{U_{i+1}}. \end{aligned}$$

Then $U_i \subseteq U_{i+1}$, $v_i \subseteq v_{i+1}$ and $\psi_i \subseteq \psi_{i+1}$. Furthermore, $\mathrm{leaf}(U_{i+1}) = U_{i+1} \setminus U_i$. We claim that U_{i+1} is a faithfully $C\mathfrak{M}$-embedded tree of T_{i+1}. First we remark that $\mathsf{r}(U) = \mathsf{r}(U_0)$ implies

$$\mathsf{r}(T_{i+1}) = \mathsf{r}(T_{t,\max}) = \psi(\mathsf{r}(U)) = \psi_0(\mathsf{r}(U_0)) = \psi_{i+1}(\mathsf{r}(U_{i+1})).$$

Fix $\mathsf{n} \in \mathrm{leaf}(U_{i+1})$. Then there is some $\mathsf{l} \in \mathrm{leaf}(U_i)$ with $\mathsf{l} = \mathrm{pred}(\mathsf{n})$. But then, since U is a sub-tree of $T_{t,\max}$, we have

$$t_{U,\mathsf{n}} \in \mathrm{term}\ \log \mathfrak{m}_{U,\psi(\mathsf{l})}.$$

In other words, the term $t_{U,\mathsf{n}}$ is the label of a leaf of the log-tree of $\mathfrak{m}_{U,\psi(\mathsf{l})}$. Hence $\mathrm{leaf}(U_i) \subseteq \mathrm{leaf}(T_i)$ implies $\mathrm{leaf}(U_{i+1}) \subseteq \mathrm{leaf}(T_{i+1})$. This finishes the inductive step and thus our construction. Conditions **ss1** and **ss2** hold by construction.

Inversely, let $(U_i)_{0 \leqslant i}$ be a sequence of faithfully $C\mathfrak{M}$-embedded trees of T_i with properties **ss1** and **ss2**. For every i, there is a mapping $\psi_i : U_i \to T_i$ with properties which realizes the fact that U_i is a faithfully $C\mathfrak{M}$-embedded sub-tree of T_i. Condition **ss2** implies $\psi_i \subseteq \psi_{i+1}$. Thus $\psi := \bigcup_i \psi_i$ defines a mapping

$$\psi : U := \bigcup_{0 \leqslant i} U_i \to T_{t,\max}.$$

Let v_i be the labeling of U_i. Again by **ss2** we have $v_i \subseteq v_{i+1}$. We let $v := \bigcup_i v_i$, which then defines a labeling of U. The labelings of the labeled trees T_i are denoted by l_i, and $l = \bigcup_i l_i$ is the labeling of $T_{t,\max}$.

First, we have to show that $\psi : U \to T_{t,\max}$ is a faithful tree-embedding. We start by noticing that $\mathsf{r}(U) = \mathsf{r}(U_0)$ and that for (S_0, ψ_0) we have

$$\psi(\mathsf{r}(U)) = \psi_0(\mathsf{r}(U_0)) = \mathsf{r}(T_{t,\mathrm{triv}}) = \mathsf{r}(T_{t,\max}).$$

TThis shows the first part of the definition of faithful embeddings. Fix $P \in \mathrm{path}(U)$ and let $\leqslant$ be the ordering of the underlying tree $T_{t,\max}$. Since $\mathsf{n}_{P,0} = \mathsf{r}(U_0)$, we have

$$\psi(P) = [\psi_0(\mathsf{n}_{P,0}), \psi_1(\mathsf{n}_{P,1}), \psi_2(\mathsf{n}_{P,2}), \dots].$$

Let $P_i = [\mathsf{n}_{P,0}, \dots, \mathsf{n}_{P,i}] \in \mathrm{path}(U_i)$. Then conditions $\mathbf{T_{max}2}$ and **ss2** inductively imply for all $i \geqslant 0$ that $\psi_i(P_i) = [\psi_0(\mathsf{n}_{P,0}), \dots, \psi_i(\mathsf{n}_{P,i})]$ is a path in T_i. If $\psi(P)$ was bounded by an element from $T_{f,\max}$, then it would be bounded by an element from some T_j. Contradiction. Furthermore, if there was some $t \in T_{f,\max}$ with

$$\psi_i(\mathsf{n}_{P,|P|+i}) \lessdot \mathsf{s} \lessdot \psi_{i+1}(\mathsf{n}_{P,|P|+i+1})$$

for some i, then the same would be true in T_i, which contradicts the fact that $\psi_{i+1}(P_{i+1})$ is a path in T_{i+1}. Hence $\psi(P)$ is a path in $T_{f,\max}$. The mapping ψ realizes thus a faithful tree-embedding. Let $\mathsf{n} \in U$, then $\mathsf{n} \in U_i$ for some $i \in \mathbb{N}$ and

$$l(\mathsf{n}) = l_i(\mathsf{n}) = v_i(\psi_i(\mathsf{n})) = v(\psi(\mathsf{n})).$$

This finishes the proof. □

3.5 Noetherian choice operators

3.5.1 Kruskal's theorem

A tree $(T, \leqslant_T)$ is **finite** iff the set T is finite. The root r of a finite tree T has only finitely many successors $\mathsf{s}_1, \dots, \mathsf{s}_k$, hence finitely many children $K_{\mathsf{s}_i} = \{\mathsf{n} \in T \mid \mathsf{s}_i \leqslant_T \mathsf{n}\}$ (with $i = 1, \dots, k$). The trees K_{s_i} are pairwise distinct. The tree T is thus completely described by r and the trees $K_{\mathsf{s}_1}, \dots, K_{\mathsf{s}_k}$. We write

$$T = \mathsf{r}[K_{\mathsf{s}_1}, \dots, K_{\mathsf{s}_k}]$$

in this case.

For a set M we let M^{T} be the set of M-labeled, finite trees. Hence for every $T \in M^{\mathsf{T}}$ there is a labeling $l_T : T \to M$. We write $T = (T, \leqslant_T, l_T)$. If M is an ordered set, then we define an ordering $\leqslant_{M^{\mathsf{T}}}$ on the set M^{T} as follows. Let $T, T' \in M^{\mathsf{T}}$, then $T \leqslant_{M^{\mathsf{T}}} T'$ iff there exists a mapping $\varphi : T \to T'$ such that

flt1. $\forall \mathsf{n}, \mathsf{m} \in T : \mathsf{n} \lessdot_T \mathsf{m} \;\Rightarrow\; \varphi(\mathsf{n}) \lessdot_{T'} \varphi(\mathsf{m})$,
flt2. $\forall \mathsf{n}, \mathsf{m} \in T : \varphi(\mathsf{n} \vee \mathsf{m}) = \varphi(\mathsf{n}) \vee \varphi(\mathsf{m})$,
flt3. $\forall \mathsf{n} \in T : l_T(\mathsf{n}) \leqslant_M l_{T'}(\varphi(\mathsf{n}))$.

The following theorem is due to Kruskal [Kru60]. We give a proof which is due to Nash-Williams [NW63].

Theorem 3.5.1 (Kruskal) *If $(M, \leqslant_M)$ is Noetherian, then so is $(M^{\mathsf{T}}, \leqslant_{M^{\mathsf{T}}})$.*

Proof: Assume that there are sequences $(T_i)_{1 \leqslant i}$ of trees in M^{T} which are not Noetherian. We call such series bad. We may assume that we have a bad sequence which is minimal in the following sense. For fixed trees $T_1, \dots, T_{i-1}$ the cardinality of T_i is minimal. (We use Zorn's lemma to show the existence of minimal bad sequences: let T_1 have minimal cardinality, and let for all $i \geqslant 2$ and fixed $T_1, \dots, T_{i-1}$, the set $\mathcal{M}_i$ be the set of trees T_i which are not comparable to $T_1, \dots, T_{i-1}$. By hypothesis, the $\mathcal{M}_i$ are non-empty, and we have $\mathcal{M}_i \supset \mathcal{M}_{i+1}$ for all i. Hence $\mathcal{N}_i = \mathcal{M}_i \setminus \mathcal{M}_{i+1}$ is a family of non-empty sets. Thus the existence of $T_1, T_2, \dots$.)

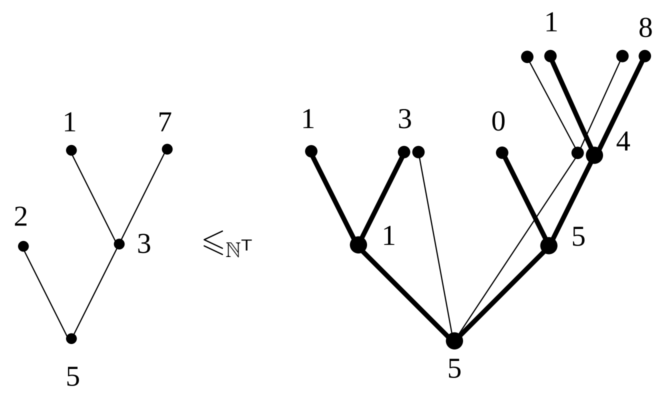


Figure 3.10: Example for $\leqslant_{\mathbb{N}^\mathsf{T}}$.

For each $i \geqslant 1$ we write $T_i = \mathsf{r}_i[T_{i,1}, \dots, T_{i,n_i}]$. For all i and all $j \leqslant n_i$ the trees $T_{i,j}$ are again in M^T. We claim that the set $S = \{T_{i,j} \,|\, 1 \leqslant i, 1 \leqslant j \leqslant n_i\}$ is Noetherian in M^T.

Suppose not, then there is a bad sequence

$$(T_{i_1,j_1}, T_{i_2,j_2}, \dots) \subseteq S.$$

Let $k \in \mathbb{N}$ be such that i_k is minimal. Then the sequence

$$(T_1, \dots, T_{i_{k-1}}, T_{i_k,j_k}, T_{i_{k+1},j_{k+1}}, \dots) \subseteq M^\mathsf{T}$$

is also bad. But the cardinality of T_{i_k,j_k} is smaller than the cardinality of T_{i_k}, which contradicts the minimality of $(T_i)_{1\leqslant i}$. This shows the claim.

Now, $M \times S^\star$ is Noetherian by Higman's theorem. Each tree T_i can be interpreted as an element of this set. This gives the desired contradiction. □

3.5.2 Labeled structures and choice operators

We extend the concept of labeled trees. Recall that to a labeled tree we could associate an underlying tree T and a labeling l which assigns a value from a given set to every node. Our present objects will be similar, only we do not demand that the underlying set is a tree, anymore.

Definition 3.5.2 *Let $X = (X, \leqslant_X)$ be an ordered set. An* ***X*-labeled structure** *is a couple $\sigma = (I_\sigma, l_\sigma)$ such that I_σ is a set (called the* **underlying structure** *of σ) and $l_\sigma : I_\sigma \to X$ is a mapping (called the* **labeling** *of σ).*

Let Σ be a set of X-labeled structures. We define an ordering on $\Sigma \times X$ by

$$(\sigma, x) < (\sigma', x') \quad :\Leftrightarrow \quad x < x'.$$

For a subset $Y \subseteq X$ we let

$$\Sigma_Y := \{\sigma \in \Sigma \mid \operatorname{im} l_\sigma \subseteq Y\},$$

that is, Σ_Y is the subset of all X-labeled structures in Σ such that all labels of σ are in Y.

A mapping $\vartheta : \Sigma \to \mathcal{P}(X)$ is called a **choice operator**. We say that $\vartheta : \Sigma \to \mathcal{P}(X)$ is **Noetherian** iff for all Noetherian sets $Y \subseteq X$ the set

$$\{(\sigma, x) \mid \sigma \in \Sigma_Y \wedge x \in \vartheta(\sigma)\} \subseteq \Sigma \times X$$

is Noetherian. A choice operator is **extensive** iff for each $\sigma \in \Sigma$:

$$\forall a \in \operatorname{im} l_\sigma : \quad \forall b \in \vartheta(\sigma) : a \leqslant_X b,$$

and ϑ is **strictly extensive** iff for all $\sigma \in \Sigma$:

$$\forall a \in \operatorname{im} l_\sigma : \quad \forall b \in \vartheta(\sigma) : a <_X b.$$

Example 3.5.3 Every M-labeled tree (T, l) is an M-labeled structure, where T is the underlying structure and l the labeling. The mapping $\vartheta(T, l) = \{l(\mathsf{n}) \mid \mathsf{n} \in \operatorname{leaf}(T)\}$ is an example for a choice operator on the set of labeled trees. Then ϑ is Noetherian, but not necessarily extensive.

In order to give an example of a Noetherian *and* extensive choice operator, let X be an ordered set, $n \in \mathbb{N}$ and $\Sigma = X^n$. More precisely, the underling set of every X-labeled structure is the set $\{1, \dots, n\}$ and the labeling of $x = (x_1, \dots, x_n) \in \Sigma$ is

$$l_x : \{1, \dots, n\} \ni i \mapsto x_i \in X.$$

Let $f : X^n \to X$ be extensive, i.e. $x_i \leqslant f(x)$ for all $1 \leqslant i \leqslant n$ and $x \in X^n$. Then

$$\vartheta : \Sigma \ni x \mapsto \{f(x)\} \in \mathcal{P}(X)$$

is a Noetherian and extensive choice operator.

3.5.3 Kruskal's theorem generalized

Given a set Σ of X-labeled structures and a choice operator $\vartheta : \Sigma \to \mathcal{P}(X)$ we can generate new sets of X-labeled structures and choice operators on these sets. In fact, we construct sets Σ^* and Σ^+ together with choice operators ϑ^* and ϑ^+, respectively. To this end, we inductively define pairwise disjoint sets $T_0, T_1, \dots$ of X-labeled sets.

The initial step: Let $\{\bullet\}$ be the one-point tree. For $x \in X$, we denote by $l_{\sigma_x^*} : \{\bullet\} \to X$ the labeling of $\{\bullet\}$ with $l_{\sigma_x^*}(\bullet) = x$. In other words, $\sigma_x^* = (\{\bullet\}, l_{\sigma_x^*})$ is the X-labeled structure where the only element of the underlying structure is labeled by x by $\bullet \mapsto x$. We let

$$T_0 := \{\sigma_x^* = (\{\bullet\}, \bullet \mapsto x) \mid x \in X\},$$

and we remark that there is a bijection between T_0 and X. For $\sigma^* \in T_0$, we let

$$\vartheta^*(\sigma^*) := \{x\} \quad \Leftrightarrow \quad \sigma^* = \sigma_x^*.$$

Then $\vartheta^* : T_0 \to \mathcal{P}(X)$ is a choice operator. This finishes the initial step.

The inductive step: Suppose that we have defined pairwise disjoint sets $T_0, \ldots, T_k$ (where $0 \leqslant k$) and for each $\sigma^* \in T_i$ $(i \leqslant k)$ an underlying structure I_{σ^*} and a labeling l_{σ^*}. Moreover, suppose that we have defined ϑ^* for σ^*, i.e. that we have $\vartheta^*(\sigma^*) \in \mathcal{P}(X)$. Hence our next step is to define the set T_{k+1} of X-labeled structures σ^*, and we have to define $\vartheta^*(\sigma^*)$. Moreover, we remark that in the construction of T_0, the set Σ has not played a role yet. It will come into play now.

Let $\sigma \in \Sigma$ with underlying set I_σ and labeling l_σ. To every point $i \in I_\sigma$ we let τ_i be an element from $T_0 \coprod \cdots \coprod T_k$, i.e.

$$\forall i \in I_\sigma : \exists l_i \leqslant k : \qquad \tau_i \in T_{l_i}.$$

Note that $l_i \leqslant k$ is uniquely determined, and we call it the **depth** of τ_i. We add an additional constraint on the choice of τ_i, namely that for each $i \in I_\sigma$ the labeling $l_\sigma(i)$ of the point i in the underlying structure is contained in $\vartheta^*(\tau_i)$, i.e.

$$\forall i \in I_\sigma : \qquad l_\sigma(i) \in \vartheta^*(\tau_i).$$

We replace each $l_\sigma(i)$ by the entire structure τ_i for each point $i \in I_\sigma$, and we write for the resulting structure

$$\sigma^* = \sigma[\tau_i]_{i \in I_\sigma}.$$

The underlying structure of σ^* is $I_{\sigma^*} = \coprod_{i \in I_\sigma} I_{\tau_i}$. The labeling l_{σ^*} is defined by

$$\forall j \in I_{\sigma^*} : \qquad l_{\sigma^*}(j) = l_{\tau_i}(j) \Leftrightarrow j \in I_{\tau_i}.$$

The structure $\sigma^* = \sigma[\tau_i]_{i \in I_\sigma}$ is X-labeled. The set of these new structures is T_{k+1}. It remains to define ϑ^* for $\sigma^* \in T_{k+1}$. For $\sigma^* = \sigma[\tau_i]_{i \in I_\sigma}$ we let

$$\vartheta^*(\sigma^*) := \vartheta(\sigma) \in \mathcal{P}(X).$$

Hence ϑ^* is a choice operator on T_{k+1}. This finishes the inductive step of the definition of the sets $T_0, T_1, \ldots$.

We let

$$\begin{aligned} \Sigma^* &:= \coprod_{0 \leqslant k} T_k \\ \Sigma^+ &:= \coprod_{1 \leqslant k} T_k. \end{aligned}$$

The function $\vartheta^+ : \Sigma^+ \to \mathcal{P}(X)$ is the restriction of ϑ^* to Σ^+, hence a choice operator on Σ^+.

Example 3.5.4 To illustrate the above construction with an easy example, let

$$X = \{x_i, y_i, z_i \mid 1 \leqslant i\}.$$

Elements from T_0 are one-point labelings like $\bullet \mapsto x_i$ (for $i \geqslant 1$). Let $\tau_1, \tau_2, \tau_3 \in T_1$ and σ the X-labeled structures in Figure 3.11. Then we have an X-labeled structure $\sigma^* = \sigma[\tau_1, \tau_2, \tau_3] \in T_2$ as shown in the same figure.

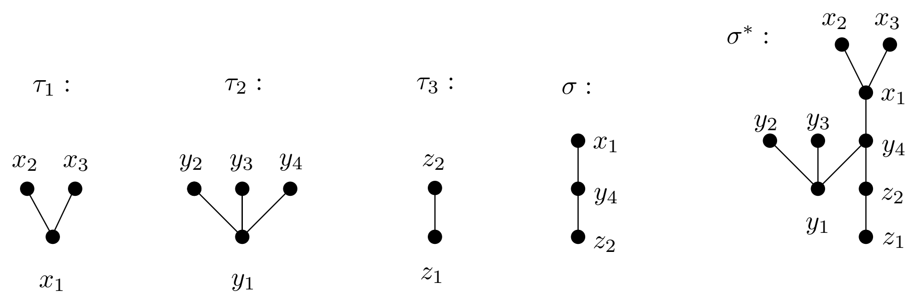


Figure 3.11: The Structures $\tau_1, \tau_2, \tau_3, \sigma$ and σ^*

We finish this section with a theorem by van der Hoeven, which generalizes Kruskal's theorem.

Theorem 3.5.5 (van der Hoeven) *Let Σ be a set of X-labeled structures and $\vartheta : \Sigma \to \mathcal{P}(X)$ a strictly extensive, Noetherian choice operator. Then $\vartheta^+ : \Sigma^+ \to \mathcal{P}(X)$ is a strictly extensive, Noetherian choice operator.*

Proof: We first show that ϑ^+ is strictly extensive. This will be done inductively. First let $\sigma^+ \in T_1$. Then there are $\sigma \in \Sigma$ and

$$(\tau_i)_{i \in I_\sigma} \subseteq X$$

such that $\sigma^+ = \sigma[\tau_i]_{i \in I_\sigma}$. From the definition of ϑ^+ we obtain $\vartheta^+(\sigma^+) = \vartheta(\sigma)$. Let $a \in \operatorname{im} l_{\sigma^+}$, then $a \in \operatorname{im} l_{\tau_i}$ for some $i \in I_\sigma$. Since $\tau_i \in T_0$, the only label is τ_i, hence $a = \tau_i$. The conditions $l_\sigma(i) \in \vartheta^+(\tau_i)$ and $\vartheta^+(\tau_i) = \{\tau_i\}$ imply $l_\sigma(i) = \tau_i$. Since ϑ is strictly extensive, we have $\tau_i < \vartheta(\sigma) = \vartheta^+(\sigma^+)$. Hence $\operatorname{im} l_{\sigma^+} < \vartheta^+(\sigma^+)$.

Now assume that for all $\hat{\sigma}^+ \in T_1 \coprod \cdots \coprod T_k$ we have shown $\operatorname{im} l_{\hat{\sigma}^+} < \vartheta^+(\hat{\sigma}^+)$. Let $\sigma^+ \in T_{k+1}$. Then there are $\sigma \in \Sigma$ and

$$(\tau_i)_{i \in I_\sigma} \subseteq T_0 \coprod \cdots \coprod T_k$$

with $\sigma^+ = \sigma[\tau_i]_{i \in I_\sigma}$. For each $a \in \operatorname{im} l_{\sigma^+}$ there is a point $i \in I_\sigma$ such that $a \in \operatorname{im} l_{\tau_i}$. The inductive hypothesis implies

$$\operatorname{im} l_{\tau_i} < \vartheta^+(\tau_i),$$

hence $a < \vartheta^+(\tau_i)$. Also, we have $l_\sigma(i) \in \vartheta^+(\tau_i)$, therefore $a < l_\sigma(i)$. In other words, if a is associated to τ_i, then the label in the structure σ for the same i is bigger than a. Since σ is strictly extensive, we have $l_\sigma(i) < \vartheta(\sigma) = \vartheta^+(\sigma^+)$. Hence $a < \vartheta^+(\sigma^+)$, thus $\operatorname{im} l_{\sigma^+} < \vartheta^+(\sigma^+)$.

Let us now show that ϑ^+ is Noetherian. Assume the contrary, and let Y be a Noetherian subset of X such that

$$\{(\sigma^+, x) \mid \sigma^+ \in \Sigma^+_Y \ \wedge \ x \in \vartheta^+(\sigma^+)\} \subseteq \Sigma^+ \times X$$

is not Noetherian. Then there is a bad sequence $((\sigma_i^+, x_i))_{1 \leqslant i}$ with

$$\begin{array}{rcl} \operatorname{im} l_{\sigma_i^+} & \subseteq & Y \\ x_i & \in & \vartheta^+(\sigma_i^+). \end{array}$$

We assume that the series is minimal in the following sense. For each $i \geqslant 1$ and fixed $x_1, \ldots, x_{i-1}$ the depth of σ_i^+ is minimal. We write each σ_i^+ as

$$\sigma_i^+ = \sigma_i[\tau_{i,j}]_{j \in I_{\sigma_i}},$$

where $\sigma_i \in \Sigma$ for all $i \geqslant 1$. Note that for all $j \in I_{\sigma_i}$ the depth of each $\tau_{i,j}$ is smaller than the depth of σ_i^+.

We claim that the induced ordering on the set $S = \{(\tau_{i,j}, l_{\sigma_i}(j)) \mid 1 \leqslant i \wedge j \in I_{\sigma_i}\}$ is Noetherian. Suppose not and let

$$((\tau_{i_1,j_1}, l_{\sigma_{i_1}}(j_1)), (\tau_{i_2,j_2}, l_{\sigma_{i_2}}(j_2)), \ldots)$$

be a bad sequence. Let $k \in \mathbb{N}$ be such that i_k is minimal. Then the sequence

$$((\sigma_1^+, x_1), \ldots, (\sigma_{k-1}^+, x_{k-1}), (\tau_{i_k,j_k}, l_{\sigma_{i_k}}(j_k)), (\tau_{i_{k+1},j_{k+1}}, l_{\sigma_{i_{k+1}}}(j_{k+1})), \ldots)$$

is also bad. But this contradicts the minimality of the sequence $((\sigma_i^+, x_i))_{1 \leqslant i}$. This shows the claim.

To finish the proof we distinguish two cases. First we assume that all σ_i^+ are elements from T_1. Then $l_{\sigma_i}(j) = \tau_{i,j}$ implies that σ_i is Y-labeled. Moreover, we have $x_i \in \vartheta^+(\sigma_i^+) = \vartheta(\sigma_i)$. Hence

$$((\sigma_1, x_1), (\sigma_2, x_2), \ldots)$$

is a sequence in $\Sigma_Y \times X$, and the theorem follows, since ϑ is Noetherian.

Secondly, assume that there is an integer $k \geqslant 1$ such that $\sigma_k^+ \notin T_1$. Then the sequence

$$((\sigma_1^+, x_1), \ldots, (\sigma_{k-1}^+, x_{k-1}), (\tau_{k,j}, l_{\sigma_k}(j)), (\sigma_{k+1}^+, x_{k+1}), \ldots)$$

is in $\Sigma_Y^+ \times X$ and cannot be bad. Then there is a strictly increasing sub-sequence in $(x_i)_{i \geqslant k+1}$. This contradiction finishes this case and the proof of the theorem. $\square$

Chapter 4

Derivations

In this chapter, we study derivations on fields of transseries. In the first part of the chapter, we axiomatize such derivations and we give an example of a transseries field admitting a derivation. We go on to discuss one possible way of extending a derivation on some transseries field to every transfinite exponential extension, and we show that the correctness of our definition depends essentially on some Noetherianity-property.

In order to prove this Noetherianity-property, we look at the problem from a different angle. Namely we show how to establish a link between derivations and tree-representations. This provides a second, more combinatorical way to define derivations. The advantage of considering tree-representations is to obtain a combinatorical proof of the Noetherianity-property.

4.1 Derivations on transseries fields

4.1.1 The notion of a derivation

Let throughout this chapter $\mathbb{T} = C[[\mathfrak{M}]]$ be a fixed transseries field. A function which acts as derivation on $\mathbb{T}$ should satisfy a number of conditions which express the compatibility between the properties of derivations and the properties of transseries fields. By that we mean for instance that C is contained in the set of constants with respect to the derivation, that the Leibniz rule holds and that the operator is strongly linear. If functions are defined, a chain rule should also be a property of the derivation.

We summarize these points in the following definition.

Definition 4.1.1 *A function $\partial : \mathbb{T} \to \mathbb{T}$ is called a* **derivation** *on $\mathbb{T}$ iff*

D1. $\forall c \in C : \partial c = 0$,
D2. $\forall f, g \in \mathbb{T} : \partial(f \cdot g) = \partial f \cdot g + f \cdot \partial g$,
D3. *if F is a Noetherian family, then so is $\partial F = (\partial f)_{f \in F}$ and $\sum \partial F = \partial \sum F$,*
D4. $\forall f \in \mathbb{T}^+ : \partial f = f \cdot \partial(\log f)$.

We will also use f' to denote ∂f, and we will write $f^{(n)}$ for $\partial_n f$. In particular, if ∂ is a derivation, then for all $f \in \mathbb{T}$ the family $(f_{\mathfrak{m}}\mathfrak{m}')_{\mathfrak{m}\in\operatorname{supp} f}$ is Noetherian and

$$f' = \sum_{\mathfrak{m}\in\operatorname{supp} f} f_{\mathfrak{m}}\mathfrak{m}'.$$

Remark 4.1.2 It would be possible to define the notion of a derivation over fields $\mathbb{S}$ of generalized power series by just using conditions **D1** and **D2**. One might thus distinguish between those "purely algebraic" derivations and our derivations which take both the strong linearity and the exponential structure in account.

Since derivations without condition **D3** are of no further interest for our purposes and since all our fields will be transseries fields, it seems reasonable to consider only operators admitting **D1** – **D4** and to call them derivations.

Remark 4.1.3 Let us remark that we can naturally extend the derivation to functions on $\mathbb{T}$ as follows. Let $\phi : \mathbb{T} \to \mathbb{T}$ be a function. Let

$$\phi^{(0)} := \phi.$$

Assume that $i \geqslant 0$ and that $\phi^{(0)}, \ldots, \phi^{(i)} : \mathbb{T} \to \mathbb{T}$ are already defined. For all $f \in \mathbb{T}$ we have $f', (\phi^{(i)} f)' \in \mathbb{T}$. If $f' \neq 0$, then we let

$$\phi^{(i+1)} f := \frac{(\phi^{(i)} f)'}{f'}.$$

We will come back to this observation in the chapter about compositions.

Remark 4.1.4 Our aim is to define derivations on fields $\mathbb{T}$. Condition **D3** suggests to define a derivation on the set of monomials. We will have to show that this function is a Noetherian mapping. Then its unique strongly linear extension to $\mathbb{T}$ will be well-defined, and **D2** for monomials implies **D2** for series by Lemma 1.6.5. The following proposition shows that condition **D4** is similarly inherited by $\mathbb{T}$ from the same property on $\mathfrak{M}$.

Proposition 4.1.5 *Let $\varphi : \mathfrak{M} \to C[[\mathfrak{N}]]$ be a Noetherian mapping such that $\varphi|_C \equiv 0$, such that $\varphi(\mathfrak{m}\mathfrak{n}) = \varphi(\mathfrak{m}) \cdot \mathfrak{n} + \mathfrak{m} \cdot \varphi(\mathfrak{n})$ and such that $\varphi(\mathfrak{m}) = \mathfrak{m} \cdot \hat{\varphi}(\log \mathfrak{m})$, where $\hat{\varphi}$ is the unique strongly linear extension of φ to $C[[\mathfrak{M}]]$. Then*

$$\hat{\varphi}(f) = f \cdot \hat{\varphi}(\log f)$$

for all $f \in C[[\mathfrak{M}]]^+$.

Proof: Let $f > 0$ and $f = c\mathfrak{d} \cdot (1 + \delta)$. Then the Leibniz rule for monomials inductively implies

$$\hat{\varphi}(\delta^i) = i \cdot \delta^{i-1} \cdot \hat{\varphi}(\delta).$$

From the linearity of $\hat{\varphi}$ and the assumption that $\hat{\varphi}$ is 0 for elements from C we then obtain

$$\hat{\varphi}(\log f) = \hat{\varphi}(\log c + \log \mathfrak{d} + l(\delta)) = \hat{\varphi}(\log \mathfrak{d}) + \hat{\varphi}(l(\delta)).$$

From the definition of $l(\delta)$ it now follows that

$$\hat{\varphi}(l(\delta)) = \sum_{1\leqslant i}(-1)^{i-1}\delta^{i-1}\cdot\hat{\varphi}(\delta) = \hat{\varphi}(\delta)\cdot\sum_{0\leqslant i}(-\delta^i) = \frac{\hat{\varphi}(\delta)}{1+\delta}.$$

Hence

$$f\cdot\hat{\varphi}(\log f) = c\mathfrak{d}\cdot\hat{\varphi}(\log\mathfrak{d})\cdot(1+\delta) + c\mathfrak{d}\cdot\hat{\varphi}(\delta) = c\varphi(\mathfrak{d})\cdot(1+\delta) + c\mathfrak{d}\cdot\hat{\varphi}(\delta).$$

The strong linearity of $\hat{\varphi}$ yields then

$$f\cdot\hat{\varphi}(\log f) = \sum_{\mathfrak{m}} f_{\mathfrak{m}}\cdot\varphi(\mathfrak{d})\cdot\frac{\mathfrak{m}}{\mathfrak{d}} + \sum_{\mathfrak{m}} f_{\mathfrak{m}}\cdot\mathfrak{d}\cdot\varphi(\frac{\mathfrak{m}}{\mathfrak{d}}) = \sum_{\mathfrak{m}} f_{\mathfrak{m}}\left(\varphi(\mathfrak{d})\cdot\frac{\mathfrak{m}}{\mathfrak{d}} + \mathfrak{d}\cdot\varphi(\frac{\mathfrak{m}}{\mathfrak{d}})\right),$$

hence by the Leibniz rule for monomials we obtain $f\cdot\hat{\varphi}(\log f) = \sum_{\mathfrak{m}} f_{\mathfrak{m}}\varphi(\mathfrak{m}) = \hat{\varphi}(f)$. This shows the proposition. □

4.1.2 Example of a derivation

Take $\mathbb{L} = C[[\log^{\mathbb{Z}^\star} x]]$. We define a function φ on the set $\log^{\mathbb{Z}^\star} x$ as follows. Fix $\log^a x$ where $a\in\mathbb{Z}^\star$. Recall that log for $\log^a x$ is defined by

$$\log(\log^a x) = \sum_{i=0}^{|a|} a_i \log_{i+1} x.$$

Then we define the function φ by

$$\begin{aligned}
\varphi(x) &:= 1\\
\varphi(\log_i x) &:= \frac{1}{x\cdots\log_{i-1} x} \qquad \text{for} \quad i\in\mathbb{N}^+\\
\varphi(\log^a x) &:= \log^a x\cdot\sum_{i=0}^{|a|} a_i\cdot\varphi(\log_{i+1} x).
\end{aligned}$$

We let $\varphi|_C \equiv 0$, thus **D1** holds by definition. The function φ verifies **D2** and **D4**. It remains to show that φ is Noetherian. To see this let $(\log^{a_i} x)_{i\in I}$ be well-ordered in $\log^{\mathbb{Z}^\star} x$. First, we observe that

$$\operatorname{supp}\varphi(\log^a x) \subseteq \log^a x\cdot\left\{1, \frac{1}{x}, \frac{1}{x\log x}, \cdots\right\},$$

hence that

$$\bigcup_{i\in I}\operatorname{supp}\varphi(\log^{a_i} x) \subseteq \{\log^{a_i} x \mid i\in I\}\cdot\left\{1, \frac{1}{x}, \frac{1}{x\log x}, \cdots\right\}.$$

The set on the right-hand side is well-ordered. Hence, by Lemma 1.1.6, the family

$$(\varphi(\log^{a_i} x))_{i\in I}$$

is Noetherian. Lemma 1.6.5 and Proposition 4.1.5 now imply that the unique strongly linear extension $\hat{\varphi}:\mathbb{L}\to\mathbb{L}$ is a derivation.

Remark 4.1.6 This example illustrates already that the derivation conditions are in general harder to prove for transseries fields than for usual power series fields. The derivation of a monomial x^n (where $n \in \mathbb{Z}$) is $n\,x^{n-1}$, thus again only one term. This is not the case in $\log^{\mathbb{Z}^\star} x$ anymore, since $\varphi(\log^a x)$ can be a finite sum. In fact, if we consider the monomial $\mathfrak{m} = \exp(x + \log x + \log_2 x + \cdots)$ in $(\log^{\mathbb{Z}^\star} x)_{\exp}$, then **D3** and **D4** imply that the derivation of this monomial has infinite support:

$$\varphi(e^{x+\log x+\log_2 x+\cdots}) = e^{x+\log x+\log_2 x+\cdots}(1 + \frac{1}{x} + \frac{1}{x\,\log x} + \cdots).$$

On the other hand, we remark that the monomials $\log_i x$ have derivations which are again monomials. Moreover, elements from $\operatorname{supp} \mathfrak{m}'$ correspond to products $\mathfrak{m}\mathfrak{n}$ where $\mathfrak{n} \in \operatorname{supp}(\log \mathfrak{m})'$.

4.1.3 Derivations and finite paths

There is a close connection between derivations and tree-representations. The aim of this section is to explain this link, which we will later use for two purposes. Firstly, it will allow us to define extensions of a given derivation to every transfinite extension of $\mathbb{T}$. In fact, this forthcoming definition will be one of two possible ways to extend derivations. Secondly, we use this correspondence in order to show that both definitions are correct.

Example 4.1.7 Let

$$t = 7e^{2e^{3x+5\log_2 x}+4x^3}$$

be a term in the field $\mathbb{L}_{\exp,\exp}$. This term has a unique relative tree-representation $T_{t,\mathbb{L}}$ with respect to $\mathbb{L}$ as shown in Figure 4.1.

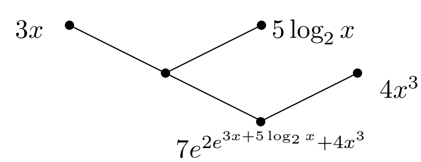


Figure 4.1: The relative tree-representation $T_{t,\mathbb{L}}$ of t over $\mathbb{L}$

Assume that we have already extended the derivation on $\mathbb{L}$ to this field, then we obtain by applying the derivation properties that

$$t' = u_1 + u_2 + u_3,$$

where u_1, u_2, u_3 are series in $\mathbb{L}_{\exp,\exp}$ with

$$\begin{aligned} u_1 &= t \cdot 2e^{3x+5\log_2 x} \cdot (3x)' \\ u_2 &= t \cdot 2e^{3x+5\log_2 x} \cdot (5\log_2 x)' \\ u_3 &= t \cdot 4e^{3\log x} \cdot (4x^3)'. \end{aligned}$$

Every term u_i corresponds to exactly one path P_i in the minimal tree-representation of the term t. For an illustration, see Figure 4.2.

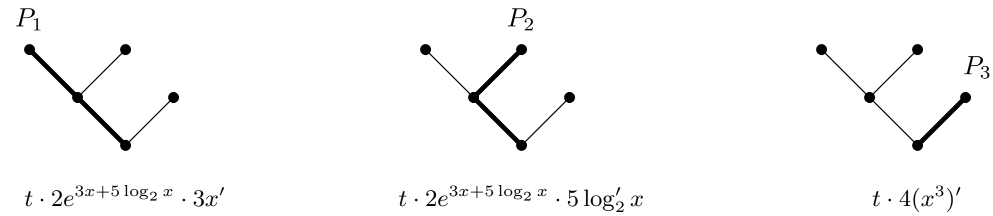


Figure 4.2: The paths P_1, P_2 and P_3 in $T_{t,\mathbb{L}}$ corresponding respectively to u_1, u_2 and u_3

Hence, in our example, we observe that to every monomial $\mathfrak{n}$ in the support of t' we find a path $P = [t_{P,0}, \dots, t_{P,|P|}]$ in the relative tree-representation of t such that

$$\mathfrak{n} = \mathfrak{m}_{P,0} \cdots \mathfrak{m}_{P,|P|-1} \cdot \mathfrak{a}$$

for some $\mathfrak{a} \in \operatorname{supp} \mathfrak{m}'_{P,|P|}$. This fact holds in general. What is more, we can write t' as

$$t' = \sum_{P \in \operatorname{path}_{\mathbb{T}(t)}} \sum_{u \in \operatorname{term} t'_{P,|P|}} t_{P,0} \cdots t_{P,|P|-1} \cdot u.$$

This observation, too, will in the following be generalized, and it will serve as main tool for the second way of defining derivations.

4.1.4 Extending derivations to transfinite extensions

Throughout this section, we fix a derivation $\partial : \mathbb{T} \to \mathbb{T}$. Our aim is to extend ∂ to every transfinite exponential extension $\mathbb{T}_\alpha$ of $\mathbb{T}$. Recall that the relative tree-representation of a series in $\mathbb{T}_\alpha$ is completely determined by its set of paths. We have introduced the forest of a series $f \in \mathbb{T}_\alpha$ as the union of all relative-minimal tree-representations $T_{t,\mathbb{T}}$, where $t \in \operatorname{term} f$. For every path $P \in T_{t,\mathbb{T}}$ we let

$$\Theta^\partial(P) := t_{P,0} \cdots t_{P,|P|-1} \cdot t'_{P,|P|}.$$

We then define a function $\partial : \mathbb{T}_\alpha \to \mathbb{T}_\alpha$ by

$$\partial(f) := f' := \sum_{\substack{t \in \operatorname{term} f: \\ P \in \operatorname{path}_{\mathbb{T}}(t)}} \Theta^\partial(P). \tag{4.1}$$

In section 4.4, we will show that the right-hand side of (4.1) is indeed the sum of a Noetherian family, which justifies this definition. We will also show there that this function, which clearly extends ∂ on $\mathbb{T}$, satisfies **D1**, **D2**, **D3** and **D4**.

4.2 Derivations and transfinite recursions

We will show that there is at most one extension of a given derivation on $\mathbb{T}$ to every transfinite exponential extension. One way is to use a transfinite induction, which will also yield an alternative way of defining such an extension.

4.2.1 Uniqueness of the extension

We start by showing that there can only be at most one such extension.

Lemma 4.2.1 *Let $\mathbb{T} \subseteq \mathbb{T}_\beta \subseteq \mathbb{T}_\alpha$ and $\partial, \partial_\alpha$ be derivations on $\mathbb{T}$ and $\mathbb{T}_\alpha$, respectively, such that ∂_α extends ∂. Let ∂_β be the restriction of ∂_α to $\mathbb{T}_\beta$. Then ∂_β is a derivation on $\mathbb{T}_\beta$.*

Proof: For every $\beta \leqslant \alpha$, the conditions **D1** – **D4** for $\mathbb{T}_\beta$ follow from the same conditions for $\mathbb{T}_\alpha$. It remains thus to show that $\operatorname{range} \partial_\beta \subseteq \mathbb{T}_\beta$.

Let β be minimal such that this is not the case. Then $0 < \beta < \alpha$. We show that $\partial_\beta(\mathfrak{m}) \in \mathbb{T}_\beta$ for all $\mathfrak{m} \in \mathfrak{M}_\beta$. From this and **D3** the necessary contradiction follows.

Let $\mathfrak{m} \in \mathfrak{M}_\beta$. Then $\log \mathfrak{m} \in \mathbb{T}_\gamma$ for some $0 \leqslant \gamma < \beta$; and

$$\partial_\beta(\log \mathfrak{m}) = \partial_\alpha(\log \mathfrak{m}) = \partial_\gamma(\log \mathfrak{m}) \in \mathbb{T}_\gamma \subseteq \mathbb{T}_\beta$$

implies $\mathfrak{m} \cdot \partial_\beta(\log \mathfrak{m}) \in \mathbb{T}_\beta$. But then by **D4** we have $\partial_\beta(\mathfrak{m}) \in \mathbb{T}_\beta$. This finishes the proof. $\square$

Proposition 4.2.2 *For every α there is at most one derivation extending $\partial : \mathbb{T} \to \mathbb{T}$.*

Proof: Let α be minimal such that the proposition fails. Then there are two extensions ∂', ∂'' of ∂. We show that they are identical on $\mathfrak{M}_\alpha$.

Let $\mathfrak{m} \in \mathfrak{M}_\alpha$. Then $\log \mathfrak{m} \in \mathbb{T}_\beta$ for some $\beta < \alpha$. By Lemma 4.2.1, the restrictions of ∂' and ∂'' to $\mathbb{T}_\beta$ are derivations. Thus they are by minimality identical. But then

$$\partial'(\mathfrak{m}) = \mathfrak{m} \cdot \partial'_{\mathbb{T}_\beta}(\log \mathfrak{m}) = \mathfrak{m} \cdot \partial''_{\mathbb{T}_\beta}(\log \mathfrak{m}) = \partial''(\mathfrak{m})$$

by **D4** for ∂' and ∂'' on $\mathbb{T}_\beta$. From the strong linearity condition **D3** on $\mathbb{T}_\alpha$ it now follows that $\partial' = \partial''$. This contradiction shows the proposition. $\square$

4.2.2 Transfinite extensions

Now that we know that there is at most one derivation ∂_α, we set out to define it. One way to do this is to use a transfinite induction. Let $\alpha > 0$ be an ordinal number, and assume that for all $\beta < \alpha$, a unique derivation $\partial_\beta : \mathbb{T}_\beta \to \mathbb{T}_\beta$ has already been defined such that

$$\begin{aligned} \partial_0 &= \partial \\ \partial_\gamma &\subseteq \partial_\beta \qquad \forall \gamma \leqslant \beta. \end{aligned}$$

Before defining a mapping $\varphi : \mathfrak{M}_\alpha \to \mathbb{T}_\alpha$ which will be used to define ∂_α, we remark that

- if $\mathfrak{m} \in \mathfrak{M}_\beta$ for some $\beta < \alpha$, then $\log \mathfrak{m} \in \mathbb{T}_\beta$, and $\log \mathfrak{m} \in \operatorname{dom} \partial_\beta$; therefore we have $\partial_\beta(\mathfrak{m}) = \mathfrak{m} \cdot \partial_\beta(\log \mathfrak{m}) \in \mathbb{T}_\alpha$;
- this is always the case, if α is a limit ordinal;
- if $\alpha = \beta + 1$ is a successor ordinal, then $\log \mathfrak{m} \in \mathbb{T}_\beta$, and again $\log \mathfrak{m} \in \operatorname{dom} \partial_\beta$; in this case $\partial_\beta(\log \mathfrak{m}) \in \mathbb{T}_\beta$ and $\mathfrak{m} \cdot \partial_\beta(\log \mathfrak{m}) \in \mathbb{T}_\alpha$;
- if $\log \mathfrak{m} \in \mathbb{T}_\gamma \cap \mathbb{T}_\beta$ for $\gamma, \beta < \alpha$, then $\mathfrak{m} \cdot \partial_\gamma(\log \mathfrak{m}) = \mathfrak{m} \cdot \partial_\beta(\log \mathfrak{m})$.

Hence, for every $\mathfrak{m} \in \mathfrak{M}_\alpha$, the mapping φ defined by

$$\varphi(\mathfrak{m}) := \mathfrak{m} \cdot \partial_\beta(\log \mathfrak{m}) \qquad \text{if } \log \mathfrak{m} \in \mathbb{T}_\beta$$

is a well-defined function $\varphi : \mathfrak{M}_\alpha \to \mathbb{T}_\alpha$, i.e. the definition does not depend on the choice of β and there is always at least one ordinal $\beta < \alpha$ with $\log \mathfrak{m} \in \mathbb{T}_\beta$. In order to extend φ to a strongly linear function $\hat{\varphi} : \mathbb{T}_\alpha \to \mathbb{T}_\alpha$, we have to show that φ is a Noetherian mapping. In fact, as the following proposition shows, this is the key to showing that $\hat{\varphi}$ is the unique extension of ∂ to $\mathbb{T}_\alpha$ as derivation.

Proposition 4.2.3 *If the above defined function $\varphi : \mathfrak{M}_\alpha \to \mathbb{T}_\alpha$ is a Noetherian mapping, then its unique linear extension*

$$\hat{\varphi} : \mathbb{T}_\alpha \to \mathbb{T}_\alpha$$

is the unique derivation on $\mathbb{T}_\alpha$ extending ∂.

Proof: We have to show the conditions **D1** – **D4** for $\hat{\varphi}$. Note that **D3** is the hypothesis, and that from $\operatorname{range} \varphi \subseteq \mathbb{T}_\alpha$ and strong linearity it follows that $\operatorname{range} \hat{\varphi} \subseteq \mathbb{T}_\alpha$.

Condition **D1** follows from **D1** on $\mathbb{T}$ and linearity. Next, we show that for $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}_\alpha$ we have

$$\varphi(\mathfrak{m}\mathfrak{n}) = \varphi(\mathfrak{m}) \cdot \mathfrak{n} + \mathfrak{m} \cdot \varphi(\mathfrak{n}).$$

For some $\beta < \alpha$, we have $\log(\mathfrak{m}\mathfrak{n}) \in \mathbb{T}_\beta$. By linearity of ∂_β we have

$$\begin{aligned}
\varphi(\mathfrak{m}\mathfrak{n}) &= \mathfrak{m}\mathfrak{n} \cdot \partial_\beta(\log \mathfrak{m} + \log \mathfrak{n}) \\
&= \mathfrak{n} \cdot \mathfrak{m} \cdot \partial_\beta(\log \mathfrak{m}) + \mathfrak{m} \cdot \mathfrak{n} \cdot \partial_\beta(\log \mathfrak{n}) \\
&= \mathfrak{n} \cdot \varphi(\mathfrak{m}) + \mathfrak{m} \cdot \varphi(\mathfrak{n}).
\end{aligned}$$

Then **D2** follows from Lemma 1.6.5. Condition **D4** follows from Proposition 4.1.5 and the definition of φ. The uniqueness follows from Proposition 4.2.2. □

Remark 4.2.4 In order to show that the function φ thus defined is Noetherian, we will have to show that for every sequence of terms

$$t_0 \succ t_1 \succ t_2 \succ$$

in $C\mathfrak{M}_\alpha$, the family $(\varphi(t_i))_{0 \leqslant i}$ is Noetherian. In other words, for every sequence $(\mathfrak{n}_i)_{0 \leqslant i}$ with $\mathfrak{n}_i \in \operatorname{supp} \varphi(t_i)$, there must be a $\prec$-decreasing sub-sequence.

Since the mapping φ admits on the set of term condition **D4**, we can in a similar way as in Example 4.1.7 show that to every $t \in C\mathfrak{M}_\alpha$ and $\mathfrak{n} \in \operatorname{supp} \varphi(t)$ there is a path P in the relative tree-representation of t over $\mathbb{T}$ such that for some $\mathfrak{a} \in \operatorname{supp} (\log \mathfrak{m}_{P,|P|})'$ we have

$$\mathfrak{n} = \mathfrak{m}_{P,0} \cdots \mathfrak{m}_{P,|P|} \cdot \mathfrak{a}.$$

Again, this reduces the problem of showing that φ is Noetherian to a problem about paths.

4.3 Path orderings

We have seen how to extend ∂ to a function $\mathbb{T}_\alpha \to \mathbb{T}_\alpha$ in two different manners. Either way we will have to show the correctness of the definitions, and we have seen that this means that we have to show certain Noetherianity conditions.

The correspondence between derivations and paths expresses the underlying combinatorial properties of the derivation, and this close connection will thus play an important role in the proof of the Noetherianity of the defined operators. To prepare the ground, we will next introduce an ordering between paths.

4.3.1 Ordering infinite paths

Remark 4.3.1 The relation which will be defined in the following does not depend on the existence of a derivation on some transseries field, nor do we need to consider transfinite extensions. Throughout this section, we fix $\mathbb{T} = C[[\mathfrak{M}]]$, a field of transseries. We recall that for a path P, the label of the node of height i is a term denoted by $t_{P,i} \in C\mathfrak{M}$ and that we write

$$t_{P,i} = c_{P,i}\, \mathfrak{m}_{P,i}.$$

Remark 4.3.2 Recall that for series f and g we symbolize the maximal common truncation by $f \;\triangle\; g$. Let $\mathfrak{m} \succ \mathfrak{n}$ be transmonomials and $t \in \operatorname{term} \log \mathfrak{n}$. Then we have

$$t \in \operatorname{term} (\log \mathfrak{m} \;\triangle\; \log \mathfrak{n}) \quad \Leftrightarrow \quad t \succ \log \mathfrak{m} - \log \mathfrak{n} \quad \Leftrightarrow \quad t \succ \log \frac{\mathfrak{m}}{\mathfrak{n}}.$$

From Remark 1.8.1 it follows that for transmonomials $\mathfrak{m}_1 \succcurlyeq \mathfrak{m}_2 \succcurlyeq \mathfrak{m}_3$, we have

$$\begin{aligned} \log \mathfrak{m}_1 \;\triangle\; \log \mathfrak{m}_3 &\trianglelefteq \log \mathfrak{m}_1 \;\triangle\; \log \mathfrak{m}_2, \\ \log \mathfrak{m}_1 \;\triangle\; \log \mathfrak{m}_3 &\trianglelefteq \log \mathfrak{m}_2 \;\triangle\; \log \mathfrak{m}_3. \end{aligned}$$

Let $\mathfrak{s}, \mathfrak{t} \in \mathfrak{M}$ be terms, and let $P \in \operatorname{paths} \mathfrak{s}$ and $Q \in \operatorname{path} \mathfrak{t}$ be paths. Then we let $P \succcurlyeq_{\!\!\!\!\!\!\!\frown\, 1} Q$ iff $t_{P,0} \succcurlyeq t_{Q,0}$ and

$$\begin{aligned} t_{P,1} \succcurlyeq t_{Q,1} \quad &\wedge \quad \mathfrak{m}_{Q,1} \in \operatorname{term} (\log \mathfrak{m}_{P,0} \;\triangle\; \log \mathfrak{m}_{Q,0}), \\ t_{P,2} \succcurlyeq t_{Q,2} \quad &\wedge \quad \mathfrak{m}_{Q,2} \in \operatorname{term} (\log \mathfrak{m}_{P,1} \;\triangle\; \log \mathfrak{m}_{Q,1}), \\ &\vdots \end{aligned}$$

and we let $P \succ_{\mathrm{I}} Q$ iff $P \succcurlyeq_{\mathrm{I}} Q$ and $P \neq Q$. Furthermore, we let $P \succ_{\mathrm{II}} Q$ iff $t_{P,0} \succcurlyeq t_{Q,0}$ and if there is an integer $i > 0$ such that

$$\left\{ \begin{array}{lrcrl} \text{and} & t_{P,1} \succcurlyeq t_{Q,1} & \wedge & t_{Q,1} & \in \operatorname{term}(\log \mathfrak{m}_{P,0} \ \vartriangle \ \log \mathfrak{m}_{Q,0}) \\ & & \vdots & & \\ \text{and} & t_{P,i-1} \succcurlyeq t_{Q,i-1} & \wedge & t_{Q,i-1} & \in \operatorname{term}(\log \mathfrak{m}_{P,i-2} \ \vartriangle \ \log \mathfrak{m}_{Q,i-2}) \\ & & \wedge & t_{Q,i} & \notin \operatorname{term}(\log \mathfrak{m}_{P,i-1} \ \vartriangle \ \log \mathfrak{m}_{Q,i-1}). \end{array} \right.$$

Moreover, we let $P \succcurlyeq_{\mathrm{II}} Q$ iff $P \succ_{\mathrm{II}} Q$ or $P = Q$. Finally, we define the relation $\succ$ on the set of paths by $P \succcurlyeq Q$ iff $P \succcurlyeq_{\mathrm{I}} Q$ or $P \succcurlyeq_{\mathrm{II}} Q$.

Remark 4.3.3 We illustrate $P \succ Q$ with Figure 4.3. The shaded area indicates the set of terms of $\log \mathfrak{m}_{P,j}$, and the bar on top of each such triangle symbolizes the maximal common truncation of $\log \mathfrak{m}_{P,j}$ and $\log \mathfrak{m}_{Q,j}$. We use the solid line for the path P and the dotted line for the path Q.

The two figures on the left-hand side show possible situations where $P \succ_{\mathrm{I}} Q$. The path Q will either coincide with P or be on the right of P. It is also possible for distict paths that they coincide up to a certain height, then split and then coincide again, as shown in (i). However, we remark that there can be only finitely many such splitting points, for otherwise the sequence $(\mathfrak{m}_{P,i})_{0 \leqslant i}$ contradicts condition **T4** of the definition of transfinite fields. Hence, the paths P and Q will coincide from some height on, which can be seen in (ii).

If $P \succ_{\mathrm{II}} Q$, then one encounters four different situations on the level i, where i is as in the definition. The first two – which correspond to (iii) and (iv) – concern the case where $t_{P,i}$ is an element of the maximal common truncation $\log \mathfrak{m}_{P,i-1} \ \vartriangle \ \log \mathfrak{m}_{Q,i-1}$. Then the term $t_{Q,i}$ cannot be an element of this truncation, and we have either that $t_{Q,i}$ is a term of $\log \mathfrak{m}_{P,i-1}$ or not.

Otherwise, the term $t_{P,i}$ is itself not in $\log \mathfrak{m}_{P,i-1} \ \vartriangle \ \log \mathfrak{m}_{Q,i-1}$, which implies that $t_{Q,i}$ is neither. Again, we can distinguish between $t_{Q,i} \in \operatorname{term} \log \mathfrak{m}_{P,i-1}$ or not. These cases are illustrated by (v) and (vi).

Proposition 4.3.4 *The relation $\succcurlyeq$ is an ordering.*

Proof: Reflexivity and anti-symmetry follow directly from the definition. We thus have to show transitivity, i.e. condition **PO3** of the definition of an ordering. Whenever in the following we have $P \succ_{\mathrm{II}} Q$, then we let i be the positive integer from the definition of the relation $\succ_{\mathrm{II}}$. Accordingly, if $Q \succ_{\mathrm{II}} R$, then we let j be the positive integer which replaces i. We have to distinguish four cases.

Case 1: $P \succ_{\mathrm{I}} Q \succ_{\mathrm{I}} R$. The relation $t_{P,k} \succcurlyeq t_{R,k}$ holds for all $k \geqslant 0$. Let m be minimal such that

$$t_{Q,m+1} \in \operatorname{term}(\log \mathfrak{m}_{P,m} \ \vartriangle \ \log \mathfrak{m}_{R,m}).$$

If m does not exists, then $P \succcurlyeq_{\mathrm{I}} R$. If it does, then $P \succ_{\mathrm{II}} Q$. Either way we have $P \succcurlyeq R$.

Case 2: $P \succ_{\mathrm{I}} Q \succ_{\mathrm{II}} R$. First we note that we have $t_{P,k} \succcurlyeq t_{R,k}$ for all $k < j$. If there is some $k < j$ with

$$t_{R,k} \notin \operatorname{term}(\log \mathfrak{m}_{P,k-1} \ \vartriangle \ \log \mathfrak{m}_{R,k-1}),$$

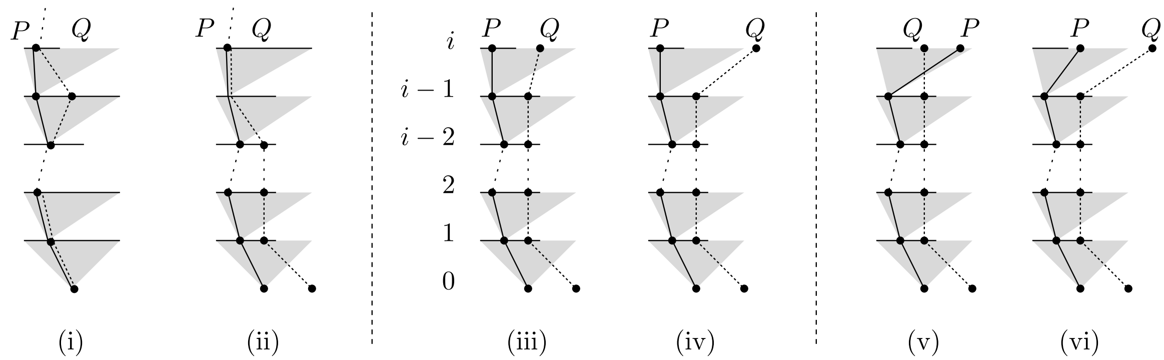


Figure 4.3: Paths with $P \succ Q$.

then we have $P \succ_{\mathrm{II}} R$. If not, then we have on the one hand

$$t_{R,j} \notin \operatorname{term}(\log \mathfrak{m}_{Q,j-1} \;\triangle\; \log \mathfrak{m}_{R,j-1}).$$

On the other hand, Remark 4.3.2 implies

$$\log \mathfrak{m}_{P,j-1} \;\triangle\; \log \mathfrak{m}_{R,j-1} \trianglelefteq \log \mathfrak{m}_{Q,j-1} \;\triangle\; \log \mathfrak{m}_{R,j-1}.$$

Hence

$$t_{R,j} \notin \operatorname{term}(\log \mathfrak{m}_{P,j-1} \;\triangle\; \log \mathfrak{m}_{R,j-1}),$$

which shows $P \succ_{\mathrm{II}} R$.

Case 3: $P \succ_{\mathrm{II}} Q \succ_{\mathrm{I}} R$. this case is symmetric to case 2. We have $t_{P,k} \succcurlyeq t_{R,k}$ for all $k < i$. If there exists a $0 < k < i$ with

$$t_{R,k} \notin \operatorname{term}(\log \mathfrak{m}_{P,k-1} \;\triangle\; \log \mathfrak{m}_{R,k-1}),$$

then we are done. Otherwise, from $t_{Q,i} \succcurlyeq t_{R,i}$ we obtain

$$t_{R,i} \notin \operatorname{term}(\log \mathfrak{m}_{P,i-1} \;\triangle\; \log \mathfrak{m}_{Q,i-1}).$$

From Remark 4.3.2 it now follows that

$$\log \mathfrak{m}_{P,i-1} \;\triangle\; \log \mathfrak{m}_{R,i-1} \trianglelefteq \log \mathfrak{m}_{P,i-1} \;\triangle\; \log \mathfrak{m}_{Q,i-1}.$$

This implies

$$t_{R,i} \notin \operatorname{term}(\log \mathfrak{m}_{P,i-1} \;\triangle\; \log \mathfrak{m}_{R,i-1}),$$

which shows $P \succ_{\mathrm{II}} R$.

Case 4: $P \succ_{\mathrm{II}} Q \succ_{\mathrm{II}} R$. The case $j < i$ can be shown as case 2, and the case $i \leqslant j$ can be shown using the proof of case 3. $\square$

4.3.2 Ordering finite paths

Let $\mathbb{T} = C[[\mathfrak{M}]]$ and $\alpha > 0$. In this section, we consider a transfinite exponential extension $\mathbb{T}_\alpha$ of $\mathbb{T}$.

Notation 4.3.5 For $P \in \operatorname{path}_{\mathbb{T}}(\mathbb{T}_\alpha)$ and $\hat{P} \in \operatorname{path}(\mathbb{T}_\alpha)$ we write

$$\hat{P} \ \vartriangleright \ P \quad \Leftrightarrow \quad \forall i \leqslant |P| : t_{P,i} = t_{\hat{P},i},$$

i.e. if P is a truncation of the path $\hat{P}$.

We let

$$\begin{aligned}
P \succcurlyeq Q \quad &:\Leftrightarrow \quad \forall \hat{P} \ \vartriangleright \ P : \exists \hat{Q} \ \vartriangleright \ Q : \hat{P} \succcurlyeq \hat{Q}, \\
P \succcurlyeq_{\mathrm{I}} Q \quad &:\Leftrightarrow \quad \forall \hat{P} \ \vartriangleright \ P : \exists \hat{Q} \ \vartriangleright \ Q : \hat{P} \succcurlyeq_{\mathrm{I}} \hat{Q}, \\
P \succ_{\mathrm{I}} Q \quad &:\Leftrightarrow \quad P \succcurlyeq_{\mathrm{I}} Q \ \wedge \ P \neq Q, \\
P \succcurlyeq_{\mathrm{II}} Q \quad &:\Leftrightarrow \quad P \succcurlyeq Q \ \wedge \ \neg(P \succcurlyeq_{\mathrm{I}} Q).
\end{aligned}$$

Proposition 4.3.6 *The relation $\succcurlyeq$ is an ordering on* $\operatorname{path}_{\mathbb{T}}(\mathbb{T}_\alpha)$*.*

Proof: Let $P, Q, R \in \operatorname{path}_{\mathbb{T}}(\mathbb{T}_\alpha)$. Reflexivity follows directly from the definition. Next suppose that $P \succcurlyeq Q \succcurlyeq P$. Fix some $\hat{P} \ \vartriangleright \ P$, then there are $\hat{Q} \ \vartriangleright \ Q$ and $\hat{P}' \ \vartriangleright \ P$ such that $\hat{P} \succcurlyeq \hat{R} \succcurlyeq \hat{P}'$. Since the labels of the roots are monomials, we get

$$t_{P,0} = t_{\hat{Q},0} = t_{P',0}.$$

Inductively, one shows that for all $i \leqslant |P|$ one has

$$t_{P,i} = t_{\hat{Q},i} = t_{P',i}.$$

Hence $t_{\hat{Q},|P|} \in \mathbb{T}$, which shows that $|Q| \leqslant |P|$. By symmetry we obtain $|P| = |Q|$ and thus $P = Q$. As for the transitivity, assume that $P \succcurlyeq Q \succcurlyeq R$. For every $\hat{P} \ \vartriangleright \ P$ there are $\hat{Q} \ \vartriangleright \ Q$ and $\hat{R} \ \vartriangleright \ R$ such that $\hat{P} \succcurlyeq \hat{Q} \succcurlyeq \hat{R}$. Thus by transitivity of $\succcurlyeq$ on $\operatorname{path}(\mathbb{T}_\alpha)$ we obtain $\hat{P} \succcurlyeq \hat{R}$, from which $P \succcurlyeq R$ follows. □

Remark 4.3.7 For all $P, Q \in \operatorname{path}(\mathbb{T}_\alpha)$ there are unique $\check{P}, \check{Q} \in \operatorname{path}_{\mathbb{T}}(\mathbb{T}_\alpha)$ with $P \ \vartriangleright \ \check{P}$ and $Q \ \vartriangleright \ \check{Q}$. We let $P \sim Q$ iff $\check{P} = \check{Q}$. Then $\sim$ is an equivalence relation on $\operatorname{path}(\mathbb{T}_\alpha)$, and the relations $\succcurlyeq$ and $\succcurlyeq_{\mathrm{I}}$ on $\operatorname{path}_{\mathbb{T}}(\mathbb{T}_\alpha)$ are obtained from the relations $\succcurlyeq$ and $\succcurlyeq_{\mathrm{I}}$ on $\operatorname{path}(\mathbb{T}_\alpha)$ by quotienting with respect to $\sim$.

Lemma 4.3.8 *Let $P, Q, R \in \operatorname{path}(\mathbb{T}_\alpha)$ with $P \succcurlyeq Q \succcurlyeq R$. Then $\check{P} \succcurlyeq \check{Q}$ or $\check{Q} \succcurlyeq \check{R}$.*

Proof: We may assume that the relations are strict. First, we consider the case $P \succ_{\mathrm{II}} Q$. Then $\mathfrak{m}_{P,0} \succcurlyeq \mathfrak{m}_{Q,0}$ and there is a minimal $i > 0$ such that

$$t_{P,i} \succcurlyeq t_{Q,i} \ \wedge \ t_{Q,i} \in \operatorname{term}\left(\log \mathfrak{m}_{P,i-1} \ \vartriangle \ \log \mathfrak{m}_{Q,i-1}\right)$$

fails. Fix some $P' \rhd \check{P}$. If $i > |\check{Q}|$, then we let

$$Q' = [\check{Q},\ t_{P',|\check{Q}|+1},\ t_{P',|\check{Q}|+2},\ \ldots]. \tag{4.2}$$

Then $Q' \rhd \check{Q}$ and $P' \succ\!\!\succ_{\mathrm{I}} Q'$. If $i \leqslant |\check{Q}|$, then every $Q' \unrhd \check{Q}$ satisfies $P' \succ\!\!\succ_{\mathrm{II}} Q'$. Similarly, one obtains from $Q \succ_{\mathrm{I}} R$ that $\check{Q} \succ\!\!\succ \check{R}$.

Next, we consider $P \succ_{\mathrm{I}} Q \succ_{\mathrm{I}} R$. Since $t_{Q,|\check{P}|+1} \in \operatorname{term} \log \mathfrak{m}_{\check{P},|\check{P}|}$ is a term in $\mathbb{T}$, we must have $|\check{Q}| \leqslant |\check{P}| + 1$. Similarly one gets that $|\check{R}| \leqslant |\check{Q}| + 1$ and, by transitivity, that $|\check{R}| \leqslant |\check{P}| + 1$. From this one of $|\check{P}| = |\check{Q}|$ or $|\check{Q}| = |\check{R}|$ follows. Assume $|\check{P}| = |\check{Q}|$ and let $P' \rhd \check{P}$. Then we let Q' as in (4.2) and obtain again $P' \succ\!\!\succ_{\mathrm{I}} Q'$. The case $|\check{Q}| = |\check{R}|$ is similar. □

Remark 4.3.9 Although $P \succ\!\!\succ Q$ does in general not imply $\check{P} \succ\!\!\succ \check{Q}$, the proof of Lemma 4.3.8 shows that at least

$$P \succ\!\!\succ_{\mathrm{II}} Q \quad \Rightarrow \quad \check{P} \succ\!\!\succ \check{Q}$$

holds. Furthermore $P \succ\!\!\succ_{\mathrm{I}} Q \succ\!\!\succ_{\mathrm{I}} R$ implies $\check{P} \succ\!\!\succ_{\mathrm{I}} \check{Q}$ or $\check{Q} \succ\!\!\succ_{\mathrm{I}} \check{R}$.

We now prove some properties of the relation $\succ$ and distinguish in particular between specific properties for $\succ\!\!\succ_{\mathrm{I}}$ and $\succ\!\!\succ_{\mathrm{II}}$.

Lemma 4.3.10 *Let α be an ordinal and $(P_\beta)_{\beta<\alpha}$ be a sequence in $\operatorname{path}_{\mathbb{T}}(\mathbb{T}_\alpha)$ such that $P_\gamma \succ\!\!\succ_{\mathrm{I}} P_\beta$ for all $\gamma < \beta$. Then there exists a sequence $\beta_0 < \beta_1 < \beta_2 < \cdots < \alpha$ such that*

$$\begin{aligned} &|P_{\beta_1}| = |P_{\beta_2}| = |P_{\beta_3}| = \cdots, \\ &\mathfrak{m}_{P_{\beta_1},|P_{\beta_1}|},\ \mathfrak{m}_{P_{\beta_2},|P_{\beta_2}|},\ \mathfrak{m}_{P_{\beta_3},|P_{\beta_3}|},\ \cdots \in \operatorname{supp} \log \mathfrak{m}_{P_{\beta_0},|P_{\beta_0}|-1}. \end{aligned}$$

Proof: Suppose that the first claim fails, then there exist $\beta_0 < \beta_1 < \cdots$ such that $|P_{\beta_0}| < |P_{\beta_1}| < \cdots$. for all $i < j$ we have

$$\mathfrak{m}_{P_{\beta_j},|P_{\beta_i}|+1} \in \operatorname{supp} \log \mathfrak{m}_{P_{\beta_i},|P_{\beta_i}|} \subseteq \mathfrak{M}.$$

This implies $|P_{\beta_i}| + 1 = |P_{\beta_j}|$. But then

$$|P_{\beta_1}| + 1 = |P_{\beta_3}| = |P_{\beta_2}| + 1 = |P_{\beta_1}| + 2.$$

This contradiction shows the first claim. The second one follows from $\hat{P}_{\beta_0} \succ\!\!\succ_{\mathrm{I}} \hat{P}_{\beta_i}$ for all i. □

Lemma 4.3.11 *Let $P, Q \in \operatorname{path}_{\mathbb{T}}(\mathbb{T}_\alpha)$ with $P \succ\!\!\succ_{\mathrm{II}} Q$. Then for all terms v with $t_{Q,|Q|} \succcurlyeq v$ we have*

$$\left(\prod_{i \leqslant |P|} t_{P,i}\right) \succ \left(\prod_{i \leqslant |Q|} t_{Q,i}\right) \cdot v.$$

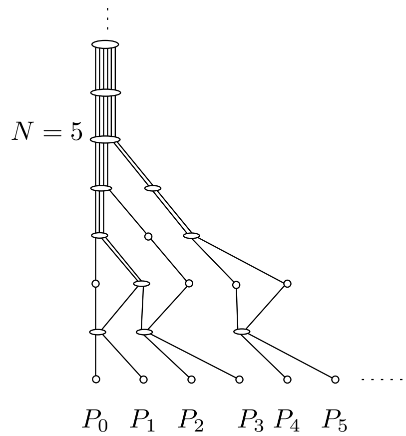


Figure 4.4: $P_0 \succ_{\mathrm{I}} P_1 \succ_{\mathrm{I}} P_2 \succ_{\mathrm{I}} \cdots$.

Proof: We first observe that $t_{P,0} \succcurlyeq t_{Q,0}$. Let $\hat{P} \vartriangleright P$ and $\hat{Q} \vartriangleright Q$ be such that $\hat{P} \succ_{\mathrm{II}} \hat{Q}$. Then there is an integer $0 < i$ such that for all $0 < j < i$ we have

$$t_{\hat{P},j} \succcurlyeq t_{\hat{Q},j} \quad \text{and} \quad t_{\hat{Q},j} \in \operatorname{term}(\log \mathfrak{m}_{\hat{P},j-1} \ \triangle \ \log \mathfrak{m}_{\hat{Q},j-1}),$$

and such that

$$t_{\hat{Q},i} \notin \operatorname{term}(\log \mathfrak{m}_{\hat{P},i-1} \ \triangle \ \log \mathfrak{m}_{\hat{Q},i-1}). \tag{4.3}$$

Notice that $i \leqslant |P|, |Q|$, for otherwise we would have $P \succ_{\mathrm{I}} Q$. Hence

$$t_{P,0} \cdots t_{P,i-2} \succcurlyeq t_{Q,0} \cdots t_{Q,i-2}. \tag{4.4}$$

On the other hand, by Remark 4.3.2, condition (4.3) implies that for all $c \in C^+$ we have

$$t_{Q,i} \preccurlyeq c \log \frac{\mathfrak{m}_{P,i-1}}{\mathfrak{m}_{Q,i-1}} = \log \left(\frac{\mathfrak{m}_{P,i-1}}{\mathfrak{m}_{Q,i-1}} \right)^c \prec \left(\frac{\mathfrak{m}_{P,i-1}}{\mathfrak{m}_{Q,i-1}} \right)^c.$$

Moreover, we notice that

$$v \ \underline{\preccurlyeq} \ t_{Q,|Q|} \prec \cdots \prec t_{Q,i} \prec \left(\frac{\mathfrak{m}_{P,i-1}}{\mathfrak{m}_{Q,i-1}} \right)^c.$$

Hence if we let $1/c = |Q| - (i-1) + 1$, then we obtain

$$t_{P,i-1} \cdots t_{P,|P|} \succcurlyeq t_{P,i-1} \succ t_{Q,i} \cdots t_{Q,|Q|} \cdot v,$$

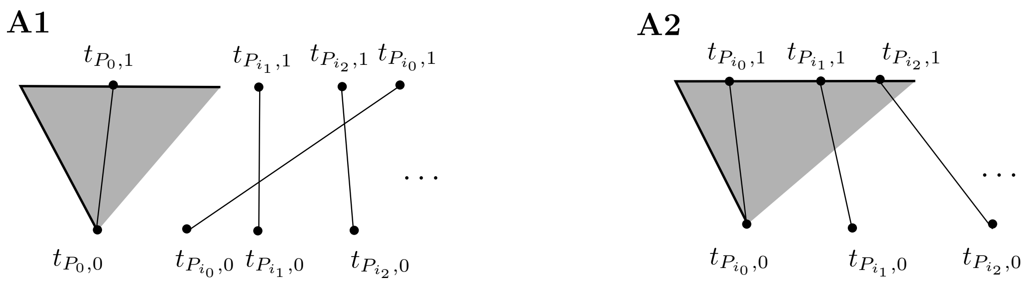
which together with inequality (4.4) proves the lemma. □

4.3.3 Noetherianity of path orderings

Throughout the rest of the section, we will work under the general assumption that we are given a sequence $\mathfrak{s}_0 \succ \mathfrak{s}_1 \succ \cdots$ of terms and that for every i there is a path P_i in the maximal tree-representation of t_i. Our aim is to show that we can extract a well-ordered sub-sequence of $(P_i)_{0\leqslant i}$ for the ordering $\succ$.

Lemma 4.3.12 *Let $\mathfrak{s}_0 \succ \mathfrak{s}_1 \succ \cdots$ and $P_i \in \mathrm{path}(\mathfrak{s}_i)$ for all $i \geqslant 0$. Then there is a sequence $(i_n)_{0\leqslant n}$ of integers with $0 \leqslant i_n < i_{n+1}$ for all $0 \leqslant n$ such that for the sequence $(P_{i_n})_{0\leqslant n}$ of paths one of the conditions* **A1** *or* **A2** *holds:*

A1. $\forall 0 \leqslant n : t_{P_{i_n},1} \notin \mathrm{term}\,(\log \mathfrak{m}_{P_0,0} \;\vartriangle\; \log \mathfrak{m}_{P_{i_n},0})$,
A2. $\forall 0 \leqslant n : t_{P_{i_n},1} \in \mathrm{term}\,(\log \mathfrak{m}_{P_{i_0},0} \;\vartriangle\; \log \mathfrak{m}_{P_{i_n},0}) \;\wedge\; t_{P_{i_n},1} \succ t_{P_{i_{n+1}},1}$.



Proof: Suppose that **A1** fails. Then there are infinitely many $i \geqslant 0$ such that

$$t_{P_i,1} \in \mathrm{term}\,(\log \mathfrak{m}_{P_0,0} \;\vartriangle\; \log \mathfrak{m}_{P_i,0}). \tag{4.5}$$

We may assume that this is the case for all $i \geqslant 0$. Since $\log \mathfrak{m}_{P_0,0}$ has well-ordered support, there is a sequence $0 \leqslant i_0 < i_1 < \cdots$ such that

$$t_{P_{i_0},1} \succ t_{P_{i_1},1} \succ t_{P_{i_2},1} \succ \cdots.$$

From Remark 4.3.2 and $\mathfrak{m}_{P_0,0} \succ \mathfrak{m}_{P_{i_0},0} \succ \mathfrak{m}_{P_{i_n},0}$ it follows for all $n \geqslant 0$ that

$$\log \mathfrak{m}_{P_0,0} \;\vartriangle\; \log \mathfrak{m}_{P_{i_n},0} \;\trianglelefteq\; \log \mathfrak{m}_{P_{i_0},0} \;\vartriangle\; \log \mathfrak{m}_{P_{i_n},0}.$$

Condition (4.5) then implies that $t_{P_{i_n},1}$ is a term of $\log \mathfrak{m}_{P_{i_0},0} \;\vartriangle\; \log \mathfrak{m}_{P_{i_n},0}$. But then condition **A2** holds. □

Proposition 4.3.13 *Let $\mathfrak{s}_0 \succ \mathfrak{s}_1 \succ \cdots$ and $P_i \in \mathrm{path}(\mathfrak{s}_i)$ for all $i \geqslant 0$. Then the ordering $\succ$ is Noetherian on $\bigcup_i \mathrm{path}(\mathfrak{s}_i)$.*

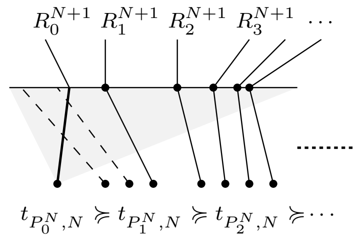


Figure 4.5: Constructing P^{N+1} from P^N.

Proof: We start by remarking that we only need to show that there are $i < j$ such that $P_i \not\succcurlyeq P_j$. To this end, we construct sequences P^N $(0 \leqslant N)$ of paths such that P^0 is the given sequence $(P_0, P_1, \dots)$ and such that P^{N+1} is a sub-sequence of P^N. We will write

$$P^N = (P_0^N, P_1^N, P_2^N, \dots).$$

During the construction we will make sure that for every integer $j \leqslant N$ we have

$$t_{P_0^N,j} \succcurlyeq t_{P_1^N,j} \succcurlyeq t_{P_2^N,j} \succcurlyeq \cdots . \tag{4.6}$$

Moreover, once we have constructed the sequence P^N, we define a sequence $R^N = (R_0^N, R_1^N, \dots)$ of paths by truncating every path P_i^N to its nodes of height $\geqslant N$. In other words, we let

$$P_i^N = [t_{P_i^N,0}, \dots, t_{P_i^N,N-1}, R_i^N].$$

Note that R^0 is the given sequence $P^0 = (P_0, P_1, \dots)$ of paths. We remark that we are done if the sequence R^0 has property **A1** of Lemma 4.3.12, for then $P_0 \not\succcurlyeq_{\mathbb{I}} P_i$ for some i.

Furthermore, our construction will yield that the proposition is shown, if there is an integer N such that R^N admits property **A1**. In fact, if there is such a N, then we stop our construction. The fact that R^N will satisfy **A2** will make sure that the sequence P^{N+1} can be constructed. We will thus assume that R^N has not property **A1** in the following.

Let us suppose that P^N has already been defined and that R^N does not satisfy property **A1**. Then we find a sequence $0 \leqslant i_0 < i_1 < \cdots$ of integers such that

$$t_{R_{i_n}^N,1} \in \operatorname{term}\left(\log \mathfrak{m}_{R_{i_0}^N,0} \ \triangle \ \log \mathfrak{m}_{R_{i_n}^N,0}\right) \ \wedge \ t_{R_{i_n}^N,1} \succcurlyeq t_{R_{i_{n+1}}^N,1}.$$

Note that this determines uniquely a sub-sequence of P^N, namely $(P_{i_0}^N, P_{i_1}^N, \dots)$. We let

$$P^{N+1} := (P_{i_0}^N,\ P_{i_1}^N,\ P_{i_2}^N, \dots).$$

(Figure 4.5 shows the situation where there are paths in P^N such that their nodes of height $N+1$ are strictly $\preccurlyeq$-bigger than $t_{P_0^N,N+1}$. In this case we do not carry these paths over to the sequence P^{N+1}.)

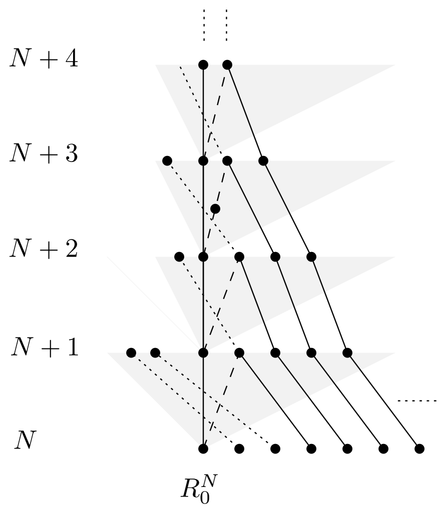


Figure 4.6: Paths converge towards R_0^N.

One notices that if $R^0, \ldots, R^N$ do not have property **A1** but the sequence R^{N+1} does, then we have $P_0^{N+1} \not\succ_{\mathrm{II}} P_m^{N+1}$ for some $m > 0$, which would stop our construction and finish the proof. If R^{N+1} does not have property **A1**, then we can continue the construction, since condition (4.6) holds now for P^{N+1}.

Suppose that we have constructed all sequences $P^0, P^1, \ldots$. Then we cannot have infinitely often $P_0^N \neq P_0^{N+1}$. To see this, we first remark that $P_0^N \neq P_0^{N+1}$ implies the existence of some minimial integer $M_N > 0$ such that

$$\mathfrak{m}_{P_0^N, M_N} \succ \mathfrak{m}_{P_0^{N+1}, M_N}.$$

Since for all $N \geqslant 0$ we have

$$\mathfrak{m}_{P_0^{N+1}, M_N} \in \operatorname{supp} \log \mathfrak{m}_{P_0^N, M_N - 1},$$

the sequence $(\mathfrak{m}_{P_0^N, N})_{0 \leqslant N}$ contradicts condition **T4** of the definition of transseries fields. This shows our claim.

Hence there is an integer $M \geqslant 0$ such that for all $m \geqslant M$ we have $P_0^m = P_0^{m+1}$ and

$$t_{P_0^m, j} \succcurlyeq t_{P_i^m, j}$$

for all $i, j \in \mathbb{N}$. But then we find for every i an integer J such that for all $j > J$

$$t_{P_0^m, j} = t_{P_i^m, j},$$

which shows $P_0^m \succcurlyeq_{\mathrm{I}} P_i^m$. This finishes our proof. (For an illustration, see Figure 4.6. In the figure, the path P_0^N will not be eliminated anymore. Moreover, we see that some paths from

P^N, P^{N+1} etc. can be eliminated, but that the remaining paths must converge towards the path P_0^N. Otherwise R_0^N would have cofinal branches to the right.) □

We can transfer the proposition to the ordering between finite paths.

Proposition 4.3.14 *The ordering* $\curlyeqsucc$ *is Noetherian on* $\bigcup_{\operatorname{supp} f} \operatorname{path}_{\mathbb{T}}(\mathfrak{s})$ *for* $f \in \mathbb{T}_\alpha$.

Proof: Suppose not, and let $(P_i)_{0 \leqslant i}$ be a sequence of pairwise incomparable elements of $\operatorname{path}_{\mathbb{T}}(\mathbb{T}_\alpha)$ such that

$$\mathfrak{m}_{P_0,0} \curlyeqsucc \mathfrak{m}_{P_1,0} \curlyeqsucc \mathfrak{m}_{P_2,0} \curlyeqsucc \cdots.$$

For every i, there is a path $\hat{P}_i \rhd P_i$. By Proposition 4.3.13 there is a sequence $0 \leqslant i_0 < i_1 < \cdots$ such that $\hat{P}_{i_m} \curlyeqsucc \hat{P}_{i_n}$, whenever $n < m$. Then by Lemma 4.3.8 we must have either $P_{i_0} \curlyeqsucc P_{i_1}$ or $P_{i_1} \curlyeqsucc P_{i_2}$, either of which contradicts the assumption on $(P_i)_{0 \leqslant i}$. □

4.4 Existence of extended derivations

We now turn back to the problem of extending a given derivation on some transseries field $\mathbb{T}$ to any given transfinite exponential extension $\mathbb{T}_\alpha = C[[\mathfrak{M}_\alpha]]$.

Proposition 4.4.1 *Let* $f \in \mathbb{T}_\alpha$, *then*

$$(\Theta^\partial(P_i))_{\substack{t \in \operatorname{term} f: \\ P \in \operatorname{path}_{\mathbb{T}}(t)}}$$

is a Noetherian family.

Proof: Let $t_0 \curlyeqsucc t_1 \curlyeqsucc \cdots$ be a sequence in $\operatorname{term} f$. Let $P_i \in \operatorname{path}_{\mathbb{T}}(t_i)$ for all i. We have to show that if the paths P_i are pairwise distinct, then the family $(\Theta^\partial(P_i))_{0 \leqslant i}$ is Noetherian. For every i, we let $\mathfrak{n}_i \in \operatorname{supp} \Theta^\partial(P_i)$. We are done, if we show that we can extract a strictly $\preccurlyeq$-decreasing sub-sequence from $(\mathfrak{n}_i)_{0 \leqslant i}$.

From Proposition 4.3.14 it follows that we may assume that modulo extracting a sub-sequence we have $P_0 \succ P_1 \succ P_2 \succ \cdots$. We claim that modulo extracting another sub-sequence we may restrict ourselves to the following two cases: either $P_0 \succ_{\mathrm{II}} P_1 \succ_{\mathrm{II}} P_2 \succ_{\mathrm{II}} \cdots$ or $\forall i, j : P_i \curlyeqsucc_{\mathrm{I}} P_j$.

Suppose not, and let $P_0 \succ P_1 \succ P_2 \succ \cdots$ be a bad sequence. Let m be maximal such that there is a sequence $i_0 < \cdots < i_m$ of integers with $P_{i_0} \curlyeqsucc_{\mathrm{II}} \cdots \curlyeqsucc P_{i_m}$. Such an m exists by badness of the sequence. From the transitivity of $\curlyeqsucc$ it follows that $P_{i_m} \curlyeqsucc P_j$ for all $j > i_m$. From the maximality of m it follows that $P_{i_m} \curlyeqsucc_{\mathrm{I}} P_j$ for all $j > i_m$. Then there are $i_m < j < k$ with $P_k \succ_{\mathrm{I}} P_k$, for otherwise the given sequence would not be bad. But then one shows $P_{i_m} \succ_{\mathrm{II}} P_k$, which contradicts the maximality of m.

Case I: $\forall i, j : P_i \curlyeqsucc_{\mathrm{I}} P_j$. Using Lemma 4.3.10, we find some $i_0 \geqslant 0$ such that for all $i \geqslant i_0$ the paths P_i have the same length N. Then we obtain for all $i \geqslant i_0$ that

$$\prod_{j \leqslant N} \mathfrak{m}_{P_i,j} \succ \prod_{j \leqslant N} \mathfrak{m}_{P_{i+1},j}.$$

Moreover, we have $\mathfrak{m}_{P_i,N} = \mathfrak{m}_{P_{i_0},N}$ for all $i \geqslant i_0$. This implies that for a sequence of integers $i_0 < i_1 < i_2 < \cdots$ we have

$$\mathfrak{a}_{i_1} \succcurlyeq \mathfrak{a}_{i_2} \succcurlyeq \mathfrak{a}_{i_3} \succcurlyeq \cdots \in \operatorname{supp}(\log \mathfrak{m}_{i_0,N})'.$$

From this $\mathfrak{n}_{i_1} \succ \mathfrak{n}_{i_2} \succ \cdots$ follows, which finishes the case I.

Case II: $P_0 \succ_{\mathrm{II}} P_1 \succ_{\mathrm{II}} P_2 \succ_{\mathrm{II}} \cdots$. We consider the set the labels of all leaves of these paths,

$$\{\mathfrak{m}_{P_0,|P_0|},\ \mathfrak{m}_{P_1,|P_1|},\ \mathfrak{m}_{P_2,|P_2|},\ \cdots\} \subseteq \mathfrak{M}.$$

Then there exists a sequence $0 \leqslant i_0 < i_1 < i_2 < \cdots$ of integers such that one of

$$\mathfrak{m}_{P_{i_0},|P_{i_0}|} \succcurlyeq \mathfrak{m}_{P_{i_1},|P_{i_1}|} \succcurlyeq \mathfrak{m}_{P_{i_2},|P_{i_2}|} \succcurlyeq \cdots \tag{4.7}$$

$$\mathfrak{m}_{P_{i_0},|P_{i_0}|} \prec \mathfrak{m}_{P_{i_1},|P_{i_1}|} \prec \mathfrak{m}_{P_{i_2},|P_{i_2}|} \prec \cdots \tag{4.8}$$

holds. If we have monomials as in (4.7), then the set

$$\bigcup_{0 \leqslant n} \operatorname{supp}(\log \mathfrak{m}_{P_{i_n},|P_{i_n}|})'$$

is well-ordered. Since for all $m \geqslant 0$ the monomial $\mathfrak{a}_{i_m}$ is an element of this union, we may — by thinning out the sequence $(i_n)_{0 \leqslant n}$, if necessary — assume that

$$\mathfrak{a}_{i_0} \succcurlyeq \mathfrak{a}_{i_1} \succcurlyeq \mathfrak{a}_{i_2} \succcurlyeq \cdots . \tag{4.9}$$

From Lemma 4.3.11 it follows that

$$\prod_{j \leqslant |P_{i_0}|} \mathfrak{m}_{P_{i_0},j} \succ \prod_{j \leqslant |P_{i_1}|} \mathfrak{m}_{P_{i_1},j} \succ \prod_{j \leqslant |P_{i_2}|} \mathfrak{m}_{P_{i_2},j} \succ \cdots . \tag{4.10}$$

Multiplying the chains of inequalities (4.9) and (4.10) shows $\mathfrak{n}_{i_0} \succ \mathfrak{n}_{i_1} \succ \mathfrak{n}_{i_2} \succ \cdots$.

If on the other hand (4.8) holds, then the sequence

$$\left(\left(\frac{1}{\mathfrak{m}_{P_{i_n},|P_{i_n}|}}\right)'\right)_{0 \leqslant n}$$

is a Noetherian family, which means that — by thinning out again, if necessary — we have

$$\frac{\mathfrak{a}_{i_0}}{\mathfrak{m}^2_{P_{i_0},|P_{i_0}|}} \succcurlyeq \frac{\mathfrak{a}_{i_1}}{\mathfrak{m}^2_{P_{i_1},|P_{i_1}|}} \succcurlyeq \frac{\mathfrak{a}_{i_2}}{\mathfrak{m}^2_{P_{i_2},|P_{i_2}|}} \succcurlyeq \cdots . \tag{4.11}$$

We use again Lemma 4.3.11 to show that

$$\left(\prod_{j \leqslant |P_{i_0}|} \mathfrak{m}_{P_{i_0},j}\right) \cdot \mathfrak{m}^2_{P_{i_0},|P_{i_0}|} \succ \left(\prod_{j \leqslant |P_{i_1}|} \mathfrak{m}_{P_{i_1},j}\right) \cdot \mathfrak{m}^2_{P_{i_1},|P_{i_1}|} \succ \cdots . \tag{4.12}$$

Multiplying chains (4.11) and (4.12) yields again $\mathfrak{n}_{i_0} \succ \mathfrak{n}_{i_1} \succ \cdots$. This finishes case II and the proof of the proposition. □

Theorem 4.4.2 *Let ∂ be a derivation on $\mathbb{T}$. Then every transfinite exponential extension $\hat{\mathbb{T}}$ of $\mathbb{T}$ admits a unique derivation which extends ∂.*

Proof: By Proposition 4.4.1, the function ∂ as defined by (4.1) (on page 79) is correct. Clearly, conditions **D1** and **D3** hold. To show **D4**, we first remark that $\operatorname{path}_{\mathbb{T}}(\mathfrak{m}) = \{[\mathfrak{m}, Q] \mid Q \in \operatorname{path}_{\mathbb{T}}(\log \mathfrak{m})\}$, hence that

$$\mathfrak{m}' = \sum_{P \in \operatorname{path}(\mathfrak{m})} P' = \sum_{Q \in \operatorname{path}(\log \mathfrak{m})} \mathfrak{m} \cdot Q' = \mathfrak{m} \cdot (\log \mathfrak{m})'.$$

Then **D4** follows from Proposition 4.1.5. By lemma 1.6.5, condition **D2** holds, if we can show it for monomials. Let $\mathfrak{m} = e^f, \mathfrak{n} = e^g \in \mathfrak{M}_\alpha$. Then by **D4**

$$\begin{aligned} (\mathfrak{m}\,\mathfrak{n})' &= e^{f+g} \cdot (f+g)' \\ &= e^f \cdot f' \cdot e^g + e^f \cdot e^g \cdot g' \\ &= \mathfrak{m}' \cdot \mathfrak{n} + \mathfrak{m} \cdot \mathfrak{n}'. \end{aligned}$$

This shows the theorem. □

Corollary 4.4.3 *For all ordinal numbers α and $\mathbb{L} = C[[\log^{\mathbb{Z}^\star} x]]$, the field $\mathbb{L}_\alpha$ admits a derivation.*

Remark 4.4.4 Let λ be a limit ordinal. The fact that $\mathbb{T}_\beta$ is stable under ∂ for each $\beta < \lambda$ implies that $\mathbb{T}_{<\lambda} = \bigcup_{\beta<\lambda} \mathbb{T}_\beta$ is also stable under ∂.

Hence $\mathbb{T}_{<\lambda}$ is a non-complete field of transseries with total exponentiation, a total logarithm on the set of positive elements and a dertivation.

Remark 4.4.5 Instead of $\succ$, one can actually choose a relation $\hat{\succ}$ on the set of paths, which is weaker than $\succ$. Essentially, one replaces all conditions about maximal common truncations by $\mathfrak{m}_{Q,j} \in \operatorname{supp} \log \mathfrak{m}_{P,j-1}$ for all $0 < j < i$, and one replaces the last condition by $\mathfrak{m}_{Q,i} \notin \operatorname{supp} \log \mathfrak{m}_{P,i-1}$. In particular, we do not demand $\mathfrak{m}_{P,i} \succcurlyeq \mathfrak{m}_{Q,i}$.

The relation $\hat{\succ}$ is not transitive, but its transitive closure is an ordering. Furthermore, it is possible to use $\hat{\succcurlyeq}$ instead of $\succcurlyeq$ in the proofs of this section.

4.5 Valuated derivations

A derivation $\partial : \mathbb{T} \to \mathbb{T}$ is **valuated** iff for all $f, g \in \mathbb{T}$ with $1 \not\asymp g$ we have

$$f \prec g \quad \Rightarrow \quad f' \prec g'.$$

Remark 4.5.1 A derivation is valued if and only if for all monomials $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$ with $\mathfrak{n} \not\asymp 1$ and $\mathfrak{m} \prec \mathfrak{n}$ we have $\mathfrak{m}'$
asy $\mathfrak{n}'$. One direction is trivial. As for the other one, let $f \prec g \not\asymp 1$. Then for all $\mathfrak{m} \in \operatorname{supp} f$ with $\mathfrak{d}_f \neq \mathfrak{m}$ we have $\mathfrak{m} \prec \mathfrak{d}_f$ and thus $\mathfrak{m}' \prec \mathfrak{d}_f'$. This implies $f' \asymp \mathfrak{d}_f'$. Similarly for g. But then $\mathfrak{d}_f$ *asy* $\mathfrak{d}_g$ implies $\mathfrak{d}_f' \prec \mathfrak{d}_g'$ and therefore $f' \prec g'$.

Example 4.5.2 The derivation on $\mathbb{L}$ is valuated: let $\log^a x \prec \log^b x$, where $b \neq 0$. Let i, j be minimal with $a_i, b_j \neq 0$. From the definition of ∂ it follows that

$$\begin{aligned}(\log^a x)' &\asymp \log^a x \cdot \frac{1}{x \cdots \log_i x},\\ (\log^b x)' &\asymp \log^b x \cdot \frac{1}{x \cdots \log_j x}.\end{aligned}$$

If $i \geqslant j$, then $x \cdot \log_i x \preccurlyeq x \cdots \log_j x$ and $\log^a x \prec \log^b x$ imply $(\log^a x)' \prec (\log^b)'$. If $i < j$, then $a_i < 0 = b_i$. From this we obtain

$$\log^a x \prec \log^b x \cdot \frac{1}{\log_{i+1} x \cdots \log_j x}.$$

Hence $(\log^a x)' \prec (\log^b)'$.

Proposition 4.5.3 *If ∂ is a valuated derivation on $\mathbb{T}$, then ∂_α is a valuated derivation on $\mathbb{T}_\alpha$.*

Proof: We use a transfinite induction. The Proposition holds for $\alpha = 0$. So let us assume $\alpha > 0$ and that the Proposition holds for all $\beta < \alpha$.

Let $\mathfrak{m} \prec \mathfrak{n} \not\asymp 1$. Then $\log \mathfrak{m}, \log \mathfrak{n} \in \mathbb{T}_\beta^\uparrow$ for some $\beta < \alpha$. Then $\log \mathfrak{m} < \log \mathfrak{n}$. Let $t = \log \mathfrak{m} \ \triangle \ \log \mathfrak{n}$ and $\log \mathfrak{m} = t + f$ and $\log \mathfrak{n} = t + g$. We claim that $f' < g'$. If $0 < f$, then $0 < g$ and thus $\mathfrak{d}_f \succcurlyeq \mathfrak{d}_g$. Hence $\delta_f' \succcurlyeq \delta_g'$ and $\tau_f' < \tau_g'$. If on the other hand $f < 0$, then $\mathfrak{d}_f \preccurlyeq \mathfrak{d}_g$. The inductive assumption implies again $\mathfrak{d}_f' \preccurlyeq \mathfrak{d}_g'$, thus $\tau_f' < \tau_g'$.

In both case, we obtain $(\log \mathfrak{m})' < (\log \mathfrak{n})'$. But then we have

By Remark 4.5.1.

Chapter 5

Compositions

Having extended derivations, we now turn our attention to compositions. First, we define the notion of compositions, and we show some basic properties. Then, we extend compositions between transseries fields to their transfinite exponential extensions. Again, we can use the framework of Noetherian operators to define such extensions and to show that our definitions are correct.

5.1 Right-compositions on transseries fields

5.1.1 Notions of compositions

As for derivations, we want to introduce a notion of composition on transseries fields. This notion should take in account both, the strongly linear and exponential nature of transseries fields and properties of compositions.

Definition 5.1.1 *Let $\mathbb{T} = C[[\mathfrak{M}]]$ and $\mathbb{U} = C[[\mathfrak{N}]]$ be transseries fields. An injective function $\vartriangle : \mathbb{T} \to \mathbb{U}$ is a* **right-composition** *iff*

RC1. $\forall c \in C : \vartriangle(c) = c$,

RC2. $\vartriangle$ *is multiplicative,*

RC3. *if F is a Noetherian family in $\mathbb{T}$, then $\vartriangle(F) = (\vartriangle(f))_{f\in F}$ is a Noetherian family in $\mathbb{U}$ and $\vartriangle(\sum F) = \sum \vartriangle(F)$,*

RC4. $\forall f \in \mathbb{T} : f \in \operatorname{dom} \exp \Rightarrow \vartriangle(\exp f) = \exp \vartriangle(f)$.

Remark 5.1.2 Right-compositions $\vartriangle : \mathbb{T} \to \mathbb{U}$ are strictly increasing, since for all $f \in \mathbb{T}$ we have by linearity $\vartriangle(-f) = -\vartriangle(f)$ and for all $0 < f$

$$\vartriangle(f) = \vartriangle(\exp \log f) = \exp \vartriangle(\log f) > 0.$$

For monomials $\mathfrak{m} \succ \mathfrak{n}$ in $\mathfrak{M}$ we have $\vartriangle(\mathfrak{m}) \succ \vartriangle(\mathfrak{n})$, since by **RC1** and **RC3** we have

$$\mathfrak{m} \succ \mathfrak{n} \;\Leftrightarrow\; \forall c \in C : 0 < \mathfrak{m} + c\,\mathfrak{n} \;\Leftrightarrow\; \forall c \in C : 0 < \vartriangle(\mathfrak{m}) + c\vartriangle(\mathfrak{n}) \;\Leftrightarrow\; \vartriangle(\mathfrak{m}) \succ \vartriangle(\mathfrak{n}).$$

Furthermore, the restriction of $\vartriangle$ to $\mathfrak{M}$ is a Noetherian mapping by **RC3**. The unique strongly linear extension of $\vartriangle|_{\mathfrak{M}}$ to $\mathbb{T}$ is $\vartriangle$.

Remark 5.1.3 Let φ be a Noetherian mapping $\mathfrak{M} \to C[[\mathfrak{N}]]$. We have seen that if φ is multiplicative, then its unique strongly linear extension $\hat{\varphi}$ to $C[[\mathfrak{M}]]$ is also multiplicative. Similarly, we have seem that if φ satisfies the Leibniz rule, then so does $\hat{\varphi}$.

Similarly, we can show for mappings φ such that $\forall \mathfrak{m} \in \mathfrak{M} : \varphi(\mathfrak{m}) = \exp \hat{\varphi}(\log \mathfrak{m})$, then the same remains true for series $f \in \operatorname{dom} \exp$. In fact, let $h \in \mathbb{L}^+$ such that $\exp f = h$. Then we have $h = c\mathfrak{d}(1+\delta)$ and

$$\begin{aligned} \hat{\varphi}(f) &= \hat{\varphi}(\log \mathfrak{d} + \log c + l(\delta)) \\ &= \hat{\varphi}(\log \mathfrak{d}) + \hat{\varphi}(\log c) + \hat{\varphi}(l(\delta)). \end{aligned}$$

From the hypotheses we obtain $\hat{\varphi}(\log \mathfrak{d}) = \log \varphi(\mathfrak{d})$ and $\hat{\varphi}(\log c) = \log \varphi(c)$. Furthermore, since $\hat{\varphi}$ is strongly linear, we have $\hat{\varphi}(l(\delta)) = l(\hat{\varphi}(\delta))$. This implies

$$\begin{aligned} \hat{\varphi}(f) &= \log \varphi(\mathfrak{d}) + \log \varphi(c) + l(\hat{\varphi}(\delta)) \\ &= \log(\varphi(c\mathfrak{d}) \cdot (1 + \hat{\varphi}(\delta))) \\ &= \log \hat{\varphi}(c\mathfrak{d}(1+\delta)). \end{aligned}$$

But then $\exp \hat{\varphi}(f) = \hat{\varphi}(\exp f)$. In other terms, if we want to show **RC4** for series, it suffices to show the property for monomials.

5.1.2 Example of a right-composition

Let $\mathbb{T}$ be a transseries field. We show that for every $g \in \mathbb{T}^+_\infty$, there is a right-composition

$$\begin{aligned} \Delta_g : \mathbb{L} &\longrightarrow \mathbb{T} \\ x &\longmapsto g. \end{aligned}$$

Let $\log^a x \in \log^{\mathbb{Z}^\star} x$ and $g \in \mathbb{T}^+_\infty$. Then

$$\varphi(\mathfrak{m}) := \log^a g := g^{a_0} \log^{a_1} g \cdots \log_n^{a_n} g \in \mathbb{T}.$$

We also write $\mathfrak{m} \circ g$ instead of $\varphi(g)$. In view of Proposition 1.6.3 we have to show that the mapping $\varphi : \log^{\mathbb{Z}^\star} x \to \mathbb{T}$ is strongly linear. This will allows us to extend φ to a mapping $\hat{\varphi}$ on all of $\mathbb{L}$.

Proposition 5.1.4 $\varphi : \log^{\mathbb{Z}^\star} x \to \mathbb{T}$ *as defined above is strongly linear.*

Proof: For each i, let us write $\log_i g = c_i \mathfrak{d}_i (1+\delta_i)$ with $c_i = c_{\log_i g}$, $\mathfrak{d}_i = \mathfrak{d}_{\log_i g}$ and $\delta_i = \delta_{\log_i g}$. Notice that Proposition 2.2.4 implies

$$\mathfrak{d}_0 \succ\!\!\succ \mathfrak{d}_1 \succ\!\!\succ \mathfrak{d}_2 \succ\!\!\succ \cdots,$$

whence $\mathfrak{d} \circ \varphi$ preserves the asymptotic ordering $\prec$. We claim that

$$S = (\operatorname{supp} \delta_0)^\star (\operatorname{supp} \delta_1)^\star \cdots$$

is well ordered. Indeed, let n be such that g is log-confluent at order n. Then

$$T = (\operatorname{supp} \delta_0)^\star \cdots (\operatorname{supp} \delta_n)^\star \left\{ \frac{1}{\mathfrak{d}_{n+1}}, \frac{1}{\mathfrak{d}_{n+1}\mathfrak{d}_{n+2}}, \ldots \right\}^\star$$

is well-ordered. Let us show by induction that

$$\operatorname{supp} \delta_i \subseteq \frac{T}{\mathfrak{d}_{n+1} \cdots \mathfrak{d}_i}, \tag{5.1}$$

for all $i \geqslant n$. This is clear for $i = n$. So assume that $i > n$ and that we have proved (5.1) for all strictly smaller i. Now $\tau_{\log_i g} = \log \tau_{\log_{i-1} g}$ implies

$$\log_i g = \log \mathfrak{d}_{i-1} + \log c_{i-1} + \log(1 + \delta_{i-1}) = \mathfrak{d}_i + \log(1 + \delta_{i-1}).$$

Consequently,

$$\operatorname{supp} \delta_i = \operatorname{supp} \frac{\log(1 + \delta_{i-1})}{\mathfrak{d}_i} \subseteq \left\{ \frac{T}{\mathfrak{d}_{n+1} \cdots \mathfrak{d}_{i-1}} \right\}^\dagger \frac{1}{\mathfrak{d}_i} \subseteq \frac{T^\star}{\mathfrak{d}_{n+1} \cdots \mathfrak{d}_i}.$$

Hence the inclusion (5.1) holds for all $i \geqslant n$. In particular, we have $\operatorname{supp} \delta_i \subseteq T$ for all $i \geqslant 0$, whence $S \subseteq T$, which proves our claim.

Now let $W \subseteq \log^{\mathbb{Z}^\star} x$ be well-ordered. For all $\mathfrak{m} \in W$ we have $\mathfrak{m} \circ g = \varphi(\mathfrak{m}) = \tau_{\varphi(\mathfrak{m})} \cdot (1 + \delta_{\varphi(\mathfrak{m})})$. From the above we conclude

$$\operatorname{supp} \varphi(\mathfrak{m}) \subseteq \{\mathfrak{d}_{\varphi(\mathfrak{m})} \mid \mathfrak{m} \in W\} \cdot (1 + S).$$

Since $\mathfrak{d} \circ \varphi$ preserves the ordering, the set $\{\mathfrak{d}_{\varphi(\mathfrak{m})} \mid \mathfrak{m} \in W\}$ is well-ordered. So is the set $1 + S$. Hence $\bigcup_{\mathfrak{m} \in W} \operatorname{supp} \varphi(\mathfrak{m})$ is contained in a well-ordered set. We have to show that for all $\mathfrak{n} \in \bigcup_{\mathfrak{m} \in W} \operatorname{supp} \varphi(\mathfrak{m})$ there are only finitely many $\mathfrak{m} \in W$ with $\mathfrak{n} \in \operatorname{supp} \varphi(\mathfrak{m})$. Suppose that for some such $\mathfrak{n}$ there is an infinite set $W_\mathfrak{n} \subseteq W$ such that $\mathfrak{n} \in \operatorname{supp} \varphi(\mathfrak{m})$ for all $\mathfrak{m} \in W_\mathfrak{n}$. Let $\mathfrak{s}_\mathfrak{m} \in (1 + S)$ such that $\mathfrak{n} = \mathfrak{d}_{\varphi(\mathfrak{m})} \cdot \mathfrak{s}_\mathfrak{m}$. Since $\{\mathfrak{d}_{\varphi(\mathfrak{m})} \mid \mathfrak{m} \in W_\mathfrak{n}\}$ is well-ordered, the set $\{\mathfrak{s}_\mathfrak{m} \mid \mathfrak{m} \in W_\mathfrak{n}\} \subseteq (1 + S)$ is decreasing in $\succ$. But $1 + S$ is well-ordered. Contradiction. Hence the family $(\varphi(\mathfrak{m}))_{\mathfrak{m} \in W}$ is Noetherian. □

Proposition 5.1.5 *Let $\varphi : \log^{\mathbb{Z}^\star} x \to \mathbb{T}$ be defined as above. Then its unique extension $\hat{\varphi} : \mathbb{L} \to \mathbb{T}$ is a right-composition. Moreover, if we let $f \circ g := \hat{\varphi}(f)$, then for all $f, h \in \mathbb{L}$ and $g \in \mathbb{T}^+_\infty$ we have $f \circ (h \circ g) = (f \circ h) \circ g$.*

Proof: Since for all $\mathfrak{m} \succ \mathfrak{n}$ in $\log^{\mathbb{Z}^\star} x$ we have $\Delta_g(\mathfrak{m}) \succ \Delta_g(\mathfrak{n})$, the function $\hat{\varphi}$ is injective. Condition **RC1** holds by strong linearity, i.e. Proposition 5.1.4, and so does **RC3**. Note that φ is multiplicative on $\log^{\mathbb{Z}^\star} x$, hence that by Lemma 1.6.5 condition **RC2** holds.

As for **RC4**, we first remark that one easily verifies

$$\forall \mathfrak{m} \in \log^{\mathbb{Z}^\star} x : \varphi(\mathfrak{m}) = \exp \hat{\varphi}(\log \mathfrak{m}).$$

Now we invoke Remark 5.1.3. The second assertion follows from Corollary 1.6.4. □

Remark 5.1.6 It follows from the proof of Proposition 5.1.4 that if $g \in \mathbb{T}_\infty^+$ is log-atomic (i.e. log-confluent at order 0) and $\mathfrak{m} \in \log^{\mathbb{Z}^\star} x$, then for all $\mathfrak{n} \in \operatorname{supp} \varphi(\mathfrak{m})$ there is a weakly decreasing function $a : \mathbb{N} \to \mathbb{N}$ such that

$$\mathfrak{n} \preccurlyeq \frac{\varphi(\mathfrak{d}_g)}{\mathfrak{d}_g^{a_0} \log^{a_1} \mathfrak{d}_g \cdots}.$$

5.1.3 Uniqueness of extensions

The next proposition shows that the real difficulty lies in showing the existence rather than in showing the uniqueness of the extension of Δ to $\mathbb{T}_\alpha$.

Proposition 5.1.7 *Let $\Delta : \mathbb{T} \to \mathbb{U}$ be a right-composition and $0 < \alpha$ an ordinal. Then there exists at most one right-composition $\Delta_\alpha : \mathbb{T}_\alpha \to \mathbb{U}_\alpha$ such that $\Delta_\alpha|_\mathbb{T} = \Delta$.*

Proof: We first notice that if α is such that Δ_α exists, then Δ_β exists for all $\beta < \alpha$. Hence, if we let α be the minimal ordinal such that there are distinct $\Delta_\alpha, \Delta'_\alpha$ with $\Delta_\alpha|_\mathbb{T} = \Delta'_\alpha|_\mathbb{T} = \Delta$, then the restrictions of Δ_α and Δ'_α to $\mathbb{T}_\beta$ exist and are identical. In particular, this is true for the monomial groups $\mathfrak{M}_\beta$.

Then α is not a limit ordinal, for otherwise for all series $f \in \mathbb{T}_\alpha$ we have

$$\Delta_\alpha(f) = \sum_\mathfrak{m} f_\mathfrak{m} \Delta_\alpha(\mathfrak{m}) = \sum_\mathfrak{m} f_\mathfrak{m} \Delta'_\alpha(\mathfrak{m}) = \Delta'_\alpha(f),$$

since the monomials are elements of some $\mathfrak{M}_\beta$ with $\beta < \alpha$. This contradicts the minimality.

Hence α is a successor ordinal $\beta + 1$. Let $\mathfrak{m} \in \mathfrak{M}_\alpha$ with $\Delta_\alpha(\mathfrak{m}) \neq \Delta'_\alpha(\mathfrak{m})$. Then $\log \mathfrak{m} \in \mathbb{T}_\beta$ and thus $\Delta_\alpha(\log \mathfrak{m}) = \Delta'_\alpha(\log \mathfrak{m})$. Therefore

$$\Delta_\alpha(\mathfrak{m}) = \Delta_\alpha(\exp \log \mathfrak{m}) = \exp \Delta'_\alpha(\log \mathfrak{m}) = \Delta'_\alpha(\exp \log \mathfrak{m}) = \Delta'_\alpha(\mathfrak{m}),$$

by **RC4**. Now **RC3** implies again $\Delta_\alpha(f) = \Delta'_\alpha(f)$, contradiction. □

5.1.4 Extending using transfinite definitions

An alternative way of defining Δ_α is the following. Assume that for all $\beta < \alpha$, a right-composition Δ_β on $\mathbb{T}_\beta$ has already been defined such that

$$\begin{aligned} \Delta_0 &= \Delta \\ \Delta_\gamma &\subseteq \Delta_\beta \qquad \forall \gamma \leqslant \beta. \end{aligned}$$

Then for all $\mathfrak{m} \in \mathfrak{M}_\alpha$ we let

$$\varphi(\mathfrak{m}) := \exp \Delta_\beta(\log \mathfrak{m}) \qquad \text{if } \log \mathfrak{m} \in \mathbb{T}_\beta.$$

Remark 5.1.8 Firstly, we notice that the definition of φ does not depend on the choice of β and that φ is totally defined on $\mathfrak{M}_\alpha$. Secondly, let us notice that in order to extend φ to $\mathbb{T}_\alpha$, we have to show that it is a Noetherian mapping. As the following proposition shows, from that it will follow that $\varphi = \Delta_\alpha$, i.e. that it is the unique right-composition on $\mathbb{T}_\alpha$ extending Δ.

Proposition 5.1.9 *If $\varphi : \mathfrak{M}_\alpha \to \mathbb{U}_\alpha$ is a Noetherian mapping, then its unique strongly linear extension*

$$\hat{\varphi} : \mathbb{T}_\alpha \to \mathbb{U}_\alpha$$

is the unique right-composition Δ_α extending Δ.

Proof: In order to show that $\hat{\varphi}$ is injective, it suffices by strong linearity of $\hat{\varphi}$ to show that $\mathfrak{m} \succ \mathfrak{n}$ implies $\varphi(\mathfrak{m}) \succ \varphi(\mathfrak{n})$. Indeed, we only need to show that $\log \varphi(\mathfrak{m}) \neq \log \varphi(\mathfrak{n})$. Suppose for a contradiction that this is not the case. Then $\Delta_\beta(\log \mathfrak{m}) = \Delta_\beta(\log \mathfrak{n})$ for some $\beta < \alpha$. Hence by linearity, $0 = \Delta_\beta(\log \mathfrak{m} - \log \mathfrak{n})$. But then the injectivity of Δ_β implies $\log \mathfrak{m} = \log \mathfrak{n}$, thus $\mathfrak{m} = \mathfrak{n}$.

It remains to show conditions **RC1** – **RC4**. Note that **RC1** holds by linearity of $\hat{\varphi}$ and that **RC3** is satisfied by hypothesis.

Let us show **RC2**. We claim that φ is multiplicative. Let $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}_\alpha$. If α is a limit ordinal, then $\mathfrak{m}, \mathfrak{n}$ are elements of some $\mathfrak{M}_\beta$ with $\beta < \alpha$, and the claim follows from $\varphi = \Delta_\beta$. If α is a successor ordinal $\alpha = \beta + 1$, then $\log \mathfrak{m}, \log \mathfrak{n} \in \mathbb{T}_\beta$. The linearity of Δ_β then implies

$$\begin{aligned} \varphi(\mathfrak{m}\mathfrak{n}) &= \exp \Delta_\beta(\log \mathfrak{m} + \log \mathfrak{n}) \\ &= \exp(\Delta_\beta(\log \mathfrak{m}) + \Delta_\beta(\log \mathfrak{n})) \\ &= \exp \Delta_\beta(\log \mathfrak{m}) \cdot \exp \Delta_\beta(\log \mathfrak{n}) \\ &= \varphi(\mathfrak{m}) \cdot \varphi(\mathfrak{n}). \end{aligned}$$

By Lemma 1.6.5, the function $\hat{\varphi}$ is multiplicative, hence **RC2**.

As for **RC4**, it suffices to notice that by definition we have $\varphi(\mathfrak{m}) = \exp \hat{\varphi}(\log \mathfrak{m})$ for every $\mathfrak{m} \in \mathfrak{M}_\alpha$. Then Remark 5.1.3 shows **RC4**. The uniqueness follows from Proposition 5.1.7. □

Remark 5.1.10 We have to show that the function Δ_α is defined on $\mathbb{T}_\alpha$. h is means that for every sequence $\mathfrak{t}_0 \succ \mathfrak{t}_1 \succ \cdots$ of monomials in $\mathfrak{M}_\alpha$ and every sequence $(\mathfrak{n}_i)_{0 \leqslant i}$ such that $\mathfrak{n}_i \in \operatorname{supp} \varphi(\mathfrak{t}_i)$, we can extract a sub-sequence $(\mathfrak{n}_{i_k})_{0 \leqslant k}$ in $\mathfrak{N}_\alpha$ with

$$\mathfrak{n}_{i_0} \succ \mathfrak{n}_{i_1} \succ \mathfrak{n}_{i_2} \succ \cdots .$$

5.2 Combinatorial representation of compositions

5.2.1 Some notations

Let $\Delta : \mathbb{T} = C[[\mathfrak{M}]] \to \mathbb{U} = C[[\mathfrak{N}]]$ be a right-composition. For every $f \in \mathbb{T}$, the series $\Delta(f)$ has the canonical decomposition into its infinite, constant and infinitesimal part. We let

$$\begin{aligned} \Delta^{\uparrow}(f) &:= (\Delta(f))^{\uparrow}, \\ \Delta^{=}(f) &:= (\Delta(f))^{=}, \\ \Delta^{\downarrow}(f) &:= (\Delta(f))^{\downarrow}, \end{aligned}$$

so that for every $\mathfrak{t} \in \mathfrak{M}$ we have

$$\Delta(\mathfrak{t}) = \exp \Delta(\log \mathfrak{t}) = \exp \Delta^{\uparrow}(\log \mathfrak{t}) \cdot \exp \Delta^{=}(\log \mathfrak{t}) \cdot e(\Delta^{\downarrow}(\log \mathfrak{t})).$$

Then $\exp \Delta^{\uparrow}(\log \mathfrak{t}) \in \mathfrak{N}$ and $\exp \Delta^{=}(\log \mathfrak{t}) \in C^*$ are the leading monomial and leading coefficient of $\Delta(\mathfrak{t})$. We let

$$\begin{aligned} \mathfrak{D}_\Delta(\mathfrak{t}) &:= \exp \Delta^{\uparrow}(\log \mathfrak{t}) \\ \mathcal{C}_\Delta(\mathfrak{t}) &:= \exp \Delta^{=}(\log \mathfrak{t}). \end{aligned}$$

Wherever Δ is clear from the context, we write $\mathfrak{D}$ and $\mathcal{C}$ instead of $\mathfrak{D}_\Delta$ and $\mathcal{C}_\Delta$.

Proposition 5.2.1 *The functions $\mathfrak{D}_\Delta$ and $\mathcal{C}_\Delta$ are multiplicative on $\mathfrak{M}$. Moreover, the function $\mathfrak{D}_\Delta$ is strictly increasing.*

Proof: The first claim follows from $(f+g)^{\uparrow} = f^{\uparrow} + g^{\uparrow}$ and

$$\exp \Delta^{\uparrow}(\log \mathfrak{t}\mathfrak{s}) = \exp \Delta^{\uparrow}(\log \mathfrak{t} + \log \mathfrak{s}) = \exp \Delta^{\uparrow}(\log \mathfrak{t}) \cdot \exp \Delta^{\uparrow}(\log \mathfrak{s}).$$

Similarly for $\mathcal{C}$. As for the second one, suppose that $1 \prec \mathfrak{t}$. Then $1 \prec \mathfrak{t}$ implies $C < \log \mathfrak{t}$. The function Δ is strictly increasing, hence $C < \Delta(\log \mathfrak{t})$. Therefore we have $1 \prec \Delta(\log \mathfrak{t})$ and $0 < \Delta(\log \mathfrak{t})$. We conclude that $\Delta^{\uparrow}(\log \mathfrak{t}) > 0$ □

Remark 5.2.2 We extend the functions $\mathcal{C}$ and $\mathfrak{D}$ to all terms $s \in C\mathfrak{M}$ by

$$\begin{aligned} \mathcal{C}(s) &:= c_s \cdot \mathcal{C}(\mathfrak{d}_s), \\ \mathfrak{D}(s) &:= \mathfrak{D}(\mathfrak{d}_s). \end{aligned}$$

The functions $\mathcal{C}$ and $\mathfrak{D}$ remain multiplicative. Throughout the rest of this paragraph, we will look at an example of a right-composition in detail.

5.2.2 Formulas for an example of composition

We let $\mathbb{T} = \mathbb{L}_{\exp,\exp} = \mathbb{L}_2$ and $\Delta_g : \mathbb{L} \to \mathbb{L}_2$ a right-composition for some $g \in \mathbb{L}_{2,\infty}^{+}$ as in Proposition 5.1.5. We extend Δ_g to a mapping $\hat{\varphi} : \mathbb{L}_2 \to \mathbb{L}_4$ as follows: let $\mathfrak{m} \in (\log^{\mathbb{Z}^\star} x)_{\exp}$, then we let

$$\varphi(\mathfrak{m}) := \exp(\Delta_g(\log \mathfrak{m})).$$

Assuming that the mapping φ is Noetherian, it extends uniquely to $\mathbb{L}_{\exp}$. We proceed similarly for monomials from $(\log^{\mathbb{Z}^\star} x)_{\exp,\exp}$. We will prove the Noetherianity assumptions later in this chapter. For the purpose of the example, we may assume that Δ_g may be extended to a right-composition on $\mathbb{L}_2$.

Let $g = x + \frac{1}{x} + \frac{1}{\exp_2 x} \in \mathbb{L}_2$ and take

$$\begin{aligned} \Delta : \mathbb{L} &\longrightarrow \mathbb{L}_2 \\ f &\longmapsto f \circ g. \end{aligned}$$

We apply Δ to $f = e^{e^{2x}+x} \in \mathbb{L}_2$. From **RC3** and **RC4** we obtain that

$$\Delta(e^{e^{2x}+x}) = \exp \Delta(e^{2x}+x), \tag{5.2}$$

$$\Delta(e^{2x}+x) = \exp \Delta(2x) + \Delta(x). \tag{5.3}$$

In order to evaluate the right-hand side of equation (5.2), we have to evaluate the right-hand side of (5.3). Applying the definition of Δ yields

$$\Delta(x) = x + \frac{1}{x} + \frac{1}{\exp_2 x}, \tag{5.4}$$

$$\exp \Delta(2x) = e^{2x} \cdot e\left(\frac{2}{x} + \frac{2}{\exp_2 x}\right). \tag{5.5}$$

The next step is to obtain $\Delta^{\uparrow}(e^{2x}+x)$, $\Delta^{=}(e^{2x}+x)$ and $\Delta^{\uparrow}(e^{2x}+x)$. Clearly from equation (5.4) it follows that $\Delta^{\uparrow}(x) = x$, $\Delta^{=}(x) = 0$ and $\Delta^{\downarrow}(x) = \frac{1}{x} + \frac{1}{\exp_2 x}$. For equation (5.5) we obtain

$$\Delta^{\uparrow}(e^{2x}) = e^{2x} \cdot \left(1 + \frac{1}{1!} \cdot \frac{2}{x} + \frac{1}{2!} \cdot \frac{2^2}{x^2} + \frac{1}{3!} \cdot \frac{2^3}{x^3} + \cdots\right), \tag{5.6}$$

$$\Delta^{=}(e^{2x}) = 0, \tag{5.7}$$

$$\Delta^{\downarrow}(e^{2x}) = e^{2x} \cdot \sum_{i=0}^{\infty} \frac{1}{i!} \sum_{j=0}^{i-1} \binom{i}{j} \frac{2^j}{x^j} \cdot \frac{2^{i-j}}{\exp_2^{i-j} x}. \tag{5.8}$$

Hence

$$\begin{aligned}
\Delta^{\uparrow}(e^{2x}+x) &= e^{2x} \cdot \left(1 + \frac{1}{1!} \cdot \frac{2}{x} + \frac{1}{2!} \cdot \frac{2^2}{x^2} + \frac{1}{3!} \cdot \frac{2^3}{x^3} + \cdots\right) + x, \\
\Delta^{=}(e^{2x}+x) &= 0, \\
\Delta^{\downarrow}(e^{2x}+x) &= \frac{1}{x} + \frac{1}{\exp_2 x} + e^{2x} \cdot \sum_{i=0}^{\infty} \frac{1}{i!} \sum_{j=0}^{i-1} \binom{i}{j} \frac{2^j}{x^j} \cdot \frac{2^{i-j}}{\exp_2^{i-j} x}.
\end{aligned}$$

Equations (5.6) – (5.8) can be used to express $\Delta(e^{e^{2x}+x})$ using the equation

$$\begin{aligned}
\Delta(e^{e^{2x}+x}) &= \exp \Delta^{=}(e^{2x}+x) \cdot \exp \Delta^{\uparrow}(e^{2x}+x) \cdot e(\Delta^{\downarrow}(e^{2x}+x)) \\
&= \mathcal{C}(e^{e^{2x}+x})\, \mathfrak{D}(e^{e^{2x}+x}) \cdot e(\Delta^{\downarrow}(e^{2x}+x)).
\end{aligned}$$

5.2.3 Combinatorial representation of the example's formulas

Let us now show how to represent terms in $\Delta(x)$, $\Delta(e^{2x}+x)$ resp. $\Delta(e^{e^{2x}+x})$ by faithfully embedded trees, whose leaves are labeled using a second labeling. We distinguish between the labeling of the faithfully embedded tree and the additional labeling of the leaves by referring to them as the **host-labeling** and the **labeling**, respectively.

Level 0. $t \in \operatorname{term} \Delta(x)$, then we take the one-point tree with host-label x. The node will get a label from $\operatorname{term} \Delta(x)$. In Figure 5.1, the two left-hand side trees illustrate the cases where the labels are x and $\frac{1}{\exp_2 x}$.

Level 1. $t \in \operatorname{term} \Delta(e^{2x}+x)$. Then t is an element of one of the sets $\operatorname{term} \Delta(x)$ or $\operatorname{term} \Delta(e^{2x})$. The former case has been treated at Level 0. If $t \in \operatorname{term} \Delta(e^{2x})$, then there is an integer $n \geqslant 0$ such that

$$t \in e^{2x} \cdot \operatorname{term} \frac{1}{n!}\left(\frac{2}{x} + \frac{2}{\exp_2 x}\right)^n.$$

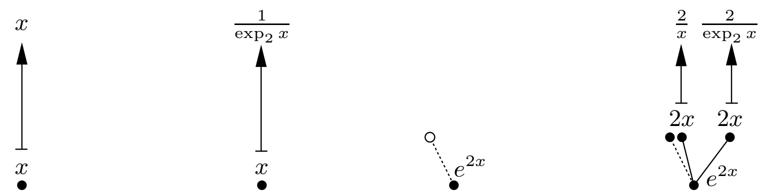


Figure 5.1: Faithful embeddings for $t = x$, $\frac{1}{\exp_2 x}$, e^{2x} and $e^{2x}\,\frac{1}{2!}\,\frac{2}{x}\,\frac{2}{\exp_2 x}$.

If $n = 0$, then we take again the one-point tree with host-label e^{2x}. No labeling is assigned in this case. Otherwise we have $n > 0$ and there are n elements $t_1, \dots, t_n$ from the set $\{\frac{2}{x}, \frac{2}{\exp_2 x}\}$ such that

$$t = e^{2x} \cdot \frac{1}{n!} t_1 \cdots t_n.$$

In this case we take a tree of height 1 with root in e^{2x} and n successor nodes with host-labels $2x$. The labeling maps every leaf to one term t_i. For an illustration of the last two situations, see the right-hand side of Figure 5.1.

Level 2. $t \in \operatorname{term}\Delta(e^{e^{2x}+x})$. We remark that $\mathcal{C}(e^{e^{2x}+x}) = e^0 = 1$. There is an integer $n \in \mathbb{N}$ such that there is a tuple $(u_1, \dots, u_n) \in (\operatorname{term}\Delta^{\downarrow}(e^{2x} + x))^n$ with

$$t = \frac{\mathcal{C}(e^{e^{2x}+x})}{n!} \cdot \mathfrak{D}(e^{e^{2x}+x}) \cdot u_1 \cdots u_n.$$

Again, if $n = 0$, then we do not assign a labeling to the tree. We consider the case $n > 0$. The terms u_i are all infinitesimal and elements from the set

$$\left\{ \frac{1}{x}, \frac{1}{\exp_2 x}, \frac{2^i}{i!} \binom{i}{j} \cdot \frac{e^{2x}}{x^j \cdot \exp_2^{i-j} x} \right\}_{0 \leqslant j < i}.$$

To every u_i we find some $s_i \in \{e^{2x}, x\}$ such that $s_i \in \operatorname{supp}\Delta^{\downarrow}(s_i)$. This gives rise to a labeled tree T as in the case of level 1. The root of T has host-label $e^{e^{2x}+x}$, and the root has exactly n successors which are respectively host-labeled by $s_1, \dots, s_n$. At a first stage, we equip the set of leaves of T with a labeling which maps the leaf labeled by s_i to u_i.

Hence, for every couple (u_i, s_i) there is a faithfully embedded sub-tree U_i in the relative tree-representation of s_i together with a labeling. We next substitute the trees U_i into the leaf of T which is labeled by s_i and which has the extra label u_i. The result is a faithfully embedded sub-tree V of the relative tree-representation of $e^{e^{2x}+x}$. On the set of leaves of V we define a labeling in the obvious way. We choose three terms from $\operatorname{term}\Delta(e^{e^{2x}+x})$ to illustrate the above. Firstly, we consider the term

$$t_1 = e^{e^{2x}\,(1 + \frac{1}{1!}\,\frac{2}{x} + \frac{1}{2!}\,\frac{2^2}{x^2} + \cdots) + x} \in \operatorname{term}\Delta(e^{e^{2x}+x}).$$

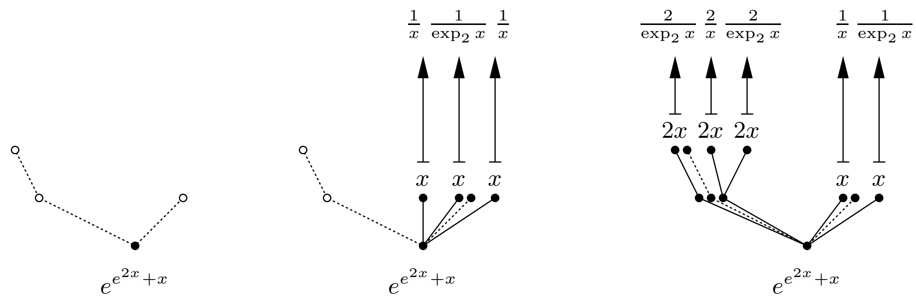


Figure 5.2: Trees that can be associates with the terms t_1, t_2 and t_3 from $\Delta(e^{e^{2x}+x})$.

The tree assigned to this tree is the one-point tree T with $t_{T,\mathsf{r}(T)} = e^{e^{2x}+x}$. This representation can be seen on the left-hand side of Figure 5.2. Secondly, we choose

$$t_2 = \frac{1}{3!} \cdot e^{e^{2x}\,(1+\frac{1}{1!}\,\frac{2}{x}+\frac{1}{2!}\,\frac{2^2}{x^2}+\cdots)+x} \cdot \frac{1}{x} \cdot \frac{1}{\exp_2 x} \cdot \frac{1}{x} \in \operatorname{term} \Delta(e^{e^{2x}+x}),$$

to which we assign the tree in the middle of the same figure. Thirdly, as a more complex term we take

$$t_3 = \frac{1}{4!} \cdot e^{e^{2x}\,(1+\frac{1}{1!}\,\frac{2}{x}+\frac{1}{2!}\,\frac{2^2}{x^2}+\cdots)+x} \cdot \left(\frac{1}{1!} \cdot e^{2x} \cdot \frac{2}{\exp_2 x}\right) \cdot \left(\frac{1}{2!} \cdot e^{2x} \cdot \frac{2}{x}\,\frac{2}{\exp_2 x}\right) \cdot \left(\frac{1}{2!} \cdot \frac{1}{x}\,\frac{1}{\exp_2 x}\right).$$

The tree for this term can be seen on the right side of Figure 5.2.

Remark 5.2.3 We see how faithfully embedded sub-trees occur naturally in this context, and what is more, we even see the reason why we had included one-point trees in the definition of such trees. Notice, though, that one-point trees were only given a labeling, if the node of the host-tree was an element of $\mathbb{T}$.

Going a step further, we can associate a term of $\Delta(e^{2x} + x)$ to every tree that is faithfully embedded into one of the children of $e^{e^{2x}+x}$ and which admits a labeling of its set of leaves. Since our trees have some special characteristics – there are always only finitely many successors and the labeling depends on Δ on $\mathbb{L}$ – we will not allow all such trees. We will come back to this point later.

5.2.4 Right-composition and well-labeled trees

Let $\Delta : \mathbb{T} \to \mathbb{U}$ be a right-composition and $\alpha > 0$ be an ordinal number. We denote the monomial groups of $\mathbb{T}$ and $\mathbb{U}$ by $\mathfrak{M}$ and $\mathfrak{N}$, respectively. Recall that the aim of this chapter is to extend Δ to $\mathbb{T}_\alpha$. This section shows how to extend our observations from the example of Section 5.2.2 to the general case.

Throughout the rest of this section, we will assume that Δ has already been extended to a right-composition $\Delta_\alpha : \mathbb{T}_\alpha \to \mathbb{U}_\alpha$. Recall that for terms $t \in C\mathfrak{M}_\alpha$ we have defined the notion of a relative tree-representation of t with respect to $\mathbb{T}$. It was denoted by $T_{t,\mathbb{T}}$. Furthermore, we have introduced faithfully $C\mathfrak{M}_\alpha$-embedded sub-trees of $T_{t,\mathbb{T}}$. There are two kinds of such sub-trees: those only consisting of a root and those with a non-empty set of leaves of positive height. In the latter case, for all $\mathsf{l} \in \operatorname{leaf}(T_{t,\mathbb{T}})$ we have $t_{T_{t,\mathbb{T}},\mathsf{l}} \in C\mathfrak{M}$. We exclude from the set of faithfully embedded trees the one -point trees where the host-label of the only node is not in $\mathbb{T}$.

We denote by $\operatorname{tree}_{\mathbb{T}}(t)$ the set of all finite faithfully $C\mathfrak{M}_\alpha$-embedded sub-trees of $T_{t,\mathbb{T}}$, and we let furthermore

$$\operatorname{tree}_{\mathbb{T}}(\mathbb{T}_\alpha) := \bigcup_{t \in C\mathfrak{M}_\alpha} \operatorname{tree}_{\mathbb{T}}(t).$$

We remark that nodes of finite trees have only finitely many successors. In the case of sub-trees of $T_{t,\mathbb{T}}$ the converse is also true, since a relative tree-representation cannot have infinite paths. Recall that we denote the label of the node n in the tree T by $t_{T,\mathsf{n}}$.

A couple (T, λ) is called a **labeled** tree associated to the term $t \in C\mathfrak{M}_\alpha$ iff $T \in \operatorname{tree}_{\mathbb{T}}(t)$ and if $\lambda : \operatorname{leaf}(T) \to C\mathfrak{N}_\alpha$ is such that for all $\mathsf{l} \in \operatorname{leaf}(T)$ we have $\lambda(\mathsf{l}) \in \operatorname{term}_\Delta(t_{T,\mathsf{l}})$. In the example of Section 5.2.2 we have seen that labeled trees occur naturally in the representation of elements of $\operatorname{term}_{\Delta_{g,2}}(e^{e^{2x}+x})$. We have also seen that not all labeled trees from $\operatorname{tree}^\lambda(e^{e^{2x}+x})$ contribute to this set of terms. We will now generalize the observations from Section 5.2.2 to elements from $\operatorname{term}_{\Delta_\alpha}(t)$ for $t \in C\mathfrak{M}_\alpha$.

Let $T_\lambda \in \operatorname{tree}^\lambda(t)$. We say that $U \subseteq T$ is a **proper** sub-tree iff there is a node $\mathsf{n} \in T \setminus \{\mathsf{r}(T)\}$ such that

$$U = \{\mathsf{t} \in T \mid \mathsf{n} \leqslant \mathsf{t}\}$$

and if the host-labeling of U is the restriction of the host-labeling of T to U, i.e. for all $\mathsf{n} \in U$ we have $t_{U,\mathsf{n}} = t_{T,\mathsf{n}}$. We let $U_\lambda = (U, \lambda|_U)$. Then U_λ is again a labeled tree. Note that one-point trees have no proper sub-trees.

We next define a function Θ^Δ on $<\cdot$. If $T = \operatorname{leaf}(T)$, then

$$\Theta^\Delta(T^\lambda) := \lambda(\mathsf{r}(T)).$$

Recall that we only allow one-point trees to be labeled, if the host-label is in $C\mathfrak{M}$.

Next let $T \neq \operatorname{leaf}(T)$. We recursively assume that Θ^Δ has been defined on the set of children $U_1, \dots, U_n$ of $\mathsf{r}(T)$. Then we let

$$\Theta^\Delta(T^\lambda) := \frac{\mathcal{C}(t_{T,\mathsf{r}(T)})}{n!} \cdot \mathfrak{D}(t_{T,\mathsf{r}(T)}) \cdot \Theta^\Delta(U_{1,\lambda}) \cdots \Theta^\Delta(U_{n,\lambda}).$$

Note that $\Theta^\Delta(T^\lambda)$ exists for all T^λ and that we have in the case $T \neq \operatorname{leaf}(T)$ that

$$\Theta^\Delta(T^\lambda) = \prod_{\mathsf{n} \in T \setminus \operatorname{leaf}(T)} \frac{\mathcal{C}(t_{T,\mathsf{n}})}{|\operatorname{succ}(\mathsf{n})|!} \cdot \mathfrak{D}(t_{T,\mathsf{n}}) \cdot \prod_{\mathsf{n} \in \operatorname{leaf}(T)} \lambda(\mathsf{n}).$$

We say that T^λ is **well-labeled** iff $\Theta^\Delta(U_\lambda) \prec 1$ for every proper sub-tree of T^λ. Instead of (T, λ) or T^λ, we will also write or $T^\bullet$. The set of well-labeled trees with root t is denoted by

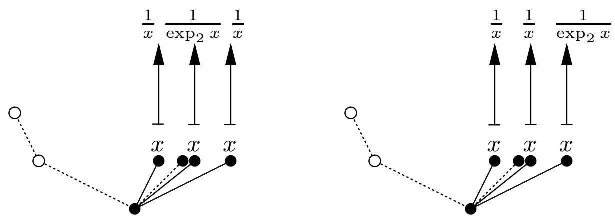


Figure 5.3: The labeled trees T_λ and $T_{\lambda'}$ from Example 5.2.5.

$\operatorname{tree}^\bullet(t)$. Similar to the case of labeled trees, we let

$$\begin{aligned}\operatorname{tree}^\bullet(f) &:= \bigcup_{t\in\operatorname{term} f} \operatorname{tree}^\bullet(t), \qquad \text{if } f \in \mathbb{T}_\alpha,\\ \operatorname{tree}^\bullet(\mathbb{T}_\alpha) &:= \bigcup_{t\in C\mathfrak{M}\alpha} \operatorname{tree}^\bullet(t).\end{aligned}$$

One-point trees are always well-labeled since they do not have any proper sub-trees.

Remark 5.2.4 If $\mathsf{l} \in \operatorname{leaf}(T)$ is such that $t_{T,\mathsf{l}} \notin C\mathfrak{M}$, then T is the one-point tree labeled by T. Labeling the root of T in this case corresponds to choosing a term from

$$\Delta_\alpha(t) = \mathcal{C}(t)\,\mathfrak{D}(t)\cdot e(\Delta_\alpha^\downarrow(\log \mathfrak{d}_t)).$$

If T is a one-point tree with root in $C\mathfrak{M}$, then Δ is defined on $t_{T,\mathsf{r}(T)}$. If T is not a one-point tree, then $t_{T,\mathsf{l}} \in C\mathfrak{M}$ for all leaves l. In this case, too, we can apply the function Δ to the label of l.

Moreover, let us point out that if $(T_1,\lambda_1) \neq (T_2,\lambda_2)$ are both labeled trees, then $T_1 \neq T_2$ or $T_1 = T_2$ and $\lambda_1 \neq \lambda_2$.

Example 5.2.5 We take $\Delta_g : \mathbb{L}_2 \to \mathbb{L}_4$ as in the example from Section 5.2.2, i.e. we have $g = x + \frac{1}{x} + \frac{1}{\exp_2 x}$. We let again $t = e^{e^{2x}+x}$. Then T is the labeled tree of height 1 with three leaves $\mathsf{l}_1, \mathsf{l}_2, \mathsf{l}_3$. The root of T is labeled by t. The leaves have all the label x. We define λ on $\operatorname{leaf}(T)$ by

$$\begin{aligned}\lambda(\mathsf{l}_1) &:= \frac{1}{x},\\ \lambda(\mathsf{l}_2) &:= \frac{1}{\exp_2 x},\\ \lambda(\mathsf{l}_3) &:= \frac{1}{x}.\end{aligned}$$

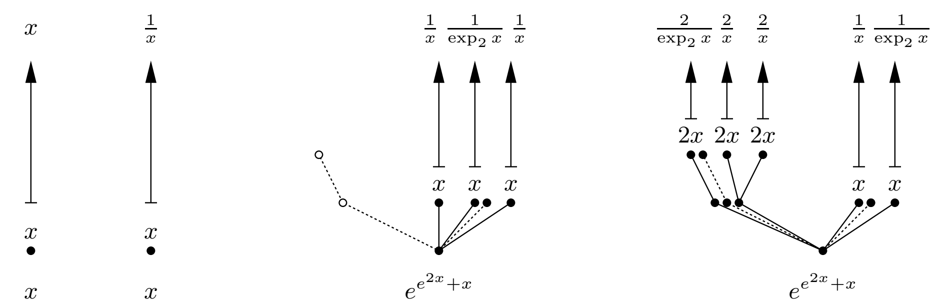


Figure 5.4: Well-labeled trees and non-well-labeled trees from Example 5.2.6.

Then T_λ is a labeled tree. For an illustration, see the tree on the left-hand side of Figure 5.3. We remark that for the same T we can define a different labeling $\lambda' : \mathrm{leaf}(T) \to C(\log^{\mathbb{Z}^\star} x)_2$ by

$$\begin{aligned}\lambda'(\mathsf{l}_1) &:= \frac{1}{x},\\ \lambda'(\mathsf{l}_2) &:= \frac{1}{x},\\ \lambda'(\mathsf{l}_3) &:= \frac{1}{\exp_2 x}.\end{aligned}$$

Then T_λ is different from $T_{\lambda'}$ although λ' is merely a permutation of the labeling λ.

Example 5.2.6 We take again the right-composition Δ_g from Section 5.2.2. First, let $t = x$. The relative tree-representation of t is in this case the one-point tree with root-leaf x. It has only one type of faithfully embedded sub-tree T. Let $T_2^{\lambda_1}, T_2^{\lambda_2} \in \mathrm{tree}^{\curlywedge}(x)$ with $T^1 = T_2 = T$, $\lambda_1(\mathsf{r}(T^1) = x$ and $\lambda_1(\mathsf{r}(T^2) = \frac{1}{x}$. Then $T_1^{\lambda_1}$ and $T_2^{\lambda_2}$ are permissible trees. Notice though that $\Theta^\Delta(T_1^{\lambda_1}) \succ 1$. See the left-hand side of Figure 5.4 for an illustration the two trees.

A more interesting example is $t = e^{e^{2x}+x}$. We choose faithfully embedded sub-trees T_3 and T_4 of its relative tree-representation as shown on the right-hand side of Figure 5.4. The labelings λ_3 and λ_4 can also be read from this figure. The tree $T_3^{\lambda_3}$ is a well-labeled tree. On the other hand, the tree $T_4^{\lambda_4}$ fails to be a well-labeled tree. Looking at it in terms of development of the terms, we can say that the label determined by the node with host-label $2x$ and labels $2/x$, $2/x$ "fails to expand down to the ground level, in other words, it "gets stuck" in exponential level at e^{2x}.

Proposition 5.2.7 *Let $t \in C\mathfrak{M}_\alpha$ and $s \in \mathrm{term}\,\Delta_\alpha(t)$. Then there is a well-labeled tree $T^\lambda \in \mathrm{tree}^\bullet(t)$ with $\mathfrak{d}_s = \mathfrak{d}(\Theta^\Delta(T^\lambda))$. In particular, if $1 \succ s$, then T^λ can only be a one-point tree if $t \in C\mathfrak{M}$.*

Proof: We show the Proposition using a transfinite induction. We start with the remark that for all $t \in C\mathfrak{M}_\alpha$ we have the equations

$$\begin{aligned} \Delta_\alpha(t) &= \mathcal{C}(t)\,\mathfrak{D}(t) \cdot e(\Delta_\alpha^{\downarrow}(\log \mathfrak{d}_t)), && (5.9)\\ \Delta_\alpha^{\downarrow}(\log \mathfrak{d}_t) &= \sum_{u \in \operatorname{term} \log \mathfrak{d}_t} \Delta_\alpha^{\downarrow}(u). && (5.10) \end{aligned}$$

The starting point of the induction is the case $t \in C\mathfrak{M}$. Then the proposition follows from $\Theta^\Delta(T^\lambda) \in \operatorname{term} \Delta(t)$. Now let $\beta > 0$ and assume that we have shown the proposition already for all terms from $C\mathfrak{M}_\gamma$ with $\gamma < \beta$. Let $t \in C\mathfrak{M}_\beta$, and we may assume that it is in no $C\mathfrak{M}_\gamma$ with $\gamma < \beta$. By equation (5.9) there exist an integer $n > 0$, terms $t_1, \dots, t_n \in \operatorname{term} \log \mathfrak{d}_t$ and terms $s_1, \dots, s_n \prec 1$ such that

$$\begin{aligned} \forall i \leqslant n : s_i &\in \operatorname{term} \Delta_\alpha(t_i),\\ s &= \frac{1}{n!}\mathcal{C}(t)\,\mathfrak{D}(t) \cdot s_1 \cdots s_n. \end{aligned}$$

By the induction hypothesis, there are well-labeled trees $T_{i,\lambda_i} \in \operatorname{tree}^\lambda(t_i)$ such that $s_i = \Theta^\Delta(T_{i,\lambda_i})$. Let T^λ be the unique labeled tree with root t and children $T_{1,\lambda_1}, \dots, T_{n,\lambda_n}$. Then

$$\Theta^\Delta(T^\lambda) = \frac{1}{n!} \cdot \mathcal{C}(t)\,\mathfrak{D}(t) \cdot \Theta^\Delta(T_{1,\lambda_1}) \cdots \Theta^\Delta(T_{n,\lambda_n}) = s.$$

This finishes the proof. □

Let $\Delta : \mathbb{T} \to \mathbb{U}$ be a right-composition as above. We wish to extend Δ to $\mathbb{T}_\alpha$ for ordinal numbers $\alpha > 0$ by

$$\Delta_\alpha(f) := \sum_{T^\lambda \in \operatorname{tree}^\bullet(f)} \Theta^\Delta(T^\lambda). \tag{5.11}$$

Remark 5.2.8 The function Δ_α defined in (5.11) is our candidate for a right-composition on $\mathbb{T}_\alpha$. It clearly extends Δ on $\mathbb{T}$. Apart from showing conditions **RC1** – **RC4**, we have to make sure that the right-hand side of the equation is defined.

5.3 Existence of extended right-compositions

The aim of this section is to make sure that the right-hand side of equation (5.11) is defined. We will then be able to prove that the resulting function is the unique right-composition on $\mathbb{T}_\alpha$ that extends Δ. More precisely, we will show the following statements.

Theorem 5.3.1 *The right-hand side of equation (5.11) is well-defined.*

Proof: We show the theorem using a transfinite induction. It clearly holds for $\alpha = 0$. In what follows, we assume that $\alpha > 0$ and that the theorem holds for all $\beta < \alpha$.

Let $f \in \mathbb{T}_\alpha$. We fix a sequence $(T_i^\bullet)_{0 \leqslant i}$ of elements from $\operatorname{tree}^\bullet(f)$. Let $\mathfrak{m}_i$ be the monomial of the term $\Theta^\Delta(T_i^\bullet) \in C\mathfrak{M}_\alpha$. We have to show that there exist $i < j$ such that

$$\mathfrak{m}_i \not\succ \mathfrak{m}_j \tag{5.12}$$

The roots of the trees T_i are labeled by terms from $C\mathfrak{M}_\alpha$. We may assume that they are monomials $\mathfrak{r}_i$. Modulo extracting a sub-sequence, we may furthermore assume that

$$\mathfrak{r}_0 \succcurlyeq \mathfrak{r}_1 \succcurlyeq \mathfrak{r}_2 \succcurlyeq \cdots .$$

We distinguish the following four cases.

Case I: All $\mathfrak{r}_i$ are from $\mathfrak{M}$.
Case II: All T_i are one-point trees.
Case III: For all $k < l$ and all $\mathsf{s} \in \operatorname{succ} \mathsf{r}(T_l)$ we have $t_{T_l,\mathsf{s}} \notin \operatorname{term}(\log \mathfrak{r}_k \;\; \vartriangle \;\; \log \mathfrak{r}_l)$.
Case IV: None of the above.

The four cases will be treated separately in sections 5.3.1, 5.3.2, 5.3.3 and 5.3.4. □

Theorem 5.3.2 *Let $\vartriangle : \mathbb{T} \to \mathbb{U}$ be a right-composition. For every transfinite exponential extension of $\mathbb{T}$, there exists a unique right-derivation extending $\vartriangle$.*

This theorem will be proved in section 5.3.5

5.3.1 First case: Root host-labeled by monomials in $\mathfrak{M}$

In this case all T_i are one-point trees with $t_{T_i,\mathsf{r}(T_i)} = \mathfrak{r}_i$. Since $\mathfrak{r}_0 \succcurlyeq \mathfrak{r}_1 \succcurlyeq \cdots$ and $\vartriangle$ is strongly linear Noetherian mapping, the family $(\vartriangle(\mathfrak{r}_i))_{0 \leqslant i}$ is Noetherian. In particular, the set

$$\bigcup_{0 \leqslant n} \operatorname{supp} \vartriangle(\mathfrak{t}_n)$$

is Noetherian. Since $\mathfrak{m}_i \in \bigcup_{0 \leqslant n} \operatorname{supp} \vartriangle(\mathfrak{r}_n)$ for all i, there indeed exist $i < j$ with $\mathfrak{m}_i \succcurlyeq \mathfrak{m}_j$. This finishes Case **I**.

5.3.2 Second case: One-point trees

We suppose that none of the $\mathfrak{r}_i$ are in $\mathfrak{M}$, for otherwise we may extract an infinite sub-sequence as in case I. Hence $\mathfrak{m}_i = \mathfrak{D}(\mathfrak{r}_i)$ for all i. Since $\mathfrak{D}$ is strictly increasing, we have

$$\mathfrak{m}_0 = \mathfrak{D}(\mathfrak{r}_0) \succcurlyeq \mathfrak{m}_1 = \mathfrak{D}(\mathfrak{r}_1) \succcurlyeq \mathfrak{m}_2 = \mathfrak{D}(\mathfrak{r}_2) \succcurlyeq \cdots .$$

In particular $\mathfrak{m}_0 \succcurlyeq \mathfrak{m}_1$. That finishes Case **II**.

5.3.3 Third case: Strong disjointness of successors of the roots

Assume that we are not in one of the cases **I** or **II**. We fix some notations. For any well-labeled tree $T^\bullet = (T, \lambda)$ we let

$$\begin{aligned}
\lfloor T \rfloor &:= \prod_{\mathsf{n} \in T \setminus \operatorname{leaf}(T)} \mathfrak{m}_{T,\mathsf{n}} \\
\lceil T \rceil &:= \prod_{\mathsf{n} \in \operatorname{leaf}(T)} \mathfrak{m}_{T,\mathsf{n}} \\
\|T^\bullet\| &:= \prod_{\mathsf{n} \in \operatorname{leaf}(T)} \mathfrak{d}_{\lambda(\mathsf{n})}.
\end{aligned}$$

Lemma 5.3.3 *Let* $T^\bullet \in \operatorname{tree}^\bullet(\mathfrak{t})$ *and* $U^\bullet \in \operatorname{tree}^\bullet(\mathfrak{u})$ *with* $\operatorname{succ}\mathsf{r}(T), \operatorname{succ}\mathsf{r}(U) \neq \emptyset$. *Suppose that* $\mathfrak{t} \succ \mathfrak{u}$ *are such that*

$$\forall \mathsf{s} \in \operatorname{succ}\mathsf{r}(U) : t_{U,\mathsf{s}} \notin \operatorname{term}(\log \mathfrak{t} \ \vartriangle \ \log \mathfrak{u}).$$

Then for all $\mathfrak{m} \prec\!\!\!\prec \mathfrak{D}(\mathfrak{t})/\mathfrak{D}(\mathfrak{u})$ *we have* $\mathfrak{D}(\lfloor T \rfloor) \succ \mathfrak{D}(\lfloor U \rfloor) \cdot \mathfrak{m}$. *In particular* $\mathfrak{D}(\lfloor T \rfloor) \succ \mathfrak{D}(\lfloor U \rfloor)$.

Proof: For $\mathsf{s} \in \operatorname{succ}\mathsf{r}(U)$ it follows from $t_{U,\mathsf{s}} \notin \operatorname{term}(\log \mathfrak{t} \ \vartriangle \ \log \mathfrak{u})$ that

$$t_{U,\mathsf{s}} \preccurlyeq \log \frac{\mathfrak{t}}{\mathfrak{u}} \prec\!\!\!\prec \frac{\mathfrak{t}}{\mathfrak{u}}.$$

Since $\mathfrak{D}$ is multiplicative and strictly increasing, one obtains $\mathfrak{D}(t_{U,\mathsf{s}}) \prec\!\!\!\prec \mathfrak{D}(\mathfrak{t})/\mathfrak{D}(\mathfrak{u})$. Moreover, for all $\mathsf{n} \in U$ with $\mathsf{s} <\!\cdot\, \mathsf{n}$ we have $t_{U,\mathsf{n}} \prec t_{U,\mathsf{s}}$, hence $\mathfrak{D}(t_{U,\mathsf{n}}) \prec \mathfrak{D}(t_{U,\mathsf{s}})$ and $\mathfrak{D}(t_{U,\mathsf{n}}) \prec\!\!\!\prec \mathfrak{D}(t_{U,\mathsf{s}})$. Consequently,

$$\frac{\mathfrak{D}(\lfloor U \rfloor)}{\mathfrak{D}(\mathfrak{u})} \prec\!\!\!\prec \frac{\mathfrak{D}(\mathfrak{t})}{\mathfrak{D}(\mathfrak{u})}$$

This implies the lemma. □

Proposition 5.3.4 *Let* $(T_i^\bullet)_{0 \leqslant i}$ *be a sequence as above and assume that for all* $0 \leqslant k < l$ *and all* $\mathsf{s} \in \operatorname{succ}\mathsf{r}(T_l)$ *we have*

$$t_{T_l,\mathsf{s}} \notin \operatorname{term}(\log \mathfrak{r}_k \ \vartriangle \ \log \mathfrak{r}_l).$$

Then there exist $i < j$ *such that* $\mathfrak{m}_i \succcurlyeq \mathfrak{m}_j$.

Proof: For all $i \geqslant 0$, we have $\lceil T_i \rceil \in \mathfrak{M}$ and $\|T_i^\bullet\| \in \operatorname{supp} \vartriangle(\lceil T_i \rceil)$. The latter follows from $\lambda(\mathsf{n}) \in \operatorname{term} \vartriangle(t_{T_i,\mathsf{n}})$ and the multiplicativity of $\mathfrak{D}$. We distinguish two cases with respect to the sequence $(\lceil T_i \rceil)_{0 \leqslant i}$: modulo taking a sub-sequence if necessary, we may assume that one of

$$\lceil T_0 \rceil \succcurlyeq \lceil T_1 \rceil \succcurlyeq \lceil T_2 \rceil \succcurlyeq \cdots \tag{5.13}$$
$$\lceil T_0 \rceil \prec \lceil T_1 \rceil \prec \lceil T_2 \rceil \prec \cdots \tag{5.14}$$

holds. Note that the assumptions about $(T_i^\bullet)_{0 \leqslant i}$ imply that $\mathfrak{r}_k \succ \mathfrak{r}_l$ for all $k < l$.

Case A: $\forall k < l : \lceil T_k \rceil \succcurlyeq \lceil T_l \rceil$. Then from **RC3** for $\vartriangle$ it follows that

$$\bigcup_{0 \leqslant i} \operatorname{supp} \vartriangle(\lceil T_i \rceil)$$

is a well-ordered set. Then there are $i < j$ such that $\|T_i^\bullet\| \succcurlyeq \|T_j^\bullet\|$. By Lemma 5.3.3 we have $\mathfrak{D}(\lfloor T_i \rfloor) \succ \mathfrak{D}(\lfloor T_j \rfloor)$. Hence

$$\mathfrak{m}_i = \mathfrak{D}(\lfloor T_i \rfloor) \cdot \|T_i^\bullet\| \succ \mathfrak{m}_j = \mathfrak{D}(\lfloor T_j \rfloor) \cdot \|T_j^\bullet\|.$$

Case B: $\forall k < l : \lceil T_k \rceil \prec \lceil T_l \rceil$. Let us start with a general observation. Let $\mathfrak{a} \in \mathfrak{M}$ and $\mathfrak{b} \in \operatorname{supp} \Delta(\mathfrak{a})$. Let $\delta \in \mathbb{T}^{\downarrow}$ with

$$\frac{1}{\Delta(\mathfrak{a})} = \mathfrak{d}_{\frac{1}{\Delta(\mathfrak{a})}} \cdot (1+\delta) = \frac{1}{\mathfrak{d}_{\Delta(\mathfrak{a})}} \cdot (1+\delta).$$

Then $\operatorname{supp} \delta \subseteq \mathfrak{d}_{\Delta(\mathfrak{a})} \cdot \operatorname{supp} \Delta(\frac{1}{\mathfrak{a}})$ and $\Delta(\mathfrak{a}) = \mathfrak{d}_{\Delta(\mathfrak{a})} \cdot (1 - \delta + \delta^2 + \cdots)$. Hence for some $k \geqslant 0$ we have $\mathfrak{b} \in \mathfrak{d}_{\Delta(\mathfrak{a})} \cdot \operatorname{supp} \delta^k$. Then there are infinitesimals $\mathfrak{c}_1, \ldots, \mathfrak{c}_k \in \operatorname{supp} \Delta(\frac{1}{\mathfrak{a}})$ such that

$$\mathfrak{b} = \mathfrak{d}_{\Delta(\mathfrak{a})}^{k+1} \cdot \mathfrak{c}_1 \cdots \mathfrak{c}_k.$$

Now consider the set of couples

$$S := \left\{ (i, \mathfrak{m}) \;\middle|\; \mathfrak{m} \in \operatorname{supp} \Delta\left(\frac{1}{\lceil T_i \rceil}\right) \right\}.$$

We order S by $(i, \mathfrak{m}) \succ (j, \mathfrak{n})$ iff $\mathfrak{m} \succ \mathfrak{n}$. From (5.14) it follows that $(S, \succcurlyeq)$ is Noetherian. By Higman's Theorem 1.1.4, the ordering $(S^\star, \succcurlyeq_{S^\star})$ is also Noetherian.

For every i we find by the above observation an integer $k_i \geqslant 0$ and infinitesimal monomials $\mathfrak{m}_{i,1}, \ldots, \mathfrak{m}_{i,k_i} \in \operatorname{supp} \Delta(\frac{1}{\lceil T_i \rceil})$ such that

$$\| T_i^\bullet \| = \mathfrak{d}_{\Delta(\lceil T_i \rceil)}^{k_i+1} \cdot \mathfrak{m}_{i,1} \cdots \mathfrak{m}_{i,k_i}.$$

By the Noetherianity of $S^\star$ there exist $i < j$ with

$$[(i, \mathfrak{m}_{i,1}), \ldots, (i, \mathfrak{m}_{i,k_i})] \succcurlyeq_{S^\star} [(j, \mathfrak{m}_{j,1}), \ldots, (j, \mathfrak{m}_{j,k_j})] \quad \text{and} \quad k_i \leqslant k_j.$$

Since all $\mathfrak{m}_{j,l}$ are infinitesimal, we obtain

$$\mathfrak{m}_{i,1} \ldots \mathfrak{m}_{i,k_i} \succcurlyeq \mathfrak{m}_{j,1} \ldots \mathfrak{m}_{j,k_j}. \tag{5.15}$$

Since on the other hand we have $t_{T_j,\mathfrak{s}} \notin \operatorname{term}(\log \mathfrak{r}_i \;\; \triangle \;\; \log \mathfrak{r}_j)$, it follows that

$$\mathfrak{d}_{\Delta(\lceil T_j \rceil)} \;\asymp\; \Delta(\lceil T_j \rceil) \;\prec\!\!\!\prec\; \frac{\mathfrak{D}(\mathfrak{r}_i)}{\mathfrak{D}(\mathfrak{r}_j)},$$

from which with Lemma 5.3.3 it follows that $\mathfrak{D}(\lfloor T_i \rfloor) \succ \mathfrak{D}(\lfloor T_j \rfloor) \cdot \mathfrak{d}_{\Delta(\lceil T_j \rceil)}^{k_j+1}$. But then

$$\mathfrak{D}(\lfloor T_i \rfloor) \cdot \mathfrak{d}_{\Delta(\lceil T_i \rceil)}^{k_i+1} \succ \mathfrak{D}(\lfloor T_j \rfloor) \cdot \mathfrak{d}_{\Delta(\lceil T_j \rceil)}^{k_j+1}. \tag{5.16}$$

Multiplying (5.15) and (5.16), we get $\mathfrak{m}_i \succ \mathfrak{m}_j$. □

5.3.4 Fourth and last case

It remains to treat the case where the sequence $(T_i^\bullet)_{0\leqslant i}$ cannot be reduced to one of the previous cases. That means that modulo extracting a sub-sequence, we suppose that no $\mathfrak{r}_i$ is an element of $\mathfrak{M}$, that no tree T_i is reduced to its proper root and that for every $k>0$ we find a successor node $\mathsf{s}_k \in \operatorname{succ} \mathsf{r}(T_k)$ such that

$$t_{T_k,\mathsf{s}_k} \in \operatorname{term}(\log \mathfrak{r}_0 \;\; \triangle \;\; \log \mathfrak{r}_k).$$

Furthermore, since all t_{T_k,s_k} are elements of $\operatorname{term} \log \mathfrak{r}_0$, we can derive from Remark 4.3.2 that for all $k<l$:

$$\log \mathfrak{r}_0 \;\; \triangle \;\; \log \mathfrak{r}_l \;\trianglelefteq\; \log \mathfrak{r}_k \;\; \triangle \;\; \log \mathfrak{r}_l.$$

Thus we also have $t_{T_l,\mathsf{s}_l} \in \operatorname{term}(\log \mathfrak{r}_k \;\; \triangle \;\; \log \mathfrak{r}_l)$ for all $0<k<l$. For sequences $(\mathsf{s}_k)_{0<k}$ with these properties, we will in the following say that the badness of $(T_i^\bullet)_{0\leqslant i}$ is realized by $(\mathsf{s}_k)_{0<k}$.

Proposition 5.3.5 *Let $(T_i^\bullet)_{0\leqslant i}$ be a bad sequence with the above properties. Then there exist $i<j$ such that $\mathfrak{m}_i \not\succcurlyeq \mathfrak{m}_j$.*

Proof: Suppose not and let $(T_i^\bullet)_{0\leqslant i}$ be a bad sequence, i.e. a counter example to the proposition. We say that the bad sequence $(T_i^\bullet)_{0\leqslant i}$ is minimal, if for every i and fixed $T_0^\bullet,\ldots,T_{i-1}^\bullet$, the number of children of $\mathsf{r}(T_i)$ is minimal. From now on, we will assume that $(T_i^\bullet)_{0\leqslant i}$ is a minimal bad sequence.

Fix for all $i>0$ a node $\mathsf{s}_i \in \operatorname{succ} \mathsf{r}(T_i)$ such that $(\mathsf{s}_i)_{1\leqslant i}$ realizes the badness of the sequence. We denote the proper sub-tree of T_i with root s_i by U_i. The trees $W_i = T_i \setminus U_i$ are non-empty and give rise to a sequence of well-labeled trees. We claim that $\{W_i^\bullet \mid 1 \leqslant i\}$ is Noetherian. Otherwise, there exists a bad sequence $W_{j_0}, W_{j_1}, \ldots$ with $j_0<j_1<\cdots$. But then

$$(T_0^\bullet,\ldots,T_{j_0-1}^\bullet, W_{j_0}^\bullet, W_{j_1}^\bullet,\ldots)$$

is also a bad sequence, which contradicts the minimality of the sequence $(T_i^\bullet)_{0\leqslant i}$. This shows our claim.

Since $\{W_i^\bullet \mid 1\leqslant i\}$ is Noetherian, there exist $i_0<i_1<\cdots$ with

$$\Theta^\Delta(T_{i_0}\setminus U_{i_0}) = \Theta^\Delta(W_{i_0}) \quad \succcurlyeq \quad \Theta^\Delta(T_{i_1}\setminus U_{i_1}) = \Theta^\Delta(W_{i_1}) \quad \succcurlyeq \quad \cdots.$$

Furthermore, all U_{i_n} have roots with host-labels in $\operatorname{term}\log \mathfrak{r}_0$. Now $\log \mathfrak{r}_0 \in \mathbb{T}_\beta$ for some $\beta<\alpha$, so that

$$\vartriangle(\log \mathfrak{r}_0) = \sum_{T^\lambda \in \operatorname{tree}^\bullet(\log \mathfrak{r}_0)} \Theta^\Delta(T^\lambda).$$

In particular, we have $U_{i_0}, U_{i_1},\ldots \in \operatorname{tree}^\bullet(\log \mathfrak{r}_0)$, so that $\Theta^\Delta(U_{i_j}) \succcurlyeq \Theta^\Delta(U_{i_k})$ for some $j<k$. We conclude that $\mathfrak{m}_{i_j} \succcurlyeq \mathfrak{m}_{i_k}$. □

5.3.5 The extension is a right-composition

Proof of Theorem 5.3.2: By Theorem 5.3.1, the function Δ_α as defined in equation (5.11) on page 107 exists. We have to show conditions **RC1** – **RC4**.

Condition **RC1** holds, since Δ_α extends Δ. Let us show **RC3** next. Fix a Noetherian family F in $\mathbb{T}_\alpha$. First we remark that for every series f we have

$$\operatorname{tree}^\bullet(f) = \coprod_{t \in \operatorname{term} f} \operatorname{tree}^\bullet(t).$$

For $T^\bullet \in \operatorname{tree}^\bullet(t)$, we let $T^\bullet_{\mathfrak{d}}$ be the tree which results from replacing the root label by $\mathfrak{d}_t$. The rest of both the host- and the labeling remain unchanged. We then have $\Theta^\Delta(T^\bullet) = c_t \cdot \Theta^\Delta(T^\bullet_{\mathfrak{d}})$. We then obtain $\sum_{f \in F} \Delta_\alpha(f)$

$$\begin{aligned}
\sum_{f \in F} \Delta_\alpha(f) &= \sum_{f \in F} \sum_{\mathfrak{m} \in \operatorname{supp} f} \sum_{T^\lambda \in \operatorname{tree}^\bullet(\mathfrak{m})} f_{\mathfrak{m}} \cdot \Theta^\Delta(T^\lambda) \\
&= \sum_{\mathfrak{m} \in \bigcup_F \operatorname{supp} f} \sum_{T^\lambda \in \operatorname{tree}^\bullet(\mathfrak{m})} \Big(\sum_{f \in F} f_{\mathfrak{m}}\Big) \cdot \Theta^\Delta(T^\lambda) \\
&= \sum_{\mathfrak{m} \in \operatorname{supp} \sum F} \sum_{T^\lambda \in \operatorname{tree}^\bullet(\mathfrak{m})} F_{\mathfrak{m}} \cdot \Theta^\Delta(T^\lambda).
\end{aligned}$$

This shows $\sum_{f \in F} \Delta_\alpha(f) = \Delta_\alpha(\sum_F f)$ and thus **RC3**. Next. we show condition **RC4**. From Remark 5.1.3 it follows that we are done if we can show that $\Delta_\alpha(\mathfrak{m}) = \exp \Delta_\alpha(\log \mathfrak{m})$ for all $\mathfrak{m} \in \mathfrak{M}_\alpha$. Let $T^\bullet \in \operatorname{tree}^\bullet(\mathfrak{m})$. For every $\mathsf{s} \in \operatorname{succ} \mathsf{r}(T^\bullet)$ we denote by $T^\bullet_{\mathsf{s}}$ the child of $\mathsf{r}(T^\bullet)$ with root s. Recall that for all $\mathsf{s} \in \operatorname{succ} \mathsf{r}(T^\bullet)$ we have $\Theta^\Delta(T^\bullet_{\mathsf{s}}) \prec 1$. Moreover, we have

$$\Theta^\Delta(T^\bullet) = \mathfrak{D}(\mathfrak{m}) \cdot \frac{1}{|\operatorname{succ} \mathsf{r}(T^\bullet)|!} \cdot \prod_{\mathsf{s} \in \operatorname{succ} \mathsf{r}(T^\bullet)} \Theta^\Delta(T^\bullet_{\mathsf{s}}).$$

From the definition of Δ_α it then follows that

$$\begin{aligned}
\Delta_\alpha(\mathfrak{m}) &= \mathfrak{D}(\mathfrak{m}) \cdot \sum_{T^\bullet \in \operatorname{tree}^\bullet(\mathfrak{m})} \frac{1}{|\operatorname{succ} \mathsf{r}(T^\bullet)|!} \prod_{\mathsf{s} \in \operatorname{succ} \mathsf{r}(T^\bullet)} \Theta^\Delta(T^\bullet_{\mathsf{s}}) \\
&= \mathfrak{D}(\mathfrak{m}) \cdot \sum_{0 \leqslant n} \frac{1}{n!} \left(\sum_{t \in \operatorname{term} (\log \mathfrak{m})} \sum_{\substack{U^\bullet \in \operatorname{tree}^\bullet(t): \\ \Theta^\Delta(U^\bullet) \prec 1}} \Theta^\Delta(U^\bullet) \right)^n .
\end{aligned}$$

On the other hand, we have

$$\sum_{t \in \operatorname{term} (\log \mathfrak{m})} \sum_{\substack{U^\bullet \in \operatorname{tree}^\bullet(t): \\ \Theta^\Delta(U^\bullet) \prec 1}} \Theta^\Delta(U^\bullet) = \left(\sum_{t \in \operatorname{term} (\log \mathfrak{m})} \sum_{U^\bullet \in \operatorname{tree}^\bullet(t)} \Theta^\Delta(U^\bullet) \right)^{\downarrow} = \Delta_\alpha^{\downarrow}(\log \mathfrak{m}).$$

This together with $\mathfrak{D}(\mathfrak{m}) = \exp \Delta_\alpha^\uparrow(\log \mathfrak{m})$ shows

$$\Delta_\alpha(\mathfrak{m}) = \exp \Delta_\alpha^\uparrow(\log \mathfrak{m}) \cdot e(\Delta_\alpha^\downarrow(\log \mathfrak{m})).$$

From $\mathcal{C}(\mathfrak{m}) = 1$ the condition **RC4** now follows. Condition **RC2** can be shown as in Proposition 5.1.9. The uniqueness follows from Proposition 5.1.7. □

Corollary 5.3.6 *For every ordinal α and every transseries field $\mathbb{T}$ and every $g \in \mathbb{T}_\infty^+$, there is a unique right-composition $\Delta_g : \mathbb{L}_\alpha \to \mathbb{T}_\alpha$ such that $x \mapsto g$.*

Chapter 6

Taylor series

In the first part of the chapter, we estblish the link between derivations and right-composition. The compatibility will be manifest in the presence of a Taylor series development.

Then we go on to show how to extend the concept to operators on transseries fields, which will lay the groundwork for the third part, in which we consider infinite iterators of functions.

6.1 Compositions on differential fields of transseries

6.1.1 Compositions and derivations

Notation 6.1.1 Let $\circ : \mathbb{T} \times \mathbb{U} \to \mathbb{U}$ be a partial function for transseries fields $\mathbb{T}, \mathbb{U}$. Let us fix the following simplifications for notations for the rest of this section. If $f \in \mathbb{T}$, then there is a (partial) unitary function $\circ(f, \cdot) : \mathbb{U} \to \mathbb{U}$ defined by $g \mapsto \circ(f, g)$. Instead of $\circ(f, \cdot)$ we write $f \circ \cdot$ or even just f. Hence we write for instance $\operatorname{dom} f$ instead of $\operatorname{dom} \circ (f, \cdot)$ and $f(g)$ instead of $\circ(f, g)$ or $f \circ g$. If ∂ is a derivation on $\mathbb{T}$, then we will write $\partial f = f'$ and $\partial_n(f) = f^{(n)}$.

Let $(\mathbb{T}, \partial_{\mathbb{T}}), (\mathbb{U}, \partial_{\mathbb{U}})$ be differential fields of transseries. A partially defined function

$$\circ : \mathbb{T} \times \mathbb{U} \longrightarrow \mathbb{U}$$

is a **composition** w.r.t. $\partial_{\mathbb{T}}$ and $\partial_{\mathbb{U}}$ (or just a composition, if the derivations are clear from the context) iff

CC1. $\forall g \in (\mathbb{U})^{+}_{\infty}$ the function $\Delta_g : \mathbb{T} \to \mathbb{U}$ with $\Delta_g(f) = f(g)$ is a right-composition,

CC2. for all $f \in \mathbb{T}^{+}_{\infty}$, the function

$$f : \mathbb{U}^{+}_{\infty} \longrightarrow \mathbb{U}$$

is strictly increasing,

CC3. $\forall f \in \mathbb{T} : \forall g \in \mathbb{U}$: if $g \in \operatorname{dom} f$, then $g \in \operatorname{dom} f'$ and

$$(f(g))' = f'(g) \cdot g',$$

CC4. $\forall f \in \mathbb{T} : \forall g, \varepsilon \in \mathbb{U}$: if $\forall \mathfrak{m} \in \operatorname{supp} f : \mathfrak{m}(g) \succ \mathfrak{m}'(g) \cdot \varepsilon$, then $g + \varepsilon \in \operatorname{dom} f$, $(f^{(n)} g \cdot \varepsilon^n)_{0 \leqslant n}$ is a Noetherian family and

$$f(g+\varepsilon) = \sum_{0 \leqslant n} \frac{1}{n!} f^{(n)}(g) \cdot \varepsilon^n.$$

If $\circ$ is a compatible composition, then for all $f \in \mathbb{T}$, we call the partial function $\Gamma_f : \mathbb{U} \to \mathbb{U}$ defined by $g \mapsto f(g)$ a **left-composition**.

Remark 6.1.2 Let $\mathbb{T} = C[[\mathfrak{M}]]$. In order to show condition **CC2**, it suffices to show that for all $\mathfrak{m} \in \mathfrak{M}^{\uparrow}$ the function $\mathfrak{m} : \mathbb{U}^{+}_{\infty} \to \mathbb{U}$ is strictly increasing. To see this, let $\mathfrak{U}$ be the monomial group of $\mathbb{U}$ and $\mathfrak{u} \in^{\uparrow}$. Then for all $\mathfrak{m} \succ \mathfrak{n}$ in $\mathfrak{M}^{\uparrow}$ we have $\mathfrak{m}(\mathfrak{u}) \succ \mathfrak{n}(\mathfrak{u})$. Hence $\mathfrak{d}_f \circ \mathfrak{u} \succ \mathfrak{m} \circ \mathfrak{u}$ for all $\mathfrak{m} \in \operatorname{supp} f \setminus \{\mathfrak{d}_f\}$. Thus for series $g_1 < g_2$ from $\mathbb{U}^{+}_{\infty}$ and for $f \in \mathbb{T}^{+}_{\infty}$ we have

$$\begin{aligned} \mathfrak{d}_{f \circ g_1} &= \mathfrak{d}_{\mathfrak{d}_f \circ \mathfrak{d}_{g_1}} \\ \mathfrak{d}_{f \circ g_2} &= \mathfrak{d}_{\mathfrak{d}_f \circ \mathfrak{d}_{g_2}}. \end{aligned}$$

Then $g_1 < g_2$ implies $\mathfrak{d}_{g_1} \preccurlyeq \mathfrak{d}_{g_2}$. If $\mathfrak{d}_{g_1} \prec \mathfrak{d}_{g_2}$, then the hypothesis implies the claim. Otherwise $c_{g_1} < c_{g_2}$ leads to $\tau_{f \circ g_1} < \tau_{f \circ g_2}$.

Remark 6.1.3 Condition **CC4** shows a taylor series development of the series f in one variable. However, we will in this chapter show that this implies a multivariable Taylor series development. In other words, we will show that we can under appropriate conditions decompose the series ε into a Noetherian family $(\varepsilon_i)_{i \in I}$ such that

$$f(g+\varepsilon) = f(g + \sum_I \varepsilon_i) = \sum_{(i_1, \ldots, i_n) \in I^\star \} \frac{1}{n!} \cdot f^{(n)} g \cdot \varepsilon_{i_1} \cdots \varepsilon_{i_n}}.$$

6.1.2 Extending compatible compositions

In this section, we start discussing the possibility of extending compatible compositions using exp-extensions. Here, we will mainly show the points which are inherited from the initial tuple $(\mathbb{T}, \mathbb{U})$ in a direct way.

Let $\mathbb{T} = C[[\mathfrak{M}]]$ and $\mathbb{U} = C[[]]$ and

$$\circ : \mathbb{T} \times \mathbb{U} \to \mathbb{U}$$

be a composition w.r.t. $\partial_{\mathbb{T}}$ and $\partial_{\mathbb{U}}$. Fix $g \in \mathbb{U}^{+}_{\infty}$. Then $\varphi_g : \mathbb{T} \to \mathbb{U}$ with $\varphi_g(f) = f(g)$ is a right-composition. Theorem 4.4.2 shows that $\partial_{\mathbb{T}}$ extends to every $\mathbb{T}_\alpha$ and that every right-composition φ_g extends uniquely to a right-composition $\varphi_g : \mathbb{T}_\alpha \to \mathbb{U}_\alpha$. Hence the function

$$\begin{aligned} \circ : \mathbb{T}_\alpha \times \mathbb{U}^{+}_{\infty} &\to \mathbb{U}_\alpha \\ (f, g) &\mapsto \varphi_g(f) \end{aligned}$$

is our candidate for a composition. Some of the conditions are satisfied by construction, so for instance **CC1**. Let us state the theorem that we want to show.

Theorem 6.1.4 *Let $\circ : \mathbb{T} \times \mathbb{U} \to \mathbb{U}$ be a composition w.r.t. $\partial_{\mathbb{T}}$ and $\partial_{\mathbb{U}}$. Then for every ordinal number α, the function*

$$\begin{array}{rcl} \circ : \mathbb{T}_\alpha \times \mathbb{U} & \to & \mathbb{U}_\alpha \\ (f, g) & \mapsto & \varphi_g(f) \end{array}$$

is a composition.

Proposition 6.1.5 *Let $\circ : \mathbb{T} \times \mathbb{U} \to \mathbb{U}$ be a composition and α an ordinal number. Suppose that for all $\beta < \alpha$, the tuple $(\mathbb{T}_\beta, \mathbb{U}_\beta)$ admits a unique composition which extends $(\mathbb{T}, \mathbb{U})$. Let $\circ$ as in Theorem 6.1.4. Then* **CC1**, **CC2** *and* **CC3** *hold.*

Proof: Condition **CC1** follows from the construction. Next, we show **CC2**. Suppose that for $\beta < \alpha$, the function $\circ : \mathbb{T}_\beta \times \mathbb{U} \to \mathbb{U}_\beta$ is already a composition. We have to show that for all $\mathfrak{m} \in \mathfrak{M}_\alpha^\uparrow$ the function $\mathfrak{m} : \mathfrak{U}^\uparrow \to \mathbb{U}_\alpha$ is increasing. If α is a limit ordinal, this holds by inductive hypothesis. Let $\alpha = \beta + 1$ and $\mathfrak{m} = \exp f$ for some $0 < f \in \mathbb{T}_\beta^\uparrow$. By **RC4**, we have $\mathfrak{m}(\mathfrak{n}) = \exp(f(\mathfrak{n}))$ for all $\mathfrak{n} \in \mathfrak{U}^\uparrow$. Hence for $\mathfrak{n}_1 \prec \mathfrak{n}_2 \in \mathfrak{U}^\uparrow$, we have to show that

$$f(\mathfrak{n}_1) < f(\mathfrak{n}_2).$$

From $\operatorname{supp} f \subseteq \mathfrak{M}_\beta^\uparrow$ and **CC2** for β we obtain

$$\forall \mathfrak{a} \in \operatorname{supp} f : \mathfrak{a}(\mathfrak{n}_1) \prec \mathfrak{a}(\mathfrak{n}_2).$$

By Remark 5.1.2, the leading term of $f(\mathfrak{n}_1)$ is the leading term of $\tau_f \circ \mathfrak{n}_1$. Similarly, for $\mathfrak{n}_2$ we obtain $\tau_{f \circ \mathfrak{n}_2} = \tau_f \circ \mathfrak{n}_2$. Condition **CC2** for β now implies $\mathfrak{d}_f \circ \mathfrak{n}_1 \prec \mathfrak{d}_f \circ \mathfrak{n}_2$. From $c_f > 0$ now $\tau_f \circ \mathfrak{n}_1 \prec \tau_f \circ \mathfrak{n}_2$ follows. Hence $\tau_{f \circ \mathfrak{n}_1} \prec \tau_{f \circ \mathfrak{n}_2}$ and therefore the inequality. This shows **CC2** for α.

The first part of condition **CC3** follows from Theorems 4.4.2 and 5.3.2. For the rest of condition **CC3**, it suffices by strong linearity to show $(\mathfrak{m}(g))' = \mathfrak{m}'(g) \cdot g'$ for monomials $\mathfrak{m} \in \mathfrak{M}_\alpha$ and series $g \in \mathbb{U}_\infty^+$. If α is a limit ordinal, this follows from $\mathfrak{m} \in \mathfrak{M}_\beta$ for some $\beta < \alpha$. If $\alpha = \beta + 1$, then $\mathfrak{m} = \exp h$ for some $h \in \mathbb{T}_\beta$. Applying **CC3**, **RC4** and **D4** yields the following equations

$$\begin{array}{rcl} (\mathfrak{m}(g))' & = & (\exp(h(g)))' = \exp h(g) \cdot (h(g))' \\ & = & \exp h(g) \cdot h'(g) \cdot g' = (\exp h \cdot h')(g) \cdot g' \\ & = & (\exp h)'(g) \cdot g' = \mathfrak{m}'(g) \cdot g'. \end{array}$$

This shows **CC3** for α. □

6.1.3 Showing compatibility

We now finish the proof of Theorem 6.1.4 by showing **CC5**. Let $\mathbb{T}, \mathbb{U}, \mathbb{U}$ and α as in the theorem. Fix $f \in \mathbb{T}_{1,\alpha}$ and $g \in \mathbb{T}_{3,\infty}^+$ such that $g \in \operatorname{dom} f$. Furthermore, fix a Noetherian family $\mathcal{E} = (\varepsilon_i)_{i \in I}$ such that

$$\forall i \in I : \forall \mathfrak{m} \in \operatorname{supp} f : \qquad \varepsilon_i \prec \frac{\mathfrak{m}(g)}{\mathfrak{m}'(g)}.$$

Let $X_{\mathcal{E}} \subseteq \mathfrak{M}_{1,\alpha} \times \mathbb{N}$ be the set

$$X_{\mathcal{E}} := \left\{ (\mathfrak{m}, m) \,|\, m \in \mathbb{N} \wedge \forall i \in I : \varepsilon_i \prec \frac{\mathfrak{m}(g)}{\mathfrak{m}'(g)} \right\} \cup \{(1, m) \,|\, m \in \mathbb{N}\}.$$

We define an ordering on $X_{\mathcal{E}}$ by

$$(\mathfrak{m}, m) \leqslant (\mathfrak{n}, n) \quad :\Leftrightarrow \quad \forall i \in I : \mathfrak{m}(g) \cdot \varepsilon_i^m \succcurlyeq \mathfrak{n}(g) \cdot \varepsilon_i^n.$$

We define a set of $X_{\mathcal{E}}$-labeled structures Σ by identifying each element from $X_{\mathcal{E}}$ with the one-point structure which is labeled with this element, i.e. for all σ we let $I_\sigma = \{\bullet\}$ and $l_\sigma(\bullet) = \sigma$. We define the function $\vartheta_{\mathcal{E}} : X_{\mathcal{E}} \to \mathcal{P}(\mathfrak{M}_{1,\alpha} \times \mathbb{N})$ by

$$\vartheta_{\mathcal{E}}((\mathfrak{m}, m)) := \{(\mathfrak{a}, m+1) \,|\, \mathfrak{a} \in \operatorname{supp} \mathfrak{m}'\}.$$

Lemma 6.1.6 *The function $\vartheta_{\mathcal{E}}$ is a choice operator on $X_{\mathcal{E}}$.*

Proof: Let $(\mathfrak{m}, m) \in X_{\mathcal{E}}$ and $\mathfrak{a} \in \operatorname{supp} \mathfrak{m}'$. Then there is a path P in $T_{\mathfrak{m},\max}$ which determines $\mathfrak{a}$, i.e. with $t_{P,i} = c_{P,i}\mathfrak{m}_{P,i}$:

$$\begin{aligned} P &= [\mathfrak{m}, t_{P,1}, \dots] \in \operatorname{path}(T_{\mathfrak{m},\max}), \\ \mathfrak{n} &\in \operatorname{supp} \log \mathfrak{m}_{P,k} \qquad \text{for some } k \in \mathbb{N}, \\ \mathfrak{a} &= \mathfrak{m}_{P,0} \cdots \mathfrak{m}_{P,k} \cdot \mathfrak{n}. \end{aligned}$$

Note that we may replace k by larger integers. By hypothesis, for all $i \in I$ the inequality $\mathfrak{a}(g) \cdot \varepsilon_i \prec \mathfrak{m}(g)$ holds. Hence for all $i \in I$ we have

$$(\mathfrak{m}_{P,1} \cdots \mathfrak{m}_{P,k} \cdot \mathfrak{n}) \circ g \prec \frac{1}{\varepsilon_i}. \tag{6.1}$$

Let $\mathfrak{b} \in \operatorname{supp} \mathfrak{a}'$. Then there is a path Q which determines $\mathfrak{b}$, i.e.

$$\begin{aligned} Q &= [\mathfrak{a}, s_{Q,1}, \dots] \in \operatorname{path}(T_{\mathfrak{a},\max}), \\ \hat{\mathfrak{n}} &\in \operatorname{supp} \log \mathfrak{m}_{Q,m} \qquad \text{for some } m \in \mathbb{N}, \\ \mathfrak{b} &= \mathfrak{m}_{Q,0} \cdots \mathfrak{m}_{Q,m} \cdot \hat{\mathfrak{n}}. \end{aligned}$$

In particular, $s_{Q,1} \in \operatorname{term} \log \mathfrak{a}$, i.e. $s_{Q,1} = (\log \mathfrak{a})_{\mathfrak{m}}\mathfrak{m}$ for some $\mathfrak{m}$. Hence there is an integer $n \in \mathbb{N}$ such that $s_{Q,1}$ is a term in $\log \mathfrak{m}_{P,n}$. But then

$$\hat{Q} = [t_{P,0}, \dots, t_{P,n}, s_{Q,1}, s_{Q,2}, \dots]$$

is an a path in $T_{\mathfrak{m},\max}$. Thus $\hat{Q}$ determines an element of $\operatorname{supp} \mathfrak{m}'$. By varying k if necessary, we may assume that $k = n$ and that inequality (6.1) holds for this monomial as well. Therefore for all $i \in I$ we have

$$\frac{1}{\varepsilon_i} \succ (\mathfrak{m}_{P,1} \cdots \mathfrak{m}_{P,n} \cdot \mathfrak{m}_{Q,1} \cdots \mathfrak{m}_{Q,m} \cdot \hat{\mathfrak{n}}) \circ g.$$

Since $1 \preccurlyeq t_{P,1} \cdots t_{P,n}$ (with equality if and only if $n = 0$) it follows from the fact that right-compositions are strictly increasing that

$$\frac{1}{\varepsilon_i} \succ (\mathfrak{m}_{Q,1} \cdots \mathfrak{m}_{Q,m} \cdot \hat{\mathfrak{n}}) \circ g = \left(\frac{\mathfrak{m}_{Q,0}\, \mathfrak{m}_{Q,1} \cdots \mathfrak{m}_{Q,m} \cdot \hat{\mathfrak{n}}}{\mathfrak{m}_{Q,0}}\right) \circ g = \frac{\mathfrak{b}(g)}{\mathfrak{a}(g)}.$$

Hence $\mathfrak{a}'(g) \cdot \varepsilon_i \prec \mathfrak{a}(g)$ and thus $(\mathfrak{a}, m+1) \in X_{\mathcal{E}}$. This shows the lemma. □

Lemma 6.1.7 *The choice operator $\vartheta_{\mathcal{E}}$ is strictly extensive and Noetherian.*

Proof: Let $\mathfrak{n} \in \operatorname{supp} \mathfrak{m}'$, then

$$\mathfrak{n} \preccurlyeq \mathfrak{m}' \prec \frac{\mathfrak{m}}{\varepsilon_i}$$

for all $i \in I$ implies $\mathfrak{n} \cdot \varepsilon_i^{m+1} \prec \mathfrak{m} \cdot \varepsilon_i^{m}$ for all $i \in I$. Hence $(\mathfrak{m}, m) < (\mathfrak{n}, m+1)$, thus the operator is strictly extensive.

Let $Y \subseteq X_{\mathcal{E}}$ be a Noetherian set. We first remark that

$$\Sigma_Y = \{\sigma \in \Sigma \,|\, \operatorname{im} l_\sigma \subseteq Y\} = \{(\mathfrak{m}, m) \in X_{\mathcal{E}} \,|\, (\mathfrak{m}, m) \in Y\} = Y,$$

hence that

$$A = \{(\sigma, x) \,|\, \sigma \in \Sigma_Y \wedge x \in \vartheta_{\mathcal{E}}(\sigma)\} = \{((\mathfrak{m}, m), (\mathfrak{n}, m+1)) \,|\, (\mathfrak{m}, m) \in Y \wedge \mathfrak{n} \in \operatorname{supp} \mathfrak{m}'\}.$$

Suppose that A is not Noetherian in the induced ordering. Let

$$((\mathfrak{m}_i, m_i), (\mathfrak{n}_i, m_i + 1))_{0 \leqslant i}$$

be a $\preccurlyeq$-increasing sequence. Then for all $i \in I$ the sequence $(\mathfrak{n}_i \cdot \varepsilon_i^{m_i+1})_{0 \leqslant i}$ is $\preccurlyeq$-increasing. But this contradicts the fact that Σ_Y is Noetherian. □

The pair $(\Sigma, \vartheta_{\mathcal{E}})$ now gives rise to a pair $(\Sigma^+, \vartheta_{\mathcal{E}}^+)$, and by van der Hoeven's Theorem, the function $\vartheta_{\mathcal{E}}^+$ is a strictly extensive, Noetherian choice operator. We will use this fact in the following.

Let $\varepsilon_i = c_i \mathfrak{d}_{\varepsilon_i}(1 + \delta_i)$, then $\bigcup_I \delta_i$ is well-ordered and purely infinitesimal. Thus $(\bigcup_I \delta_i)^\star$ is well-ordered. We have to show that

$$(f^{(n)} g \cdot \varepsilon_i)_{0 \leqslant n, i \in I^n}$$

is Noetherian. Since

$$\bigcup_{0 \leqslant n} \bigcup_{i \in I^n} \operatorname{supp} f^{(n)} g \cdot \varepsilon_i \subseteq \bigcup_{0 \leqslant n} \bigcup_{i \in I^n} \operatorname{supp} f^{(n)} g \cdot \mathfrak{d}_{\varepsilon_i} \cdot \delta_i \subseteq \bigcup_{0 \leqslant n} \bigcup_{i \in I^n} \operatorname{supp} f^{(n)} g \cdot \mathfrak{d}_{\varepsilon_i} \cdot (\bigcup_I \delta_i)^\star,$$

we only have to show that

$$\bigcup_{0 \leqslant n} \bigcup_{i \in I^n} \operatorname{supp} f^{(n)} g \cdot \mathfrak{d}_{\varepsilon_i}$$

is well-ordered.

Recall that $\Sigma^+ = T_1 \coprod T_2 \coprod \cdots$. Let us determine the sets $T_1, T_2, \ldots$ in this application of van der Hoeven's Theorem. We have identified T_0 with $X_{\mathcal{E}}$ by looking at $X_{\mathcal{E}}$ as the set of one-point structures labeled with the elements from $X_{\mathcal{E}}$. Elements from T_1 are determined by some $\sigma \in \Sigma$ and structures $\tau_i \in T_0$ for all $i \in I_\sigma$. But since $I_\sigma = \{1\}$, the structure σ is a one-point structure and elements τ_1 from T_1 are of the form $(\mathfrak{m}, m)[(\mathfrak{n}, n)]$. Moreover, the condition $l_\sigma(1) \in \vartheta^*(\tau_1) = \vartheta(\sigma) = \vartheta_{\mathcal{E}}(\mathfrak{m}, m)$ implies that

$$\tau_1 = (\mathfrak{m}_1, m+1)[(\mathfrak{m}, m)],$$

where $\mathfrak{m}_1 \in \operatorname{supp} \mathfrak{m}'$. Inductively, we see that every element $\tau_k \in T_k$ is a one-point structure of the form

$$\tau_k = (\mathfrak{m}_k, m+k)[(\mathfrak{m}_{k-1}, m+k-1)[\ldots(\mathfrak{m}_1, m+1)[(\mathfrak{m}_0, m)]]\ldots]$$

where $\mathfrak{m}_{j+1} \in \operatorname{supp} \mathfrak{m}'_j$ for all $j < k$. For the series f, the set $Y = \operatorname{supp} f \times \{0\}$ is Noetherian in $X_{\mathcal{E}}$, hence $(X_{\mathcal{E}})_Y$ is Noetherian. But then so is the set

$$\bigcup_{0 \leqslant n} \bigcup_{i \in I^n} \operatorname{supp} f^{(n)} g \cdot \mathfrak{d}_{\varepsilon_i}.$$

This shows the first part of **CC5**.

Let $\mathfrak{m} \in \mathfrak{M}_{1,\alpha}$ and assume that **CC5** holds for all series from $\mathbb{T}_{1,\beta}$ with $\beta < \alpha$. Then in particular, the equation holds for $\log \mathfrak{m}$. We show that this fact implies the equation for $\mathfrak{m}$ and that from this the condition follows.

We have to show

$$\mathfrak{m}(g + \textstyle\sum_I \varepsilon_i) = \sum_{0 \leqslant n} \frac{1}{n!} \mathfrak{m}^{(n)} g \cdot \sum_{i \in I^n} \varepsilon_i.$$

This is equivalent to

$$\log(\mathfrak{m}(g + \textstyle\sum_I \varepsilon_i)) = \log \sum_{0 \leqslant n} \frac{1}{n!} \mathfrak{m}^{(n)} g \cdot \sum_{i \in I^n} \varepsilon_i = \log \mathfrak{m}(g) + l\left(\sum_{1 \leqslant n} \frac{1}{n!} \frac{\mathfrak{m}^{(n)} g}{\mathfrak{m}(g)} \cdot \sum_{i \in I^n} \varepsilon_i\right).$$

Since $\log \mathfrak{m} \in \mathbb{T}_{1,\beta}$, we obtain from **RC4** and the inductive hypothesis for $h = \log \mathfrak{m}$

$$\log(\mathfrak{m}(g + \textstyle\sum_I \varepsilon_i)) = h(g + \sum_I \varepsilon_i) = \sum_{0 \leqslant n} \frac{1}{n!} h^{(n)} g \cdot \sum_{I^n} \varepsilon_i.$$

On the other hand, from the definition of $l(X)$ we obtain

$$\begin{aligned} l\left(\sum_{1 \leqslant n} \frac{1}{n!} \frac{\mathfrak{m}^{(n)} g}{\mathfrak{m}(g)} \cdot \sum_{I^n} \varepsilon_i\right) &= \sum_{1 \leqslant j} \frac{(-1)^{j+1}}{j} \left(\sum_{1 \leqslant n} \frac{1}{n!} \frac{\mathfrak{m}^{(n)} g}{\mathfrak{m}(g)} \cdot \sum_{I^n} \varepsilon_i\right)^j \\ &= \sum_{1 \leqslant n} \frac{1}{n!} \sum_{I^n} \varepsilon_i \cdot \sum_{j=1}^{n} \frac{(-1)^{j+1}}{j} \sum_{L \in T^*(j,n)} \frac{n!}{L!} \cdot \frac{\mathfrak{m}^{(L)} g}{\mathfrak{m}(g)^j}. \end{aligned}$$

Hence, we are done, if we can show

$$(\log \mathfrak{m})^{(n)} g = \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i} \sum_{L \in T^*(i,n)} \frac{n!}{L!} \cdot \frac{\mathfrak{m}^{(L)} g}{\mathfrak{m}(g)^i}$$

for all $n \geqslant 1$. This clearly holds for $n = 1$. The general case can be shown inductively using the equation

$$\left(\frac{\mathfrak{m}^{(L)} g}{\mathfrak{m}(g)^i} \right)' = \frac{\mathfrak{m}^{(L)} g}{\mathfrak{m}(g)^i} \cdot \sum_{j=1}^{i} \frac{\mathfrak{m}^{(L_j+1)} g}{\mathfrak{m}^{(L_j)} g} \cdot g' - i \cdot \frac{\mathfrak{m}^{(L)} g}{\mathfrak{m}(g)^i} \cdot \frac{\mathfrak{m}'(g)}{\mathfrak{m}(g)} \cdot g'.$$

Now by strong linearity we have

$$\begin{aligned} f(g + \textstyle\sum_I \varepsilon_i) &= \sum_{\mathfrak{m} \in \mathfrak{M}_{1,\alpha}} f_{\mathfrak{m}} \mathfrak{m}(g + \textstyle\sum_I \varepsilon_i) \\ &= \sum_{\mathfrak{m} \in \mathfrak{M}_{1,\alpha}} f_{\mathfrak{m}} \sum_{0 \leqslant n} \frac{1}{n!} \mathfrak{m}^{(n)} g \cdot \textstyle\sum_{I^n} \varepsilon_i \\ &= \sum_{0 \leqslant n} \frac{1}{n!} \cdot \Big(\sum_{\mathfrak{m} \in \mathfrak{M}_{1,\alpha}} f_{\mathfrak{m}} \cdot \mathfrak{m}^{(n)} g \Big) \cdot \textstyle\sum_{I^n} \varepsilon_i \\ &= \sum_{0 \leqslant n} \frac{1}{n!} f^{(n)} g \cdot \textstyle\sum_{I^n} \varepsilon_i. \end{aligned}$$

This shows **CC5** and finishes the proof of Theorem 6.1.4.

6.2 Taylor families of operators

6.2.1 Definition of Taylor families

Let $\mathcal{F}$ be a set of partially defined functions $\Phi : \mathbb{T} \to \mathbb{T}$, where $\mathbb{T} = C[[\mathfrak{M}]]$ is a transseries field. Then $\mathcal{F} = (\mathcal{F}, ')$ is a **Taylor family** iff

Tf1. $\forall \Phi \in \mathcal{F} : \Phi' \in \mathcal{F}$ and $\operatorname{dom} \Phi' \supseteq \operatorname{dom} \Phi$,

Tf2. $\forall f, \varepsilon \in \mathbb{T} : \forall \Phi \in \mathcal{F}$: if $f, f + \varepsilon \in \operatorname{dom} \Phi$ and if

$$\left(\frac{1}{n!} \cdot \Phi^{(n)} f \cdot \varepsilon^n \right)_{0 \leqslant n}$$

is a Noetherian family, then

$$\Phi(f + \varepsilon) = \sum_{0 \leqslant n} \frac{1}{n!} \cdot \Phi^{(n)} f \cdot \varepsilon^n.$$

Remark 6.2.1 First let us remark that we use again the convention $\Phi' = \Phi^{(1)}$ and $(\Phi^{(n)})' = \Phi^{(n+1)}$ in the above definition. The usage of the derivation notation is justified since the function $' : \mathcal{F} \to \mathcal{F}$ will in applications always be a derivation.

The second point to notice concerns condition **Tf2**. In fact, the condition states that $\Phi(f+\varepsilon)$ is actually independent from the decomposition into f and ε. That is, if we find g, δ with $f+\varepsilon = g+\delta$ such that the hypotheses of **Tf2** hold for g and δ in place of f and ε, then

$$\sum_{0\leqslant n} \frac{1}{n!} \cdot \Phi^{(n)} f \cdot \varepsilon^n = \sum_{0\leqslant n} \frac{1}{n!} \cdot \Phi^{(n)} g \cdot \delta^n.$$

If a couple (f, ε), satisfies the hypotheses of condition **Tf2**, then we call

$$\mathcal{R}_\Phi(f,\varepsilon) := \sum_{1\leqslant i} \frac{1}{i!} \Phi^{(i)} f \cdot \varepsilon^i$$

the **restricted Taylor series** of Φ in (f, ε). The series $\mathcal{T}_\Phi(f,\varepsilon) := \Phi f + \mathcal{R}_\Phi(f,\varepsilon)$ is called the **Taylor series** of Φ in (f, ε).

Example 6.2.2 Let $\alpha \geqslant 0$ be an ordinal number. Fix a transseries field $\mathbb{T}$. Then $\mathbb{T} \subseteq \mathbb{T}_\alpha$. For a series $\Phi(x) \in \mathbb{L}_\alpha$ we let

$$\mathcal{F}_\Phi := \{\Phi, \Phi', \Phi'', \dots\}$$

with the derivation in $\mathbb{L}_\alpha$, which exists by Theorem 4.4.2. Recall that then for all $f \in \mathbb{T}^+_\infty$ we have $f \in \operatorname{dom} \Phi^{(n)}$. This is by Theorem 5.3.2. In other words, the set $\mathcal{F}_\Phi$ satisfies **Tf1**. Theorem 6.1.4 shows that **Tf2** holds as well.

6.2.2 Saturated Taylor families

A Taylor family $\mathcal{F}$ is said to be **saturated** iff for all $\Phi \in \mathcal{F}$ and all $f, \varepsilon \in \mathbb{T}$ such that $f \in \operatorname{dom} \Phi$ and such that

$$\left(\frac{1}{n!} \cdot \Phi^{(n)} f \cdot \varepsilon^n \right)_{0\leqslant n}$$

is a Noetherian family, we have $f + \varepsilon \in \operatorname{dom} \Phi$. Note that the family $\mathcal{F}_\Phi$ from Example 6.2.2 is saturated.

Our first aim will be to show that every Taylor family can be extended to a saturated family. This of course means that the domains of the functions $\Phi \in \mathcal{F}$ will be extended. The proof of this property requires some preliminary steps.

Let $\Phi \in \mathcal{F}$ and $f, \varepsilon \in \mathbb{T}$ be such that $(\frac{1}{n!}\Phi^{(n)} f \cdot \varepsilon^n)_{0\leqslant n}$ is a Noetherian family. Then Corollary 1.5.8 implies that for every $\delta \preccurlyeq \varepsilon$ in $\mathbb{T}$, the sequence

$$\left(\frac{1}{n!} \cdot \Phi^{(n)} f \cdot \delta^n \right)_{0\leqslant n}$$

is a Noetherian family.

Lemma 6.2.3 *Let $\mathcal{F}$ be a Taylor family and $\Phi \in \mathcal{F}$. Suppose that $f \in \mathbb{T}$ and that $(\varepsilon_i)_{i\in I}$ is a Noetherian family in $\mathbb{T}$ such that f and $f + \sum_I \varepsilon_i$ are in the domain of Φ. Furthermore, suppose that for all $i \in I$ the sequence $(\frac{1}{n!} \cdot \Phi^{(n)} f \cdot \varepsilon_i^n)_{0\leqslant n}$ is a Noetherian family. Then*

$$\left(\frac{1}{n!} \cdot \Phi^{(n)} f \cdot \varepsilon_{i_1} \cdots \varepsilon_{i_n}\right)_{(i_1,\dots,i_n)\in I^\star}$$

is a Noetherian family and

$$\Phi(f + \textstyle\sum_I \varepsilon_i) = \displaystyle\sum_{(i_1,\dots,i_n)\in I^\star} \frac{1}{n!} \cdot \Phi^{(n)} f \cdot \varepsilon_{i_1} \cdots \varepsilon_{i_n}.$$

Proof: Let $j \in I$ be such that $\varepsilon_j \succcurlyeq \sum_I \varepsilon_i$. Then we let $g = \sum_I \varepsilon_i$ and apply the above observation to $(\frac{1}{n!} \cdot \Phi^{(n)} f \cdot \varepsilon_j^n)_{0\leqslant n}$ and g. Then the sequence $(\frac{1}{n!} \cdot \Phi^{(n)} f \cdot g^n)_{0\leqslant n}$ is a Noetherian family. Then by Remark 1.5.4, the set

$$\{(n, \mathfrak{m}\,\mathfrak{n}_1 \cdots \mathfrak{n}_n) \mid \mathfrak{m} \in \operatorname{supp} \Phi^{(n)} f \ \wedge\ \forall i \leqslant n : \mathfrak{n}_i \in \operatorname{supp} \sum_I \varepsilon_i\}$$

is Noetherian for the ordering $(i, \mathfrak{a}) \succ (j, \mathfrak{b}) \Leftrightarrow \mathfrak{a} \succ \mathfrak{b}$. Since

$$\bigcup_{n\in\mathbb{N}} \{n\} \times \operatorname{supp} \Phi^{(n)} f \cdot \varepsilon_{i_1} \cdots \varepsilon_{i_n}$$

is contained in this set, we obtain the first part of the lemma.

As for the second assertion, we remark that by generalized associativity for Noetherian families, we have

$$\sum_{(i_1,\dots,i_n)\in I^\star} \frac{1}{n!} \cdot \Phi^{(n)} f \cdot \varepsilon_{i_1} \cdots \varepsilon_{i_n} = \sum_{0\leqslant n} \frac{1}{n!} \cdot \Phi^{(n)} f \cdot \left(\sum_{(i_1,\dots,i_n)\in I^\star} \varepsilon_{i_1} \cdots \varepsilon_{i_n}\right).$$

Since for Noetherian families F, G we have $\sum F\,G = (\sum F)(\sum G)$, we obtain

$$\sum_{(i_1,\dots,i_n)\in I^\star} \frac{1}{n!} \cdot \Phi^{(n)} f \cdot \varepsilon_{i_1} \cdots \varepsilon_{i_n} = \sum_{0\leqslant n} \frac{1}{n!} \cdot \Phi^{(n)} f \cdot \left(\sum_{i\in I} \varepsilon_i\right)^n = \Phi(f + \textstyle\sum_I \varepsilon_i).$$

This finishes the proof. □

Proposition 6.2.4 *Every Taylor family $(\mathcal{F},')$ can be extended to a (minimal) saturated Taylor family $\mathcal{F}_{\mathrm{hor}}$, called the* **horizontal closure** *of $\mathcal{F}$.*

Proof: For $\Phi \in \mathcal{F}$ we let

$$\begin{aligned} X_\Phi &:= \{(f, \varepsilon) \mid f \in \operatorname{dom}_{\mathcal{F}} \Phi \ \wedge\ (\Phi^{(n)} f \cdot \varepsilon^n)_{0\leqslant n} \quad \text{is a Noetherian family}\}, \\ Y_\Phi &:= \{f + \varepsilon \mid (f, \varepsilon) \in X_\Phi\} \supseteq \operatorname{dom} \Phi_{\mathcal{F}}. \end{aligned}$$

if $(f,\varepsilon) \in X_\Phi$, then $(f,\varepsilon) \in X_{\Phi'}$. Hence $Y_\Phi \subseteq Y_{\Phi'}$. We extend every $\Phi \in \mathcal{F}$ to Y_Φ by

$$\Phi(f+\varepsilon) := \sum_{0 \leqslant n} \frac{1}{n!} \cdot \Phi^{(n)} f \cdot \varepsilon^n.$$

Let us show that this definition indeed only depends on the sum $f+\varepsilon$ and not on the choice of $(f,\varepsilon) \in X_\Phi$. Let $(f,\varepsilon),(g,\delta) \in X_\Phi$ with $f+\varepsilon = g+\delta$. We have $\delta \preccurlyeq \varepsilon$ or $\varepsilon \preccurlyeq \delta$. We will assume that $\delta \preccurlyeq \varepsilon$; the othercase is treated similarly.

The couple $(\varepsilon-\delta,\delta)$ is a Noetherian family. From $\varepsilon - \delta \preccurlyeq \varepsilon$ and Lemma 1.5.8 it follows that the sequence $(\Phi^{(n)} f \cdot (\varepsilon-\delta)^n)_{0 \leqslant n}$ is a Noetherian family. Then so is $(\Phi^{(n+i)} f \cdot (\varepsilon-\delta)^n)_{0 \leqslant n}$ for all $0 \leqslant i$. Since $f+\varepsilon-\delta = g \in \operatorname{dom}\Phi$ in $\mathcal{F}$, it follows from **Tf2** for $\mathcal{F}$ that

$$\Phi^{(i)} g = \sum_{0 \leqslant n} \frac{1}{n!} \cdot \Phi^{(n+i)} f \cdot (\varepsilon-\delta)^n. \tag{6.2}$$

Moreover, Lemma 6.2.3 implies that the sequence $(\frac{1}{n!m!} \cdot \Phi^{(m+n)} f \cdot (\varepsilon-\delta)^m \cdot \delta^n)_{0 \leqslant n,m}$ is Noetherian. By generalized associativity for Noetherian families

$$\sum_{0 \leqslant n} \frac{1}{n!m!} \cdot \Phi^{(m+n)} f \cdot (\varepsilon+\delta)^m \cdot \delta^n = \sum_{0 \leqslant s} \frac{1}{s!} \cdot \Phi^{(s)} f \cdot \sum_{n=0}^{s} \binom{s}{n} (\varepsilon-\delta)^{s-n}\, \delta^n = \sum_{0 \leqslant s} \frac{1}{s!} \cdot \Phi^{(s)} f \cdot \varepsilon^s. \tag{6.3}$$

By generalized associativity and (6.2) we have

$$\sum_{0 \leqslant n} \frac{1}{n!m!} \cdot \Phi^{(m+n)} f \cdot (\varepsilon+\delta)^m \cdot \delta^n = \sum_{0 \leqslant n} \frac{1}{n!} \cdot \sum_{0 \leqslant m} \frac{1}{m!} \cdot \Phi^{(m+n)} f \cdot (\varepsilon+\delta)^m \cdot \delta^n = \sum_{0 \leqslant n} \frac{1}{n!} \cdot \Phi^{(n)} g \cdot \delta^n. \tag{6.4}$$

Equations (6.3) and (6.4) show that $\Phi(f+\varepsilon)$ is well-defined in $\tilde{\mathcal{F}}$. We have to show **Tf1** and **Tf2**. Condition **Tf1** follows from $Y_\Phi \subseteq Y_{\Phi'}$. Now, let $f \in \operatorname{dom}\Phi_{\tilde{\mathcal{F}}} = Y_\Phi$ and $\varepsilon \in \mathbb{T}$ such that

$$(\frac{1}{n!} \cdot \Phi^{(n)} f \cdot \varepsilon^n)_{0 \leqslant n}$$

is a Noetherian family. We first claim that $f+\varepsilon \in Y_\Phi$. Let $h,\rho \in \mathbb{T}$ with $h \in \operatorname{dom}\Phi_{\mathcal{F}}$ such that

$$(\frac{1}{m!} \cdot \Phi^{(m)} h \cdot \rho^m)_{0 \leqslant m}$$

and $f = h+\rho$. Then for all n the family $(\frac{1}{m!} \cdot \Phi^{(m+n)} h \cdot \rho^m)_{0 \leqslant m}$ is Noetherian, and we have

$$\Phi^{(n)} f = \sum_{0 \leqslant m} \frac{1}{m!} \cdot \Phi^{(m+n)} h \cdot \rho^m.$$

By the above, this definition is correct. Then $(\frac{1}{m!\,n!} \cdot \Phi^{(n+m)} h \cdot \rho^n \cdot \varepsilon^m)_{0 \leqslant m,n}$ is a Noetherian family. But then so is by generalized associativity the family

$$(\frac{1}{n!} \cdot \varepsilon^n \sum_{0 \leqslant m} \frac{1}{m!} \cdot \Phi^{(n+m)} h \cdot \rho^m)_{0 \leqslant n} = (\frac{1}{n!} \cdot \Phi^{(n)} f \cdot \varepsilon^n)_{0 \leqslant n}.$$

This shows that $f+\varepsilon \in Y_\Phi$. But then from the above it follows that $\Phi(f+\varepsilon)=\Phi(g+\delta)$ for all $(g,\delta)\in X_\Phi$ with $g+\delta=f+\varepsilon$. This shows **Tf2**. □

Remark 6.2.5 Also, it should be noticed that Lemma 6.2.3 allows a Taylor development in the widest possible way. In fact, if $f+\varepsilon$ can be developed into the series $\sum_{0\leqslant n}\frac{1}{n!}\cdot\Phi^{(n)}f\cdot\varepsilon^n$, then we can actually take the Noetherian family $(\varepsilon_i)_{i\in I} := \operatorname{term}\varepsilon$. In particular, we are not confined to finite decompositions of ε.

6.3 Taylor series expansions of iterators

6.3.1 Stirling polynomials

We are interested in constructing transseries fields admitting super-logarithmic functions, that is, functions L such that the functional equation $Lf = L\log f + 1$ is satisfied whenever both sides are defined. More generally, we are interested in solving functional equations of the form

$$\Phi f = \Phi\phi f + 1. \tag{6.5}$$

Remark 6.3.1 We use Taylor families for constructing such functions. Let $\mathcal{F}$ be a Taylor family and $\phi,\Phi\in\mathcal{F}$ such that

$$\forall f\in\operatorname{dom}\phi:\quad \phi f\in\operatorname{dom}\Phi \quad\Rightarrow\quad f\in\operatorname{dom}\Phi\ \wedge\ \Phi f=\Phi\phi f+1.$$

The right-hand side will be used in order to extend Φ to all series f such that $\phi f\in\operatorname{dom}\Phi$. Similarly, if $f\in\operatorname{dom}\phi$ is such that $f\in\operatorname{dom}\Phi$, then we extend Φ to ϕf by letting $\Phi\phi f=\Phi f-1$. However, we will have to extend the domains of $\Phi',\Phi'',\ldots$ as well if f resp. ϕf is not in their domains yet.

On the other hand, the function $'$ will in all our applications be a derivation on some transseries field. One effect is that equation (6.5) determines already $\Phi' f,\Phi'' f,\ldots$. Indeed, using the chain rule for derivations, we obtain

$$\Phi' f = \Phi'\phi f\cdot\phi' f. \tag{6.6}$$

Similarly, applying a derivation and the chain rule again, equation (6.6) leads to

$$\begin{aligned}
\Phi'' f &= \Phi''\phi f\cdot(\phi' f)^2+\Phi'\phi f\cdot\phi'' f, && (6.7)\\
\Phi''' f &= \Phi'''\phi f\cdot(\phi' f)^3+\Phi''\phi f\cdot 3\phi' f\cdot\phi'' f+\Phi'\phi f\cdot\phi''' f, && (6.8)\\
& && (6.9)
\end{aligned}$$

In other terms, we have a dependence between $\Phi^{(n)}f$ on the one side, and the terms $\Phi^{(i)}\phi f$ and $\phi^{(i)}f$ (where $i\leqslant n$) on the other side. We formalize this connection in the following.

Notation 6.3.2 We denote by $\bar{1}$ and $\bar{0}$ the tuples $(1,\ldots,1)$ and $(0,\ldots,0)$ respectively. Recall that addition and subtraction between tuples is pointwise. If $k\in\mathbb{N}^n$, then $X^k = X_1^{k_1}\cdots X_n^{k_n}$. Recall that for integers i,n we denote the set $\{k\in(\mathbb{N}^+)^i | k_1+\cdots+k_i=n\}$ by $T^*(i,n)$. To keep the subscripts of some sums short, we sometimes will write $\sum_{T^*(i,n)} Y_L$ instead of $\sum_{L\in T^*(i,n)} Y_L$. Since in general the summation can only be over one index, this should not lead to any confusion.

Let $X_1, X_2, \dots$ be indeterminates. The formal derivation with respect to X_i will be denoted by $\frac{\partial}{\partial X_i}$, i.e. $\frac{\partial}{\partial X_i} X_j$ is 1 if $i = j$, and 0 otherwise. We construct polynomials $S_{n,i}$ in n indeterminates $X_1, \dots, X_n$ with coefficients in $\mathbb{N}$ for all $n, i \in \mathbb{N}$ as follows. If $i = 0$ or $i > n$, then we let $S_{n,i} := 0$. For $n = 1$ we let

$$S_{1,1}[X_1] := X_1.$$

For all $k \in \mathbb{N}^n$ we denote the coefficient of X^k in the series $S_{n,i}$ by $c^k_{n,i}$, i.e. for all $i, n \in \mathbb{N}$ we have

$$S_{n,i}[X] = \sum_{k \in \mathbb{N}^n} c^k_{n,i} X^k.$$

Then for all $i, n \in \mathbb{N}$ we recursively define $S_{n,i}$ by

$$S_{n+1,i} := S_{n,i-1} \cdot X_1 + \sum_{j=1}^{n} \frac{\partial S_{n,i}}{\partial X_j} \cdot X_{j+1}.$$

One shows recursively that all $S_{n,i}$ are polynomials. They generalize Stirling numbers. For that reason, we call them **Stirling polynomials**.

Example 6.3.3 One verifies that $S_{1,1} = X_1$, $S_{2,2} = X_1^2$ and $S_{3,3} = X_1^3$. Also we have $S_{2,1} = X_2$, $S_{3,1} = X_3$ and $S_{3,2} = 3X_1X_2$.

Using Stirling polynomials, we can express equations (6.6), (6.6) and (6.8) by

$$\begin{aligned}
\Phi' f &= \Phi'\phi f \cdot S_{1,1}[\phi' f], \\
\Phi'' f &= \Phi'\phi f \cdot S_{2,1}[\phi' f, \phi'' f] + \Phi''\phi f \cdot S_{2,2}[\phi' f, \phi'' f], \\
\Phi'' f &= \Phi'\phi f \cdot S_{3,1}[\phi' f, \phi'' f, \phi''' f] + \Phi''\phi f \cdot S_{3,2}[\phi' f, \phi'' f, \phi''' f] + \Phi'''\phi f \cdot S_{3,3}[\phi' f, \phi'' f, \phi''' f].
\end{aligned}$$

Lemma 6.3.4 *For all $n, i \in \mathbb{N}$, the series $S_{n,i}[X]$ are polynomials in n indeterminates over $\mathbb{N}$ such that for all $n \geqslant 1$*

(1) $S_{n,1}[X] = X_n$ and $S_{n,n}[X] = X_1^n$,

(2) for all $1 \leqslant i \leqslant n$: if $c^k_{n,i} \neq 0$, then $\sum_{j=1}^{n} j \cdot k_j = n$ and $\sum_{j=1}^{n} k_j = i$,

(3) for all $1 \leqslant i \leqslant n$:

$$S_{n,i}[X] = \frac{n!}{i!} \sum_{L \in T^*(i,n)} \frac{X_L}{L!}.$$

Proof: The properties (1) and (2) follow easily from the recursive definition. Let us show (3) in detail. We first remark that for all $n \geqslant 1$ we have $T^*(1, n) = \{n\}$ and $T^*(n, n) = \{\bar{1} \in \mathbb{N}^n\}$, hence that $X_n = S_{n,1}$ and $X_1^n = S_{n,n}$. Suppose that (3) holds for all integers $\leqslant n$. Let $1 < i < n + 1$. From the recursive condition one infers that

$$S_{n+1,i} = X_1 \cdot S_{n,i-1} + \sum_{k \in \mathbb{N}^n} \sum_{j=1}^{n} k_j \cdot c^k_{n,i} \cdot X^k \cdot \frac{X_{j+1}}{X_j}.$$

From $n+1 = L_1 + \cdots + L_i$ we obtain

$$\frac{(n+1)!}{i!} \sum_{T^*(i,n+1)} \frac{X_L}{L!} = \frac{n!}{i!} \sum_{T^*(i,n+1)} \frac{(n+1)X_L}{L!} = \frac{n!}{i!} \sum_{j=1}^{i} \sum_{T^*(i,n+1)} \frac{L_j X_L}{L!}.$$

From the hypothesis about $S_{n,i-1}$ and

$$\sum_{j=1}^{i} \sum_{\substack{T^*(i,n+1):\\ L_j=1}} \frac{L_j X_L}{L!} = X_1 \cdot i \sum_{T^*(i-1,n)} \frac{X_L}{L!}$$

we then infer

$$\frac{(n+1)!}{i!} \sum_{T^*(i,n+1)} \frac{X_L}{L!} = X_1 \cdot S_{n,i-1} + \frac{n!}{i!} \sum_{j=1}^{i} \sum_{\substack{T^*(i,n+1):\\ L_j>1}} \frac{L_j X_L}{L!}.$$

One verifies that

$$\sum_{j=1}^{i} \sum_{\substack{T^*(i,n+1):\\ L_j>1}} \frac{L_j X_L}{L!} = \sum_{T^*(i,n)} \frac{X_L}{L!} \left(\frac{X_{L_1+1}}{X_{L_1}} + \cdots \frac{X_{L_i+1}}{X_{L_i}} \right).$$

Fix $L \in T^*(i,n)$. For $1 \leqslant j \leqslant n$ we let $k_j^L = |\{m \leqslant i | L_m = j\}|$. Then $X_L = X^{k(L)}$ where $k(L) = (k_1^L, \ldots, k_n^L)$. From the hypothesis about $S_{n,i}$ we obtain

$$\sum_{T^*(i,n)} \frac{n!}{i!L!} X_L = \sum_{k \in \mathbb{N}^n} c_{n,i}^k X^k \quad \text{and thus} \quad c_{n,i}^k = \sum_{\substack{T^*(i,n):\\ k(L)=k}} \frac{n!}{i!L!}.$$

This implies

$$\sum_{T^*(i,n)} \sum_{j=1}^{i} \frac{n!}{i!L!} X_L \cdot \frac{X_{L_j+1}}{X_{L_j}} = \sum_{k \in \mathbb{N}^n} \sum_{j=1}^{n} c_{n,i}^k X^k \cdot k_j \frac{X_{j+1}}{X_j}.$$

Then we obtain

$$\frac{(n+1)!}{i!} \sum_{T^*(i,n+1)} \frac{X_L}{L!} = X_1 \cdot S_{n,i-1} + \sum_{k \in \mathbb{N}^n} \sum_{j=1}^{n} k_j c_{n,i}^k X^k \cdot \frac{X_{j+1}}{X_j} = S_{n+1,i}.$$

This finishes the proof. □

Remark 6.3.5 An alternative route to show (3) of Lemma 6.3.4 uses the fact that for every $k \in \mathbb{N}^n$, the coefficient $c_{n,i}^k$ equals the number of possibilities to partition a set of n points into k_1 sets of size 1, k_2 sets of size 2 and so forth. A similar inductive argument can then be used.

6.3.2 Vertical extensions of Taylor families

Let $\mathcal{F}$ be a Taylor family and $\phi, \Phi \in \mathcal{F}$. Suppose that $\mathcal{F}$ is horizontally closed for ϕ. Then Φ is an **infinite iterator** of ϕ if for all $f \in \operatorname{dom}\Phi$ we have $f \in \operatorname{dom}\phi$ and $\phi f \in \operatorname{dom}\Phi$ and

$$\Phi f = \Phi\phi f + 1 \tag{6.10}$$

$$\Phi^{(N)} f = \sum_{j=1}^{N} \Phi^{(j)}(\phi f) \cdot S_{N,j}[\phi' f, \ldots, \phi^{(N)} f]. \tag{6.11}$$

A Taylor family is **vertically closed** for the infinite iterator Φ of $\phi \in \mathcal{F}$ iff for all $f \in \operatorname{dom}\phi$ we have $f \in \operatorname{dom}\Phi \Leftrightarrow \phi f \in \operatorname{dom}\Phi$. The family $\mathcal{F}$ is vertically closed iff it is vertically closed for all infinite iterators $\Phi \in \mathcal{F}$ of operators $\phi \in \mathcal{F}$.

Proposition 6.3.6 *If $\phi, \Phi \in \mathcal{F}$ and Φ is an infinite iterator of ϕ, then the same holds in $\mathcal{F}_{\text{hor}}$.*

Proof: Denote the extension of Φ to $\mathcal{F}_{\text{hor}}$ by $\tilde{\Phi}$. Let $f \in \operatorname{dom}\tilde{\Phi}$. We have to show three points, namely that $f \in \operatorname{dom}\phi$, that $\phi f \in \operatorname{dom}\tilde{\Phi}$ and that the equations (6.10) and (6.11) hold.

In order to show that $f \in \operatorname{dom}\tilde{\Phi}$, let $f = h+\delta$ such that $h \in \operatorname{dom}\Phi$, such that $(\Phi^{(N)} h \cdot \delta^N)_{0 \leqslant N}$ is a Noetherian family and such that

$$\tilde{\Phi} f = \sum_{0 \leqslant N} \frac{1}{N!} \Phi^{(N)} h \cdot \delta^N.$$

From $h \in \operatorname{dom}\Phi$ it follows that $h \in \operatorname{dom}\phi, \phi', \ldots$. We claim that $(\phi^{(N)} h \cdot \delta^N)_{0 \leqslant N}$ is a Noetherian family. The series $\Phi'(\phi h) \cdot \phi^{(N)} h \cdot \delta^N$ is a component of the N-th element of the sequence $(\Phi^{(N)} h \cdot \delta^N)_{0 \leqslant N}$. To see this recall that

$$\Phi^{(N)} h = \sum_{j=1}^{N} \Phi^{(j)}(\phi h) \cdot S_{N,j}[\phi' h, \ldots, \phi^{(N)} h].$$

Hence the sequence $(\Phi'(\phi h) \cdot \phi^{(N)} h \cdot \delta^{)}_{0 \leqslant N}$ is Noetherian. But then so is $(\phi^{(N)} h \cdot \delta^{)}_{0 \leqslant N}$. Thus $h + \delta \in \operatorname{dom}\phi$, which shows our claim.

In order to show that $\phi f \in \operatorname{dom}\tilde{\Phi}$, we let h, δ as above. Then $\phi f = \phi h + \rho$, where $\rho = \phi' h \cdot \frac{1}{2!}\phi'' h \cdot \delta^2 + \cdots$. From $h \in \operatorname{dom}\Phi$, it follows that $\phi \in \operatorname{dom}\Phi, \Phi', \ldots$. We claim that $(\Phi^{(j)}\phi h \cdot \rho^j)_{0 \leqslant j}$ is a Noetherian family. Since $(\Phi^{(N)} h \cdot \delta^N)_{0 \leqslant N}$ is a Noetherian family, so is

$$(\sum_{j=1}^{N} \Phi^{(j)}(\phi h) \cdot S_{N,j}[\phi' h, \ldots, \phi^{(N)} h] \cdot \delta^N)_{0 \leqslant N}$$

For every $j \geqslant 1$ we let

$$F_j := (\Phi^{(j)}(\phi h) \cdot S_{N,j}[\phi' h, \ldots, \phi^{(N)} h] \cdot \delta^N)_{j \leqslant N}$$

Then every F_j is a Noetherian family and we have $\sum F_j = \Phi^{(j)}(\phi h)\cdot\rho^j$. The claim follows from the fact that $(\sum F_j)_{1\leqslant j}$ is Noetherian. On the other hand, we have

$$\sum_{1\leqslant j}\frac{1}{j!}\cdot\sum F_j = \sum_{1\leqslant j}\frac{1}{j!}\cdot\Phi^{(j)}(\phi h)\cdot\rho^j.$$

From $\Phi(\phi h) = \Phi h - 1$ it then follows that $\tilde{\Phi}(\phi f) = \tilde{\Phi} f - 1$, from which equation (6.10) follows. The equations (6.11) follow from a similar argument. □

A Taylor family is not necessarily vertically closed. For $f \in \operatorname{dom}\phi$ one can have $f \notin \operatorname{dom}\Phi \ \wedge\ \phi f \in \operatorname{dom}\Phi$. We add ϕf resp. f to the domains of $\Phi, \Phi', \Phi'', \ldots$ via (6.10) and (6.11). Since $\phi f \in \operatorname{dom}\Phi$, all $\Phi^{(N)} f$ are defined in $\mathbb{T}$, since all terms on the right-hand side of (6.10) and (6.11) exist. We extend $\mathcal{F}$ by all $f \in \operatorname{dom}\phi$ with one of the above properties. Let

$$Y \quad := \quad \{f \in \operatorname{dom}\phi \mid f \notin \operatorname{dom}\Phi \ \wedge\ \phi f \in \operatorname{dom}\Phi\},$$

and we extend $\mathcal{F}$ to $\tilde{\mathcal{F}}$ by adding X and Y to $\operatorname{dom}\Phi, \Phi', \Phi'', \ldots$ using the equations (6.10) and (6.11). We have to show that $\tilde{\mathcal{F}}$ is again a Taylor family.

Lemma 6.3.7 *Suppose $\mathcal{F} = \mathcal{F}_{\mathrm{hor}}$. Then $\tilde{\mathcal{F}}$ is a Taylor family, and Φ is an infinite iterator of ϕ in $\tilde{\mathcal{F}}$.*

Proof: Let $f \in \operatorname{dom}\phi$. Since for all $N \geqslant 0$ we have

$$\operatorname{dom}_{\tilde{\mathcal{F}}}\Phi^{(N)} = \operatorname{dom}_{\mathcal{F}}\Phi^{(N)} \ \cup\ Y,$$

the condition **Tf1** holds in $\tilde{\mathcal{F}}$. Next, let $f \in \operatorname{dom}_{\tilde{\mathcal{F}}}\Phi$. If $f \in \operatorname{dom}_{\mathcal{F}}\Phi$, then **Tf2** follows from the same condition in $\mathcal{F}$. We may thus suppose that $f \in X$ or $f \in Y$. Let $\varepsilon \in \mathbb{T}$ be such that $f + \varepsilon \in \operatorname{dom}_{\tilde{\mathcal{F}}}\Phi$ and such that

$$(\Phi^{(n)} f\cdot\varepsilon^n)_{0\leqslant n}$$

is a Noetherian family. If $f \in X$, then $f \in \operatorname{dom}_{\mathcal{F}}\Phi, \Phi', \ldots$. We are done by **Tf2**, since $\mathcal{F} = \mathcal{F}_{\mathrm{hor}}$.

If $f \in Y$, then it follows from the definition of $\Phi^{(n)} f$ that

$$\begin{aligned}\sum_{0\leqslant n}\frac{1}{n!}\cdot\Phi^{(n)} f\cdot\varepsilon^n &= \Phi f + \sum_{1\leqslant n}\frac{1}{n!}\cdot\Phi^{(n)} f\cdot\varepsilon^n\\ &= 1 + \Phi\phi f + \sum_{1\leqslant n}\frac{1}{n!}\varepsilon^n\cdot\sum_{i=1}^{n}\Phi^{(i)}(\phi f)\cdot S_{n,i}[\phi' f, \ldots, \phi^{(n)} f]\end{aligned}$$

From Lemma 6.3.4 it then follows that

$$\begin{aligned}\sum_{0\leqslant n}\frac{1}{n!}\cdot\Phi^{(n)} f\cdot\varepsilon^n &= 1 + \Phi\phi f + \sum_{1\leqslant n}\frac{1}{n!}\varepsilon^n\cdot\sum_{i=1}^{n}\Phi^{(i)}(\phi f)\cdot\frac{n!}{i!}\sum_{L\in T^*(i,n)}\frac{1}{L!}\phi^{(L)} f\\ &= 1 + \Phi\phi f + \sum_{1\leqslant i}\frac{1}{i!}\Phi^{(i)}(\phi f)\cdot\left(\sum_{1\leqslant k}\frac{1}{k!}\phi^{(k)} f\cdot\varepsilon^k\right)^i\\ &= 1 + \Phi(\phi f + \sum_{1\leqslant k}\frac{1}{k!}\phi^{(k)} f\cdot\varepsilon^k).\end{aligned}$$

This shows **Tf2** for $\tilde{\mathcal{F}}$ and $\Phi(f+\varepsilon) = \Phi\phi(f+\varepsilon)+1$. The fact that Φ is an infinite iterator of ϕ on $\tilde{\mathcal{F}}$ as well follows with the same argument. $\square$

Proposition 6.3.8 *Every Taylor family $(\mathcal{F},')$ can be extended to a (minimal) vertically closed Taylor family $\mathcal{F}_{\mathrm{ver}}$.*

Proof: If $\mathcal{F}$ is not vertically closed, then extend one of the domains of Φ as in Lemma 6.3.7. The result is a Taylor family, and we can apply Proposition 6.3.8. We thus obtain a saturated Taylor family, where the domains of the functions $\Phi, \Phi', \ldots$ contain all series which could be obtained by applying the functional equations.

Since $\mathbb{T}$ remains unchanged throughout the extension process, this process will lead to a horizontally and vertically closed Taylor family. $\square$

Remark 6.3.9 We call $\mathcal{F}_{\mathrm{ver}}$ from Prposition 6.2.4 the **vertical closure**. Alternating horizontal and vertical closures, Zorn's lemma implies the existence of infinite iterator functions which are both horizontally and vertically closed.

6.3.3 Application to logarithmic functions

Let us look at the described closures in an example. Let $\mathbb{T} \supseteq \mathbb{L}$ be a transseries field containing x and at least a partially defined infinite iterator Φ of log.

We first remark that $x, \log x, \log_2 x, \ldots$ are all elements of $\mathbb{T}$. We define first $\Phi', \Phi'', \ldots$.

Remark 6.3.10 Since Φ is the infinite iterator of log, we have

$$\Phi' x = \frac{1}{x} \cdot \Phi' \log x = \frac{1}{x \log x} \cdot \Phi' \log_2 x = \cdots .$$

We will thus let

$$\Phi' x = \frac{1}{x \log x \log_2 x \cdots}.$$

Then $\Phi' x \in \mathbb{L}_{\exp}$. Since we have a derivation on $\mathbb{L}_{\exp}$, we obtain the functions $\Phi'', \Phi''', \ldots$ recursively by applying equation (6.11). All of them are again elements of $\mathbb{L}_{\exp}$. We have in this case $\Phi^{(N)} : \mathbb{L} \to \mathbb{L}_{\exp}$ for all $n \geqslant 1$.

Note that $(\{\Phi', \Phi'', \ldots\}, \partial_{\mathbb{L}})$ is a Taylor family. We extend the field $\mathbb{L}$ by monomials Lx, $L_2 x$, $L_3 x, \ldots$ where we demand

$$\begin{aligned} Lx &= \Phi x \\ L_2 x &= \Phi L x \\ L_3 x &= \Phi L_2 x \\ &\vdots \end{aligned}$$

Hence from now on, we will denote Φ by L, too. Then the family $(\{L, L', L'', \dots\}, \partial_{\mathbb{L}})$ is again a Taylor family. Now assume that for some i, the monomial $\log_i x$ is in the domain of L. If $i > 0$ and $\log_{i-1} x$ is not in the domain of L, then we can apply the vertical extension step and let

$$L \log_{i-1} x := L \log_i x + 1.$$

This way we can extend L to all monomials $x, \dots, \log_i x$. If on the other hand $j > i$ and $\log_j x \notin \operatorname{dom} L$, then we have in the vertical closure

$$L \log_j x = L \log_i x - (j - i) = Lx - j.$$

The action of the horizontal closure is similar. Suppose that ε is a series such that $(Lx, L'x \cdot \varepsilon, L''x \cdot \varepsilon^2, \dots)$ is a Noetherian family. Then we have in the horizontal closure

$$L(x + \varepsilon) = Lx + \frac{1}{1!} \cdot L'x \cdot \varepsilon + \frac{1}{2!} \cdot L''x \cdot \varepsilon^2 + \cdots .$$

We can thus define L for certain $x + \varepsilon$. The fact that the horizontal closure is saturated tells us now that if we can splitt ε into a sum $\varepsilon = \varepsilon_1 + \varepsilon_2$ such that $(L^{(n)} x \cdot \varepsilon_1^n)_{0 \leqslant n}$ and $(L^{(m)}(x + \varepsilon_1) \cdot \varepsilon_2^m)_{0 \leqslant m}$ are Noetherian families, then

$$L(x + \varepsilon) = \sum_{0 \leqslant m} \frac{1}{m!} L^{(m)}(x + \varepsilon_1) \cdot \varepsilon_2^m.$$

In other words, we do not obtain any incoherences from different possibilities of developing the series $L(x + \varepsilon)$. Similarly, the fact that the vertical closure is a Taylor family again allows us to extend L to series like $3x + \frac{1}{x}$. Indeed, we have

$$\begin{aligned} L(3x + \frac{1}{x}) &= L \log(3x + \frac{1}{x}) + 1 \\ &= L(\log x + \log(3 + \frac{1}{x^2})) \\ &= \sum_{0 \leqslant n} \frac{1}{n!} L^{(n)} \log x \cdot \log^n (3 + \frac{1}{x^2}). \end{aligned}$$

What is more, we might have applied log not just once, but as many times as we like to the series $3x + \frac{1}{x}$ and then have developed the resulting series. That would have yielded the same result.

6.4 Inverse functions

We will finish with some remarks on the inverse function of Φ, if it exists. In view of the applications we have in mind for the pair (ϕ, Φ), namely the pair $(\log, L)$, this provides information concerning the construction of extensions of a given field. In particular, we will use the following facts in the construction of super-exponential functions.

As before, we will work under some assumptions about the functions ϕ and Φ as well as about $\mathbb{T}$. Let ϕ be strictly monotone. This ensures that ϕ admits an inverse function on its image range ϕ. We remark that this assumption holds for the example $\phi = \log$.

Also, we assume that Φ is strictly monotone wherever it is defined. Following the notations for iterations of functions, the inverse functions are denoted by ϕ_{-1} and Φ_{-1} respectively. However, we also use ψ and Ψ. In other words, we have $\phi \circ \psi = \psi \circ \phi = \mathrm{id}$ and $\Phi \circ \Psi = \Psi \circ \Phi = \mathrm{id}$. Recall that for all $f \in \mathbb{T}$ with $f \in \operatorname{dom} \phi$ and $f, \phi f \in \operatorname{dom} \Phi$ we have $\Phi \phi f = \Phi f - 1$. It follows that ψ and Ψ satisfy a functional equation.

Lemma 6.4.1 *Let $\mathbb{T}$ be atransseries field and ψ and Ψ the inverse operators of ϕ and Φ respectively. Let $f \in \mathbb{T}$ be such that $f, f+1 \in \operatorname{dom} \Psi$ and $\Psi f \in \operatorname{dom} \psi$. Then $\Psi(f+1) = \psi \Psi f$.*

Proof: Let $y \in \mathbb{T}$ be such that $f+1 = \Phi y$. Then $\Psi(f+1) = y$. From $\Phi y - 1 = \Phi \phi y = f$ one obtains $\Psi f = \phi y$ and therefore $y = \psi \Psi f$. □

As for Φ, we can now introduce operators $\Psi^{(i)}$. Once we have fixed the operator Ψ', the choice of the operator $\Psi^{(i)}$ ($i \geqslant 2$) can be made in the same way as it was done for Φ. Since the series are thought to stand for derivatives, we use the equation $1 = \Phi' \Psi \cdot \Psi'$ to let

$$\Psi' f := \frac{1}{\Phi'(\Psi f)}$$

Note that is $f \in \operatorname{dom} \Psi$, then $\Psi f \in \operatorname{dom} \Phi$. Since $\Phi \in \mathcal{F}$ is an operator from a Taylor family, this implies $\Psi f \in \operatorname{dom} \Phi^{(n)}$ for all $n \geqslant 0$. In particular, this means that the right-hand side of the above definition of $\Psi' f$ is defined, and we have $\operatorname{dom} \Psi = \operatorname{dom} \Psi'$.

Example 6.4.2 Let E be a super-exponential function on $\mathbb{T}$, i.e. E satisfies the functional equation $\exp Ef = E(f+1)$. Then $E' f = Ef \cdot E(f-1) \cdot E(f-2) \cdots$.

Suppose that we have already defined the operators $\Psi', \ldots, \Psi^{(i)}$. Recall that for all $j \geqslant 2$ the polynomials $S_{i+1,j}[X]$ do not contain the indeterminate X_{i+1}. This makes the following definition possible: if $f \in \operatorname{dom} \Psi$, then

$$\Psi^{(i+1)} f := -\Psi' f \sum_{j=2}^{i+1} \Phi^{(j)}(\Psi f) \cdot S_{i+1,j}[\Psi' f, \ldots, \Psi^{(i+1)} f].$$

Again, we find $\operatorname{dom} \Psi^{i+1} = \operatorname{dom} \Psi$. Since all $S_{i,j}$ are polynomials, all series $\Psi^{(i)} f$ are elements of $\mathbb{T}$. As for Φ, it is now possible to define the restricted Taylor series.

Lemma 6.4.3 *$(\{\Psi, \Psi', \ldots\}, ')$ is a Taylor family.*

Proof: For the condition **Tf1** it suffices to remark that by the construction we have $\operatorname{dom} \Psi = \operatorname{dom} \Psi^n$. We have thus to show **Tf2**. The proof of this condition is similar to the proof of Lemma 6.3.7 by inserting the definitions of $\Psi^{(n)} f$ and invoking Lemma 6.3.4. □

The functional equation $\psi \Psi f = \Psi(f+1)$ is again a potential source of incoherences. However, as the next lemma shows, the conditions imposed on both Φ and Ψ prevent contradictions. Moreover, we show that Ψ is also the inverse operator in the horizontal closure of $\mathcal{F}$.

Lemma 6.4.4 *Let f, ε be series in $\mathbb{T}$ such that $f, f+1 \in \operatorname{dom}\Psi$ and such that both $(\Psi^{(n)} f \cdot \varepsilon^n)_{0 \leqslant n}$ and $(\Psi^{(n)}(f+1) \cdot \varepsilon^n)_{0 \leqslant n}$ are Noetherian families. Suppose that*

$$\sum \frac{1}{n!} \cdot \Psi^{(n)} f \cdot \varepsilon^n \in \operatorname{dom} \phi.$$

Then in the horizontal closure of $\mathcal{F}$ we have $\Phi\Psi = id$ and $\psi\Psi(f+\varepsilon) = \Psi(f+1+\varepsilon)$.

Proof: We first claim that for all $i \geqslant 2$:

$$0 = \sum_{j=1}^{i} \Phi^{(j)}(\Psi f) \cdot S_{i,j}[\Psi' f, \ldots, \Psi^{(i)} f].$$

From the definition of $\Psi' f$ we get $(\Psi' f)^{-1} \cdot \Psi^{(i)} f = \Phi'(\Psi f) \cdot \Psi' f$. From the definition of $\Psi^{(i)} f$ it now follows

$$-\Phi'(\Psi f) \cdot \Psi' f = \sum_{j=2}^{i+1} \Phi^{(j)}(\Psi f) \cdot S_{i,j}[\Psi' f, \ldots, \Psi^{(i+1)} f],$$

from which the claim follows. From the definition of Φ we obtain

$$\Phi(\Psi f + \mathcal{R}_\Psi(f,\varepsilon)) = \Phi\Psi f + \sum_{1 \leqslant i} \frac{1}{i!} \Phi^{(i)}(\Psi f) \cdot \left(\sum_{1 \leqslant k} \frac{1}{k!} \Psi^{(k)} f \cdot \varepsilon^k \right)^i,$$

hence

$$\Phi(\Psi f + \mathcal{R}_\Psi(f,\varepsilon)) = f + \sum_{1 \leqslant n} \frac{\varepsilon^n}{n!} \sum_{i=1}^{n} \Phi^{(i)} f \frac{n!}{i!} \sum_{K \in T^*(i,n)} \frac{\Psi^{(K)} f}{K!}.$$

We apply Lemma 6.3.4 and obtain

$$\Phi(\Psi f + \mathcal{R}_\Psi(f,\varepsilon)) = f + \sum_{1 \leqslant n} \frac{\varepsilon^n}{n!} \sum_{i=1}^{n} \Phi^{(i)} f \cdot S_{n,i}[\Psi' f, \ldots, \Psi^{(n)} f].$$

From $\Phi'(\Psi f) \cdot \Phi' f = 1$ and the above claim it now follows that $\Phi(\Psi f + \mathcal{R}_\Psi(f,\varepsilon)) = f + \varepsilon$. Since ψ is the inverse operator of ϕ, we have for all series $h \in \operatorname{dom}\psi$ with $\psi h \in \operatorname{dom}\Phi$ that $\Phi\psi h = 1 + \Phi h$. Let $h = \Psi(f+\varepsilon)$, then the second assertion follows from the first one. □

Chapter 7

Transseries fields of positive strength

7.1 Two aims of extending fields

Recall that one of our objectives is to construct fields $\mathcal{K}$ of generalized power series such that there are functions E and L with

- $E \circ L = L \circ E = \mathrm{id}$,
- $\mathcal{K}_\infty^+ \subseteq \operatorname{dom} E, \operatorname{dom} L$,

such that for all $f \in \mathcal{K}_\infty^+$ the functional equations

$$\begin{aligned} \exp E f &= E(f+1) \\ L \log f &= Lf - 1. \end{aligned}$$

hold. More generally, let us call $E = \mathrm{e}_\omega$ and $L = \mathrm{l}_\omega$, and let us suppose that this construction has been carried out already. That means we have solved the case $n = 0$ of the following generalization of the above: construct a field $\mathcal{K}$ of generalized power series such that there are functions $\mathrm{e}_{\omega^n}, \mathrm{e}_{\omega^{n+1}}, \mathrm{l}_{\omega^n}$ and $\mathrm{l}_{\omega^{n+1}}$ with

- $\mathrm{e}_{\omega^n} \circ \mathrm{l}_{\omega^n} = \mathrm{e}_{\omega^{n+1}} \circ \mathrm{l}_{\omega^{n+1}} = \mathrm{id}$,
- $\mathcal{K}_\infty^+ \subseteq \operatorname{dom} \mathrm{e}_{\omega^n}, \operatorname{dom} \mathrm{e}_{\omega^{n+1}}, \operatorname{dom} \mathrm{l}_{\omega^n}, \operatorname{dom} \mathrm{l}_{\omega^{n+1}}$,

such that for all $f \in \mathcal{K}_\infty^+$ the functional equations

$$\mathrm{e}_{\omega^n} \circ \mathrm{e}_{\omega^{n+1}} f = \mathrm{e}_{\omega^{n+1}}(f+1) \tag{7.1}$$
$$\mathrm{l}_{\omega^{n+1}} \circ \mathrm{l}_{\omega^n} f = \mathrm{l}_{\omega^{n+1}} f - 1. \tag{7.2}$$

hold.

Both topics are closely related. We will use the tools developed in Chapter 6 to tackle them. As we will see, many properties of the fields have their origin in the functional equations (7.1) and (7.2). We have therefore decided not to distinguish between the construction of the structures $\langle \mathcal{K}, E, L \rangle$ and $\langle \mathcal{K}, \mathrm{e}_{\omega^n}, \mathrm{l}_{\omega^n} \rangle$. Many of the necessary lemmas and properties are proved in the same way for the initial and general case (although sometimes the generalized version requires more care; but one can always simplify the generalized proof to the case $(\exp, E)$).

Before we go into the details, let us briefly sketch the structure of this chapter.

• In the case $(\exp, E)$, all properties and proofs can be given using these functions and their inverse functions $(\log, L)$. Since we want to treat the general case $(\exp, \ldots, \mathrm{e}_{\omega^n})$, this becomes difficult, if one wants to keep the proofs readable. We therefore introduce a new notation. This will be done in Section 7.2.

• Then we revise some properties of transseries fields and show that they provide the initial conditions for a notion of transseries fields of higher strength. Indeed, usual transseries fields will then be of strength 0. The definition of strength $n+1$ requires the definition of strength n. We show that the process of increasing the strength has a starting point. We will apply the new notations to the results of Chapter 6. Section 7.3 will cover this topic.

• Section 7.4 will provide general properties of transseries fields of strength n. Most of the properties in this section will be needed to extend a given field of strength n to a larger field of strength n.

• More properties of transseries fields of positive strength are shown in Section 7.5. This time, however, the focus of the properties is to provide tools that will help to go from strength n to strength $n+1$. Centrepiece of this section is a partial composition result similar to Proposition 5.1.5.

• Finally, we show in Section 7.6 the existence of fields of arbitrary positive strength, and we give a simple but useful application of the properties of transseries fields shown in this chapter.

7.2 Ordinal notations

We start with some recalls about ordinal numbers. Let in what follows $\alpha, \beta, \gamma, \ldots$ be ordinal numbers. We use λ to denote limit ordinals. The total ordering on the class of ordinal numbers is defined by $\alpha < \beta$ iff $\alpha \in \beta$. The smallest limit ordinal is denoted by ω. Let $+$ be the addition on the ordinals which is defined by

$$\begin{aligned} \alpha + 0 &:= \alpha, \\ \alpha + (\beta + 1) &:= (\alpha + \beta) + 1, \\ \alpha + \lambda &:= \bigcup_{\beta < \lambda} \alpha + \beta. \end{aligned}$$

Similarly, one defines a multiplication $\cdot$ on the class of ordinal numbers:

$$\begin{aligned} \alpha \cdot 1 &:= \alpha, \\ \alpha \cdot (\beta + 1) &:= \alpha \cdot \beta + \alpha, \\ \alpha \cdot \lambda &:= \bigcup_{\beta < \lambda} \alpha \cdot \beta. \end{aligned}$$

The addition and multiplication are not commutative. Standard examples are $1 + \omega < \omega + 1$ and $2 \cdot \omega < \omega \cdot 2$. Let $\omega^\omega = \bigcup_{\mathbb{N}} \omega^n$. A frequently used result about countable ordinals $< \omega^\omega$ is Cantor's theorem: let $\alpha < \omega^\omega$, then there are $n \in \mathbb{N}$ and $a_0, \ldots, a_n \in \mathbb{N}$ with $a_n \neq 0$ and

$$\alpha = \omega^n \cdot a_n + \cdots + \omega \cdot a_1 + a_0.$$

We remark that if $\alpha, \beta < \omega^\omega$ are such that

$$\begin{array}{rcll} \alpha & = & \omega^n \cdot a_n + \cdots + \omega^m \cdot a_m & (n \geqslant m) \quad \text{and} \\ \beta & = & \omega^m \cdot b_m + \cdots + \omega \cdot b_1 + b_0 & (b_m \geqslant 0) \end{array}$$

then

$$\alpha + \beta = \omega^n \cdot a_n + \cdots + \omega^m \cdot (a_m + b_m) + \omega^{m-1} \cdot b_{m-1} + \cdots + b_0.$$

Recall that our aim is to construct functions $\log, L, \mathcal{L}, \ldots$ and $\exp, E, \mathcal{E}, \ldots$ such that the functional equations

$$\begin{array}{rcrl} L \log x = Lx - 1 & \qquad & \exp Ex = E(x+1) \\ \mathcal{L}Lx = \mathcal{L}x - 1 & & E\mathcal{E}x = \mathcal{E}(x+1) \\ \vdots & & \vdots \end{array}$$

hold. Let ψ, ϕ, Ψ and Φ be functions with $\psi \circ \phi = \Psi \circ \Phi = \mathrm{id}$ and $\Phi\phi x = \Phi x - 1$, then we let $\psi_\omega := \Psi$ and $\phi_\omega = \Phi$. Hence with $1 = \omega^0$ we obtain

$$\begin{array}{rcr} \log = \mathrm{l}_1 & \qquad & \exp = \mathrm{e}_1 \\ L = \mathrm{l}_\omega & & E = \mathrm{e}_\omega \\ \mathcal{L} = \mathrm{l}_{\omega^2} & & \mathcal{E} = \mathrm{e}_{\omega^2} \\ \vdots & & \vdots \end{array}$$

Thus for all $n \geqslant 0$ we have

$$\begin{array}{rcl} \mathrm{l}_{\omega^{n+1}} \circ \mathrm{l}_{\omega^n} x & = & \mathrm{l}_{\omega^{n+1}} x - 1 \\ \mathrm{e}_{\omega^n} \circ \mathrm{e}_{\omega^{n+1}} x & = & \mathrm{e}_{\omega^{n+1}}(x+1). \end{array}$$

For countable ordinals $\alpha = \omega^n \cdot a_n + \cdots + a_0$ and for functions ϕ we then have

$$\phi_\alpha x = \phi_{\alpha_0} \circ \phi_{\omega \cdot a_1} \circ \cdots \circ \phi_{\omega^n \cdot a_n} x.$$

For instance the term $\log_5 L_3 \mathcal{L}_7 x$ then can be written as $\mathrm{l}_{\omega^2 \cdot 7 + \omega \cdot 3 + 5} x$. Also, from the above remark we obtain that $\mathrm{l}_\beta \circ \mathrm{l}_\alpha x = \mathrm{l}_{\alpha+\beta} x$, if $\alpha = \omega^n \cdot a_n + \cdots + \omega^m \cdot a_m$ and $\beta < \omega^{m+1}$.

7.3 Fields of positive strength

In Chapter 2, we have introduced transseries fields. We will now extend this concept to fields of generalized power series with functions $\mathrm{e}_\omega, \mathrm{e}_{\omega^2}$ and so forth. We will speak of them as transseries fields of strength n. In this sense, all transseries fields will be of strength 0; and our aim is to introduce transseries fields of strength $n > 0$.

Warning 7.3.1 The definition of positive strength of $\mathbb{T}$ will require that $\mathbb{T}$ has strength $n-1$. Hence the notion of transseries fields will serve as starting point from which we will define strength 1. Then – using properties of transseries fields of strength 1 – we will define transseries fields of strength 2 and so on. The reader should always be aware of this inductive method and of the fact that the case $n = 0$ plays a special role.

7.3.1 Exponential fields of positive strength

As in the case of transseries fields (of strength 0), we start with a general definition of fields admitting functions e_{ω^n} and l_{ω^n}. Let C be a totally ordered field. We say that C is an **exponential field of strength 0** iff it is an exponential field. For $n > 0$, the field C is an **exponential field of strength $\boldsymbol{n}$** iff there are functions $\exp, \ldots, \mathrm{e}_{\omega^n}$ such that

E1. C is an exponential field of strength $n-1$ for the functions $\exp, \ldots, \mathrm{e}_{\omega^{n-1}}$,

E2. $\exists c_n \in C : \forall c_n \leqslant x < y$:

(i) $x \in \operatorname{dom} \mathrm{e}_{\omega^n}$,

(ii) $x + 1 < \mathrm{e}_{\omega^n} x$ and $\mathrm{e}_{\omega^n} x < \mathrm{e}_{\omega^n} y$,

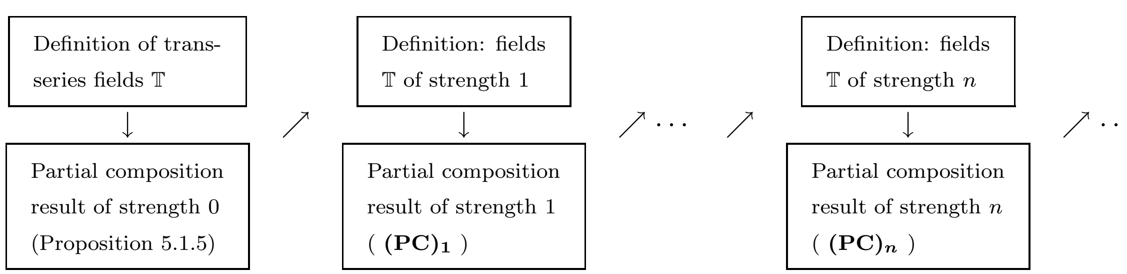
(iii) $\mathrm{e}_{\omega^n} x \in \operatorname{dom} \mathrm{e}_{\omega^{n-1}}$ and $\mathrm{e}_{\omega^{n-1}} \circ \mathrm{e}_{\omega^n} x = \mathrm{e}_{\omega^n}(x+1)$.

Remark 7.3.2 One example of an exponential field of positive strength are the real numbers. In the interest of this chapter, we will not dwell on explicitely describing such examples. This will be done in Appendix A. There we also show some analytical properties of exponential functions of positive strength.

7.3.2 Dependencies during the construction

Recall from Chapter 2 that transseries fields $\mathbb{T}$ admit by Proposition 5.1.5 a partial composition with series from $C[[\log^{\mathbb{Z}^\star} x]]$. In other words, we have a partial composition result for the strength 0 which allows for all $f \in \mathbb{T}^+_\infty$ to define series $L'f$, $L''f, \ldots$ in $\mathbb{T}_{\exp}$. In order to define transseries fields of strength $n > 0$, we need a similar partial composition result for $n-1$.

The case of strength 0 provides again the initial step for an inductive argument. Indeed, we will assume that we already have an appropriate partial composition result for strength $n - 1 \geqslant 0$. This will allow to define series $\mathrm{l}^{(i)}_{\omega^n} f$ for all $f \in \mathbb{T}^+_\infty$ and all $i \geqslant 1$ and, eventually, the definition of transseries fields of strength n. We then have to show that transseries fields of strength n admit a partial composition result. This will be done in Section 7.5. Hence, we have the following dependencies:



We first define the partial composition result for strength $n \geqslant 0$ such that the case $n = 0$ coincides with Proposition 5.1.5. Then we will assume that for some $n > 0$ the notion of transseries fields of strength $n - 1$ has already been defined and that the partial composition result holds for such fields. We then give the definition for transseries fields of strength n.

7.3.3 Logarithmic iterators

In the construction of functions l_{ω^n} with positive n we will apply the results from Chapter 6. In particular, we are interested in our definitions to be coherent. Also, Chapter 6 provides information about the derivatives. This section will be concerned with these questions.

In the case $n = 0$, we have already seen how to define a field of purely logarithmic transseries. This field, $\mathbb{L} = C[[\log^{\mathbb{Z}^\star} x]]$, has properties which made it possible to define a composition with transseries fields. More precisely, it was possible to define a derivation on $\mathbb{L}$ and to define a partial composition for transseries fields. As a result we obtained a compatible composition, which could be extended by taking exp-extensions.

The general case $n \geqslant 0$ will need similar properties. We define for all $n \geqslant 0$ sets $\mathfrak{B}_n$. Let $\mathfrak{B}_0 := \log^{\mathbb{Z}^\star} x$, i.e.

$$\mathfrak{B}_0 = \left\{ \prod_{i<n} \log_i^{a_i} x \mid n \in \omega \ \wedge \ a : n \to \mathbb{Z} \right\}.$$

From Section 2.3 it follows that $\mathfrak{B}_0$ admits an ordered group structure. Let for $n \in \mathbb{N}$

$$\mathfrak{B}_n := \left\{ \log^a x = \prod_{\beta<\alpha} \mathrm{l}_\beta^{a_\beta} x \mid \alpha \in \omega^{n+1} \ \wedge \ a : \alpha \to \mathbb{Z} \right\}.$$

We introduce on $\mathfrak{B}_n$ a multiplicative group structure by $\log^a x \cdot \log^b x = \log^{a+b} x$. An ordering on $\mathfrak{B}_n$ is defined by $1 \preccurlyeq \log^a x \Leftrightarrow 0 \leqslant a$. Let $\mathbb{B}_n = C[[\mathfrak{B}_n]]$. Note that in the case $n = 0$, this is a transseries field. We will later see, that also for $n > 0$, the field $\mathbb{B}_n$ is a transseries field. Moreover, let us assume that we have a derivation ∂ on $\mathbb{B}_n$. This assumption, too, is true for $n = 0$.

Let $\mathbb{T}$ be such that there are functions $\log, \ldots, \mathrm{l}_{\omega^n}$ on $\mathbb{T}^+_\infty$. In particular, assume that $\mathbb{T}$ is of strength n. We say that the partial composition result $\mathbf{(PC)}_{\boldsymbol{n}}$ of strength $n \geqslant 0$ holds for $\mathbb{T}$ iff

- $\mathbf{(PC1)}_{\boldsymbol{n}}$ $\mathfrak{B}_n$ is an ordered group structure extending $\mathfrak{B}_{n-1}$, if $n > 0$,
- $\mathbf{(PC2)}_{\boldsymbol{n}}$ $\forall \mathfrak{n} \in \mathfrak{B}_n : \forall f \in \mathbb{T}^+_\infty : \mathfrak{n}(f) \in \mathbb{T}$,
- $\mathbf{(PC3)}_{\boldsymbol{n}}$ Let $(\mathfrak{n}_i)_{i\in I} \subseteq \mathfrak{B}_n$ be well-ordered and $f \in \mathbb{T}^+_\infty$, then $(\mathfrak{n}_i(f))_{i\in I}$ is a Noetherian family in $\mathbb{T}$.

In other words, if we let $\mathbb{B}_n = C[[\mathfrak{B}_n]]$, then the composition $\circ : \mathbb{B}_n \times \mathbb{T}^+_\infty \to \mathbb{T}$ is defined for fields $\mathbb{T}$ of strength n. Note that $\mathbf{(PC)_0}$ follows from Proposition 5.1.5. Let us remark that the hard part in showing $\mathbf{(PC)}_{\boldsymbol{n}}$ is condition $\mathbf{(PC3)}_{\boldsymbol{n}}$.

In the case $n = 0$, we have seen that $\mathbb{B}_0$ admits a derivation ∂ and that every positive infinite series f of a field $\mathbb{T}$ determines a right-composition φ_f such that

$$\begin{aligned} \circ : \mathbb{B}_0 \times \mathbb{T} &\longrightarrow \mathbb{T} \\ (g, f) &\longmapsto \varphi_f(g) = g \circ f \end{aligned}$$

is a compatible composition. We will therefore assume that for fields $\mathbb{T}$ of strength n the function

$$\begin{array}{rcl} \circ : \mathbb{B}_n \times \mathbb{T} & \to & \mathbb{T} \\ (g, f) & \mapsto & \varphi_f(g) = g \circ f \end{array}$$

is also a compatible composition. This will make the definition of $\mathrm{l}_{\omega^{n+1}}$ coherent.

Lemma 7.3.3 *Let $n \geqslant 0$. For all $i \geqslant 1$ and all $\alpha \in (\omega^n)^{i-1}$ there are integers a_α, b_α with*

(1) $\mathrm{l}'_{\omega^n} x = \prod_{\gamma < \omega^n} \frac{1}{\mathrm{l}_\gamma x}$;

(2) $\mathrm{e}'_{\omega^n} x = \prod_{\gamma < \omega^n} \mathrm{l}_\gamma(\mathrm{e}_{\omega^n} x)$;

(3) $\mathrm{l}^{[i]}_{\omega^n} x = \mathrm{l}'_{\omega^n} x \cdot \sum_{\alpha \in (\omega^n)^{i-1}} a_\alpha \cdot \mathrm{l}'_{\alpha + \bar{1}} x \qquad (a_\alpha \in \mathbb{Z})$;

(4) $\mathrm{e}^{[i]}_{\omega^n} x = (\mathrm{e}'_x)^i \cdot \sum_{\alpha \in (\omega^n)^{i-1}} b_\alpha \cdot \mathrm{l}'_{\alpha + \bar{1}}(\mathrm{e}_{\omega^n} x) \qquad (b_\alpha \in \mathbb{Z})$.

Proof: We start with $\mathrm{l}'_{\omega^n} x$. For $n = 0$ we have $\omega^0 = 1$. With $\log' x = x^{-1}$ we obtain the initial case. Now suppose that we have shown the claimed equality for $\phi = \mathrm{l}_{\omega^n}$. Then for $\Phi = \mathrm{l}_{\omega^{n+1}}$ we have $\Phi' x = \prod_{n < \omega} \phi' \phi_n x$. For any $m \in \mathbb{N}$ we have $\phi_m = \mathrm{l}_{\omega^n \cdot m} x$ and thus

$$\phi' \phi_m x = \prod_{\gamma < \omega^n} \frac{1}{\mathrm{l}_\gamma \mathrm{l}_{\omega^n \cdot m} x} = \prod_{\gamma < \omega^n} \frac{1}{\mathrm{l}_{\omega^n \cdot m + \gamma} x}.$$

Therefore

$$\Phi' x = \mathrm{l}'_{\omega^{n+1}} x = \prod_{m < \omega} \prod_{\gamma < \omega^n} \frac{1}{\mathrm{l}_{\omega^n \cdot m + \gamma} x} = \prod_{\gamma < \omega^{n+1}} \frac{1}{\mathrm{l}_\gamma x}.$$

This shows the equation (1). Equation (2) follows from $1 = \mathrm{e}'_{\omega^n} x \cdot \mathrm{l}'_{\omega^n}(\mathrm{e}_{\omega^n} x)$. In order to show the third equation, we start with the case $n = 0$. The initial case follows from $\log^{(i)} x = (-1)^{i-1} \cdot x^{-i}$ and $x^{-(i-1)} = \log_{\bar{1}} x$ for $\bar{1} \in \mathbb{N}^{i-1}$. Now assume that we have shown the equation for $n \geqslant 0$.

Let $\mathfrak{m} \in \operatorname{supp} \Phi^{(N)} x$. From the definition of $\Phi^{(N)}$ it follows that there are $i \geqslant 0$ and $1 \leqslant j \leqslant N - 1$ such that

$$\mathfrak{m} \in \operatorname{supp} \Phi^{(j)} \phi_{i+1} x \cdot S_{N,j}[\phi' \phi_i x, \cdots, \phi^{(N)} \phi_i x] \cdot (\prod_{l=0}^{i-1} \phi' \phi_l x)^N.$$

We first remark that $\prod_{l \leqslant i-1} \phi' \phi_l x = \phi'_i x$. Let $\mathfrak{a} \in \operatorname{supp} \Phi^{(j)} \phi_{i+1} x$ then by inductive hypothesis there is a tuple $\beta \in (\omega^{n+1})^{j-1}$ such that $\mathfrak{a} = \Phi' \phi_{i+1} x \cdot \mathrm{l}'_{\alpha + \bar{1}}(\phi_{i+1} x)$. Then there is an $\hat{\alpha} \in (\omega^{n+1})^{j-1}$ such that

$$\mathrm{l}'_{\alpha + \bar{1}}(\phi_{i+1} x) = \frac{\mathrm{l}'_{\hat{\alpha} + \bar{1}} x}{(\phi'_{i+1} x)^{j-1}}.$$

From $(\Phi\phi_{i+1}x)' = \Phi'\phi_{i+1}x \cdot (\phi_{i+1}x)'$ we then obtain that

$$\mathfrak{a} = \frac{\Phi' x \cdot \mathrm{l}'_{\hat{\alpha}+\bar{1}}x}{(\phi'_{i+1}x)^j}.$$

On the other hand, let $\mathfrak{b} \in \operatorname{supp} S_{N,j}[\phi'\phi_i x, \cdots, \phi^{(N)}\phi_i x]$. Then by Lemma 6.3.4, there are integers $0 \leqslant k_1, \ldots, k_N$ such that

$$\begin{aligned} j &= k_1 + \cdots + k_N \\ N &= k_1 + 2k_2 + \cdots N \cdot k_N \end{aligned}$$

and such that

$$\mathfrak{b} \in \operatorname{supp} (\phi'\phi_i x)^{k_1} \cdots (\phi^{(N)}\phi_i x)^{k_N}.$$

Now we apply the inductive hypothesis for ϕ. We obtain

$$\mathfrak{b} \in \operatorname{supp} (\phi'\phi_i x)^{k_1} \cdot \left(\phi'\phi_i x \cdot \sum_{(\omega^{n-1})^1} \mathrm{l}'_{\beta+\bar{1}}\phi_i x\right)^{k_2} \cdots \left(\phi'\phi_i x \cdot \sum_{(\omega^{n-1})^{N-1}} \mathrm{l}'_{\beta+\bar{1}}\phi_i x\right)^{k_N},$$

thus there is a $\beta \in (\omega^{n+1})^{N-j}$ such that $\mathfrak{b} = (\phi'\phi_i x)^j \cdot \mathrm{l}_{\beta+\bar{1}}(\phi_i x)$. But then

$$\mathfrak{b} = \frac{(\phi_{i+1}x)^j}{(\phi'_i x)^N} \cdot \mathrm{l}_{\hat{\beta}+\bar{1}}x.$$

Hence $\mathfrak{m} = \Phi' x \cdot \mathrm{l}_{(\hat{\alpha},\hat{\beta})+\bar{1}}x$, where $(\hat{\alpha}, \hat{\beta}) \in (\omega^{n+1})^{N-1}$. This shows (3).

Assume that we have shown the equation for $j \leqslant i-1$. From the equation for $\phi^{(j)}(\psi x)$ and $1 = \psi' x \cdot \phi'(\psi x)$ we obtain

$$\psi' x \cdot \phi^{(j)}(\psi x) = \sum_{\alpha \in (\omega^n)^{j-1}} a_\alpha \cdot \mathrm{l}'_{\alpha+\bar{1}}(\psi x).$$

On the other hand, we have

$$S_{i,j}[\psi' x, \ldots, \psi^{(i)}x] = \sum_{k \in \mathbb{N}^i} c^k_{i,j} \cdot (\psi' x)^{k_1} \cdots (\psi^{(i)}x)^{k_i}.$$

Note that $k_i = 0$. From the hypothesis we obtain that for all $1 \leqslant m \leqslant i-1$

$$(\psi^{(m)}x)^{k_m} = (\psi' x)^{m \cdot k_m} \cdot \left(\sum_{\alpha \in \mathbb{N}^{m-1}} b_\alpha \cdot \mathrm{l}'_{\alpha+\bar{1}}\psi x\right)^{k_m}.$$

From $k_1 + 2 \cdot k_2 \cdots i \cdot k_i = i$ and $k_1 + \cdots + k_i = j$ (by Lemma 6.3.4) we obtain

$$S_{i,j}[\psi' x, \ldots, \psi^{(i)}x] = (\psi' x)^i \cdot \sum_{\alpha \in \mathbb{N}^{i-j}} \hat{b}_\alpha \cdot \mathrm{l}'_{\alpha+\bar{1}}(\psi x).$$

Hence

$$\psi' x \cdot \phi^{(j)}(\psi x) \cdot S_{i,j}[\psi' x, \dots, \psi^{(i)} x] = (\psi' x)^i \cdot \sum_{\alpha \in \mathbb{N}^{i-1}} \tilde{b}_\alpha \cdot \mathrm{l}'_{\alpha+\bar{1}}(\psi x)$$

for integers $\tilde{b}_\alpha$. Substituting these terms into the definition of $\psi^{(i)}$ yields the claimed equation (4). □

Corollary 7.3.4 *For each countable ordinal $\alpha < \omega^\omega$ we have* $\mathrm{l}'_\alpha x = \prod_{\beta<\alpha} \frac{1}{\mathrm{l}_\beta x}$.

Proof: The corollary holds for $\alpha \leqslant \omega$. Hence assume that $\alpha > \omega$ and that the corollary holds for all $\beta < \alpha$. If $\alpha = \gamma + 1$, then

$$\mathrm{l}'_{\gamma+1} x = \log' \mathrm{l}_\gamma x \cdot \mathrm{l}'_\gamma x = \frac{1}{\mathrm{l}_\gamma x} \cdot \prod_{\beta<\gamma} \frac{1}{\mathrm{l}_\beta x} = \prod_{\beta<\gamma+1} \frac{1}{\mathrm{l}_\beta x}.$$

Now assume that α is a limit ordinal. If $\alpha = \omega^n$, then the corollary follows from Lemma 7.3.3. Otherwise we have $\alpha = \omega^n \cdot a_n + \cdots + \omega^m \cdot a_m$ with $m > 0$. If $n > m$, then $a_m > 0$; and if $n = m$, then $a_n > 1$. In both cases there is an ordinal $\hat{\alpha} < \alpha$ such that $\alpha = \hat{\alpha} + \omega^m$. Hence

$$\mathrm{l}'_\alpha x = \mathrm{l}'_{\omega^m}(\mathrm{l}_{\hat{\alpha}} x) \cdot \mathrm{l}'_{\hat{\alpha}} x = \prod_{\beta<\omega^m} \frac{1}{\mathrm{l}_{\hat{\alpha}+\beta} x} \cdot \prod_{\beta<\hat{\alpha}} \frac{1}{\mathrm{l}_\beta x} = \prod_{\beta<\alpha} \frac{1}{\mathrm{l}_\beta x}.$$

This shows the corollary. □

Remark 7.3.5 Note that for all $i \geqslant 1$, the series $\mathrm{l}^{(i)}_{\omega^n} x$ are in $\mathbb{B}_{n-1,\exp}$. Moreover, for all $\varepsilon \prec f$, the family

$$\left(\frac{1}{i!} \mathrm{l}^{(i)}_{\omega^n} f \cdot \varepsilon^i \right)_{1 \leqslant i}$$

is Noetherian, thus its sum exists and is an element from $\mathbb{T}_{\exp}$. Chapter 6 implies the coherence of a possible definition of l_{ω^n} in $\mathbb{T}$, that is, if l_{ω^n} is already partially defined on $\mathbb{T}$.

7.3.4 Definition of positive strength for transseries

Assume that $n > 0$ and that we have already defined strength $n - 1$ for transseries fields such that the partial composition result $\mathbf{(PC)}_{\boldsymbol{n-1}}$ holds for such fields $\mathbb{T}$. From the definition of exponential fields of strength $n - 1$ it follows that for all $f \in \mathbb{T}^+_\infty$ the function $\mathrm{l}_{\omega^{n-1}}$ is defined for the series f and that $\mathrm{l}_{\omega^{n-1}} f \in \mathbb{T}^+_\infty$. Hence for all $i \geqslant 0$ we have

- $f \in \operatorname{dom} \mathrm{l}_{\omega^{n-1} \cdot i}$ and
- $\mathrm{l}_{\omega^{n-1} \cdot i} f \in \mathbb{T}^+_\infty$.

We define a relation $\prec_{\mathrm{l}_{\omega^{n-1}}}$ on $\mathbb{T}$ as follows. Let $f, g \in T$ such that $1 \not\asymp f$. Then we let

$$g \prec_{\mathrm{l}_{\omega^{n-1}}} f \quad \text{iff} \quad \|g\| \prec \mathrm{l}_{\omega^{n-1}\cdot i}\|f\|$$

for all $i \geqslant 0$. Note that if $1 \not\asymp f \in \mathbb{T}$, then the unary relation $\cdot \prec_{\mathrm{l}_{\omega^{n-1}}} f$ is totally defined.

Let $n \geqslant 0$. A series $f \in \mathbb{T}^+_\infty$ is $\mathbf{l}_{\boldsymbol{\omega}^{\boldsymbol{n}}}$**-confluent** at order $k \in \mathbb{N}$ iff for all $i \geqslant 0$

$$\begin{aligned} \mathfrak{d}_{\mathrm{l}_{\omega^n\cdot(k+i)}f} &= \mathrm{l}_{\omega^n\cdot i}(\mathfrak{d}_{\mathrm{l}_{\omega^n\cdot k}f}) \\ 1 &= c_{\mathrm{l}_{\omega^n\cdot(k+i)}f}. \end{aligned}$$

We say that f is $\mathbf{l}_{\boldsymbol{\omega}^{\boldsymbol{n}}}$**-confluent** iff it is l_{ω^n}-confluent at some order $k \in \mathbb{N}$. A subset S of $\mathbb{T}^+_\infty$ is $\mathbf{l}_{\boldsymbol{\omega}^{\boldsymbol{n}}}$**-confluent** (at some order k) iff every series $f \in S$ is l_{ω^n}-confluent (at order k). Instead of l_{ω^n}-confluent at order 0 we also say l_{ω^n}-atomic.

Definition 7.3.6 *The transseries field $\mathbb{T} = C[[\mathfrak{M}]]$ is of* **strength $\boldsymbol{n > 0}$** *iff C is an exponential field of strength n, if $\mathbb{T}$ is of strength $n-1$ and if there is a partial function logarithmic function*

$$\mathrm{l}_{\omega^n} : \mathbb{T} \longrightarrow \mathbb{T}$$

of strength n with

$\mathbf{T}^{\boldsymbol{n}}$1 $\mathbb{T}^+_\infty \subseteq \operatorname{dom} \mathrm{l}_{\omega^n}$,

$\mathbf{T}^{\boldsymbol{n}}$2 *if e_{ω^n} denotes the inverse function of l_{ω^n}, then*

$$\forall f \in \operatorname{dom} \mathrm{e}_{\omega^n} : \operatorname{supp} f^{\downarrow} \prec_{\mathrm{l}_{\omega^{n-1}}} \mathrm{e}_{\omega^n} f \Rightarrow \mathrm{e}_{\omega^n} f \in \mathfrak{M},$$

$\mathbf{T}^{\boldsymbol{n}}$3 *for all $f \in \mathbb{T}^+_\infty$ there is some $k \in \mathbb{N}$ with*

- *f is $\mathrm{l}_{\omega^{n-1}}$-confluent at order k,*
- $\mathfrak{m} = \mathfrak{d}_{\mathrm{l}_{\omega^{n-1}\cdot k}f} \in \operatorname{dom} \mathrm{l}_{\omega^n}$,
- $\mathrm{l}'_{\omega^n}\mathfrak{m} \in \mathbb{T}$,
- *for $R \in \mathbb{T}^{\downarrow}$ with $\mathrm{l}_{\omega^{n-1}\cdot k} f = \mathfrak{m} + R$ we have $\mathrm{l}_{\omega^n} f = k + \mathcal{T}_{\mathrm{l}_{\omega^n}}(\mathfrak{m}, R)$,*

$\mathbf{T}^{\boldsymbol{n}}$4 $\mathbb{T}^+_\infty$ *is l_{ω^n}-confluent.*

Remark 7.3.7 Condition $\mathbf{T}^{\boldsymbol{n}}\mathbf{2}$ is a strong property for monomials. Indeed, it is only a sufficient condition for being a monomial. Assume that $f \in \mathbb{T}^+_\infty$ satisfies this condition. Then for all $k \in \mathbb{N}$ we have $f^{\downarrow} = (f-k)^{\downarrow}$ and $f - k \in \operatorname{dom} \mathrm{e}_{\omega^n}$. The latter property follows from $f \in \operatorname{dom} \mathrm{e}_{\omega^n}$ and $\mathrm{e}_{\omega^n} f \in \operatorname{dom} \mathrm{l}_{\omega^{n-1}\cdot k}$ for all $k \geqslant 0$. Hence we can apply $\mathbf{T}^{\boldsymbol{n}}\mathbf{2}$ and obtain that $\mathrm{e}_{\omega^n}(f-k)$ is again a monomial. Note that this is in general not the case. Take for instance the case $n = 1$ and the monomial $\mathfrak{m}^2$ for $\mathfrak{m} \in \mathfrak{M}$.

Remark 7.3.8 Let $n > 0$ and $\mathbb{T}$ of strength $n-1$. Let $f, g \in \mathbb{T}^+_\infty$ such that $\operatorname{supp} f^{\downarrow} \prec_{\mathrm{l}_{\omega^{n-1}}} g$. Then we have

$$\forall i \geqslant 0 : \|\operatorname{supp} f^{\downarrow}\| \prec \mathfrak{d}_{\mathrm{l}_{\omega^{n-1}\cdot i}g}.$$

In particular, if $f \in \operatorname{dom} \mathrm{e}_{\omega^n}$, then for $g = \mathrm{e}_{\omega^n} f$ this implies $\|\operatorname{supp} f^{\downarrow}\| \prec \mathfrak{d}_{\mathrm{e}_{\omega^n}(f-i)}$. We will use this observation in proofs that a field has strength n.

Notation 7.3.9 We generalize the notion of exp-log-substructures. Let $n > 0$ and $\mathbb{T}_1 = C[[\mathfrak{M}_1]]$, $\mathbb{T}_2 = C[[\mathfrak{M}_2]]$ be of strength n. Denote the exponential functions of strength n of the fields $\mathbb{T}_1$ and $\mathbb{T}_2$ with $\mathrm{e}_{\omega^n}^{\mathbb{T}_1}$ and $\mathrm{e}_{\omega^n}^{\mathbb{T}_2}$, respectively. Then $\mathbb{T}_1$ is an **$\mathbf{e}_{\omega^n}$-$\mathbf{l}_{\omega^n}$-substructure** of $\mathbb{T}_2$ iff

(1) $\mathbb{T}_1$ is an $\mathrm{e}_{\omega^{n-1}}$-$\mathrm{l}_{\omega^{n-1}}$-substructure of $\mathbb{T}_2$ and
(2) $\mathrm{e}_{\omega^n}^{\mathbb{T}_1} f = \mathrm{e}_{\omega^n}^{\mathbb{T}_2} f$ for all $f \in \operatorname{dom} \mathrm{e}_{\omega^n}^{\mathbb{T}_1}$.

We remark that (1) implies $\mathfrak{M}_1 \subseteq \mathfrak{M}_2$, thus $\mathbb{T}_1 \subseteq \mathbb{T}_2$. By **T1** in the definition of transseries fields it then follows that $\operatorname{dom} \mathrm{e}_{\omega^n}^{\mathbb{T}_1} \subseteq \operatorname{dom} \mathrm{e}_{\omega^n}^{\mathbb{T}_2}$. In other words, if $\mathcal{L}_{\mathrm{l}_{\omega^n}, \mathrm{e}_{\omega^n}}$ denotes the first-order language of ordered rings with function symbols $\exp, \ldots, \mathrm{e}_{\omega^n}$ and $\log, \ldots, \mathrm{l}_{\omega^n}$, then $\mathbb{T}_1$ is an $\mathcal{L}_{\mathrm{e}_{\omega^n}, \mathrm{l}_{\omega^n}}$-substructure of $\mathbb{T}_2$. We further notice that if $\mathbb{T}_1$ is an e_{ω^n}-l_{ω^n}-substructure of $\mathbb{T}_2$ and $\mathbb{T}_2$ is an e_{ω^n}-l_{ω^n}-substructure of $\mathbb{T}_3$, then $\mathbb{T}_1$ is an e_{ω^n}-l_{ω^n}-substructure of $\mathbb{T}_3$.

7.4 Basic properties of fields of positive strength

In this section, we will prove a number of properties for transseries fields of positive strength. The results will mainly be used to extend a given field $\mathbb{T}$ of strength $n > 0$ to a field $\hat{\mathbb{T}} \supseteq \mathbb{T}$ which again will be of strength n. Some of the following lemmas, however, only serve to show the properties. They need in turn the properties in a lower-strength version. Throughout this section, we will try to make the dependencies of the different lemmas clear.

Lemma 7.4.1 *Let $\mathbb{T}$ be of strength $n \geqslant 0$. Suppose that (f, ε) is an e_{ω^n}-Taylor couple of series from $\mathbb{T}$. Then $\mathrm{e}_{\omega^n}(f + \varepsilon) \asymp \mathrm{e}_{\omega^n} f$.*

Proof: Since $\mathbb{T}$ is of strength n, it follows from Lemma 7.3.3 that for every $i \geqslant 1$ there is a series $\delta_i \in \mathbb{T}^{\downarrow}$ such that

$$\mathrm{e}_{\omega^n}^{(i)} f = (\mathrm{e}_{\omega^n}' f)^i \cdot \left(\frac{1}{\mathrm{e}_{\omega^n} f}\right)^{i-1} \cdot (1 + \delta_i).$$

Since (f, ε) is an e_{ω^n}-Taylor couple, the sequence

$$(g_i)_{1 \leqslant i} = \left((\mathrm{e}_{\omega^n}' f \cdot \varepsilon)^i \cdot \left(\frac{1}{\mathrm{e}_{\omega^n} f}\right)^{i-1} \right)_{1 \leqslant i}$$

must be a Noetherian family. This implies $g_1 \succ g_2 \succ \cdots$. Hence

$$\mathrm{e}_{\omega^n}' f \cdot \varepsilon \succ (\mathrm{e}_{\omega^n}' f \cdot \varepsilon)^2 \cdot \frac{1}{\mathrm{e}_{\omega^n} f},$$

which implies $\mathrm{e}_{\omega^n} f \succ \mathrm{e}_{\omega^n}' f \cdot \varepsilon$. From $\mathrm{e}_{\omega^n}' f \cdot \varepsilon \asymp \mathcal{R}_{\mathrm{e}_{\omega^n}}(f, \varepsilon)$ the lemma follows. □

Lemma 7.4.2 *Let $\mathbb{T}$ of strength $n > 0$. Let $f, g, \varepsilon \in \mathbb{T}$ with $f \in \mathbb{T}_{\infty}^{+}$. Then:*

(1) If $f \in \operatorname{dom} \mathrm{e}_{\omega^n}^{(i)}$ for all $i \geqslant 0$ and $1 \succ \mathrm{e}_{\omega^n}' f \cdot \varepsilon$, then (f, ε) is an e_{ω^n}-Taylor couple.
(2) If $|g| < f$, then (f, g) is a l_{ω^n}-Taylor couple.

Proof: (1) From the partial composition result for fields of strength $n-1$ one concludes that the family $(\sum_{\alpha\in(\omega^n)^{i-1}} b_\alpha \cdot \mathrm{l}'_{\alpha+\bar{1}} f)_{1\leqslant i}$ is Noetherian (where b_α are the integers from Lemma 7.3.3). From $1 \succ \mathrm{e}'_{\omega^n} f \cdot \varepsilon$ it follows that the sequence $((\mathrm{e}'_{\omega^n} f \cdot \varepsilon)^i)_{1\leqslant i}$ is Noetherian. Hence the sequence

$$(\mathrm{e}^{(i)}_{\omega^n} f \cdot \varepsilon^i)_{1\leqslant i} = \left((\mathrm{e}'_{\omega^n} f)^i \cdot \sum_{\alpha\in(\omega^n)^{i-1}} b_\alpha \cdot \mathrm{l}'_{\alpha+\bar{1}} f \cdot \varepsilon^i \right)_{1\leqslant i}$$

is Noetherian, from which (1) follows.

(2) From the partial composition result for $n-1$ it follows that

- $\forall i \geqslant 1 : \exists \varepsilon_i \prec 1 : \mathrm{l}'_{\omega^n} f \cdot \sum_{\alpha\in(\omega^n)^{i-1}} a_\alpha \cdot \mathrm{l}'_{\alpha+\bar{1}} f = \frac{1}{f^i} \mathrm{l}'_{\omega^n}(\log f) \cdot (1+\varepsilon_i)$ and
- the family $(\varepsilon_i)_{1\leqslant i}$ is Noetherian.

First assume that $g \prec f$. From $|g| < f$ is follows that $((g/f)^i)_{1\leqslant i}$ is Noetherian, thus the sequence

$$\left(\left(\frac{g}{f}\right)^i \cdot \mathrm{l}'_{\omega^n}(\log f) \cdot (1+\varepsilon_i) \right)_{1\leqslant i}$$

is Noetherian. Hence $(\mathrm{l}^{(i)}_{\omega^n} f \cdot g^i)_{1\leqslant i}$ is a Noetherian family. Now let $g \asymp f$. We are done if we can show that for $h = c + \varepsilon$ with $c \in C, |c| < 1$ and $\varepsilon \prec 1$ the sum $\sum_{0\leqslant i} h^i$ is defined in $\mathbb{T}$. We first remark that

$$\bigcup_{0\leqslant i} \operatorname{supp} h^i \subseteq \bigcup_{0\leqslant i} \bigcup_{j=0}^{i} \operatorname{supp} \varepsilon^j \subseteq \bigcup_{0\leqslant j} \operatorname{supp} \varepsilon^j,$$

which is a well-ordered set, since $\varepsilon \prec 1$. Hence for each $\mathfrak{m} \in \bigcup_{0\leqslant i} \operatorname{supp} h^i$ there is a $k \in \mathbb{N}$ such that $\mathfrak{m} \in \operatorname{supp} \varepsilon^j$ implies $j \leqslant k$. Then the sum $\sum_{0\leqslant i} (h^i)_\mathfrak{m}$ is bounded if and only if the sum $\sum_{k\leqslant i} (h^i)_\mathfrak{m}$ is bounded. But

$$\sum_{k\leqslant i} (h^i)_\mathfrak{m} = \sum_{k\leqslant i} (c+\varepsilon)^i_\mathfrak{m} = \sum_{k\leqslant i} \sum_{l=0}^{k} \binom{i}{l} c^{i-l} \cdot (\varepsilon^l)_\mathfrak{m} = \sum_{l=0}^{k} \cdot \sum_{0\leqslant j} \binom{i+j}{l} c^{i-l+j}.$$

We remark that $\sum_{0\leqslant j} \binom{i+j}{l} X^{i-l+j}$ converges for all $|X| < 1$. Thus the last sum is bounded. Now apply this fact for $h = g/f$. This shows (2). □

Corollary 7.4.3 *Let $\mathbb{T}$ be of strength $n \geqslant 0$. Then the function l_{ω^n} is strictly increasing, and the function e_{ω^n} is defined on* $\operatorname{range} \mathrm{l}_{\omega^n}$.

Proof: Let $f, g \in \mathbb{T}^+_\infty$ such that $f < g$. Then $0 < \varepsilon = g - f$ and $|-\varepsilon| < g$. Then (g, ε) is a l_{ω^n}-Taylor couple. From the horizontal coherence, we obtain

$$\mathrm{l}_{\omega^n} f = \mathrm{l}_{\omega^n}(g - \varepsilon) = \mathrm{l}_{\omega^n} g + \mathcal{R}_{\mathrm{l}_{\omega^n}}(g, -\varepsilon).$$

From $0 > \mathcal{R}_{\mathrm{l}_{\omega^n}}(g, -\varepsilon)$ the corollary follows. $\square$

We remark that until now we only needed strength n. The next lemma uses Lemma 7.4.2, therefore it also only needs strength n. The lemma will have applications in later sections.

Lemma 7.4.4 *Let $n > 0$ and $\mathfrak{M} \subseteq \mathfrak{N}$ totally ordered groups such that*

- *the field $\mathbb{T}_{\mathfrak{M}} = C[[\mathfrak{M}]]$ is of strength n and*
- *the field $\mathbb{T}_{\mathfrak{N}} = C[[\mathfrak{N}]]$ is of strength $n-1$.*

Let (f, g) be a l_{ω^n}-Taylor couple in $\mathbb{T}_{\mathfrak{N}}$ (i.e. $f, g \in \mathbb{T}_{\mathfrak{N}}$) such that $f \succ g$. Then

$$\mathcal{R}_{\mathrm{l}_{\omega^n}}(f, g) \in \mathbb{T}_{\mathfrak{M}} \ \wedge \ f \in \mathbb{T}_{\mathfrak{M}} \Rightarrow g \in \mathbb{T}_{\mathfrak{M}}.$$

Proof: Suppose not. Then let $h \trianglelefteq g$ be the maximal truncation of g such that $h \in \mathbb{T}_{\mathfrak{M}}$. From $g \notin \mathbb{T}_{\mathfrak{M}}$ it follows that h is a proper truncation of g, hence that $\hat{h} = g - h \neq 0$. In particular, $\mathfrak{d}_{\hat{h}} \in \mathfrak{N} \setminus \mathfrak{M}$. We claim that $\mathcal{R}_{\mathrm{l}_{\omega^n}}(f, h) \in \mathbb{T}_{\mathfrak{M}}$. If $h = 0$, then this is true since $\mathcal{R}_{\mathrm{l}_{\omega^n}}(f, 0) = 0$. If $h \neq 0$, then $h \asymp g$ and Lemma 7.4.2 imply that (f, h) is a l_{ω^n}-Taylor couple in $\mathbb{T}_{\mathfrak{M}}$. Thus $\mathcal{R}_{\mathrm{l}_{\omega^n}}(f, h)$ is defined since $\mathbb{T}_{\mathfrak{M}}$ is of strength n. Hence it is an element of $\mathbb{T}_{\mathfrak{M}}$, which shows the claim.

Then

$$\mathcal{R}_{\mathrm{l}_{\omega^n}}(f, g) = \mathcal{R}_{\mathrm{l}_{\omega^n}}(f, h + \hat{h}) = \sum_{1 \leqslant i} \frac{1}{i!} \mathrm{l}_{\omega^n}^{(i)} f \cdot (h + \hat{h})^i$$

implies $\mathcal{R}_{\mathrm{l}_{\omega^n}}(f, g) = \mathcal{R}_{\mathrm{l}_{\omega^n}}(f, h) + \mathrm{l}_{\omega^n}' f \cdot \hat{h} \cdot (1 + \mu)$ where $\mu \in \mathbb{T}_{\mathfrak{N}}^{\downarrow}$ is of the form

$$\mu = \sum_{2 \leqslant i} \frac{1}{i!} \cdot \frac{\mathrm{l}_{\omega^n}^{(i)} f}{\mathrm{l}_{\omega^n}' f} \cdot \sum_{j=0}^{i-1} \binom{i}{j} h^j \cdot \hat{h}^{i-j-1} \prec 1.$$

Now from $\mathcal{R}_{\mathrm{l}_{\omega^n}}(f, g), \mathcal{R}_{\mathrm{l}_{\omega^n}}(f, h), \mathrm{l}_{\omega^n}' f \in \mathbb{T}_{\mathfrak{M}}$ we obtain

$$\frac{\mathcal{R}_{\mathrm{l}_{\omega^n}}(f, g) - \mathcal{R}_{\mathrm{l}_{\omega^n}}(f, h)}{\mathrm{l}_{\omega^n}' f} = \hat{h} \cdot (1 + \mu) \in \mathbb{T}_{\mathfrak{M}}.$$

This implies $\mathfrak{d}_{\hat{h}} \in \mathfrak{M}$. This contradiction shows the lemma. $\square$

The next two lemmas will have applications in proofs of other lemmas in this section.

Lemma 7.4.5 *Let $\mathbb{T}$ be of strength $n - 1 \geqslant 0$. Recall that l_{ω^n}' is totally defined on $\mathbb{T}_{\infty}^{+}$. Let $\phi = \mathrm{l}_{\omega^{n-1}}$ and $\Phi = \mathrm{l}_{\omega^n}$. Then for all $f \in \mathbb{T}_{\infty}^{+}$:*

(1) $1 \succ \phi' f$.

(2) *If* $\forall i \geqslant 0 : \phi_i f \in \operatorname{dom} \Phi'$, *then* $\forall i \geqslant 0 : 1 \succ \Phi' \phi_{i+1} f \succ \Phi' \phi_i f$.

(3) *If* $\forall i \geqslant 0 : \phi_i f \in \operatorname{dom} \Phi'$, *then* $\forall i \geqslant 0 : \Phi' \phi_{i+1} f \succ \phi' \phi_i f \succ \Phi' \phi_i f$.

(4) $\forall i \geqslant 0 : \phi' \phi_{i+1} f \succ \dfrac{1}{\phi_i f} \succ \phi' \phi_i f$.

Proof: For all $\alpha < \omega^{n-1}$ we have $1 \prec \mathrm{l}_\alpha f \prec f$. Then from $\phi' f = \prod_{\alpha<\omega^{n-1}} \mathrm{l}_\alpha^{-1} f$ the part (1) follows. From $\Phi'\phi_i f = \phi'(\phi_i f) \cdot \Phi'\phi_{i+1} f$ and $\phi_i f \in \mathbb{T}_\infty^+$ it follows – using (1) – that

$$\Phi'\phi_i f \prec \Phi'\phi_{i+1} f.$$

The inequality $\Phi'\phi_i f \prec 1$ can be shown as part (1). Hence (2). Since $1 \succ \Phi'\phi_{i+1} f$, the equation $\Phi'\phi_i f = \phi'(\phi_i f) \cdot \Phi'\phi_{i+1} f$ implies that $\Phi'\phi_i f \prec \phi'\phi_i f$. On the other hand,

$$\Phi'\phi_{i+1} f = \prod_{0 \leqslant m} \phi'\phi_{i+1+m} f \prec 1$$

leads to $\log \Phi'\phi_{i+1} f < 0$. From $\phi'\phi_i f \prec \phi'\phi_{i+1+m} f \prec 1$ we obtain $\log \phi'\phi_i f < \log \phi'\phi_{i+1+m} f < 0$ and thus

$$\log \phi'\phi_i f < \sum_{0 \leqslant m} \log \phi'\phi_{i+1+m} f < 0$$

But this shows $\phi'\phi_i f \prec \Phi'\phi_{i+1} f$, hence (3). We are done if we can show that for all $i \geqslant 0$ the inequality

$$0 > \log \phi'\phi_{i+1} f > -\log \phi_i f > \log \phi'\phi_i f \tag{7.3}$$

holds. Recall that for all $g \in \mathbb{T}_\infty^+$ we have

$$\log \phi' g = -\sum_{\alpha<\omega^{n-1}} \mathrm{l}_{\alpha+1} g = -\log g - \hat{g}$$

where $\hat{g} \in \mathbb{T}_\infty^+$ with $\hat{g} \prec \log g$. Applying this to $\phi_i f$ and $\phi_{i+1} f$ leads to

$$\begin{aligned} \log \phi'\phi_{i+1} f &= -\sum_{\alpha<\omega^{n-1}} \mathrm{l}_{\alpha+1}\phi_{i+1} f = -\log \phi_{i+1} f - h_1 \\ \log \phi'\phi_i f &= -\sum_{\alpha<\omega^{n-1}} \mathrm{l}_{\alpha+1}\phi_i f = -\log \phi_i f - h_2 \end{aligned}$$

with $h_1, h_2 \in \mathbb{T}_\infty^+$ and $h_1 \prec \log \phi_{i+1} f$ and $h_2 \prec \log \phi_i f$. From this we obtain

$$0 > -\log \phi_{i+1} f - h_1 > -\log \phi_i f > -\log \phi_i f - h_2,$$

from which inequality (7.3) follows. □

Lemma 7.4.6 *Let $\mathbb{T}$ be of strength $n > 0$. Let $\phi = \mathrm{l}_{\omega^{n-1}}$ and $\Phi = \mathrm{l}_{\omega^n}$. Then for all $i > 0$ and all $f \in \mathbb{T}_\infty^+$:*

$$1 \succ \Phi'\phi_{i+1} f \succ \frac{1}{\phi_i f} \succ \Phi'\phi_{i-1} f.$$

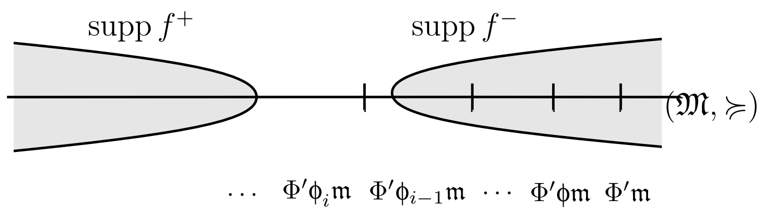


Figure 7.1: Determining f^+ and f^-

Proof: Apply Lemma 7.4.5 and note that for all $i \geqslant 0$ we have $\phi_i f \in \operatorname{dom} \Phi'$. □

The next lemma will be used frequently in proofs of strength n. Essentially, we show that for a special class of monomials $\mathfrak{m}$ the support of $\mathrm{l}_{\omega^n}\mathfrak{m}$ is strictly bigger than the entire support of the restricted Taylor development $\mathcal{R}_{\mathrm{l}_{\omega^n}}(\mathfrak{m}, \cdot)$. In the proof, we use the Lemmas 7.4.1, 7.4.2 and 7.4.6. Hence the proof entirely relies on the strength n of $\mathbb{T}$.

Lemma 7.4.7 *Let $\mathbb{T} = C[[\mathfrak{M}]]$ be of strength $n > 0$. Let $\mathfrak{m} \in \mathfrak{M}$ be such that $\mathrm{l}_{\omega^{n-1}.i}\mathfrak{m} \in \mathfrak{M}$ for all $i \geqslant 0$. Then* $\operatorname{supp} \mathrm{l}_{\omega^n}\mathfrak{m} \succ \mathrm{l}'_{\omega^n}\mathrm{l}_{\omega^{n-1}.i}\mathfrak{m}$ *for all $i \geqslant 0$.*

Proof: Let $\phi = \mathrm{l}_{\omega^{n-1}}$ and $\Phi = \mathrm{l}_{\omega^n}$, and let $\Psi = \mathrm{e}_{\omega^n}$. We define two series f^+ and f^- in $\mathbb{T}$ as follows. (See Figure 7.1.)

(1) $\Phi\mathfrak{m} = f^+ + f^-$,
(2) $f^+ \trianglelefteq \Phi\mathfrak{m}$,
(3) $\forall i \geqslant 0 : \operatorname{supp} f^+ \succ \Phi'\phi_i\mathfrak{m}$ and
(4) $\forall \mathfrak{n} \in \operatorname{supp} f^- : \exists i \geqslant 0 : \Phi'\phi_i\mathfrak{m} \succ \mathfrak{n}$.

The series f^+ and f^- are uniquely determined by these conditions, and we have to show that $f^- = 0$. We remark that $\operatorname{supp} f^-$ is well-ordered in $\mathfrak{M}$, therefore $\operatorname{supp} f^-$ cannot contain a strictly $\succcurlyeq$-decreasing sequence. Since the sequence $(\Phi'\phi_i\mathfrak{m})_{0 \leqslant i}$ is strictly $\succcurlyeq$-decreasing, the sequence of leading monomials is strictly $\succcurlyeq$-decreasing, and therefore there must be an $I \in \mathbb{N}$ such that $\Phi'\phi_I\mathfrak{m} \succ \operatorname{supp} f^-$. We may assume that $I \geqslant 1$.

From $\Phi'\phi_I\mathfrak{m} = \Phi'\Psi(\Phi\mathfrak{m} - I)$ and $\Phi'\phi_I\mathfrak{m} \succ f^-$ it follows then by Lemma 7.4.2 that

$$(\Psi^{(i)}(\Phi\mathfrak{m} - I) \cdot (-f)^i)_{0 \leqslant i}$$

is a Noetherian family. From $\Phi\mathfrak{m} - I - f^- = f^+ - I$ it follows now that $\Psi(f^+ - I) \in \mathbb{T}$. In other words, we have $f^+ - I \in \operatorname{dom} \Psi$. We now show that $\Psi(f^+ - I) \in \mathfrak{M}$.

First, we note that from the definition of the series f^+ and f^- it follows that

$$\operatorname{supp}(f^+ - I)^\downarrow = \operatorname{supp}(f^+)^\downarrow \succ \Phi'\phi_i\mathfrak{m} \qquad (\forall i \geqslant 0).$$

Fix $\mathfrak{n} \in \operatorname{supp}(f^+ - I)^\downarrow$, then both $\|\mathfrak{n}\| = \mathfrak{n}^{-1}$ and $\mathfrak{n} \succ \Phi'\phi_i\mathfrak{m}$ imply

$$\forall i \geqslant 0 : \|\mathfrak{n}\| \prec \frac{1}{\Phi'\phi_i\mathfrak{m}}.$$

Since $\mathbb{T}$ is of strength n, it is also of strength $n-1$, and we can apply Lemma 7.4.6. Then $\forall i \geqslant 0 : \mathfrak{n} \succ \Phi' \phi_i \mathfrak{m}$ implies $\forall i \geqslant 0 : \|\mathfrak{n}\| \prec \phi_i \mathfrak{m}$. In particular, we have for all $i \geqslant 0$ that $\|\mathfrak{n}\| \prec \phi_i \phi_I \mathfrak{m}$. From Lemma 7.4.1 we obtain $\Psi(f^+ - I) = \Psi(\Phi \phi_I \mathfrak{m} - f^-) \asymp \phi_I \mathfrak{m}$. Therefore we have

$$\forall i \geqslant 0 : \|\mathfrak{n}\| \prec_\phi \Psi(f^+ - I).$$

Since $\mathbb{T}$ is of strength n, we conclude that $\Psi(f^+ - I) \in \mathfrak{M}$. From the hypothesis about $\mathfrak{m}$ we get $\phi_I \mathfrak{m} \in \mathfrak{M}^\uparrow$ and

$$\phi_I \mathfrak{m} = \Psi((f^+ - I) + f^-) = \Psi(f^+ - I) + \mathcal{R}_\Psi(f^+ - I, f^-).$$

This means $\mathcal{R}_\Psi(f^+ - I, f^-) = 0$, which shows $f^- = 0$. □

We have seen in Chapter 2 that in transseries fields the logarithm of a series can only be a monomial, if the series itself is a monomial. This result can be generalized to fields of positive strength. Note in particular, that we use the fact that the forthcomming lemma is true in the case $n = 0$. If $n > 0$, we may therefore assume that the lemma holds in the case $n-1$, and we can use the lemma in this case in order to show the case n.

Lemma 7.4.8 *Let $\mathbb{T} = C[[\mathfrak{M}]]$ be of strength $n \geqslant 0$. If $f \in \mathbb{T}$ is such that $\mathrm{l}_{\omega^n} f \in \mathfrak{M}$, then $f \in \mathfrak{M}$.*

Proof: The case $n = 0$ holds by **T2** for transseries fields. We therefore assume that in the following $n \geqslant 1$. Note that in particular, we can apply Lemma 7.4.7 for the case n.

Let $\phi = \mathrm{l}_{\omega^{n-1}}$ and $\Phi = \mathrm{l}_{\omega^n}$, assume that we have already shown the lemma in the case $n-1$. Let $f \in \mathbb{T}$ such that $\Phi f \in \mathfrak{M}$. By $\mathbf{T^n 3}$ there is a $k \in \mathbb{N}$ such that f is ϕ-confluent at order k. We show that $k = 0$. Suppose $k > 0$ and let $\phi_k f = \mathfrak{m} + \varepsilon$ such that $\mathfrak{m} \in \mathfrak{M}$, $\varepsilon \preccurlyeq 1$ and $\phi_i f \in \mathfrak{M}^\uparrow$ for all $i \geqslant 0$. Applying Lemma 7.4.7 for n yields $\operatorname{supp} \Phi \mathfrak{m} \succ \mathcal{R}_\Phi(\mathfrak{m}, \varepsilon)$. On the other hand, we have

$$\Phi f = k + \Phi \phi_k f = k + \Phi \mathfrak{m} + \mathcal{R}_\Phi(\mathfrak{m}, \varepsilon) \in \mathfrak{M}.$$

Hence $\varepsilon = 0$ and $\Phi \mathfrak{m} = \Phi \phi_k f$. This means $\mathfrak{m} = \phi_k f = \phi(\phi_{k-1} f)$. Applying the lemma in the case $n-1$ implies $\phi_{k-1} f \in \mathfrak{M}$. This contradicts the minimality of k. Hence $k = 0$ and $f = \mathfrak{m} \in \mathfrak{M}$. □

Another frequently used tool in proofs that certain fields are of positive strength will be the following lemma. Recall that atomic means to be of confluence at order 0.

Lemma 7.4.9 *Let $n > 0$ and $\mathbb{T}$ be of strength n. If $f \in \mathbb{T}^+_\infty$ is l_{ω^n}-atomic, then f is $\mathrm{l}_{\omega^{n-1}}$-atomic. Consequently, the series f is l_{ω^i}-atomic for all $i \leqslant n$.*

Proof: Let again $\phi = \mathrm{l}_{\omega^{n-1}}$ and $\Phi = \mathrm{l}_{\omega^n}$. Fix $f \in \mathbb{T}^+_\infty$ such that f is Φ-atomic. For all ϕ-Taylor couples (g, ε) we have $\phi g \succ \mathcal{R}_\phi(g, \varepsilon)$, hence

$$\tau_{\phi(g+\varepsilon)} = \tau_{\phi g}.$$

This observation implies that for $\mathfrak{n} = \mathfrak{d}_f$ and all $k \geqslant 0$ we have

$$\tau_{\phi_k f} = \tau_{\phi_k \mathfrak{n}}.$$

As an immediate consequence we obtain that for all $k \geqslant 0$ the series f is ϕ-confluent at order k if and only if $\mathfrak{n}$ is ϕ-confluent at order k. Hence instead of showing that f is ϕ-atomic, we show that $\mathfrak{n}$ is ϕ-atomic.

Suppose that this is not the case and let $k > 0$ be minimal such that $\mathfrak{n}$ is ϕ-confluent at order k. Such an integer k exists by $\mathbf{T^n 3}$. We can also assume that $\phi_k \mathfrak{n} = \mathfrak{m} + \rho$ such that $\phi_i \mathfrak{m} \in \mathfrak{M}^{\uparrow}$ for all $i \geqslant 0$. Hence, we can apply Lemma 7.4.7 and obtain $\operatorname{supp} \Phi\mathfrak{m} \succ \mathcal{R}_{\Phi}(\mathfrak{m}, \rho)$. But we have

$$\Phi\mathfrak{m} + \mathcal{R}_{\Phi}(\mathfrak{m}, \rho) = \Phi\phi_k \mathfrak{n} = \Phi\mathfrak{n} - k \in \mathfrak{M} - \mathbb{N}.$$

Thus $(\Phi(\mathfrak{m} + \rho))^{\downarrow} = 0$, i.e. $\mathcal{R}_{\Phi}(\mathfrak{m}, \rho) = 0$ and therefore $\rho = 0$. Hence $\phi_k \mathfrak{n} = \mathfrak{m} \in \mathfrak{M}$. We apply Lemma 7.4.8 to conclude $\phi_{k-1} \mathfrak{n} \in \mathfrak{M}$. But this is a contradiction to the minimality of k. □

Remark 7.4.10 We point out that throughout this section, we never needed the condition that $\mathbb{T}_{\infty}^{+}$ is l_{ω^n}-confluent, if $\mathbb{T}$ is of strength n. All properties can therefore be shown in more general fields. Furthermore, we notice that Lemma 7.4.8 uses a lower-strength version of itself, and that Lemma 7.4.9 needs Lemma 7.4.8. Apart from this, all lemmas follow from the fact that the field $\mathbb{T}$ is of strength n. Hence there are no loops in the dependencies of the lemmas.

7.5 The partial composition result for positive strength

This section provides the proof of the partial composition result $\mathbf{(PC)_{\mathit{n}}}$ for fields of strength $n \geqslant 0$. This will enable us to define structures of strength $n+1$. In particular, we will use the fact that $\mathbb{T}_{\infty}^{+}$ is l_{ω^n}-confluent, if $\mathbb{T}$ is of strength n.

Let in the following $\mathbb{T}$ be of strength $n \geqslant 0$. We need the following lemma in for the proofs of $\mathbf{(PC1)_{\mathit{n}}}$ – $\mathbf{(PC3)_{\mathit{n}}}$.

Lemma 7.5.1 *Let $\mathbb{T} = C[[\mathfrak{M}]]$ be of strength $n \geqslant 0$. Let $\mathfrak{m} \in \mathfrak{M}^{\uparrow}$ be l_{ω^n}-atomic. Then*

(1) $\forall \alpha \in \omega^{n+1} : \mathrm{l}_{\alpha}\mathfrak{m} \in \mathfrak{M}^{\uparrow}$.
(2) $\{\mathrm{l}'_{\alpha}\mathfrak{m} \mid \alpha \in (\omega^{n+1})^i, 0 \leqslant i\} \subseteq \mathfrak{M}^{\downarrow}$ is well-ordered.
(3) $\{(\mathrm{l}'_{\omega^n}\mathrm{l}_{\omega^n \cdot l}\mathfrak{m})^{-1} \mid l \geqslant 0\} \subseteq \mathfrak{M}^{\uparrow}$ is well-ordered.

Proof: Let $\Phi = \mathrm{l}_{\omega^n}$. The case $n = 0$ is clear. Assume $n > 0$. If $\mathfrak{m}$ is Φ-atomic, then so is $\Phi_k \mathfrak{m}$ for all $k \geqslant 0$. Hence if $\alpha = \omega^n a_n + \cdots + a_0$, then $\Phi_{a_n}\mathfrak{m}$ is Φ-atomic. By Lemma 7.4.9, the monomial $\Phi_{a_n}\mathfrak{m}$ is l_{ω^i}-atomic for all $i < n$. Since $\mathbb{T}$ is of strength $n-1$ and $\beta = \omega^{n-1} a_{n-1} + \cdots + a_0 < \omega^n$, we can apply this lemma for the case $n-1$ and obtain $\mathrm{l}_{\alpha}\mathfrak{m} = \mathrm{l}_{\beta}(\mathrm{l}_{\omega^n \cdot a_n}\mathfrak{m}) \in \mathfrak{M}^{\uparrow}$. This shows (1).

Recall that for $\alpha = (\alpha_1, \ldots, \alpha_i) \in (\omega^{n+1})^i$ we have

$$\mathrm{l}'_{\alpha}\mathfrak{m} = \mathrm{l}'_{\alpha_1}\mathfrak{m} \cdots \mathrm{l}'_{\alpha_i}\mathfrak{m} = \prod_{j=1}^{i} \prod_{\beta < \alpha_j} \frac{1}{\mathrm{l}_{\beta}\mathfrak{m}}.$$

From (1) it follows that $\mathrm{l}'_\alpha \in \mathfrak{M}^{\downarrow\!\uparrow}$ (note that we allow $i = 0$ here). Also, we remark that for every α there exists a function $a : \omega^{n+1} \to \mathbb{N}$ with

- $\mathrm{l}'_\alpha \mathfrak{m} = \prod_{\beta<\omega^{n+1}} \mathrm{l}_\beta^{-a_\beta} \mathfrak{m} = \log^{-a} \mathfrak{m}$,
- the function a is weakly decreasing.

By Lemma 1.7.7 the set $\{a : \omega^{n+1} \to \mathbb{N} \,|\, a \text{ is weakly decreasing}\}$ is well-ordered in the lexicographic ordering. Since $a <_{\mathrm{lex}} b$ if and only if $\log^{-a} \mathfrak{m} \succ \log^{-b} \mathfrak{m}$, it follows that

$$\{\log^{-a} \mathfrak{m} \,|\, a : \omega^{n+1} \to \mathbb{N} \text{ weakly decreasing}\} \subseteq \mathfrak{M}^{\downarrow\!\uparrow}$$

is well-ordered. Hence (2).

From $\Phi_l \mathfrak{m} \succ \Phi_{l+1} \mathfrak{m}$ follows $\Phi' \Phi_l \mathfrak{m} \prec \Phi' \Phi_{l+1} \mathfrak{m} \prec 1$, thus (3). □

We now show $\mathbf{(PC)}_{\boldsymbol{n}}$ and start with $\mathbf{(PC1)}_{\boldsymbol{n}}$. Let $\mathfrak{m}, \mathfrak{n} \in \mathfrak{B}_n$ with

$$\begin{aligned} \mathfrak{n} = \quad & \log^a x = \prod_{\gamma<\alpha} \mathrm{l}_\gamma^{a_\gamma} x \qquad && (\alpha \in \omega^{n+1} \text{ and } a : \alpha \to \mathbb{Z}) \\ \mathfrak{n} = \quad & \log^b x = \prod_{\gamma<\beta} \mathrm{l}_\gamma^{b_\gamma} x \qquad && (\beta \in \omega^{n+1} \text{ and } b : \beta \to \mathbb{Z}) \end{aligned}$$

Note that $1 = \log^0 x \in \mathfrak{B}_n$. We may assume that $\alpha = \beta$, for if not, then for $\alpha < \beta$ we extend the function a to β be letting $a_\gamma = 0$ for all $\gamma \geqslant \alpha$; similarly if $\beta < \alpha$). Let $a + b : \alpha \to \mathbb{Z}$ be the pointwise sum of a and b, and let $-a$ the function with $(-a)_\gamma = -a_\gamma$ for $\gamma < \alpha$. Then we let

$$\begin{aligned} \mathfrak{n} \cdot \mathfrak{m} := \quad & \log^{a+b} x \\ \mathfrak{n}^{-1} := \quad & \log^{-a} x. \end{aligned}$$

Hence $\mathfrak{n} \cdot \mathfrak{m}, \mathfrak{n}^{-1} \in \mathfrak{B}_n$ and $\mathfrak{n} \cdot \mathfrak{n}^{-1} = 1$. This defines a multiplicative group structure on $\mathfrak{B}_n$. We remark that $\mathfrak{B}_0 = \log^{\mathbb{Z}^\star} x$ and that for $n \geqslant 1$ the group structure of $\mathfrak{B}_n$ extends the group structure of $\mathfrak{B}_{n-1}$.

The second step is to define an ordering $\succcurlyeq$ on $\mathfrak{B}_n$. For $\mathfrak{n} = \log^a x$ we let

$$M_\mathfrak{n} := \min\{\gamma < \omega^{n+1} \,|\, a_\gamma \neq 0\}.$$

Then we let $\mathfrak{n} \succ 1$ iff $a_{M_\mathfrak{n}} > 0$. For $\mathfrak{n}, \mathfrak{m} \in \mathfrak{B}_n$ we let $\mathfrak{m} \succcurlyeq \mathfrak{n}$ iff $\mathfrak{m} \cdot \mathfrak{n}^{-1} \succcurlyeq 1$. The ordering $\succcurlyeq$ on $\mathfrak{B}_n$ extends the ordering $\succcurlyeq$ on $\mathfrak{B}_{n-1}$; and for $n = 0$ the ordering coincides with the ordering from Section 2.3. This shows $\mathbf{(PC1)}_{\boldsymbol{n}}$.

Next, we show $\mathbf{(PC2)}_{\boldsymbol{n}}$. Let $\mathbb{T}$ be of strength n and $f \in \mathbb{T}^+_\infty$. Let $\mathfrak{n} \in \mathfrak{B}_n$ with $\mathfrak{n} = \log^a x$ and $a : \alpha < \omega^{n+1} \to \mathbb{Z}$. In view of the infinite-product notation we let

$$\mathfrak{n}(f) := \exp \sum_{\gamma<\alpha} a_\gamma \cdot \mathrm{l}_{\gamma+1} f.$$

In order to show that $\mathfrak{n}(f) \in \mathbb{T}$, we have to show that

(i) $\forall \gamma < \alpha : \mathrm{l}_{\gamma+1} f \in \mathbb{T}$,
(ii) $(\mathrm{l}_{\gamma+1} f)_{\gamma<\alpha}$ is a Noetherian family,
(iii) $\sum_{\gamma<\alpha} a_\gamma \cdot \mathrm{l}_{\gamma+1} f \in \mathrm{dom}\ \exp$.

For all integers $l \geqslant 0$ we let $f_l := \mathrm{l}_{\omega^n \cdot l} f$. Since $\alpha < \omega^{n+1}$, we have $\alpha = \omega^n\, a_n + \cdots + a_0$. Hence for every $\gamma < \alpha$ there are integers $g_0, \ldots, g_n \in \mathbb{N}$ such that $g_n \leqslant a_n$ and

$$\mathrm{l}_\gamma x = \log_{g_0} \circ\, \mathrm{l}_{\omega \cdot g_1} \circ \cdots \circ \mathrm{l}_{\omega^n \cdot g_n} x.$$

Since $\mathbb{T}$ is of strength n, we have $f_d \in \mathbb{T}^+_\infty$ for all $d \leqslant a_n$. Inductively invoking the strengths $i < n$ yields

$$\log_{g_0} \circ\, \mathrm{l}_{\omega \cdot g_1} \circ \cdots \circ \mathrm{l}_{\omega^n \cdot g_n} f \in \mathbb{T}^+_\infty.$$

Hence $\mathrm{l}_{\gamma+1} f \in \mathbb{T}$. This shows (i). In order to show (ii) we remark that from $d_n \leqslant a_n$ it follows that

$$(\mathrm{l}_{\gamma+1} f)_{\gamma<\alpha} = (\mathrm{l}_{\gamma+1} f_d)_{\substack{\gamma<\omega^n \\ d \leqslant a_n}}.$$

For each $d \leqslant a_n$, the family $(\mathrm{l}_{\gamma+1} f_d)_{\gamma<\omega^n}$ is Noetherian. This follows from $\mathbf{(PC)}_{\boldsymbol{n-1}}$. As a finite union of Noetherian families, the sequence $(\mathrm{l}_{\gamma+1} f)_{\gamma<\alpha}$ is itself Noetherian. Hence (ii). $\mathbf{(PC)}_{\boldsymbol{n-1}}$ also implies that

$$\sum_{\gamma<\omega^n} a_\gamma \cdot \mathrm{l}_{\gamma+1} f_d \in \mathrm{dom}\ \exp$$

for $d \leqslant a_n$. Thus

$$\sum_{\gamma<\alpha} a_\gamma \cdot \mathrm{l}_{\gamma+1} f = \sum_{d=0}^{a_n} \sum_{\gamma<\omega^n} a_\gamma \cdot \mathrm{l}_{\gamma+1} f_d \in \mathrm{dom}\ \exp\,.$$

Hence (iii) and therefore $\mathbf{(PC2)}_{\boldsymbol{n}}$.

We show $\mathbf{(PC3)}_{\boldsymbol{n}}$ in three steps. In a first step, we show that we can reduce the statement to series $f \in \mathbb{T}^+_\infty$ which are l_{ω^n}-atomic. The second step consists in showing properties of the support of $\mathrm{l}_{\omega^n \cdot i} f$ for $i \geqslant 0$ assuming that f is l_{ω^n}-atomic. In a final step, we apply the properties from the second step to conclude the proof. In the following, we let $\Phi = \mathrm{l}_{\omega^n}$ and $\Psi = \mathrm{e}_{\omega^n}$.

Step 1: We show that we can restrict ourselves to series which are Φ-atomic. Since $\mathbb{T}$ is of strength n, the series $f \in \mathbb{T}^+_\infty$ is Φ-confluent at order $k \in \mathbb{N}$. For $\mathfrak{n} \in \mathfrak{B}_n$ with $\mathfrak{n} = \log^a x$ and $a : \alpha < \omega^{n+1} \to \mathbb{Z}$ there is a sequence $(\mathfrak{n}_l)_{l<\omega} \subseteq \mathfrak{B}_{n-1}$ such that

$$\mathfrak{n} = \prod_{l<\omega} \mathfrak{n}_l(\Phi_l x).$$

Let $\mathfrak{a} = \prod_{l<k} \mathfrak{n}_l(\Phi_l x)$ and $\mathfrak{b} = \prod_{k\leqslant l} \mathfrak{n}_l(\Phi_l x)$, then $\mathfrak{a}\cdot\mathfrak{b} = \mathfrak{n}$ and $\mathfrak{a}(f)\cdot\mathfrak{b}(f) = \mathfrak{n}(f)$. For the sequence $(\mathfrak{n}_i)_{i\in I}$ we now obtain $\mathfrak{a}_i(f)\cdot\mathfrak{b}_i(f) = \mathfrak{n}_i(f)$, where

$$\begin{aligned}\mathfrak{a}_i(f) &= \prod_{l<k} \mathfrak{n}_{i,l}(\Phi_l f)\\ \mathfrak{b}_i(f) &= \prod_{k\leqslant l} \mathfrak{n}_{i,l}(\Phi_l f).\end{aligned}$$

If we have shown $\mathbf{(PC3)}_{\boldsymbol{n}}$ for series are Φ-atomic, then $(\mathfrak{b}_i(f))_{i\in I}$ is a Noetherian family. In this case it remains to show that $(\mathfrak{a}_i(f))_{i\in I}$ is a Noetherian family. From $\mathbf{(PC)}_{\boldsymbol{n-1}}$ it follows that for all $l<k$ the sequence

$$s_l = (\mathfrak{n}_{i,l}(\Phi_l f))_{i\in I}$$

is a Noetherian family. From $(\mathfrak{a}_i(f))_I \subseteq s_0\cdots s_{k-1}$ it now follows that $(\mathfrak{a}_i(f))_I$ is a Noetherian family. This finishes the first step.

Step 2: Assume that $f = \mathfrak{m}+\varepsilon \in \mathbb{T}_\infty^+$ is Φ-atomic. By Lemma 7.4.9, the series f is l_{ω^i}-atomic for all $i \leqslant n$. Hence for all $\alpha < \omega^{n+1}$ by Lemma 7.5.1 we must have $(\mathrm{l}_\alpha f)^\uparrow = \mathrm{l}_\alpha \mathfrak{m} \in \mathfrak{M}$. In order to show that $(\mathfrak{n}_i(f))_{i\in I}$ is a Noetherian family, we have to consider the family $(\log \mathfrak{n}_i(f))_{i\in I}$. We start with the following lemma.

Lemma 7.5.2 *Let $\mathbb{T}$ be of strength $n \geqslant 0$. Let $f \in \mathbb{T}_\infty^+$ be l_{ω^n}-atomic, $\mathfrak{m} = \mathfrak{d}_f$ and $\varepsilon = R_f$. Then for all $\alpha < \omega^{n+1}$:*

$$\operatorname{supp}(\mathrm{l}_{\alpha+1} f)^{\updownarrow} \subseteq (\operatorname{supp}\varepsilon)^\dagger \cdot \{\log^{-a}\mathfrak{m} \mid a : \alpha+1 \to \mathbb{N}^+,\ \textit{weakly decreasing}\}.$$

Proof: For $\alpha < \omega$, the lemma follows from Remark 5.1.6. Now let $i \leqslant n$, and suppose that the lemma holds for all ordinals $\alpha < \omega^i$. We first treat the case $\alpha = \omega^i$ and then by induction the case $\omega^i < \alpha < \omega^{i+1}$. By Lemma 7.4.9, the monomial $\mathfrak{m}$ is l_{ω^i}-atomic for $i \leqslant n$.

From $\mathrm{l}_{\omega^i+1} f = \log \mathrm{l}_{\omega^i} f$ and $\mathrm{l}_{\omega^i} f = \mathcal{T}_{\mathrm{l}_{\omega^i}}(\mathfrak{m},\varepsilon)$ we obtain that

$$\mathrm{l}_{\omega^i+1} f = \log \mathcal{T}_{\mathrm{l}_{\omega^i}}(\mathfrak{m},\varepsilon).$$

Thus

$$(\mathrm{l}_{\omega^i+1} f)^{\updownarrow} = \left(\log\circ \mathrm{l}_{\omega^i}\mathfrak{m}\cdot\left(1+\frac{\mathcal{R}_{\mathrm{l}_{\omega^i}}(\mathfrak{m},\varepsilon)}{\mathrm{l}_{\omega^i}\mathfrak{m}}\right)\right)^{\updownarrow} = l\left(\frac{\mathcal{R}_{\mathrm{l}_{\omega^i}}(\mathfrak{m},\varepsilon)}{\mathrm{l}_{\omega^i}\mathfrak{m}}\right),$$

where $l(X)$ is the formal power series $\sum_{1\leqslant i}\frac{(-1)^{i-1}}{i}X^i$. This implies

$$\operatorname{supp}(\mathrm{l}_{\omega^i+1} f)^{\updownarrow} \subseteq \left(\operatorname{supp}\frac{\mathcal{R}_{\mathrm{l}_{\omega^i}}(\mathfrak{m},\varepsilon)}{\mathrm{l}_{\omega^i}\mathfrak{m}}\right)^\dagger.$$

Note that

$$\operatorname{supp}\frac{\mathcal{R}_{\mathrm{l}_{\omega^i}}(\mathfrak{m},\varepsilon)}{\mathrm{l}_{\omega^i}\mathfrak{m}} = \operatorname{supp}\sum_{1\leqslant j}\frac{1}{j!}\cdot\frac{\mathrm{l}_{\omega^i}^{(j)}\mathfrak{m}}{\mathrm{l}_{\omega^i}\mathfrak{m}}\cdot\varepsilon^j \subseteq (\operatorname{supp}\varepsilon)^\dagger\cdot\bigcup_{1\leqslant j}\operatorname{supp}\frac{\mathrm{l}_{\omega^i}^{(j)}\mathfrak{m}}{\mathrm{l}_{\omega^i}\mathfrak{m}}.$$

If $\mathfrak{a} \in \operatorname{supp} l_{\omega^i}^{(j)} \mathfrak{m}$ for some integer $j \geqslant 1$, then there is $\alpha \in (\omega^i)^{j-1}$ such that $\mathfrak{a} = l'_{\omega^i} \mathfrak{m} \cdot l'_{\alpha} \mathfrak{m}$, hence there is a function $\hat{a} : \omega^i \to \mathbb{N}^+$ which is weakly decreasing such that $\mathfrak{a} = \log^{-\hat{a}} \mathfrak{m}$. Thus for $\mathfrak{b} \in \operatorname{supp} l_{\omega^i}^{(j)} \mathfrak{m} / l_{\omega^i} \mathfrak{m}$ there is a function $a : \omega^{i+1} \to \mathbb{N}^+$, weakly decreasing, such that $\mathfrak{b} = \log^{-a} \mathfrak{m}$. Hence

$$\begin{aligned} \operatorname{supp}(l_{\omega^i+1} f)^{\downarrow} \subseteq & \quad ((\operatorname{supp} \varepsilon)^{\dagger} \cdot \{\log^{-a} \mathfrak{m} \mid a : \omega^{i+1} \to \mathbb{N}^+, \text{ weakly decreasing}\})^{\dagger} \\ = & \quad (\operatorname{supp} \varepsilon)^{\dagger} \cdot \{\log^{-a} \mathfrak{m} \mid a : \omega^{i+1} \to \mathbb{N}^+, \text{ weakly decreasing}\} \end{aligned}$$

This shows the case $\alpha = \omega^i$.

Now suppose that $\alpha = \beta + 1$ where $\omega^i \leqslant \beta < \omega^{i+1}$, and suppose that the lemma is true for β. From $l_\alpha f = l_\alpha \mathfrak{m} + \varepsilon_\alpha = l_\alpha \mathfrak{m} + \varepsilon_{\beta+1}$ we obtain

$$(l_{\alpha+1} f)^{\downarrow} = \left(\log \circ l_\alpha \mathfrak{m} \cdot \left(1 + \frac{\varepsilon_{\beta+1}}{l_\alpha \mathfrak{m}}\right)\right)^{\downarrow} = l\left(\frac{\varepsilon_{\beta+1}}{l_\alpha \mathfrak{m}}\right),$$

thus

$$\operatorname{supp}(l_{\alpha+1} f)^{\downarrow} \subseteq \left(\operatorname{supp}\left(\frac{\varepsilon_{\beta+1}}{l_\alpha \mathfrak{m}}\right)\right)^{\dagger}.$$

Now let $\mathfrak{a} \in \operatorname{supp}(l_{\alpha+1} f)^{\downarrow}$, then for some integer $j \geqslant 1$ there are $\mathfrak{b}_1, \ldots, \mathfrak{b}_j \in \operatorname{supp} \varepsilon_{\beta+1}$ such that

$$\mathfrak{a} = \frac{\mathfrak{b}_1}{l_\alpha \mathfrak{m}} \cdots \frac{\mathfrak{b}_j}{l_\alpha \mathfrak{m}}.$$

By the inductive hypothesis for all $1 \leqslant m \leqslant j$ there are weakly decreasing functions $a_m : \alpha \to \mathbb{N}^+$ such that $\mathfrak{b}_m \in \log^{-a_m} \mathfrak{m} \cdot (\operatorname{supp} \varepsilon)^{\dagger}$. The function $a_1 + \cdots + a_j$ is weakly decreasing, and for all $\gamma < \alpha$ we have $(a_1 + \cdots + a_j)_\gamma \geqslant j$. Hence the function $a : \alpha + 1 \to \mathbb{N}^+$ with

$$a_\gamma := \begin{cases} (a_1 + \cdots + a_j)_\gamma & \text{if } \gamma < \alpha \\ j & \text{if } \gamma = \alpha \end{cases}$$

is weakly decreasing and $\mathfrak{a} \in \log^{-a} \mathfrak{m} \cdot (\operatorname{supp} \varepsilon)^{\dagger}$. This shows the remaining case and finishes the proof. □

Step 3: Note that the set

$$\bigcup_{\alpha < \omega^{n+1}} \{\log^{-a} \mathfrak{m} \mid a : \alpha + 1 \to \mathbb{N}^+, \text{ weakly decreasing}\}$$

is a subset of the set

$$\{\log^{-a} \mathfrak{m} \mid a : \omega^{n+1} \to \mathbb{N}^+, \text{ weakly decreasing}\}.$$

Then from Lemma 7.5.2 it follows that

$$\begin{aligned} \bigcup_{\alpha < \omega^{n+1}} \operatorname{supp}(l_{\alpha+1} f)^{\downarrow} & \subseteq \quad (\operatorname{supp} \varepsilon)^{\dagger} \cdot \bigcup_{\alpha < \omega^{n+1}} \{\log^{-a} \mathfrak{m} \mid a : \alpha + 1 \to \mathbb{N}^+, \text{ weakly decreasing}\} \\ & \subseteq \quad (\operatorname{supp} \varepsilon)^{\dagger} \cdot \{\log^{-a} \mathfrak{m} \mid a : \omega^{n+1} \to \mathbb{N}^+, \text{ weakly decreasing}\}. \end{aligned}$$

The set $\{\log^{-a}\mathfrak{m} \mid a : \omega^{n+1} \to \mathbb{N}^+,\ \text{weakly decreasing}\}$ is by Lemma 7.5.1 well-ordered in $\mathfrak{M}$. The set $(\operatorname{supp}\varepsilon)^\dagger$ is by Lemma 1.1.5 well-ordered. Hence the set $\bigcup_{\alpha<\omega^{n+1}} \operatorname{supp}(\mathrm{l}_{\alpha+1}f)^{\downarrow}$ is well-ordered in $\mathfrak{M}$.

Now let $(\mathfrak{n}_i)_{i\in I} \subseteq \mathfrak{B}_n$ be a well-ordered sequence with $\forall i \in I : \exists\alpha_i \in \omega^{n+1}$ and a function $a_i : \alpha_i \to \mathbb{Z}$ such that $\mathfrak{n}_i = \log^a x$. Then $\mathfrak{n}_i(f) = \log^a f$ implies

$$\mathfrak{n}_i(f) = \exp \sum_{\beta<\alpha_i} a_{i,\beta} \cdot \mathrm{l}_{\beta+1} f = \exp \sum_{\beta<\alpha_i} a_{i,\beta} \cdot (\mathrm{l}_{\beta+1}\mathfrak{m} + (\mathrm{l}_{\beta+1}f)^{\downarrow}),$$

hence $\mathfrak{n}_i(f) = \mathfrak{n}_i(\mathfrak{m}) \cdot e(\sum_{\beta<\alpha_i} a_{i,\beta} \cdot (\mathrm{l}_{\beta+1}f)^{\downarrow})$. Therefore

$$\operatorname{supp}\mathfrak{n}_i(f) \subseteq \mathfrak{n}_i(\mathfrak{m}) \cdot \left(\operatorname{supp} \sum_{\beta<\alpha_i} a_{i,\beta} \cdot (\mathrm{l}_{\beta+1}f)^{\downarrow}\right)^{\Diamond}.$$

From the above we conclude

$$\bigcup_{i\in I} \operatorname{supp}\mathfrak{n}_i(f) \subseteq \{\mathfrak{n}_i(\mathfrak{m}) \mid i \in I\} \cdot (\operatorname{supp}\varepsilon)^{\Diamond} \cdot \{\log^{-a}\mathfrak{m} \mid a : \omega^{n+1} \to \mathbb{N}^+,\ \text{weakly decreasing}\}.$$

Thus $\bigcup_{i\in I} \operatorname{supp}\mathfrak{n}_i(f)$ is contained in a well-ordered set and therefore well-ordered itself. Lemma 1.1.6 now implies that for all monomials $\mathfrak{a}$ from $\bigcup_{i\in I} \operatorname{supp}\mathfrak{n}_i(f)$ there are only finitely many $i \in I$ such that $\mathfrak{a} \in \operatorname{supp}\mathfrak{n}_i(f)$. Thus $(\mathfrak{n}_i(f))_{i\in I}$ is a Noetherian family. This finishes the proof of $\mathbf{(PC3)}_{\boldsymbol{n}}$. We therefore have proved

Proposition 7.5.3 *The partial composition result* $\mathbf{(PC)}_{\boldsymbol{n}}$ *holds for fields of strength* $n \geqslant 0$.

We finish this section with the following corollary.

Corollary 7.5.4 *Let* $\mathbb{T}$ *be of strength* $n \geqslant 0$. *Suppose that* $\mathfrak{m} \in \mathfrak{M}^{\uparrow}$ *is* l_{ω^n}*-atomic. Then:*

(1) $\mathrm{l}'_{\omega^{n+1}}\mathfrak{m} \in \mathfrak{M}_{\exp}$.
(2) For all $i \geqslant 1$ *and all* $\mathfrak{n} \in \mathrm{l}^{(i)}_{\omega^{n+1}}\mathfrak{m}$ *there is a weakly decreasing function* $a : \omega^{n+1} \to \mathbb{N}$ *such that* $\mathfrak{n} = \mathrm{l}'_{\omega^{n+1}}\mathfrak{m} \cdot \log^{-a}\mathfrak{m}$.
(3) For all $\mathfrak{n} \in \operatorname{supp}\mathcal{R}_{\mathrm{l}_{\omega^{n+1}}}(\mathfrak{m}, \varepsilon)$ *with* $\varepsilon \preccurlyeq 1$ *there is a weakly decreasing function* $a : \omega^{n+1} \to \mathbb{N}$ *such that* $\mathfrak{n} \preccurlyeq \mathrm{l}'_{\omega^{n+1}}\mathfrak{m} \cdot \log^{-a}\mathfrak{m}$.

Proof: (1) follows from $\log \mathrm{l}'_{\omega^{n+1}}\mathfrak{m} = -\sum_{\alpha<\omega^{n+1}} \mathrm{l}_{\alpha+1}\mathfrak{m} \in \mathbb{T}^{\uparrow}$. In order to show (2), recall that

$$\mathrm{l}^{(i)}_{\omega^{n+1}}\mathfrak{m} = \mathrm{l}'_{\omega^{n+1}}\mathfrak{m} \cdot \sum_{\alpha\in(\omega^{n+1})^{i-1}} a_\alpha \cdot \mathrm{l}'_{\alpha+\bar{1}}\mathfrak{m}$$

for integers a_α. Hence for $\mathfrak{n} \in \operatorname{supp}\mathrm{l}^{(i)}_{\omega^{n+1}}\mathfrak{m}$ there is an $\alpha \in (\omega^{n+1})^{i-1}$ such that $\mathfrak{n} = \mathrm{l}'_{\omega^{n+1}}\mathfrak{m} \cdot \mathrm{l}'_{\alpha+\bar{1}}\mathfrak{m}$. This shows (2). (3) follows from (2). □

7.6 Examples of fields of positive strength

We will give examples of fields of arbitrary strength. In fact, the example of a field of strength $n \geqslant 0$ will be contained in the example of the field of strength $m > n$. This is due to our choice of the monomial group.

Recall from Section 7.3.4 that for all $n \geqslant 0$ we have defined totally ordered, multiplicative groups $\mathfrak{B}_n$. We extend these groups to sets $\mathfrak{L}_n = \{\log^a x \,|\, a : \omega^{n+1} \to \mathbb{Z}\}$. Note that $\mathfrak{L}_0 = \log^{\mathbb{Z}^{\star\star}} x$. We define a group structure on each $\mathfrak{L}_n$ as follows. Let $a, b : \omega^{n+1} \to \mathbb{Z}$, then

$$\begin{aligned} 1 &:= \log^0 x \\ \log^a x \cdot \log^b x &:= \log^{a+b} x \\ (\log^a x)^{-1} &:= \log^{-a} x. \end{aligned}$$

We let $1 \prec \mathfrak{m} = \log^a x$ iff for $M_{\mathfrak{m}} = \min\{\gamma \,|\, a_\gamma \neq 0\}$ we have $0 < a_{M_{\mathfrak{m}}}$. Hence each $\mathfrak{L}_n$ is a totally ordered group, and $\mathfrak{L}_n$ is a subgroup of $\mathfrak{L}_{n+1}$ with $\mathfrak{L}_0 \subsetneq \mathfrak{L}_1 \subsetneq \mathfrak{L}_2 \subsetneq \cdots$. The field $\mathbb{L}_n := C[[\mathfrak{L}_n]]$ will be our example for strength n. Note that

$$\mathbb{L}_0 \subsetneq \mathbb{L}_1 \subsetneq \mathbb{L}_2 \subsetneq \cdots .$$

Fix an integer $n \geqslant 0$. In the following, we will equip $\mathbb{L}_n$ with functions $\log = \mathrm{l}, \mathrm{l}_\omega, \ldots, \mathrm{l}_{\omega^n}$ such that for each $i < n$ the structure

$$\langle \mathbb{L}_n, \log, \ldots, \mathrm{l}_{\omega^i} \rangle$$

is of strength i. We then use the partial composition result $\mathbf{(PC)}_{\boldsymbol{i}}$ to define a function $\mathrm{l}_{\omega^{i+1}}$. This will eventually lead to a field of strength n.

We begin by defining a logarithmic function log on $\mathbb{L}_n$. Let $\log^a x \in \mathfrak{L}_n$ and $f = c\mathfrak{m} \cdot (1+\delta) \in \mathbb{L}_n$, then we let

$$\begin{aligned} \log(\log^a x) &:= \sum_{\gamma < \omega^{n+1}} a_\gamma \cdot \mathrm{l}_{\gamma+1} x \\ \log f &:= \log \mathfrak{m} + \log c + l(\delta). \end{aligned}$$

One verifies that $\log \mathfrak{m} \prec \mathfrak{m}$ for all $\mathfrak{m} \in \mathfrak{L}_n$. Hence, $\mathbb{L}_n$ is an exponential field. From the definition of the ordering on $\mathfrak{L}_n$ we obtain that $\{\mathrm{l}_\alpha x \,|\, \alpha < \omega^{n+1}\} \subseteq \mathfrak{L}_n^{\uparrow}$ is well-ordered. Thus $\log \mathfrak{L}_n \subseteq \mathbb{L}_n^{\uparrow}$. Remark that each $f \in (\mathbb{L}_n)^+_\infty$ is log-confluent at order 2. For the same reason, condition **T4** of the definition of transseries fields holds. We have therefore proved

Lemma 7.6.1 *Let $n \geqslant 0$ and $\mathfrak{L}_n = \{\log^a x \mid a : \omega^{n+1} \to \mathbb{Z}\}$. Define an ordered group structure on $\mathfrak{L}_n$ as above. Then $\mathbb{L}_n = C[[\mathfrak{L}_n]]$ is a transseries field, i.e. of strength* 0.

The next step is to define functions $\mathrm{l}_\omega, \ldots, \mathrm{l}_{\omega^n}$ such that $\langle \mathbb{L}_n, \log, \ldots, \mathrm{l}_{\omega^i} \rangle$ is of strength i. In fact for every $i < n$ we will show that for our choice of l_{ω^i} we have

- $\langle \mathbb{L}_n, \log, \ldots, \mathrm{l}_{\omega^i}\rangle$ is of strength i,
- $(\mathbb{L}_n)^+_\infty$ admits a partial function $\mathrm{l}_{\omega^{i+1}}$.

Having these properties will suffice to extend the structure $\langle \mathbb{L}_n, \log, \ldots, \mathrm{l}_{\omega^i}\rangle$ to a field of strength $i+1$.

Lemma 7.6.2 *Let $n \geqslant 0$, then there are functions* $\log, \ldots, \mathrm{l}_{\omega^n}$ *such that $\mathbb{L}_n$ is of strength n.*

Proof: The lemma holds for $n = 0$. Now suppose that $i < n$ is such that

$(\mathbf{l}_{\omega^i})_\mathbf{1}$ $\langle \mathbb{L}_n, \log, \ldots, \mathrm{l}_{\omega^i}\rangle$ is of strength i,

$(\mathbf{l}_{\omega^i})_\mathbf{2}$ $(\mathbb{L}_n)^+_\infty$ is l_{ω^i}-confluent at order 2, and for $f \in (\mathbb{L}_n)^+_\infty$ there is an ordinal

$$\beta = \omega^n\, b_n + \cdots + \omega^i\, b_i < \omega^{n+1}$$

with $b_i \geqslant 2$ and $\mathfrak{d}_{\mathrm{l}_{\omega^i.2} f} = \mathrm{l}_\beta x$,

$(\mathbf{l}_{\omega^i})_\mathbf{3}$ $(\mathbb{L}_n)^+_\infty$ admits a partial function $\mathrm{l}_{\omega^{i+1}}$ such that for all $f \in (\mathbb{L}_n)^+_\infty$ we have $\mathfrak{d}_{\mathrm{l}_{\omega^i.2} f} \in \operatorname{dom} \mathrm{l}_{\omega^{i+1}}$.

We start with the case $i = 0$. Condition $(\mathbf{log})_\mathbf{1}$ follows from Lemma 7.6.1. Note that $(\mathbb{L}_n)^+_\infty$ is log-confluent at order 2. From $\mathfrak{d}_{\log_2 f} = \mathfrak{d}_{\log_2 \mathfrak{d}_f}$ it follows that we only need to consider $\mathfrak{L}_n^\uparrow$. Let $\mathfrak{m} = \log^a x \in \mathfrak{L}_n^\uparrow$, then

$$\log_2 \mathfrak{m} = \log(a_\alpha \cdot \mathrm{l}_{\alpha+1} x \cdot (1+\rho)) = \mathrm{l}_{\alpha+2} x + \varepsilon$$

where $\alpha < \omega^{n+1}$ and $\varepsilon \preccurlyeq 1$. From $\alpha = \omega^n\, a_n + \cdots + a_0$ we obtain $\alpha + 2 = \omega^n\, a_n + \cdots + (a_0 + 2)$. Hence the monomial $\mathrm{l}_{\alpha+2} x$ is log-atomic and $a_0 + 2 \geqslant 2$. Thus $(\mathbf{log})_\mathbf{2}$. Finally we let

$$\begin{aligned} \beta &:= \omega^n\, a_n + \cdots + \omega\,(a_1 + 1) \\ \mathrm{l}_\omega(\mathrm{l}_{\alpha+2} x) &:= \mathrm{l}_\beta x - (a_0 + 2). \end{aligned}$$

This shows $(\mathbf{log})_\mathbf{3}$ and therefore the initial case.

Now suppose that $(\mathbf{l}_{\omega^i})_\mathbf{1}$ – $(\mathbf{l}_{\omega^i})_\mathbf{3}$ are satisfied for $i < n$. We define the function $\mathrm{l}_{\omega^{i+1}}$ for a series $f \in (\mathbb{L}_n)^+_\infty$ by

$$\mathrm{l}_{\omega^{i+1}} f := 2 + \mathrm{l}_{\omega^{i+1}}(\mathrm{l}_{\omega^i.2} f) = 2 + \mathrm{l}_{\omega^{i+1}}(\mathfrak{d}_{\mathrm{l}_{\omega^i.2} f} + \varepsilon) = 2 + \mathcal{T}_{\mathrm{l}_{\omega^{i+1}}}(\mathfrak{d}_{\mathrm{l}_{\omega^i.2} f}, \varepsilon).$$

From Chapter 6 it now follows that this is a coherent definition of a logarithmic function of strength $i+1$, which proves $\mathbf{T^{i+1}1}$ and $\mathbf{T^{i+1}1}$. In order to show $\mathbf{T^{i+1}3}$, let $f \in \operatorname{dom} \mathrm{e}_{\omega^{i+1}}$ with $\operatorname{supp} f^\downarrow \prec_{\mathrm{l}_{\omega^i}} \mathrm{e}_{\omega^{i+1}} f$. Then there is some $h \in (\mathbb{L}_n)^+_\infty$ with $f = \mathrm{l}_{\omega^{i+1}} h$. From the definition of $\mathrm{l}_{\omega^{i+1}}$ we then obtain

$$\begin{aligned} f &= 2 + \mathrm{l}_{\omega^{i+1}}(\mathfrak{d}_{\mathrm{l}_{\omega^i.2} h} + \varepsilon) \\ &= \mathrm{l}_{\omega^{i+1}}(\mathfrak{d}_{\mathrm{l}_{\omega^i.2} h}) + 2 + \mathcal{R}_{\mathrm{l}_{\omega^{i+1}}}(\mathfrak{d}_{\mathrm{l}_{\omega^i.2} h}, \varepsilon) \end{aligned}$$

where $\varepsilon \preccurlyeq 1$. Let $\beta = \omega^n\, b_n + \cdots + \omega^i\, b_i$ such that $\mathfrak{d}_{\mathrm{l}_{\omega^i.2} h} = \mathrm{l}_\beta x$ and $b_i \geqslant 2$. Then from

$$\mathrm{l}_{\omega^{i+1}}(\mathrm{l}_\beta x) + 2 = \mathrm{l}_{\omega^n\, b_n + \cdots + \omega^{i+1}\,(b_{i+1}+1)} x - (b_i - 2) \in \mathfrak{L}_n^\uparrow + \mathbb{Z}$$

we conclude that $f^{\downarrow} = \mathcal{R}_{\mathrm{l}_{\omega^{i+1}}}(\mathrm{l}_\beta x, \varepsilon)$. From Lemma 7.5.4 it now follows that for all $\mathfrak{n} \in \operatorname{supp} f^{\downarrow}$ there is a weakly decreasing function $a : \omega^{i+1} \to \mathbb{N}$ such that

$$\mathfrak{n} \preccurlyeq \mathrm{l}'_{\omega^{i+1}}(\mathrm{l}_\beta x) \cdot \log^{-a}(\mathrm{l}_\beta x).$$

Hence

$$\forall j \geqslant 1 : \qquad \|\mathfrak{n}\| = \mathfrak{n}^{-1} = (\mathrm{l}_\beta x)^{a_0+1}(\mathrm{l}_{\beta+1} x)^{a_1+1} \cdots \succ \mathrm{l}_{\beta+j} x.$$

From $\mathrm{e}_{\omega^{i+1}} f = h$ we obtain for $j \geqslant 3$ that $\mathrm{l}_{\omega^i \cdot j}(\mathrm{e}_{\omega^{i+1}} f) = \mathrm{l}_{\omega^i \cdot j} h \asymp \mathrm{l}_{\omega^i \cdot j}(\mathrm{l}_\beta x)$. This contradicts the assumption $\operatorname{supp} f^{\downarrow} \prec_{\mathrm{l}_{\omega^i}} \mathrm{e}_{\omega^{i+1}} f$. Therefore $\varepsilon = 0$ and

$$\mathrm{e}_{\omega^{i+1}} f = \mathrm{l}_{\omega^i \cdot 2}(\mathrm{l}_\beta x) \in \mathfrak{L}_n^{\uparrow},$$

which shows $\mathbf{T^{i+1}3}$. As for $\mathbf{T^{i+1}4}$ we remark that fom the definition of $\mathrm{l}_{\omega^{i+1}}$ for a series f we obtain that

$$(\mathrm{l}_{\omega^{i+1}} f)^{\uparrow} = \mathrm{l}_{\omega^{i+1}}(\mathfrak{d}_{\mathrm{l}_{\omega^i \cdot 2} f})^{\uparrow} = (\mathrm{l}_{\omega^{i+1}} \mathrm{l}_\beta x)^{\uparrow}$$

where $\beta = \omega^n\, b_n + \cdots + \omega^i\, b_i$. Thus $\mathrm{l}_{\omega^{i+1}} \circ \mathrm{l}_\beta x = \mathrm{l}_\gamma x - b_i$ with $\gamma = \omega^n\, b_n + \cdots + \omega^{i+1} \cdot (b_{i+1} + 1)$. Now $\mathrm{l}_\gamma x = (\mathrm{l}_{\omega^{i+1}} \circ \mathrm{l}_\beta x)^{\uparrow}$ is $\mathrm{l}_{\omega^{i+1}}$-atomic. Hence every $f \in (\mathbb{L}_n)^+_\infty$ is $\mathrm{l}_{\omega^{i+1}}$-atomic. This shows $(\mathbf{l}_{\boldsymbol{\omega}^{i+1}})_{\mathbf{1}}$ and $(\mathbf{l}_{\boldsymbol{\omega}^{i+1}})_{\mathbf{2}}$, and what is more, we have $\mathbf{T^{i+1}4}$.

We have to define a function $\mathrm{l}_{\omega^{i+2}}$ for all $\mathrm{l}_\beta x$ with $\beta = \omega^n\, b_n + \cdots + \omega^{i+1}\, b_{i+1}$. We let

$$\mathrm{l}_{\omega^{i+2}}(\mathrm{l}_\beta x) := \mathrm{l}_{\omega^n\, b_n + \cdots + \omega^{i+2}\,(b_{i+2}+1)} x - b_{i+1}.$$

This shows $(\mathbf{l}_{\boldsymbol{\omega}^{i+1}})_{\mathbf{3}}$ and completes the proof. □

Now we have examples of fields of positive strength. Let us generate more such fields. In the following, we give first applications of the properties shown in Sections 7.4 and 7.5.

Lemma 7.6.3 *Let $\mathbb{T} = C[[\mathfrak{M}]]$ be of strength $n \geqslant 0$. Then its exp-extension $\mathbb{T}_{\exp}$ is of strength i for all $i \leqslant n$. In particular, the field $\mathbb{T}_{\exp}$ is of strength n.*

During the proof of the lemma, we will use the following fact, which will also have applications in the next chapter.

Fact 7.6.4 *Let $i \geqslant 0$ and $\mathbb{T} \subseteq \hat{\mathbb{T}}$ fields of generalized power series such that $\mathbb{T}$ is of strength $i+1$ and such that $\hat{\mathbb{T}}$ is of strength i. Let $f, h \in \hat{\mathbb{T}}^+_\infty$ be such that*

(1) there is a $k \in \mathbb{N}$ such that $\mathrm{l}_{\omega^i \cdot k} h = \mathfrak{m} + \varepsilon$, $\varepsilon \preccurlyeq 1$ and such that $(\mathfrak{m}, \varepsilon)$ is a $\mathrm{l}_{\omega^{i+1}}$-Taylor couple.

(2) $f = k + \mathcal{T}_{\mathrm{l}_{\omega^{i+1}}}(\mathfrak{m}, \varepsilon)$.

We let in this case $\mathrm{e}_{\omega^{i+1}} f := h$. If $\operatorname{supp} f^{\downarrow} \prec_{\mathrm{l}_{\omega^i}} \mathrm{e}_{\omega^{i+1}} f$. Then either we have $\operatorname{supp} \mathcal{R}_{\mathrm{l}_{\omega^{i+1}}}(\mathfrak{m}, \varepsilon) \subseteq \operatorname{supp} \mathrm{l}_{\omega^{i+1}} \mathfrak{m}$ or we have $\mathcal{R}_{\mathrm{l}_{\omega^{i+1}}}(\mathfrak{m}, \varepsilon) = 0$.

Proof: Suppose that $0 \neq \mathfrak{n} \in \operatorname{supp} \mathcal{R}_{\mathrm{l}_{\omega^{i+1}}}(\mathfrak{m}, \varepsilon) \setminus \operatorname{supp} \mathrm{l}_{\omega^{i+1}} \mathfrak{m}$. We show $\varepsilon = 0$. Remark first that from Corollary 7.5.4 it follows that there is a weakly decreasing function $a : \omega^{i+1} \to \mathbb{N}$ such that

$$\mathfrak{n} \preccurlyeq \mathrm{l}'_{\omega^{i+1}} \mathfrak{m} \cdot \log^{-a} \mathfrak{m} \prec \frac{1}{\mathfrak{m}}.$$

Thus $\mathfrak{m} \prec \|\mathfrak{n}\|$. From $\|\mathfrak{n}\| \prec_{\mathrm{l}_{\omega^i}} \mathrm{e}_{\omega^{i+1}} f = h$ it follows that $\|\mathfrak{n}\| \prec \mathrm{l}_{\omega^i \cdot j} h$. In particular, for $j = k$, we obtain $\|\mathfrak{n}\| \prec \mathfrak{m} + \varepsilon$, hence

$$\mathfrak{m} \prec \|\mathfrak{n}\| \prec \mathfrak{m} + \varepsilon \asymp \mathfrak{m}.$$

This contradiction shows $\mathcal{R}_{\mathrm{l}_{\omega^{i+1}}}(\mathfrak{m}, \varepsilon) = 0$, thus the fact. □

We can now prove Lemma 7.6.3.

Proof: For $n = 0$ the lemma follows from Section 2.3.2. We assume from now on that $n > 0$. Recall that $\mathbb{T}_{\exp}$ is a transseries field. We have to define a function l_ω on $(\mathbb{T}_{\exp})^+_\infty$.

Let $f \in (\mathbb{T}_{\exp})^+_\infty$ with $f = ce^g \cdot (1 + \delta)$ where $0 < g \in \mathbb{T}^\uparrow$. Then $\mathfrak{d}_{\log f} = \mathfrak{d}_g \in \mathfrak{M}$, and there is an integer $k \in \mathbb{N}$ such that g is log-confluent at order k with $\mathfrak{d}_{\log_k g} \in \operatorname{dom} \mathrm{l}_\omega$. Hence $\log_{k+1} f = \mathfrak{m} + \varepsilon$ such that $\mathfrak{m} \in \operatorname{dom} \mathrm{l}_\omega$ and $\varepsilon \in \mathbb{T}^\downarrow_{\exp}$. Then we let

$$\mathrm{l}_\omega f := (k+1) + \mathrm{l}_\omega \mathfrak{m} + \mathcal{R}_{\mathrm{l}_\omega}(\mathfrak{m}, \varepsilon),$$

which shows $\mathbf{T^1 1}$ and $\mathbf{T^1 3}$.

In order to show $\mathbf{T^1 2}$ for $\mathbb{T}_{\exp}$, we fix $f \in \operatorname{dom} \mathrm{e}_\omega$ with $\operatorname{supp} f^\downarrow \prec_{\mathrm{l}_\omega} \mathrm{e}_\omega f$. Let $h \in \mathbb{T}_{\exp}$ such that $f = \mathrm{l}_\omega h$. Let $\mathfrak{m}, \varepsilon$ be as above with $\log_{k+1} h = \mathfrak{m} + \varepsilon$, i.e.

$$f = \mathrm{l}_\omega h = \mathrm{l}_\omega \mathfrak{m} + (k+1) + \mathcal{R}_{\mathrm{l}_\omega}(\mathfrak{m}, \varepsilon).$$

From Lemma 7.4.7 it now follows that $\operatorname{supp} \mathrm{l}_\omega \mathfrak{m} \succ \mathcal{R}_{\mathrm{l}_\omega}(\mathfrak{m}, \varepsilon)$. Hence

$$\operatorname{supp} f^\downarrow = \operatorname{supp} (\mathrm{l}_\omega \mathfrak{m})^\downarrow \coprod \operatorname{supp} \mathcal{R}_{\mathrm{l}_\omega}(\mathfrak{m}, \varepsilon).$$

Fact 7.6.4 implies $\mathrm{e}_\omega f = h = \exp_{k+1} \mathfrak{m} \in \mathfrak{M}_{\exp}$. Note that for $f \in \mathbb{T}_{\exp}$ we have $\tau_{\mathrm{l}_\omega f} \in \mathfrak{M}$. Thus $(\mathbb{T}_{\exp})^+_\infty$ is l_ω-confluent, i.e. $\mathbf{T^1 4}$ holds. This shows strength 1.

We show strength $i > 0$ inductively. Assume that $\mathbb{T}_{\exp}$ is of strength $i - 1$. Let $\phi = \mathrm{l}_{\omega^{i-1}}$, $\Psi = \mathrm{e}_{\omega^i}$ and $\Phi = \mathrm{l}_{\omega^i}$. Assume that for $i > 0$ the conditions

$\mathbf{(a)}_\phi$ $\forall f \in (\mathbb{T}_{\exp)})^+_\infty : \exists \mathfrak{n} \in \mathfrak{M}^\uparrow, \varepsilon \in \mathbb{T}^\downarrow_{\exp} : \phi f = \phi \mathfrak{n} + \varepsilon$,

$\mathbf{(b)}_\phi$ $\mathfrak{d}_{\phi_k \mathfrak{n}} \in \operatorname{dom} \Phi$ for some $k \in \mathbb{N}$.

hold. Note that these conditions are satisfied in the case $i = 1$. We have to define Φ on $(\mathbb{T}_{\exp})^+_\infty$. Fix $f \in (\mathbb{T}_{\exp})^+_\infty$ and let $k \in \mathbb{N}$, $\mathfrak{n}$ and ε as above. Then $\mathfrak{n}$ is ϕ-confluent at order k and

$$\phi_k f = \phi_{k-1}(\phi \mathfrak{n} + \varepsilon) = \phi_k \mathfrak{n} + \mathcal{R}_{\phi_{k-1}}(\phi \mathfrak{n}, \varepsilon).$$

We let

$$\Phi f := k + \Phi \phi_k \mathfrak{n} + \mathcal{R}_\Phi(\phi_k \mathfrak{n}, \mathcal{R}_{\phi_{k-1}}(\phi \mathfrak{n}, \varepsilon)).$$

With this definition we have $\mathbf{T}^{\boldsymbol{i+1}}\mathbf{1}$ and $\mathbf{T}^{\boldsymbol{i+1}}\mathbf{1}$. Moreover $\phi_k \mathfrak{n} = \hat{\mathfrak{n}} + \mu$ for some $\mu \in \mathbb{T}^{\mapsdown}$. From $\mathcal{R}_{\phi_{k-1}}(\phi\mathfrak{n}, \varepsilon) \in \mathbb{T}^{\mapsdown}_{\exp}$ and $\hat{\mathfrak{n}} \in \operatorname{dom}\Phi$ it now follows that for

$$\rho = \mu + \mathcal{R}_{\phi_{k-1}}(\phi\mathfrak{n}, \varepsilon) \in \mathbb{T}^{\mapsdown}_{\exp}$$

we have $\phi_k f = \hat{\mathfrak{n}} + \rho$ and

$$\Phi f = \Phi\hat{\mathfrak{n}} + k + \mathcal{R}_{\Phi}(\hat{\mathfrak{n}}, \rho) = \Phi\hat{\mathfrak{n}} + \hat{\varepsilon},$$

where $\hat{\varepsilon} \in \mathbb{T}^{\mapsdown}_{\exp}$. Since $\mathfrak{n} \in \mathfrak{M}^{\uparrow}$ and since $\mathbb{T}$ is of strength i, it follows that $\hat{\mathfrak{n}} \in \mathfrak{M}^{\uparrow}$ is Φ-confluent at order $l \in \mathbb{N}$ such that $\mathfrak{d}_{\Phi_l \hat{\mathfrak{n}}} \in \operatorname{dom} \mathrm{e}_{\omega^{i+2}}$. This shows (a)$_\Phi$ and (b)$_\Phi$.

We have to show $\mathbf{T}^{\boldsymbol{i+1}}\mathbf{2}$. The fact that $f \in (\mathbb{T}_{\exp})^{+}_{\infty}$ is l_{ω^i}-confluent follows from $\mathbf{T}^{\boldsymbol{i}}\mathbf{4}$ for $\mathbb{T}$. Hence is suffices to show $\mathbf{T}^{\boldsymbol{i+1}}\mathbf{2}$. Let $f \in \operatorname{dom} \mathrm{e}_{\omega^{i+1}}$. Then there is a series $h \in \mathbb{T}_{\exp}$ such that $f = \mathrm{l}_{\omega^{i+1}} h = \mathrm{l}_{\omega^{i+1}} \hat{\mathfrak{n}} + \hat{\varepsilon}$ as above. Again, Lemma 7.4.7 implies that

$$\operatorname{supp} \mathrm{l}_{\omega^{i+1}} \hat{\mathfrak{n}} \succ \mathcal{R}_{\mathrm{l}_{\omega^{i+1}}}(\hat{\mathfrak{n}}, \rho).$$

Hence $\operatorname{supp} f^{\downarrow} = \operatorname{supp} (\mathrm{l}_{\omega^{i+1}} \hat{\mathfrak{n}})^{\downarrow} \coprod \operatorname{supp} \mathcal{R}_{\mathrm{l}_{\omega^{i+1}}}(\hat{\mathfrak{n}}, \rho)$. Applying Fact 7.6.4 yields $\rho = 0$. Thus $f = \mathrm{l}_{\omega^{i+1}} \hat{\mathfrak{n}} + k \in \mathbb{T}$. Since $\mathbb{T}$ is a transseries field, we then obtain $\mathrm{e}_{\omega^{i+1}} f \in \mathfrak{M} \subseteq \mathfrak{M}_{\exp}$. This finishes the proof. $\square$

Chapter 8

Extending transseries fields of positive strength

In Chapter 7, we have defined the notion of transseries fields of positive strength, we have shown some basic properties of such fields, and we have given some examples. We have also shown that the exponential extension of a transseries field of strength $n \geqslant 0$ is again of strength n. The present chapter is concerned with generalizing the latter result.

8.1 The general outline of the extension process

Recall that for transseries fields $\mathbb{T} = C[[\mathfrak{M}]]$, the logarithm is totally defined on $\mathbb{T}^+$, but that the exponential function is not total on $\mathbb{T}^+$. An immediate consequence is that the same remains true for logarithmic and exponential functions of positive strength. We have seen, however, that we can construct a field of generalized power series such that the logarithmic and exponential function are total on the set of positive elements of this set. This field was called the exponential closure, and it was constructed as the inductive limit of a chain of transeries fields. We recall that the exponential closure is not of the form $C[[\mathfrak{N}]]$ anymore.

In what follows, we will employ the same idea to construct fields of generalized power series with total logarithmic and exponential function of positive strength n on the sets of positive and infinite elements. Again, the resulting field cannot be of the form $C[[\mathfrak{N}]]$.

The construction requires a number of steps; and we will treat the steps separately and add remarks about the motivation of the definitions in every step. Although this might lengthen the construction, we have chosen to do so because we think that knowing what motivates the definitions makes it easier to follow the necessary proofs. However, the reader may always skip the explanations and go straight to the definitions.

Recall that in order to construct an exponential extension of some transseries field, we have first defined a set of new monomials (which included the monomials of the field which was to be extended), that we had to define a multiplication and an ordering on the set of new monomials and that in a third step we had to define a logarithm on the extended field.

In the case of positive strength, the method will be carried out along the same lines. We have, however, to be more careful when choosing the set of monomials in the first step. As a result, the definitions of the multiplication and the ordering become slightly more difficult. The third step will then be broken down into a number of sub-tasks. We have to define functions $\log = \mathrm{l}, \mathrm{l}_\omega, \ldots$ such that the new field is of strength i for the function l_{ω^i}. Hence we have to start with strength 0, then we treat the case of strength 1 and so on.

Fix a transseries field $\mathbb{T} = C[[\mathfrak{M}]]$ of strength $n > 0$. We will define the e_{ω^n}-extension of $\mathbb{T}$. Moreover, we will show that if $\mathbb{T}$ is of strength $N > n$, then its e_{ω^n}-extension is also of strength N. Note that that the case $n = 0$ has been treated in Section 7.6. For integers $n > 0$ we will now

- define the extended set of monomials $\mathfrak{M}_{\mathrm{e}_{\omega^n}} \supseteq \mathfrak{M}$;
- define a multiplication and a total ordering on $\mathfrak{M}_{\mathrm{e}_{\omega^n}}$ such that $\mathfrak{M}$ is a totally ordered subgroup of $\mathfrak{M}_{\mathrm{e}_{\omega^n}}$;
- define a logarithm on $\mathbb{T}_{\mathrm{e}_{\omega^n}} = C[[\mathfrak{M}_{\mathrm{e}_{\omega^n}}]]$ such that $\langle \mathbb{T}_{\mathrm{e}_{\omega^n}}, \log \rangle$ is a transseries field;
- inductively define functions $\mathrm{l}_\omega, \ldots, \mathrm{l}_{\omega^N}$ such that the structure $\langle \mathbb{T}_{\mathrm{e}_{\omega^n}}, \log, \ldots, \mathrm{l}_{\omega^i} \rangle$ is of strength i for all $i \leqslant N$.

8.2 New monomials of strength n

Recall that for all $f \in \mathbb{T}^+_\infty$ we eventually want an extension $\hat{\mathbb{T}} \supseteq \mathbb{T}$ of strength N such that $f \in \operatorname{dom} \mathrm{e}_{\omega^n}$ in $\hat{\mathbb{T}}$. In the case of the exponential extension $\mathbb{T}_{\exp}$, we have chosen a truncation of f (namely its infinite part $f^\uparrow$) and we have added the exponential of this truncation as a new monomial.

We had thus obtained a set $\mathfrak{M}_{\exp} = \{\exp f^\uparrow \mid f \in \mathbb{T}^+_\infty\}$ which could be equipped with a multiplication and an ordering in a canonical way. Moreover, we have seen that $\mathfrak{M} \subseteq \mathfrak{M}_{\exp}$. Note that $\mathfrak{M}_{\exp} = \{\exp f \mid f \in \mathbb{T}^+_\infty : f = f^\uparrow\}$.

In the case of strength $n > 0$ we will proceed similarly. Given a series $f \in \mathbb{T}^+_\infty$, we determine a truncation $t_f \trianglelefteq f$ and let $\mathrm{e}_{\omega^n}(t_f)$ be a new monomial. Instead of determining the truncations t_f we can as well give the condition of when f coincides with this truncation (in analogy with the second way of writing the monomial group $\mathfrak{M}_{\exp}$ in the case $n = 0$).

Hence the question is: when is $\mathrm{e}_{\omega^n} f$ a new monomial? Generally speaking, we have to avoid two different kinds of instability which we call (in accordance with the coherence) horizontal and vertical instability.

- **The horizontal instability:** Suppose that we want to add $\mathrm{e}_{\omega^n} f$ as a monomial and that $g \triangleleft f$ is a proper truncation of f such that $f = g + \varepsilon$ and such that $\mathrm{e}_{\omega^n} g$ is also defined in the extended field. Then we have to make sure that (g, ε) is not an e_{ω^n}-Taylor couple, for otherwise $\mathrm{e}_{\omega^n} f$ could be developed into a series and therefore would not be a monomial.
- **The vertical instability:** Suppose that $f \notin \operatorname{dom} \mathrm{e}_{\omega^n}$ in $\mathbb{T}$, but that for some $k \in \mathbb{N}$ we have $f - k \in \operatorname{dom} \mathrm{e}_{\omega^n}$ in $\mathbb{T}$. In this case, we might let $\mathrm{e}_{\omega^n} f = \mathrm{e}_{\omega^{n_1}.k} \circ \mathrm{e}_{\omega^n}(f - k)$. Assuming that we already have defined the notion of $\mathrm{e}_{\omega^{n-1}}$-extensions, the series $\mathrm{e}_{\omega^{n-1}.k} \circ \mathrm{e}_{\omega^n}(f - k)$ would eventually be defined in an extension of $\mathbb{T}$. Hence vertical stability means that we have to make sure that $\mathrm{e}_{\omega^n}(f - k)$ is not horizontally instable for any $k \in \mathbb{N}$.

8.2.1 The criterion for new monomials

We start by introducing a new notation. For all $i \leqslant n$ and all $f \in \mathbb{T}_\infty^+$ we define the **exponential depth of strength i** in $\mathbb{T}$ by

$$\pi_{i,\mathbb{T}} f := \begin{cases} \min\{k \in \mathbb{N} \,|\, f - k \in \operatorname{dom} \mathrm{e}_{\omega^i}\} & \text{if there is a } k \in \mathbb{N} : f - k \in \operatorname{dom} \mathrm{e}_{\omega^i} \text{ in } \mathbb{T}, \\ \\ \infty & \text{otherwise.} \end{cases}$$

Example 8.2.1 We consider series from the field $\mathbb{T} = C[[\mathfrak{L}_1]]$. Let $L = \mathrm{l}_\omega$. First notice that the exponential depth of strength 1 of the series $Lx, L_2x, \ldots$ is 0, i.e. $\pi_{1,\mathbb{T}}(L_jx) = 0$ for all $j \geqslant 1$. On the other hand we have $\pi_{1,\mathbb{T}}(x) = \infty$. For all $N \in \mathbb{N}$ we have $\pi_{1,\mathbb{T}}(Lx + N) = N$. We remark that for all $f \in \mathbb{T}_\infty^+$ such that $f^{\uparrow}$ is not a singleton from the set $\{L_jx \,|\, j > 0\}$ we always have $\pi_{1,\mathbb{T}} f = \infty$.

In the following, we will only consider series f with $\pi_{n,\mathbb{T}} f = \infty$. Let $\hat{\mathbb{T}} \supseteq \mathbb{T}$ be of strength n such that f is in the domain of e_{ω^n} in $\hat{\mathbb{T}}$. For $\mathrm{e}_{\omega^n} f$ to be a new monomial, we demand that $\mathrm{e}_{\omega^n} f$ is a monomial in $\hat{\mathbb{T}}$. In other words, the series f is neither vertically nor horizontally instable. Consequently, in a first version, the criterion can be formulated (in $\hat{\mathbb{T}}$) as follows: $\mathrm{e}_{\omega^n} f$ is a new monomial iff

$$\forall k \in \mathbb{N} : \forall g \lhd f : (g - k, f - g) \text{ is not an } \mathrm{e}_{\omega^n}\text{-Taylor couple.} \qquad \textbf{C1.}$$

Let $k \in \mathbb{N}$, $g \lhd f$ and $\varepsilon = f - g$. Then $(g - k, \varepsilon)$ is an e_{ω^n}-Taylor couple if and only if the sequence $(\mathrm{e}_{\omega^n}^{(i)}(g - k) \cdot \varepsilon^i)_{0 \leqslant i}$ is a Noetherian family. If this is the case, then

$$(\mathrm{e}_{\omega^n}^{(i)}(g - k) \cdot \varepsilon^i)_{1 \leqslant i} \tag{8.1}$$

is a Noetherian family. From Lemma 7.3.3 it follows that there are series $\delta_i \in \hat{\mathbb{T}}^{\downarrow}$ such that

$$\mathrm{e}_{\omega^n}^{(i)}(g - k) \cdot \varepsilon^i = (\mathrm{e}_{\omega^n}^{'}(g - k) \cdot \varepsilon)^i \cdot (1 + \delta_i).$$

Hence, sequence (8.1) is Noetherian if and only if $1 \succ \mathrm{e}_{\omega^n}^{'}(g - k) \cdot \varepsilon$. Thus, if

$$1 \preccurlyeq \mathrm{e}_{\omega^n}^{'}(g - k) \cdot \varepsilon, \tag{8.2}$$

then the families are not Noetherian, and $\mathrm{e}_{\omega^n} f$ is therefore a new monomial. Note that if $1 \preccurlyeq \varepsilon$, then this is always the case. Hence we can restrict our criterion to all $\varepsilon \prec 1$. For such ε we have $f^{\updownarrow} = g^{\updownarrow}$. Since inequality (8.2) must hold for all $k \in \mathbb{N}$, we can restate the criterion **C1** in $\hat{\mathbb{T}}$ as follows: $\mathrm{e}_{\omega^n} f$ is a new monomial iff

$$\forall g \lhd f : \forall k \in \mathbb{N} : \forall \mathfrak{n} \in \operatorname{supp}(f - g)^{\downarrow} : \|\mathfrak{n}\| \prec \mathrm{e}_{\omega^n}^{'}(g - k). \qquad \textbf{C2.}$$

We remark that $\|\mathfrak{n}\| \prec \mathrm{e}_{\omega^n}^{'}(g - k)$ implies

$$\log \|\mathfrak{n}\| < \log \mathrm{e}_{\omega^n}(g - k) + \log \mathrm{e}_{\omega^n}(g - k - 1) + \cdots .$$

Since $\log \mathrm{e}_{\omega^n}(g-k-1) \prec \log \mathrm{e}_{\omega^n}(g-k)$ for all k, we obtain $\log \|\mathfrak{n}\| < \log \mathrm{e}_{\omega^n}(g-k)$. Thus criterion **C2** is in $\hat{\mathbb{T}}$ equivalent to the following criterion: $\mathrm{e}_{\omega^n} f$ is a new monomial iff

$$\forall g \lhd f : \forall k \in \mathbb{N} : \forall \mathfrak{n} \in \operatorname{supp}(f-g)^{\downarrow} : \|\mathfrak{n}\| \prec \mathrm{e}_{\omega^n}(g-k) \qquad \mathbf{C3.}$$

Since $\hat{\mathbb{T}}$ admits a strictly increasing function l_{ω^n}, criterion **C3** is in $\hat{\mathbb{T}}$ equivalent to: $\mathrm{e}_{\omega^n} f$ is a new monomial iff

$$\forall g \lhd f : \forall k \in \mathbb{N} : \forall \mathfrak{n} \in \operatorname{supp}(f-g)^{\downarrow} : \mathrm{l}_{\omega^n} \|\mathfrak{n}\| < g-k. \qquad \mathbf{C4.}$$

From $f^{\updownarrow} = g^{\updownarrow}$ it now follows that $\forall k \in \mathbb{N} : \mathrm{l}_{\omega^n}\|\mathfrak{n}\| < g-k$ if and only if $\forall k \in \mathbb{N} : \mathrm{l}_{\omega^n}\|\mathfrak{n}\| < f-k$. Moreover, the function l_{ω^n} from $\hat{\mathbb{T}}$ coincides with l_{ω^n} in $\mathbb{T}$. Since $\operatorname{supp} f \subseteq \mathfrak{M}$, the criterion can finally be formulated in $\mathbb{T}$ as follows:

$$\mathrm{e}_{\omega^n} f \text{ is a new monomial} \quad \text{iff} \quad \mathrm{l}_{\omega^n} \|\operatorname{supp} f^{\downarrow}\| < f - \mathbb{N}. \qquad \mathbf{C5.}$$

Example 8.2.2 Let $n=1$ and $E = \mathrm{e}_\omega$. We consider the series f, g with

$$\begin{aligned} f &= \quad Ex + \frac{1}{\exp_i Ex} \qquad (i \geqslant 0) \\ g &= \quad Ex + \frac{1}{\log_i E_2 x} \qquad (i \geqslant 0) \end{aligned}$$

Note that $f^{\uparrow} = g^{\uparrow} = Ex$. We claim that Ef verifies **C5**. To see this note that $\|\operatorname{supp} f^{\downarrow}\| = \{\exp_i Ex\}$ and that $L\|\operatorname{supp} f^{\downarrow}\| = x + i$. Thus

$$L\|\operatorname{supp} f^{\downarrow}\| = x + i < Ex - \mathbb{N}.$$

Hence $E(Ex + \frac{1}{\exp_i Ex})$ will be a monomial for every $i \geqslant 0$. On the other hand, Eg does not verify the criterion **C5**, since $\|\operatorname{supp} g^{\downarrow}\| = \{\log_i E_2 x\}$ implies

$$L\|\operatorname{supp} g^{\downarrow}\| = Ex - i =\not< g - \mathbb{N}.$$

Indeed, one develops $E(g-(i+1))$ as a series

$$E(g-(i+1)) = E(Ex-(i+1)) + \frac{E'(Ex-(i+1))}{\log_i E_2 x} + \frac{1}{2!}\frac{E''(Ex-(i+1))}{\log_i^2 E_2 x} + \cdots.$$

8.2.2 Extending the group of new monomials

Let

$$\mathcal{F}_{n,\mathbb{T}} := \{f \in \mathbb{T}_\infty^+ \,|\, \pi_{n,\mathbb{T}} f = \infty \ \wedge\ \mathrm{l}_{\omega^n}\|\operatorname{supp} f^{\downarrow}\| < f - \mathbb{N}\},$$

be the set of positive, infinite series f in $\mathbb{T}$ such that $\mathrm{e}_{\omega^n} f$ is a new monomial. Note in particular that $\mathfrak{M} \cap \mathrm{e}_{\omega^n} \mathcal{F}_{n,\mathbb{T}} = \emptyset$.

In order to extend $\mathbb{T}$ to a transseries field of strength n, the multiplicative group generated by $\mathrm{e}_{\omega^n}\mathcal{F}_{n,\mathbb{T}}$ does not suffice yet. Take for instance the problem of defining the function l_{ω^i} for some $i < n$. We might consider a series $g = \mathrm{e}_{\omega^n}f + \varepsilon$ where $f \in \mathcal{F}_{n,\mathbb{T}}$ and $\varepsilon \prec \mathrm{e}_{\omega^n}f$. Then $\mathrm{l}_{\omega^i}g$ would be defined as

$$\mathrm{l}_{\omega^i}(\mathrm{e}_{\omega^n}f + \varepsilon) = \mathrm{l}_{\omega^i}(\mathrm{e}_{\omega^n}f) + \mathrm{l}'_{\omega^i}(\mathrm{e}_{\omega^n}f)\cdot\varepsilon + \frac{1}{2!}\mathrm{l}^{(2)}_{\omega^i}(\mathrm{e}_{\omega^n}f)\cdot\varepsilon^2 + \cdots .$$

Recall that if $\mathrm{l}'_{\omega^i}(\mathrm{e}_{\omega^n}f) \in \mathbb{T}$, then also $\mathrm{l}^{(n)}_{\omega^i}(\mathrm{e}_{\omega^n}f) \in \mathbb{T}$ for all $n \geqslant 1$. However, the expression $\mathrm{l}'_{\omega^i}(\mathrm{e}_{\omega^n}f)$ cannot be defined in $\mathbb{T}$ yet, for otherwise its logarithm would be in $\mathbb{T}^{\uparrow}$, and thus $\mathrm{e}_{\omega^n}f \in \mathfrak{M}$. We have therefore to add all $\mathrm{l}'_{\omega^i}(\mathrm{e}_{\omega^n}f)$.

Remark 8.2.3 From $\pi_{n,\mathbb{T}}f = \infty$ and $f^{\downarrow} = (f-k)^{\downarrow}$ it follows that $f - k \in \mathcal{F}_{n,\mathbb{T}}$. Thus $\mathrm{l}_{\omega^{n-1}.k}(\mathrm{e}_{\omega^n}f) = \mathrm{e}_{\omega^n}(f-k)$ is a new monomial for all k. Generalizing this result to every $\alpha < \omega^n$, we will let $\mathrm{l}_\alpha(\mathrm{e}_{\omega^n}f)$ to be a new monomial. We have, however, to make sure that this does not lead to incoherences. In particular, we have to make sure that $\mathrm{l}_\alpha(\mathrm{e}_{\omega^n}f)$ is not a series with non-zero remainder or leading coefficient different from 1.

Lemma 8.2.4 *Let $f \in \mathcal{F}_{n,\mathbb{T}}$ and $\alpha < \omega^n$. Suppose that $\hat{\mathbb{T}} = C[[\hat{\mathfrak{M}}]]$ is of strength n and that $\mathrm{l}_\alpha(\mathrm{e}_{\omega^n}f) \in \hat{\mathbb{T}}$. Then $\mathrm{e}_{\omega^n}(f-k) \in \hat{\mathbb{T}}$ for some $k \in \mathbb{N}$ and $\mathrm{l}_\alpha(\mathrm{e}_{\omega^n}f) \in \hat{\mathfrak{M}}$.*

Proof: Let $a_0, \ldots, a_{n-1} \in \mathbb{N}$ be such that $\alpha = \omega^{n-1}a_{n-1} + \cdots + a_0$. For $i \leqslant n-1$ we let

$$\alpha_i := \omega^{n-1}a_{n-1} + \cdots + \omega^i a_i + \omega^i.$$

Since $\hat{\mathbb{T}}$ is of strengh $\geqslant 1$, we have

$$\mathrm{l}_\omega \circ \mathrm{l}_\alpha(\mathrm{e}_{\omega^n}f) = \mathrm{l}_{\alpha_1}(\mathrm{e}_{\omega^n}f) - a_0 \in \hat{\mathbb{T}},$$

hence $\mathrm{l}_{\alpha_1}(\mathrm{e}_{\omega^n}f) \in \hat{\mathbb{T}}$. Applying $\mathrm{l}_{\omega^2}, \ldots, \mathrm{l}_{\omega^{n-1}}$, we inductively obtain $\mathrm{l}_{\alpha_i}(\mathrm{e}_{\omega^n}f) \in \hat{\mathbb{T}}$ for all $i \leqslant n-1$. In particular, this implies $\mathrm{e}_{\omega^n}(f - a_{n-1}) \in \hat{\mathbb{T}}$.

To show the second assertion, we assume that $\mathrm{e}_{\omega^i}h$ is a monomial for all $i < n$ and all $0 < h \in \hat{\mathbb{T}}^{\updownarrow}$. Note that this is true for $n = 1$ which therefore provides the initial step of the following inductive argument.

Note that $f \in \mathcal{F}_{n,\mathbb{T}}$ implies that for all $a_{n-1} \in \mathbb{N}$ we have $f - a_{n-1} \in \mathcal{F}_{n,\mathbb{T}}$, hence that $\mathrm{l}_{\omega^{n-1}a_{n-1}}(\mathrm{e}_{\omega^n}f)$ is a monomial. We therefore only need to show the lemma for $\alpha < \omega^{n-1}$. Let $k \in \mathbb{N}$ and $g = \mathrm{l}_{\omega^{n-2}.k}(\mathrm{e}_{\omega^n}f)$. Then $\mathrm{l}_{\omega^{n-1}}g = \mathrm{l}_{\omega^{n-1}}(\mathrm{e}_{\omega^n}f) - k = \mathrm{e}_{\omega^n}(f-1) - k$. Since $\mathrm{e}_{\omega^n}(f-1) - k$ has no infinitesimal part, we conclude that $g = \mathrm{e}_{\omega^{n-1}}(\mathrm{e}_{\omega^n}(f-1) - k)$ is a monomial in $\hat{\mathfrak{M}}$. We can repeat the same argument for

$$\mathrm{l}_{\omega^{n-3}\cdot a_{n-3}} \circ \mathrm{l}_{\omega^{n-2}\cdot a_{n-2}}(\mathrm{e}_{\omega^n}f)$$
$$\mathrm{l}_{\omega^{n-4}\cdot a_{n-4}} \circ \mathrm{l}_{\omega^{n-3}\cdot a_{n-3}} \circ \mathrm{l}_{\omega^{n-2}\cdot a_{n-2}}(\mathrm{e}_{\omega^n}f)$$

and so forth. This shows that $\mathrm{l}_\alpha(\mathrm{e}_{\omega^n}f) \in \hat{\mathfrak{M}}$ for all $\alpha < \omega^{n-1}$. □

By Remark 8.2.3 the group of new monomials must contain the multiplicative group generated by the set

$$\{\mathrm{l}_\alpha(\mathrm{e}_{\omega^n} f) \,|\, \alpha < \omega^n \ \wedge\ f \in \mathcal{F}_{n,\mathbb{T}}\}.$$

However, the multiplicative closure of this set is still too small. Let again g be a series in the extended field with leading term $\mathrm{l}_\alpha(\mathrm{e}_{\omega^n} f)$, i.e. for a series $\varepsilon \prec \mathrm{l}_\alpha(\mathrm{e}_{\omega^n} f)$ we have $g = \mathrm{l}_\alpha(\mathrm{e}_{\omega^n} f)+\varepsilon$. In order to define $\mathrm{l}_{\omega^i} g$ using Taylor-series developments, we need

$$\begin{gathered}\mathrm{l}_{\omega^i}\mathrm{l}_\alpha(\mathrm{e}_{\omega^n} f)\\ \mathrm{l}'_{\omega^i}\mathrm{l}_\alpha(\mathrm{e}_{\omega^n} f)\\ \mathrm{l}^{[2]}_{\omega^i}\mathrm{l}_\alpha(\mathrm{e}_{\omega^n} f)\\ \vdots\end{gathered}$$

The derivatives of l_{ω^i} are not captured by the multiplicative closure of the above set. We therefore need the closure under l'_{ω^i} as well.

Finally, we define the set of new monomials $\mathfrak{N}_{n,\mathbb{T}}$ as follows. Recall that for any countable ordinal number α we have $\mathrm{l}'_\alpha x = \prod_{\beta<\alpha} 1/\mathrm{l}_\beta x$. We let

$$\mathfrak{N}_{n,\mathbb{T}} := \left\{ \prod_{l=1}^{N} \mathrm{l}'_{\alpha_l}(\mathrm{e}_{\omega^n} f_l)^{n_l} \,|\, N \in \mathbb{N} \wedge \forall 1 \leqslant l \leqslant N : \alpha_l \leqslant \omega^n, n_l \in \mathbb{Z}^*, f_l \in \mathcal{F}_{n,\mathbb{T}} \right\}.$$

Remark 8.2.5 Note that $\mathfrak{M} \cap \mathfrak{N}_{n,\mathbb{T}} = \emptyset$. To see this, we remark that if there was some

$$\mathfrak{m} = \prod_{l=1}^{N} \mathrm{l}'_{\alpha_l}(\mathrm{e}_{\omega^n} f_l)^{n_l} \in \mathfrak{M} \cap \mathfrak{N}_{n,\mathbb{T}},$$

then $\log \mathfrak{m} \in \mathbb{T}$, and thus $\mathfrak{d}_{\log \mathfrak{m}} \in \mathfrak{M}$. But then there are $f \in \mathcal{F}_{n,\mathbb{T}}$ and $\alpha < \omega^n$ such that $\mathfrak{d}_{\log \mathfrak{m}} = \mathrm{l}_\alpha(\mathrm{e}_{\omega^n} f)$. By Lemma 8.2.4 this implies $\mathrm{e}_{\omega^n}(f-1) \in \mathfrak{M}$, which contradicts $f \in \mathcal{F}_{n,\mathbb{T}}$.

Since $\mathfrak{N}_{n,\mathbb{T}}$ does not contain the monomial group $\mathfrak{M}$, we have to add $\mathfrak{M}$ and define the **e_{ω^n}-extension** of $\mathfrak{M}$ by $\mathfrak{M}_{\mathrm{e}_{\omega^n}} := \mathfrak{M} \cdot \mathfrak{N}_{n,\mathbb{T}}$.

Remark 8.2.6 First we remark that for all $\alpha < \omega^n$ and all $f \in \mathcal{F}_{n,\mathbb{T}}$ we have

$$\mathrm{l}_\alpha(\mathrm{e}_{\omega^n} f) = \frac{\mathrm{l}'_\alpha(\mathrm{e}_{\omega^n} f)}{\mathrm{l}'_{\alpha+1}(\mathrm{e}_{\omega^n} f)} \in \mathfrak{N}_{n,\mathbb{T}}.$$

Thus the multiplicative group generated by the set $\{\mathrm{l}_\alpha(\mathrm{e}_{\omega^n} f) \,|\, \alpha < \omega, f \in \mathcal{F}_{n,\mathbb{T}}\}$ is contained in $\mathfrak{N}_{n,\mathbb{T}}$. Furthermore, if we fix a countable ordinal $\gamma < \omega^n$ with

$$\gamma = \omega^{n-1}\, a_{n-1} + \cdots + \omega^i\, a_i$$

where $0 < a_i$ and $0 \leqslant a_{i+1}, \cdots, a_{n-1}$, then for all $\alpha < \omega^{i+1}$ we have

$$\mathrm{l}_{\gamma+\alpha}(\mathrm{e}_{\omega^n} f) = \mathrm{l}_\alpha \circ \mathrm{l}_\gamma(\mathrm{e}_{\omega^n} f) \in \mathfrak{N}_{n,\mathbb{T}}.$$

Thus $\mathrm{l}'_{\gamma+\alpha}(\mathrm{e}_{\omega^n} f)$ and $\mathrm{l}'_\gamma(\mathrm{e}_{\omega^n} f)$ are elements of $\mathfrak{N}_{n,\mathbb{T}}$ and

$$\frac{\mathrm{l}'_{\gamma+\alpha}(\mathrm{e}_{\omega^n} f)}{\mathrm{l}'_\gamma(\mathrm{e}_{\omega^n} f)} = \prod_{\beta<\alpha} \frac{1}{\mathrm{l}_\beta \circ \mathrm{l}_\gamma(\mathrm{e}_{\omega^n} f)} = \mathrm{l}'_\alpha(\mathrm{l}_\gamma(\mathrm{e}_{\omega^n} f)) \in \mathfrak{N}_{n,\mathbb{T}}.$$

for all $\alpha \leqslant \omega^i \leqslant \omega^{n-1} a_{n-1} + \cdots + \omega^i a_i < \omega^n$. In other words, $\mathfrak{N}_{n,\mathbb{T}}$ has the desired closure properties.

Example 8.2.7 In the case $n = 1$, we let $E = \mathrm{e}_\omega$ and $L = \mathrm{l}_\omega$. Then $\alpha \leqslant \omega$ is either an integer or ω. If $\alpha \in \mathbb{N}$, then

$$\mathrm{l}'_\alpha(Ef) = (Ef \cdot E(f-1) \cdots E(f-\alpha+1))^{-1} = \frac{1}{Eg_1 \cdots Eg_\alpha}$$

for series $g_1, \cdots, g_\alpha \in \mathcal{F}_{1,\mathbb{T}}$. As in Remark 8.2.6, we can write each Eg_i as $L'E(g_i - 1)/L'Eg_i$. If $\alpha = \omega$, then $\mathrm{l}'_\alpha Ef = L'Ef$. Hence from $L'Ef = 1/E'f$ we then obtain

$$\mathfrak{N}_{1,\mathbb{T}} = \left\{ \prod_{l=1}^{N} (E' f_l)^{n_l} \,|\, N \in \mathbb{N} \ \wedge \ \forall 1 \leqslant l \leqslant N : n_l \in \mathbb{Z}^*, f_l \in \mathcal{F}_{1,\mathbb{T}} \right\}.$$

Let us summarize the three important sets defined in this section. Recall that $\mathbb{T} = C[[\mathfrak{M}]]$ is of strength n. Then

$$\begin{aligned}
\mathcal{F}_{n,\mathbb{T}} &:= \{ f \in \mathbb{T}^+_\infty \,|\, \pi_{n,\mathbb{T}} f = \infty \ \wedge \ \mathrm{l}_{\omega^n} \|\mathrm{supp}\, f^\downarrow\| < f - \mathbb{N} \} \\
\mathfrak{N}_{n,\mathbb{T}} &:= \left\{ \prod_{l=1}^{N} \mathrm{l}'_{\alpha_l}(\mathrm{e}_{\omega^n} f_l)^{n_l} \,|\, N \in \mathbb{N} \ \wedge \ \forall 1 \leqslant l \leqslant N : \alpha_l \leqslant \omega^n, n_l \in \mathbb{Z}^*, f_l \in \mathcal{F}_{n,\mathbb{T}} \right\} \\
\mathfrak{M}_{\mathrm{e}_{\omega^n}} &:= \mathfrak{M} \cdot \mathfrak{N}_{n,\mathbb{T}}
\end{aligned}$$

8.3 The group structure of the extended set of monomials

The next step after having defined the set $\mathfrak{M}_{\mathrm{e}_{\omega^n}}$ is to define a multiplication and a total ordering on this set such that $\mathfrak{M}_{\mathrm{e}_{\omega^n}}$ is a totally ordered, multiplicative group containing $\mathfrak{M}$. We start with the multiplication. Let $N, M \in \mathbb{N}$, and for all $1 \leqslant l \leqslant N$ resp. M let $m_l, n_l \in \mathbb{Z}$, $\alpha_l, \beta_l \leqslant \omega^n$ and $f_l, g_l \in \mathcal{F}_{n,\mathbb{T}}$. Let $\mathfrak{m}, \mathfrak{n} \in \mathfrak{N}_{n,\mathbb{T}}$ with

$$\mathfrak{m} = \prod_{l=1}^{M} \mathrm{l}'_{\alpha_l}(\mathrm{e}_{\omega^n} f_l)^{m_l} \qquad \text{and} \qquad \mathfrak{n} = \prod_{l=1}^{N} \mathrm{l}'_{\beta_l}(\mathrm{e}_{\omega^n} g_l)^{n_l}.$$

For $M < l \leqslant M+N$ we let

$$\begin{aligned} m_l &:= n_{l-M} \\ \alpha_l &:= \beta_{l-M} \\ f_l &:= g_{l-M}. \end{aligned}$$

Then for $\mathfrak{m}, \mathfrak{n} \in \mathfrak{N}_{n,\mathbb{T}}$ we define $\mathfrak{m} \cdot \mathfrak{n}$ and $\mathfrak{m}^{-1} \in \mathfrak{N}_{n,\mathbb{T}}$ by

$$\begin{aligned} \mathfrak{m} \cdot \mathfrak{n} &:= \prod_{l=1}^{M+N} \mathrm{l}'_{\alpha_l}(\mathrm{e}_{\omega^n} f_l)^{m_l} \\ \mathfrak{m}^{-1} &:= \prod_{l=1}^{M} \mathrm{l}'_{\alpha_l}(\mathrm{e}_{\omega^n} f_l)^{-m_l}. \end{aligned}$$

If $M = 0$, then we let $\mathfrak{m} = 1$. This defines an abelian multiplication on $\mathfrak{N}_{n,\mathbb{T}}$. Note that this multiplication defines a group structure on $\mathfrak{N}_{n,\mathbb{T}}$. For $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}$ and $\mathfrak{m}, \mathfrak{n} \in \mathfrak{N}_{n,\mathbb{T}}$ we let

$$(\mathfrak{a}\,\mathfrak{m}) \cdot (\mathfrak{b}\,\mathfrak{n}) := \mathfrak{a}\mathfrak{b} \cdot \mathfrak{m}\mathfrak{n} \in \mathfrak{M}_{\mathrm{e}_{\omega^n}}.$$

Hence $\mathfrak{M}_{\mathrm{e}_{\omega^n}}$ is a group which extends the group $\mathfrak{M}$.

Next, we define the total ordering. We start by defining an ordering on $\mathfrak{N}_{n,\mathbb{T}}$. Let $f, g \in \mathcal{F}_{n,\mathbb{T}}$ and $\alpha, \beta < \omega^{n-1}$, then we let

$$\mathrm{l}_\beta(\mathrm{e}_{\omega^n} g) \succ \mathrm{l}_\alpha(\mathrm{e}_{\omega^n} f) \quad :\Leftrightarrow \quad \begin{cases} \text{if } g > f \text{ or} \\ \text{if } f = g \text{ and } \beta < \alpha. \end{cases}$$

Note that $\succ$ totally defined on

$$\{\mathrm{l}_\alpha(\mathrm{e}_{\omega^n} f) \mid \alpha < \omega^n \ \wedge \ f \in \mathcal{F}_{n,\mathbb{T}}\},$$

since for every $\alpha < \omega^n$ there are $a \in \mathbb{N}$ and $\hat{\alpha} < \omega^{n-1}$ such that $\alpha = \omega^{n-1} a + \hat{\alpha}$. Then $\mathrm{l}_\alpha(\mathrm{e}_{\omega^n} f) = \mathrm{l}_{\hat{\alpha}}(\mathrm{e}_{\omega^n}(f-a))$, where $f - a \in \mathcal{F}_{n,\mathbb{T}}$.

Remark 8.3.1 Let us explain why $\succ$ is the canonical choice for an ordering of the set of monomials $\mathrm{l}_\alpha(\mathrm{e}_{\omega^n} f)$. Suppose that $f < g$ are series from $\mathcal{F}_{n,\mathbb{T}}$ and that $\alpha, \beta < \omega^{n-1}$ are such that $\mathrm{l}_\alpha(\mathrm{e}_{\omega^n} f) \succ \mathrm{l}_\beta(\mathrm{e}_{\omega^n} g)$. Let $a_i, b_i \in \mathbb{N}$ be such that

$$\begin{aligned} \alpha &= \omega^{n-2} a_{n-2} + \cdots + a_0 \\ \beta &= \omega^{n-2} b_{n-2} + \cdots + b_0. \end{aligned}$$

We let for $i \leqslant n-2$

$$\begin{aligned} \hat{\alpha}_i &= \omega^{n-2} a_{n-2} + \cdots + \omega^i (a_i + 1) \\ \hat{\beta}_i &= \omega^{n-2} b_{n-2} + \cdots + \omega^i (b_i + 1). \end{aligned}$$

Since log is strictly monotone, we obtain

$$\mathrm{l}_{\hat{\beta}_1}(\mathrm{e}_{\omega^n} g) - b_0 < \mathrm{l}_{\hat{\alpha}_1}(\mathrm{e}_{\omega^n} f) - a_0.$$

Note that $\mathrm{l}_{\hat{\beta}_1}(\mathrm{e}_{\omega^n} g)$ and $\mathrm{l}_{\hat{\alpha}_1}(\mathrm{e}_{\omega^n} f)$ are monomials. Hence

$$\mathrm{l}_{\hat{\beta}_1}(\mathrm{e}_{\omega^n} g) < \mathrm{l}_{\hat{\alpha}_1}(\mathrm{e}_{\omega^n} f).$$

Inductively applying the strictly monotone functions l_{ω^i} (where $i \leqslant n-2$) leads to

$$\mathrm{l}_{\hat{\beta}_{n-2}}(\mathrm{e}_{\omega^n} g) < \mathrm{l}_{\hat{\alpha}_{n-2}}(\mathrm{e}_{\omega^n} f).$$

Applying $\mathrm{l}_{\omega^{n-1}}$ yields $\mathrm{e}_{\omega^n}(g-1) < \mathrm{e}_{\omega^n}(f-1)$, which implies $g < f$. The asymptotic behaviour of iterated logarithmic functions provides the motivation for the definition in the case $f = g$.

Let $1 \neq \mathfrak{n} \in \mathfrak{N}_{n,\mathbb{T}}$. Then there are $N \in \mathbb{N}^+$ and $n_l \in \mathbb{Z}^*$, $f_l \in \mathcal{F}_{n,\mathbb{T}}$, $\alpha_l \leqslant \omega^n$ for $1 \leqslant l \leqslant N$ such that

$$\mathfrak{n} = \prod_{l=1}^{N} \mathrm{l}_{\alpha_l}^{'}(\mathrm{e}_{\omega^n} f_l)^{n_l}.$$

To each $\mathfrak{v} \in \{\mathrm{l}_{\beta+1}(\mathrm{e}_{\omega^n} f_l) \,|\, 1 \leqslant l \leqslant N \,\wedge\, \beta < \alpha_l\}$, there is a set

$$S(\mathfrak{v}) = \{(l, \beta) \,|\, \mathfrak{v} = \mathrm{l}_{\beta+1}(\mathrm{e}_{\omega^n} f_l) \,\wedge\, 1 \leqslant l \leqslant N \,\wedge\, \beta < \alpha_l\}.$$

We let

$$\begin{aligned} n_{\mathfrak{v}} &:= \sum_{(l,\beta)\in S(\mathfrak{v})} n_l \\ \mathfrak{v}^* &:= \max\{\mathfrak{v} \,|\, n_{\mathfrak{v}} \neq 0\} \\ n^* &:= n_{\mathfrak{v}^*}. \end{aligned}$$

Then we let $\mathfrak{n} \succ 1$ iff $0 > n^*$.

Remark 8.3.2 We motivate again our definition. Suppose that we have a logarithmic function on $\mathfrak{N}_{n,\mathbb{T}}$. Then from the product rule we obtain

$$\log \mathfrak{n} = \sum_{l=1}^{N} n_l \cdot \log \mathrm{l}_{\alpha_l}^{'}(\mathrm{e}_{\omega^n} f_l).$$

The support of this series is contained in the set $\{\mathrm{l}_{\beta+1}(\mathrm{e}_{\omega^n} f_l) \,|\, 1 \leqslant l \leqslant N \,\wedge\, \beta < \alpha_l\}$. The leading term is $-n^*\mathfrak{v}^*$. This together with $\log \mathfrak{n} > 0 \Leftrightarrow \mathfrak{n} \succ 1$ motivates the definition of the ordering.

Example 8.3.3 Let $\alpha < \omega^n$ and $f \in \mathcal{F}_{n,\mathbb{T}}$. Then $\mathrm{l}_\alpha(\mathrm{e}_{\omega^n} f)$ is by Remark 8.2.6 equal to $\mathrm{l}'_\alpha(\mathrm{e}_{\omega^n} f)^{+1} \cdot \mathrm{l}'_{\alpha+1}(\mathrm{e}_{\omega^n} f)^{-1}$. We have to consider

$$\mathfrak{v} \in \{\mathrm{l}_{\beta+1}(\mathrm{e}_{\omega^n} f) \,|\, \beta < \alpha + 1\}.$$

Note that $S(\mathfrak{v}) = 0$ if and only if $\beta < \alpha$. Thus $\mathfrak{v}^* = \mathrm{l}_{\alpha+1}(\mathrm{e}_{\omega^n} f)$ and $n^* = -1$. Hence $\mathrm{l}_\alpha(\mathrm{e}_{\omega^n} f) \succ 1$.

Proposition 8.3.4 *Let $\mathfrak{n}, \mathfrak{m} \in \mathfrak{N}_{n,\mathbb{T}}$. Then $\mathfrak{n}, \mathfrak{m} \succ 1$ implies $\mathfrak{n}\mathfrak{m} \succ 1$.*

Proof: Let $n^* < 0$ be as in the definition of $\mathfrak{n} \succ 1$. Similarly, we let

$$m_{\mathfrak{w}} = \sum_{(l,\beta)\in S(\mathfrak{w})} m_l,$$

and $\mathfrak{w}^* = \max\{\mathfrak{w} \,|\, m_{\mathfrak{w}} \neq 0\}$ and $m^* = m_{\mathfrak{w}^*}$. Then $m^* < 0$. The proposition follows from $0 > m^*, n^*, m^* + n^*$. □

Finally, we have to define an ordering on $\mathfrak{M}_{\mathrm{e}_{\omega^n}}$. Let $\mathfrak{a} \in \mathfrak{M}$ and $\mathfrak{n} \in \mathfrak{N}_{n,\mathbb{T}}$. We consider the case $\mathfrak{a} \neq 1$, for otherwise we have already defined the ordering. As in the definition of $\succ$ on $\mathfrak{N}_{n,\mathbb{T}}$, we assume that

$$\begin{aligned} \mathfrak{n} &= \prod_{l=1}^{N} \mathrm{l}'_{\alpha_l}(\mathrm{e}_{\omega^n} f_l)^{n_l} \\ \mathfrak{v}^* &= \mathrm{l}_{\beta+1}(\mathrm{e}_{\omega^n} f_l). \end{aligned}$$

Let $n^* = n_{\mathfrak{v}^*}$. There are $\alpha < \omega^{n-1}$ and $g \in \mathcal{F}_{n,\mathbb{T}}$ with $\mathfrak{v}^* = \mathrm{l}_\alpha(\mathrm{e}_{\omega^n} f)$. We let

$$\mathfrak{a}\,\mathfrak{n} \succcurlyeq 1 \quad :\Leftrightarrow \quad \begin{cases} \mathfrak{a}, \mathfrak{n} \succcurlyeq 1 \\ \mathfrak{a} \succcurlyeq 1 \succcurlyeq \mathfrak{n} \text{ and } f < \mathrm{l}_{\omega^n}\left(\frac{\log \mathfrak{a}}{n^*}\right) \\ \mathfrak{n} \succcurlyeq 1 \succcurlyeq \mathfrak{a} \text{ and } f > \mathrm{l}_{\omega^n}\left(\frac{\log \mathfrak{a}}{n^*}\right), \end{cases}$$

with equality iff $1 = \mathfrak{a} = \mathfrak{n}$. Note in particular that $\log \mathfrak{a} \in \mathbb{T}^\uparrow$, thus that $\mathrm{l}_{\omega^n}(\log \mathfrak{a}/n^*)$ is defined in $\mathbb{T}$. Hence the conditions in the definition can be verified in the exponential extension of $\mathbb{T}$.

Remark 8.3.5 We have chosen this definition with a similar motivation as in the case of the ordering of $\mathfrak{N}_{n,\mathbb{T}}$. If $\mathfrak{a}, \mathfrak{n} \succ 1$, then we will let $\mathfrak{a}\mathfrak{n} \succ 1$. Similarly, if $1 \succ \mathfrak{a}, \mathfrak{n}$, we will have $1 \succ \mathfrak{a}\mathfrak{n}$. In the cases $\mathfrak{n} \succ 1 \succ \mathfrak{a}$ and $\mathfrak{a} \succ 1 \succ \mathfrak{n}$ we have to give a separate condition in order to decide whether $1 \succ \mathfrak{a}\mathfrak{n}$ or $\mathfrak{a}\mathfrak{n} \succ 1$. Also, this condition must be such that we can verify it in $\mathbb{T}$ already.

Clearly, if a logarithm is defined for the product $\mathfrak{a}\mathfrak{n}$, then $1 \prec \mathfrak{a}\,\mathfrak{n}$ if and only if $0 < \log \mathfrak{a} + \log \mathfrak{n}$. According to Remark 8.3.2, the leading term of $\log \mathfrak{n}$ is $-n^* \mathrm{l}_{\beta+1}(\mathrm{e}_{\omega^n} f_l)$. Since it is different from the leading term of $\log \mathfrak{a}$ (this follows from $\pi_{n,\mathbb{T}} f = \infty$), it suffices to compare $\log \mathfrak{a}/n^*$ and $\mathrm{l}_{\beta+1}(\mathrm{e}_{\omega^n} f)$. Since $\mathbb{T}$ is of strength n, the functions which are necessary to define these series exist. This motivates the definition.

Proposition 8.3.6 *For $\mathfrak{a},\mathfrak{b}\in\mathfrak{M}$ and $\mathfrak{m},\mathfrak{n}\in\mathfrak{N}_{n,\mathbb{T}}$ we let*

$$\mathfrak{a}\,\mathfrak{n}\succcurlyeq\mathfrak{b}\,\mathfrak{m}\quad:\Leftrightarrow\quad\mathfrak{a}\mathfrak{b}^{-1}\cdot\mathfrak{n}\mathfrak{m}^{-1}\succcurlyeq 1,$$

with equality iff $\mathfrak{a}=\mathfrak{b}$ and $\mathfrak{n}=\mathfrak{m}$. Then $\succcurlyeq$ is a total ordering on $\mathfrak{M}\mathfrak{e}_{\mathrm{e}_{\omega^n}}$ which is compatible with the group structure.

Proof: We have to show that $\mathfrak{a}\mathfrak{m}\succ 1$ and $\mathfrak{b}\mathfrak{n}\succ 1$ implies $\mathfrak{a}\mathfrak{b}\,\mathfrak{m}\mathfrak{n}\succ 1$. Before going through the different cases of the definition, let us fix some notations. Let $\mathfrak{v}^*$, n^* and f be elements of $\mathfrak{N}_{n,\mathbb{T}}$, $\mathbb{Z}$ and $\mathcal{F}_{n,\mathbb{T}}$ respectively as in the definition of $\mathfrak{a}\mathfrak{n}\succ 1$. Then, similarly, we have $\mathfrak{w}^*\in\mathfrak{N}_{n,\mathbb{T}}$, $m^*\in\mathbb{Z}$ and $g\in\mathcal{F}_{n,\mathbb{T}}$ with respect to $\mathfrak{m}$. Also, we remark that we only have to consider monomials such that the inequalities are strict. Finally let $\mathfrak{z}^*$, k^* and h be the corresponding elements of $\mathfrak{N}_{n,\mathbb{T}}$, $\mathbb{Z}$ and $\mathcal{F}_{n,\mathbb{T}}$ with respect to $\mathfrak{m}\mathfrak{n}$. Then there are ordinals β,γ,δ such that

$$\begin{aligned}\mathfrak{v}&=\mathrm{l}_{\beta+1}(\mathrm{e}_{\omega^n}f)\\ \mathfrak{w}&=\mathrm{l}_{\gamma+1}(\mathrm{e}_{\omega^n}g)\\ \mathfrak{z}&=\mathrm{l}_{\delta+1}(\mathrm{e}_{\omega^n}h).\end{aligned}$$

We distinguish three main-cases relative to the definition of $\mathfrak{a}\mathfrak{n}\succ 1$ and in each main-case three sub-cases relative to the definition of $\mathfrak{b}\mathfrak{m}\succ 1$.

Case I: $\mathfrak{a},\mathfrak{n}\succ 1$.

Sub-case I.1: $\mathfrak{b},\mathfrak{m}\succ 1$. Then $\mathfrak{a}\mathfrak{b}\succ 1$, and by Lemma 8.3.4 we have $\mathfrak{n}\mathfrak{m}\succ 1$. Hence the claim.

Sub-case I.2: $\mathfrak{b}\succ 1$ and $\mathfrak{m}\prec 1$. Then

$$g<\mathrm{l}_{\omega^n}\left(\frac{\log\mathfrak{b}}{m^*}\right)$$

and $\mathfrak{a}\mathfrak{b}\succ 1$. If $\mathfrak{v}\prec\mathfrak{w}$, then $k^*=m^*$ and $\mathfrak{z}=\mathfrak{w}$. In particular, $h=g$. Thus $0<\log\mathfrak{a}$ implies

$$h=g<\mathrm{l}_{\omega^n}\left(\frac{\log\mathfrak{b}}{k^*}\right)<\mathrm{l}_{\omega^n}\left(\frac{\log\mathfrak{a}\mathfrak{b}}{k^*}\right),$$

and we are done. If $\mathfrak{v}\succ\mathfrak{w}$, then $k^*=n^*<0$ and thus $\mathfrak{n}\mathfrak{m}\succ 1$. This also shows the lemma. Finally, if $\mathfrak{v}=\mathfrak{w}$, then we have to distinguish two cases. First, if $\mathfrak{z}=\mathfrak{v}$, then $k^*=m^*+n^*$ and $f=g=h$. Thus

$$0<\frac{\log\mathfrak{a}\mathfrak{b}}{k^*}=\frac{\log\mathfrak{a}+\log\mathfrak{b}}{n^*+m^*}<\frac{\log\mathfrak{a}}{n^*}+\frac{\log\mathfrak{b}}{m^*}.$$

Otherwise, we have $n^*+m^*=0$ and $\mathfrak{z}\prec\mathfrak{v}$. Note that in this case we must have $h<g$, hence $k\mathfrak{z}<m\mathfrak{v}$ for all k,m. Letting $m=k^*$ yields

$$h<g<\mathrm{l}_{\omega^n}\left(\frac{\log\mathfrak{b}}{k^*}\right)<\mathrm{l}_{\omega^n}\left(\frac{\log\mathfrak{a}\mathfrak{b}}{k^*}\right),$$

since $1\prec\mathfrak{a}$. This finishes case I.2.

Sub-case I.3: $\mathfrak{b} \prec 1$ and $\mathfrak{m} \succ 1$. Then $\mathfrak{n}\mathfrak{m} \succ 1$. In particular, this means $g \leqslant h$ and $k^* < 0$. Again, the lemma follows immediately if $\mathfrak{a}\mathfrak{b} \succcurlyeq 1$. However, if $\mathfrak{a}\mathfrak{b} \prec 1$, then $\log \mathfrak{a}$ and $-\log \mathfrak{b}$ have a non-empty common truncation. In other words, $\mathfrak{d}_{\log \mathfrak{a}\mathfrak{b}} \prec \mathfrak{d}_{\log \mathfrak{b}}$. Hence

$$\frac{\log \mathfrak{a}\mathfrak{b}}{k^*} < \frac{\log \mathfrak{b}}{m^*}.$$

Then the hypothesis of this case and the fact that l_{ω^n} is strictly monotone imply

$$\mathrm{l}_{\omega^n}\left(\frac{\log \mathfrak{a}\mathfrak{b}}{k^*}\right) < \mathrm{l}_{\omega^n}\left(\frac{\log \mathfrak{b}}{m^*}\right) < g \leqslant h.$$

This finishes case I.3 and thus case I.

Case II: $\mathfrak{a} \succ 1$ and $\mathfrak{n} \prec 1$.

Sub-case II.1: $\mathfrak{b}, \mathfrak{m} \succ 1$. This case is equivalent to I.2.

Sub-case II.2: $\mathfrak{b} \succ 1$ and $\mathfrak{m} \prec 1$. Then $\mathfrak{a}\mathfrak{b} \succ 1$ and $\mathfrak{m}\mathfrak{n} \prec 1$. If $\mathfrak{v} \prec \mathfrak{w}$, then $k^* = m^*$ and $h = g$. In this case

$$h = g < \mathrm{l}_{\omega^n}\left(\frac{\log \mathfrak{b}}{m^*}\right) < \mathrm{l}_{\omega^n}\left(\frac{\log \mathfrak{a}\mathfrak{b}}{k^*}\right).$$

Similarly, on treats the case $\mathfrak{w} \prec \mathfrak{v}$. If $\mathfrak{v} = \mathfrak{w}$, then $f = g = h$ and $h^* = n^* + m^*$. In this case we obtain

$$h < \mathrm{l}_{\omega^n}\left(\frac{\log \mathfrak{a}\mathfrak{b}}{h^*}\right)$$

as in case I.2.

Sub-case II.3: $\mathfrak{b} \prec 1$ and $\mathfrak{m} \succ 1$. Suppose that $\mathfrak{a}\mathfrak{b} \succ 1$. We are done if $\mathfrak{m}\mathfrak{n} \succcurlyeq 1$. So let us suppose $\mathfrak{m}\mathfrak{n} \prec 1$. Then $h \leqslant f$ and $k^* > 0$. Notice that the hypotheses imply $\mathfrak{d}_{\log \mathfrak{a}} \succ \mathfrak{d}_{\log \mathfrak{b}}$, and thus $\log \mathfrak{a} \asymp \log \mathfrak{a}\mathfrak{b} \succ \log \mathfrak{b}$. Then

$$\frac{\log \mathfrak{a}\mathfrak{b}}{k^*} \asymp \frac{\log \mathfrak{a}}{n^*}.$$

Applying l_{ω^n} then yields

$$h \leqslant f < \mathrm{l}_{\omega^n}\left(\frac{\log \mathfrak{a}}{n^*}\right) \asymp \mathrm{l}_{\omega^n}\left(\frac{\log \mathfrak{a}\mathfrak{b}}{k^*}\right).$$

If $1 \prec \mathfrak{m}\mathfrak{n}$ and $1 \succ \mathfrak{a}\mathfrak{b}$, then on the one hand, we obtain $h < f = g$ and $k^* < 0$. On the other hand, as in case I.3, the series $\log \mathfrak{a}$ and $-\log \mathfrak{b}$ have a proper common truncation and thus

$$\frac{\log \mathfrak{a}}{n^*} < \frac{\log \mathfrak{a}\mathfrak{b}}{k^*}.$$

Again, we obtain

$$h < f < \mathrm{l}_{\omega^n}\left(\frac{\log \mathfrak{a}}{n^*}\right) < \mathrm{l}_{\omega^n}\left(\frac{\log \mathfrak{a}\mathfrak{b}}{k^*}\right).$$

It remains the case $\mathfrak{a}\mathfrak{b} \succ 1$ and $1 \prec \mathfrak{m}\mathfrak{n}$, which can be treated similarly. This finishes case II.

Case III: $\mathfrak{a} \prec 1$ and $\mathfrak{n} \succ 1$.

Sub-case III.1: $\mathfrak{b}, \mathfrak{m} \succ 1$. This case is equivalent to I.3.

Sub-case III.2: $\mathfrak{b} \succ 1$ and $\mathfrak{m} \prec 1$. This case is equivalent to II.3.

Sub-case III.3: $\mathfrak{b} \prec 1$ and $\mathfrak{m} \succ 1$. This case is equivalent to II.2.

Showing conditions **PO1** – **PO3** is now straightforward. This finishes the proof of the proposition. □

8.4 Logarithms of positive strength on the extended field

Recall that $\mathbb{T}$ is of strength n. Fix an integer $0 \leqslant m \leqslant n$. Then $\mathbb{T}$ is of strength m, too. The field $\mathbb{T}_{\mathrm{e}_{\omega^m}}$ exists therefore.

Of the programme outlined in Section 8.1 we have so far covered the first two points, i.e. we have defined a set $\mathfrak{M}_{\mathrm{e}_{\omega^m}}$ of new monomials and we have defined a group structure and a compatible total ordering on the set. We use this group to enlarge $\mathbb{T}$ to the field $\mathbb{T}_{\mathrm{e}_{\omega^m}}$. The remaining two points of the programme now consist of defining functions $\log, \mathrm{l}_\omega, \ldots, \mathrm{l}_{\omega^n}$ such that the structure

$$\langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^i} \rangle$$

is of strength i for every $i \leqslant n$.

Before we start the construction of the functions l_{ω^i} (where $0 \leqslant i \leqslant n$), let us explain the method of the construction.

In a first step, we will define a function

$$\log : \mathbb{T}^+_{\mathrm{e}_{\omega^m}} \longrightarrow \mathbb{T}_{\mathrm{e}_{\omega^m}}$$

such that $\langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log \rangle$ is a transseries field. Then we will show how to define a partial function l_ω on $\mathfrak{M}^{\uparrow}_{\mathrm{e}_{\omega^m}}$ which has a large enough domain to allow a Taylor-series like definition of l_ω on the set $(\mathbb{T}_{\mathrm{e}_{\omega^m}})^+_\infty$. In other words, we show that the case $i = 0$ of the following two conditions holds:

$(\mathbb{T}_{\mathbf{e}_{\omega^m}}\mathbf{1})_{\boldsymbol{i}}$ $\langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^i} \rangle$ is of strength i.

$(\mathbb{T}_{\mathbf{e}_{\omega^m}}\mathbf{2})_{\boldsymbol{i}}$ There is a partial function $\mathrm{l}_{\omega^{i+1}} : \mathfrak{M}^{\uparrow}_{\mathrm{e}_{\omega^m}} \to \mathbb{T}^{\uparrow}_{\mathrm{e}_{\omega^m}}$ such that for all $\mathfrak{a} \in \mathfrak{M}_{\mathrm{e}_{\omega^m}}$

- if $\mathfrak{a}, \mathrm{l}_{\omega^i}\mathfrak{a} \in \operatorname{dom} \mathrm{l}_{\omega^{i+1}}$, then $\mathrm{l}_{\omega^{i+1}} \circ \mathrm{l}_{\omega^i}\mathfrak{a} = \mathrm{l}_{\omega^{i+1}}\mathfrak{a} - 1$,
- $\exists k \in \mathbb{N} : \mathfrak{a}$ is l_{ω^i}-confluent at order k and $\mathfrak{d}_{\mathrm{l}_{\omega^i \cdot k}\mathfrak{a}} \in \operatorname{dom} \mathrm{l}_{\omega^{i+1}}$.

If the two conditions hold for $i \geqslant 0$, then we say that $(\mathbb{T}_{\mathbf{e}_{\omega^m}})_{\boldsymbol{i}}$ holds. Note that condition $(\mathbb{T}_{\mathbf{e}_{\omega^m}}\mathbf{2})_{\boldsymbol{i}}$ implies $\mathbf{T}^{i+1}1$ and $\mathbf{T}^{i+1}3$ for $\mathbb{T}_{\mathrm{e}_{\omega^m}}$. From the first step it will therefore follow that $(\mathbb{T}_{\mathbf{e}_{\omega^m}})_{\mathbf{0}}$ holds. The results about transseries fields for positive strength then imply that we can define a function

$$\mathrm{l}_{\omega^{i+1}} : (\mathbb{T}_{\mathrm{e}_{\omega^m}})^+_\infty \longrightarrow \mathbb{T}_{\mathrm{e}_{\omega^m}}$$

such that $\langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^{i+1}} \rangle$ is of strength $i+1$. Hence $(\mathbb{T}_{\mathbf{e}_{\omega^m}}\mathbf{1})_{\boldsymbol{i+1}}$ will follow from $(\mathbb{T}_{\mathbf{e}_{\omega^m}})_{\boldsymbol{i}}$. We will use the fact that $\mathbb{T}$ is a transseries field and the construction of $\mathfrak{M}_{\mathrm{e}_{\omega^m}}$ to show that also $(\mathbb{T}_{\mathbf{e}_{\omega^m}}\mathbf{2})_{\boldsymbol{i+1}}$ follows. Hence, our work breaks into two main parts.

- Showing that $(\mathbb{T}_{\mathrm{e}_{\omega^m}})_{\mathbf{0}}$ holds.
- Showing that $(\mathbb{T}_{\mathrm{e}_{\omega^m}})_{\boldsymbol{i+1}}$, if $i < n$ and if $(\mathbb{T}_{\mathrm{e}_{\omega^m}})_{\boldsymbol{i}}$ holds.

Once this is done, we will obtain the chain

$$\langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log\rangle \hookrightarrow \cdots \hookrightarrow \langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^i}\rangle \hookrightarrow \cdots \hookrightarrow \langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^n}\rangle.$$

This eventually leads to a field $\mathbb{T}_{\mathrm{e}_{\omega^m}}$ of strength n.

Remark 8.4.1 To show the $\mathrm{l}_{\omega^{i+1}}$-confluence requires some care, and indeed, we will see that the proofs of the condition are different in the cases $i < n$ and $i \geqslant n$. On the other hand, the l_{ω^i}-atomic monoials will prove to be appropriate to define a function $\mathrm{l}_{\omega^{i+1}}$. Hence the condition $(\mathbb{T}_{\mathrm{e}_{\omega^m}}\mathbf{1})_{\boldsymbol{i+1}}$ will be used to show $(\mathbb{T}_{\mathrm{e}_{\omega^m}}\mathbf{2})_{\boldsymbol{i+1}}$.

8.4.1 Extensions of positive strength are transseries fields

We start by defining a function $\log : \mathfrak{M}_{\mathrm{e}_{\omega^m}} \to \mathbb{T}_{\mathrm{e}_{\omega^m}}$ and use this function to define a function

$$\log : \mathbb{T}^{+}_{\mathrm{e}_{\omega^m}} \longrightarrow \mathbb{T}_{\mathrm{e}_{\omega^m}}.$$

Let $f = c\mathfrak{d}\,(1+\delta) \in \mathbb{T}^{+}_{\mathrm{e}_{\omega^m}}$ where $c\mathfrak{d} = \tau_f \in C\mathfrak{M}_{\mathrm{e}_{\omega^m}}$ is the leading term of f. Let $\mathfrak{a} \in \mathfrak{M}$ and $\mathfrak{n} \in \mathfrak{N}_{m,\mathbb{T}}$ be such that $\mathfrak{d} = \mathfrak{a}\mathfrak{n}$ and $\mathfrak{n} = \prod_{l=1}^{N} \mathrm{l}'_{\alpha_l}(\mathrm{e}_{\omega^m} f_l)^{n_l}$. Then we let

$$\begin{aligned}
\log \mathfrak{n} &:= -\sum_{l=1}^{N}\sum_{\beta<\alpha_l} n_l \cdot \mathrm{l}_{\beta+1}(\mathrm{e}_{\omega^m} f_l)\\
\log \mathfrak{d} &:= \log \mathfrak{a} + \log \mathfrak{n}\\
\log f &:= \log \mathfrak{d} + \log c + l(\delta).
\end{aligned}$$

Lemma 8.4.2 *Let $\mathbb{T}$ be of strength $n > 0$. For $0 \leqslant m \leqslant n$ let $\mathbb{T}_{\mathrm{e}_{\omega^m}}$ and* log *be defined as above. Then*

(1) If $\mathfrak{m} \in \mathfrak{M}_{\mathrm{e}_{\omega^m}}$ is such that $\mathfrak{d}_{\log \mathfrak{m}} \notin \mathfrak{M}$, then there is a series $g \in \mathcal{F}_{m,\mathbb{T}}$ and an ordinal $\beta < \omega^m$ such that $\mathfrak{d}_{\log \mathfrak{m}} = \mathrm{l}_{\beta+1}(\mathrm{e}_{\omega^m} g)$.

(2) $\langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log\rangle$ is a transseries field. In particular, the set $\mathfrak{M}^{\uparrow}_{\mathrm{e}_{\omega^m}}$ is log*-confluent, and if for $\mathfrak{m} \in \mathfrak{M}^{\uparrow}_{\mathrm{e}_{\omega^m}}$ we have $\mathfrak{d}_{\log \mathfrak{m}} \notin \mathfrak{M}$, then $\mathfrak{m}$ is* log*-confluent at order 2 and $\mathfrak{d}_{\log_2 \mathfrak{m}} = \mathrm{l}_{\alpha+2}(\mathrm{e}_{\omega^m} g)$ for some $\alpha < \omega^m$ and $g \in \mathcal{F}_{m,\mathbb{T}}$.*

Proof: Throughout the proof, whenever we write $\mathfrak{m} = \mathfrak{a}\mathfrak{n}$, then we mean by that $\mathfrak{a} \in \mathfrak{M}$ and

$$\mathfrak{n} = \prod_{l=1}^{N} \mathrm{l}'_{\alpha_l}(\mathrm{e}_{\omega^m} f_l)^{n_l} \in \mathfrak{N}_{m,\mathbb{T}}.$$

We start with (1). We have either $\mathfrak{d}_{\log \mathfrak{m}} \in \mathfrak{M}$ or $\mathfrak{d}_{\log \mathfrak{m}} \notin \mathfrak{M}$. In the latter case we have $\mathfrak{d}_{\log \mathfrak{m}} = \mathfrak{d}_{\log \mathfrak{n}}$ for some $\mathfrak{n} \in \mathfrak{N}_{m,\mathbb{T}}$. From the definition of $\log \mathfrak{n}$ it follows then that for some $1 \leqslant l \leqslant N$ there is an ordinal $\beta < \alpha_l \leqslant \omega^m$ such that $\mathfrak{d}_{\log \mathfrak{n}} = \mathrm{l}_{\beta+1}(\mathrm{e}_{\omega^m} f_l)$. This shows (1).

(2) We have to show that $\mathbb{T}_{\mathrm{e}_{\omega^m}}$ is an exp-log field and that $\log \mathfrak{M}_{\mathrm{e}_{\omega^m}} \subseteq \mathbb{T}^{\uparrow}_{\mathrm{e}_{\omega^m}}$. To show the first point we remark that from the definition of log we obtain that for all $\mathfrak{n}_1, \mathfrak{n}_2 \in \mathfrak{N}_{m,\mathbb{T}}$ we have $\log(\mathfrak{n}_1\,\mathfrak{n}_2) = \log \mathfrak{n}_1 + \log \mathfrak{n}_2$. Since $\mathbb{T}$ is an exp-log field, this means that for all $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathfrak{M}_{\mathrm{e}_{\omega^m}}$ we have $\log(\mathfrak{m}_1\,\mathfrak{m}_2) = \log \mathfrak{m}_1 + \log \mathfrak{m}_2$, thus that $\log(fg) = \log f + \log g$ for all $f, g \in \mathbb{T}^{+}_{\mathrm{e}_{\omega^m}}$. This shows **e1**.

As for **e2**, it suffices to show that log is monotone on $\mathfrak{N}_{m,\mathbb{T}}$. But this follows directly from the definition of $\succ$ on $\mathfrak{N}_{m,\mathbb{T}}$.

We have to show that for all $f \in \mathbb{T}_{\mathrm{e}_{\omega^m}}$ in the domain of exp the inequality $f+1 \leqslant \exp f$ holds. For $f = f^{\downarrow}$, this follows from the definition of the basic exp-log structure (see Example 2.1.3 in Section 2). It remains to show the claim for infinite series f. Let $g \in (\mathbb{T}_{\mathrm{e}_{\omega^m}})^{+}_{\infty}$ such that $f = \log g$. We have to show $\log g + 1 \leqslant g$. We distinguish the cases $\mathfrak{d}_{\log g} \in \mathfrak{M}$ and $\mathfrak{d}_{\log g} \in \mathfrak{N}_{m,\mathbb{T}}$. Let $\mathfrak{a}\mathfrak{n} = \mathfrak{d}_g$, then in the first case we have $\mathfrak{a} \succ \mathfrak{n}$ and $\mathfrak{a} \succ \log \mathfrak{a}$. The latter holds since $\mathbb{T}$ is an exp-log field. Moreover, we have $\log \mathfrak{a} \asymp \log g$. In the second case, we have $\mathfrak{n} \succ \mathfrak{a}$ and $\mathfrak{n} \succ \log \mathfrak{n}$, where the last inequality follows from the definition of $\succ$ is $\mathfrak{N}_{m,\mathbb{T}}$. Hence, $\mathfrak{n} \asymp \log g$. In both cases we have

$$g \asymp \mathfrak{d}_g = \mathfrak{a}\mathfrak{n} \succ \log g,$$

which shows the inequality and thus **e3**. Hence $\mathbb{T}_{\mathrm{e}_{\omega^m}}$ is an exp-log field.

T1 and **T3** hold by construction. Since $\log \mathfrak{a} \in \mathbb{T}^{\uparrow}$, we have to show that $\log \mathfrak{n} \in \mathbb{T}^{\uparrow}_{\mathrm{e}_{\omega^m}}$. But for all $\mathfrak{n} \in \mathfrak{N}_{m,\mathbb{T}}$ we have

$$\operatorname{supp} \log \mathfrak{n} \subseteq \{\mathrm{l}_{\alpha}(\mathrm{e}_{\omega^m} f) \,|\, \alpha < \omega^m \wedge f \in \mathcal{F}_{m,\mathbb{T}}\} \succ 1.$$

This shows **T2**. As for **T4**, we remark that the case $\mathfrak{d}_{\log \mathfrak{m}} \in \mathfrak{M}$ follows from the same property in $\mathbb{T}$. Hence let $\mathfrak{d}_{\log \mathfrak{m}} \notin \mathfrak{M}$. Then $\mathfrak{d}_{\log_2 \mathfrak{m}} = \mathrm{l}_{\beta+2}(\mathrm{e}_{\omega^m} g)$ is log-atomic. This shows **T4**. □

Let $f \in (\mathbb{T}_{\mathrm{e}_{\omega^m}})^{+}_{\infty}$ and $\mathfrak{d}_f = \mathfrak{a}\mathfrak{n}$, where $\mathfrak{a} \in \mathfrak{M}$ and $\mathfrak{n} \in \mathfrak{N}_{m,\mathbb{T}}$. By Lemma 8.4.2, the series f is log-confluent. We will use this fact to define l_{ω} for f. To do so, we distinguish two cases, $\mathfrak{d}_{\log f} \in \mathfrak{M}$ and $\mathfrak{d}_{\log f} \notin \mathfrak{M}$. In the first case $\mathfrak{d}_{\log f} = \mathfrak{d}_{\log \mathfrak{a}} \in \mathfrak{M}$. Since $\mathbb{T}$ is of strength 1, there is a $k \in \mathbb{N}$ such that

- f is log-confluent at order k,
- $\mathfrak{d}_{\log_k f} \in \operatorname{dom} \mathrm{l}_{\omega}$ and $\mathrm{l}'_{\omega}(\mathfrak{d}_{\log_k f}) \in \mathbb{T}$.

We let $R \preccurlyeq 1$ be the remainder of $\log_k f$, i.e. $\log_k f = \mathfrak{d}_{\log_k f} + R$. Then $(\mathfrak{d}_{\log_k f}, R)$ is a l_{ω}-Taylor couple and we let

$$\mathrm{l}_{\omega} f := k + \mathcal{T}_{\mathrm{l}_{\omega}}(\mathfrak{d}_{\log_k f}, R).$$

In the second case, there are $\beta = \omega^{m-1}\, b_{m-1} + \cdots + b_0$ and $g \in \mathcal{F}_{m,\mathbb{T}}$ such that for some $R \preccurlyeq 1$ we have $\log_2 f = \mathrm{l}_{\beta+2}(\mathrm{e}_{\omega^m} g) + R$. Then we let

$$\begin{aligned} \hat{\beta} &:= \omega^{m-1}\, b_{m-1} + \cdots + \omega \cdot (b_1 + 1) \\ \mathrm{l}_{\omega}(\mathrm{l}_{\beta+2}(\mathrm{e}_{\omega^m} g)) &:= \mathrm{l}_{\hat{\beta}}(\mathrm{e}_{\omega^m} g) - (b_0 + 2) \\ \mathrm{l}_{\omega} f &:= 2 + \mathcal{T}_{\mathrm{l}_{\omega}}(\mathfrak{d}_{\log_2 f}, R). \end{aligned}$$

Note that by Remark 8.2.6 we have

$$\mathrm{l}_\omega'(\mathrm{l}_{\beta+2}(\mathrm{e}_{\omega^m} g)) \in \mathfrak{N}_{m,\mathbb{T}},$$

hence that $(\mathrm{l}_{\beta+2}(\mathrm{e}_{\omega^m} g), R)$ is a l_ω-Taylor couple. Thus the definition of $\mathrm{l}_\omega f$ is correct. We remark in particular that $(\mathrm{l}_\omega f)^\uparrow$ is a monomial of the form $\mathrm{l}_\gamma(\mathrm{e}_{\omega^m} g)$, where

$$\gamma = \omega^{m-1}\, g_{m-1} + \cdots + \omega\, g_1$$

is an ordinal with $g_1 \geqslant 1$. Hence $\boldsymbol{(\mathbb{T}_{e_{\omega^m}})_0}$ holds.

8.4.2 The logarithmic functions of strength $< m$

The properties of the transseries field

$$\langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log\rangle$$

shown in Lemma 8.4.2 provide the initial step for the inductive process in which we define functions $\log, \ldots, \mathrm{l}_{\omega^{m-1}}$ such that $\langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^{m-1}}\rangle$ is of strength $m-1$. Moreover, the structure will have properties which allow to continue a similar process beyond $m-1$. This will be done in the next section. Here we are only concerned with logarithmic functions l_{ω^i} of strength $i < m$.

The inductive step will require several assumptions on $\mathbb{T}_{\mathrm{e}_{\omega^m}}$ and the function l_{ω^i}. These assumptions will – as in the case $i = 0$ – make sure that the function $\mathrm{l}_{\omega^{i+1}}$ can be defined. We will, however, have to make sure that the new structure $\langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^{i+1}}\rangle$ possesses the same properties to make the induction work.

We say that the inductive assumptions $\boldsymbol{(\mathrm{IA}^{<m})_i}$ hold for $\mathbb{T}_{\mathrm{e}_{\omega^m}}$ in the case $i < m$ iff there are functions $\log, \ldots, \mathrm{l}_{\omega^i}$ such that:

$\boldsymbol{(\mathrm{IA}_0^{<m})_i}$ For all $j < i$ the inductive assumptions $\boldsymbol{(\mathrm{IA}^{<m})_j}$ hold.

$\boldsymbol{(\mathrm{IA}_1^{<m})_i}$ If $\mathfrak{m} \in \mathfrak{M}^\uparrow_{\mathrm{e}_{\omega^m}}$ is such that $\mathfrak{d}_{\log \mathfrak{m}} \notin \mathfrak{M}$, then there are $g \in \mathcal{F}_{m,\mathbb{T}}$ and $\alpha < \omega^m$ such that

$$\begin{aligned} \alpha &= \omega^{m-1}\, a_{m-1} + \cdots + \omega^i\, a_i \qquad (0 < a_i) \\ \mathfrak{d}_{\mathrm{l}_{\omega^i} \mathfrak{m}} &= \mathrm{l}_\alpha(\mathrm{e}_{\omega^m} g). \end{aligned}$$

$\boldsymbol{(\mathrm{IA}_2^{<m})_i}$ $\langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^i}\rangle$ is of strength i.

$\boldsymbol{(\mathrm{IA}_3^{<m})_i}$ In particular, for all $\mathfrak{m} \in \mathfrak{M}^\uparrow_{\mathrm{e}_{\omega^m}}$ we have

- if $\mathfrak{d}_{\log \mathfrak{m}} \notin \mathfrak{M}$, then $\mathfrak{m}$ is l_{ω^i}-confluent at order 2,
- if $\mathfrak{d}_{\log \mathfrak{m}} \in \mathfrak{M}$, then there is a $k \in \mathbb{N}$ such that $\mathfrak{d}_{\mathrm{l}_{\omega^i \cdot k} \mathfrak{m}} \in \operatorname{dom} \mathrm{l}_{\omega^{i+1}}$ in $\mathbb{T}$.

Lemma 8.4.2 implies that $\boldsymbol{(\mathrm{IA}^{<m})_0}$ holds for $\mathbb{T}_{\mathrm{e}_{\omega^m}}$. Now assume that $\boldsymbol{(\mathrm{IA}^{<m})_i}$ holds for $\langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \mathrm{l}_\omega, \ldots, \mathrm{l}_{\omega^i}\rangle$. Before we show that then $\boldsymbol{(\mathrm{IA}^{<m})_{i+1}}$ holds for $\mathbb{T}_{\mathrm{e}_{\omega^m}}$, we have to define a function $\mathrm{l}_{\omega^{i+1}}$. Let $f \in (\mathbb{T}_{\mathrm{e}_{\omega^m}})^+_\infty$ with $\mathfrak{m} = \mathfrak{d}_f$. As in the case $i = 0$, we have to distinguish the two cases $\mathfrak{d}_{\log \mathfrak{m}} \in \mathfrak{M}$ and $\mathfrak{d}_{\log \mathfrak{m}} \notin \mathfrak{M}$.

- $\mathfrak{d}_{\log \mathfrak{m}} \in \mathfrak{M}$ **:** From $(\mathbf{IA}^{<m})_i$ it follows that for some $k \in \mathbb{N}$ we have $\mathfrak{d}_{\mathrm{l}_{\omega^i \cdot k}\mathfrak{m}} \in \operatorname{dom} \mathrm{l}_{\omega^{i+1}}$ in $\mathbb{T}$. Since $\mathfrak{d}_{\mathrm{l}_{\omega^i \cdot k}\mathfrak{m}} = \mathfrak{d}_{\mathrm{l}_{\omega^i \cdot k} f}$, this means that for a series $\varepsilon \in \mathbb{T}^{\downarrow}_{\mathrm{e}_{\omega^m}}$ and a monomial $\mathfrak{a} \in \operatorname{dom} \mathrm{l}_{\omega^{i+1}} \cap \mathfrak{M}$ we have $\mathrm{l}_{\omega^i \cdot k} f = \mathfrak{a} + \varepsilon$. Then we let

$$\mathrm{l}_{\omega^{i+1}} f := k + \mathcal{T}_{\mathrm{l}_{\omega^{i+1}}}(\mathfrak{a}, \varepsilon).$$

- $\mathfrak{d}_{\log \mathfrak{m}} \notin \mathfrak{M}$ **:** From $(\mathbf{IA}^{<m})_i$ it follows that the monomial $\mathfrak{d}_{\mathrm{l}_{\omega^i \cdot 2} f} = \mathfrak{d}_{\mathrm{l}_{\omega^i \cdot 2}\mathfrak{m}}$ is of the form $\mathrm{l}_\alpha(\mathrm{e}_{\omega^m} g)$ where $g \in \mathcal{F}_{m,\mathbb{T}}$ and $\alpha = \omega^{m-1} a_{m-1} + \cdots + \omega^i a_i$ with $a_i > 1$. Let $\varepsilon \in \mathbb{T}^{\downarrow}_{\mathrm{e}_{\omega^m}}$ be such that $\mathrm{l}_{\omega^i \cdot 2} f = \mathrm{l}_\alpha(\mathrm{e}_{\omega^m} g) + \varepsilon$. Then we let

$$\mathrm{l}_{\omega^{i+1}} f := 2 + \mathcal{T}_{\mathrm{l}_{\omega^{i+1}}}(\mathrm{l}_\alpha(\mathrm{e}_{\omega^m} g), \varepsilon).$$

  Note in particular that

$$\begin{aligned} \mathrm{l}_{\omega^{i+1}}(\mathrm{l}_\alpha(\mathrm{e}_{\omega^m} g)) &= \mathrm{l}_{\omega^{m-1} a_{m-1} + \cdots + \omega^{i+1}(a_{i+1}+1)}(\mathrm{e}_{\omega^m} g) - a_i \\ \mathrm{l}'_{\omega^{i+1}}(\mathrm{l}_\alpha(\mathrm{e}_{\omega^m} g)) &= \prod_{\beta < \omega^{i+1}} \frac{1}{\mathrm{l}_{\alpha+\beta}(\mathrm{e}_{\omega^m} g)} \end{aligned}$$

  are elements of $\mathfrak{M}_{\mathrm{e}_{\omega^m}} - \mathbb{N}$ and $\mathfrak{M}_{\mathrm{e}_{\omega^m}}$, respectively.

Lemma 8.4.3 *Let $i < m$. If condition $(\mathbf{IA}^{<m})_i$ holds for $\langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^i} \rangle$, then condition $(\mathbf{IA}^{<m})_{i+1}$ holds for $\langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^{i+1}} \rangle$, where $\mathrm{l}_{\omega^{i+1}}$ is the function defined above.*

Proof: Condition $(\mathbf{IA}_0^{<m})_{i+1}$ clearly holds. Condition $(\mathbf{IA}_1^{<m})_{i+1}$ follows directly from conditions $(\mathbf{IA}_1^{<m})_i$, $(\mathbf{IA}_3^{<m})_i$ and the definition of $\mathrm{l}_{\omega^{i+1}}$ in the case $\mathfrak{d}_{\log \mathfrak{m}} \notin \mathfrak{M}$. Hence $\mathbf{T}^{i+1}\mathbf{1}$ and $\mathbf{T}^{i+1}\mathbf{3}$ hold. In order to show $\mathbf{T}^{i+1}\mathbf{2}$, we let $f \in \operatorname{dom} \mathrm{e}_{\omega^{i+1}}$ with $\operatorname{supp} f^{\downarrow} \prec_{\mathrm{l}_{\omega^i}} \mathrm{e}_{\omega^{i+1}} f$. Then there is a series $h \in \mathbb{T}_{\mathrm{e}_{\omega^m}}$ such that $f = \mathrm{l}_{\omega^{i+1}} h$. Let $\mathfrak{m} = \mathfrak{d}_h$. We have to distinguish the cases $\mathfrak{d}_{\log \mathfrak{m}} \in \mathfrak{M}$ and $\mathfrak{d}_{\log \mathfrak{m}} \notin \mathfrak{M}$.

First assume that $\mathfrak{d}_{\log \mathfrak{m}} \in \mathfrak{M}$. Then by $(\mathbf{IA}^{<m})_i$ there is a $k \in \mathbb{N}$ such that h is l_{ω^i}-confluent at order k, and we can assume that k is large enough such that $\mathrm{l}_{\omega^i \cdot k} h = \mathfrak{a} + \varepsilon$ where $\mathfrak{a} \in \mathfrak{M} \cap \operatorname{dom} \mathrm{l}_{\omega^{i+1}}$ and $\varepsilon \preccurlyeq 1$. Then by Lemma 7.4.7 and the fact that $\mathbb{T}$ is of strength $i+1$, we have

$$\operatorname{supp} \mathrm{l}_{\omega^{i+1}} \mathfrak{a} \succ \mathrm{l}'_{\omega^{i+1}} \mathfrak{a} \succcurlyeq \mathcal{R}_{\mathrm{l}_{\omega^{i+1}}}(\mathfrak{a}, \varepsilon).$$

Hence $\operatorname{supp} f^{\downarrow} = (\operatorname{supp} \mathrm{l}_{\omega^{i+1}} \mathfrak{a})^{\downarrow} \coprod \operatorname{supp} \mathcal{R}_{\mathrm{l}_{\omega^{i+1}}}(\mathfrak{a}, \varepsilon)$. We apply Fact 7.6.4 and obtain $\varepsilon = 0$. Whence $f = k + \mathrm{l}_{\omega^{i+1}} \mathfrak{a} \in \mathbb{T}$, and we can apply $\mathbf{T}^{i}\mathbf{2}$ for $\mathbb{T}$. We obtain $\mathrm{e}_{\omega^{i+1}} f \in \mathfrak{M} \subseteq \mathfrak{M}_{\mathrm{e}_{\omega^m}}$. This finishes the case $\mathfrak{d}_{\log \mathfrak{m}} \in \mathfrak{M}$.

Now let us assume that $\mathfrak{d}_{\log \mathfrak{m}} \notin \mathfrak{M}$. By $(\mathbf{IA}^{<m})_i$ the series h is l_{ω^i}-confluent at order 2 and $\mathfrak{d}_{\mathrm{l}_{\omega^i \cdot 2} h} = \mathrm{l}_\alpha(\mathrm{e}_{\omega^m} g)$ for some $g \in \mathcal{F}_{m,\mathbb{T}}$ and $\alpha = \omega^{m-1} a_{m-1} + \cdots + \omega^i a_i$. Let $\mathrm{l}_{\omega^i \cdot 2} h = \mathrm{l}_\alpha(\mathrm{e}_{\omega^m} g) + \varepsilon$, then

$$\begin{aligned} f^{\uparrow} &= \mathrm{l}_{\omega^{m-1} a_{m-1} + \cdots + \omega^{i+1}(a_{i+1}+1)}(\mathrm{e}_{\omega^m} g) = \mathrm{l}_{\hat{\alpha}}(\mathrm{e}_{\omega^m} g) \\ f^{\downarrow} &= \mathcal{R}_{\mathrm{l}_{\omega^{i+1}}}(\mathrm{l}_\alpha(\mathrm{e}_{\omega^m} g), \varepsilon). \end{aligned}$$

Again Fact 7.6.4 shows that $\varepsilon = 0$. Thus we obtain $f = \mathrm{l}_{\hat{\alpha}}(\mathrm{e}_{\omega^m} g) - a_i$. Since $f \in \operatorname{dom} \mathrm{e}_{\omega^{i+1}}$, this implies

$$\begin{aligned} \mathrm{e}_{\omega^{i+1}} f &= \mathrm{e}_{\omega^{i+1}}(\mathrm{l}_{\hat{\alpha}}(\mathrm{e}_{\omega^m} g) - a_i) \\ &= \mathrm{l}_{\omega^i \cdot a_i} \mathrm{e}_{\omega^{i+1}} \mathrm{l}_{\hat{\alpha}}(\mathrm{e}_{\omega^m} g) \\ &= \mathrm{l}_{\gamma}(\mathrm{e}_{\omega^m} g) \end{aligned}$$

where

$$\gamma = \omega^{m-1}\, a_{m-1} + \cdots + \omega^{i+1}\, a_{i+1} + \omega^i\, a_i = \alpha.$$

This shows $\mathrm{e}_{\omega^{i+1}} f = \mathrm{l}_{\alpha}(\mathrm{e}_{\omega^m} g)$. Hence we obtain $\mathrm{e}_{\omega^{i+1}} f \in \mathfrak{M}_{\mathrm{e}_{\omega^m}}$ and therefore the case $\mathfrak{d}_{\log \mathfrak{m}} \notin \mathfrak{M}$, thus $\boldsymbol{(\mathrm{IA}_2^{<m})_{i+1}}$ and $\mathbf{T^{i+1}2}$.

Finally, we remark that if $\mathfrak{d}_{\log \mathfrak{m}} \in \mathfrak{M}$, then the $\mathrm{l}_{\omega^{i+1}}$-confluence follows from $\mathbf{T^{i+2}4}$ for $\mathbb{T}$. If $\mathfrak{d}_{\log \mathfrak{m}} \notin \mathfrak{M}$, then $\boldsymbol{(\mathrm{IA}_3^{<m})_{i+1}}$ follows directly from the definition of $\mathrm{l}_{\omega^{i+1}}$. This finishes the proof. □

Remark 8.4.4 From Lemmas 8.4.2 and 8.4.3 it follows that there are functions l_{ω^i} for $i \leqslant m-1$ such that $\langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^i} \rangle$ is of strength $i \leqslant m-1$. Hence we have a chain

$$\langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log \rangle \hookrightarrow \cdots \hookrightarrow \langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^i} \rangle \hookrightarrow \cdots \hookrightarrow \langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^{n-1}} \rangle.$$

Moreover, the inductive assumption $\boldsymbol{(\mathrm{IA}^{<m})_{m-1}}$ allows to define a function l_{ω^m} as before. Note that always $\mathfrak{d}_{\mathrm{l}_{\omega^m} f} \in \mathfrak{M}$ and that $(\mathbb{T}_{\mathrm{e}_{\omega^m}})^+_\infty$ is therefore l_{ω^m}-confluent. We will use this fact in the following section to extend the chain beyond $m-1$.

8.4.3 The logarithmic functions of strength $\geqslant m$

We show now how to add functions $\mathrm{l}_{\omega^n}, \ldots, \mathrm{l}_{\omega^m}$ to the field $\mathbb{T}_{\mathrm{e}_{\omega^m}}$ such that for all $i \geqslant m$ the structure $\langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^i} \rangle$ is of strength i. We start by defining a set of inductive assumptions $\boldsymbol{(\mathrm{IA}^{\geqslant m})_i}$ similar to the case $i < m$.

We say that for $\langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^i} \rangle$ with $i \geqslant m$ the inductive assumption $\boldsymbol{(\mathrm{IA})_i^{\geqslant m}}$ holds iff

$\boldsymbol{(\mathrm{IA}_0^{\geqslant m})_i}$ If $m < i$, then for all $m \leqslant j < i$ the inductive assumption $\boldsymbol{(\mathrm{IA}^{\geqslant m})_j}$ holds.
$\boldsymbol{(\mathrm{IA}_1^{\geqslant m})_i}$ $(\mathbb{T}_{\mathrm{e}_{\omega^m}})^+_\infty$ is l_{ω^i}-confluent.
$\boldsymbol{(\mathrm{IA}_2^{\geqslant m})_i}$ $\langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^i} \rangle$ is of strength i.
$\boldsymbol{(\mathrm{IA}_3^{\geqslant m})_i}$ For all $f \in (\mathbb{T}_{\mathrm{e}_{\omega^m}})^+_\infty$ there is a $k \in \mathbb{N}$ such that $\mathfrak{d}_{\mathrm{l}_{\omega^i \cdot k} f} \in \operatorname{dom} \mathrm{l}_{\omega^{i+1}} \cap \mathfrak{M}$.

As in the case $i < m$, our inductive process breaks down into two parts:

- Showing that $\langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^m} \rangle$ is of strength m.
- Showing that for all $m \leqslant i < n$ such that $\boldsymbol{(\mathrm{IA}^{\geqslant m})_i}$ holds for $\mathbb{T}_{\mathrm{e}_{\omega^m}}$, there is a function $\mathrm{l}_{\omega^{i+1}}$ on $(\mathbb{T}_{\mathrm{e}_{\omega^m}})^+_\infty$ such that $\boldsymbol{(\mathrm{IA}^{\geqslant m})_{i+1}}$ holds for $\langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^{i+1}} \rangle$.

Lemma 8.4.5 *Let $0 < m \leqslant n$ and $\mathbb{T}$ of strength n Let $\mathbb{T}_{\mathrm{e}_{\omega^m}}$ be defined as above. Let l_{ω^m} be the function defined in Remark 8.4.4. Then $\langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^m} \rangle$ satisfies the inductive assumption* $\mathbf{(IA^{\geqslant m})_m}$.

Proof: In view of Remark 8.4.4 it suffices to show that $\mathbb{T}_{\mathrm{e}_{\omega^m}}$ is of strength m. In fact, we only need to show $\mathbf{T^m 2}$.

Fix a series $f \in \operatorname{dom} \mathrm{e}_{\omega^m}$ with $\operatorname{supp} f^{\downarrow} \prec_{\mathrm{l}_{\omega^{m-1}}} \mathrm{e}_{\omega^m} f$. Then for some $h \in \mathbb{T}_{\mathrm{e}_{\omega^m}}$ we have $f = \mathrm{l}_{\omega^m} h$. The series h is $\mathrm{l}_{\omega^{m-1}}$-confluent at order $k \in \mathbb{N}$ such that $\mathfrak{d}_{\mathrm{l}_{\omega^{m-1}\cdot k}} h = \mathrm{l}_{\omega^{m-1}\cdot a}(\mathrm{e}_{\omega^m} g)$ for a series $g \in \mathcal{F}_{m,\mathbb{T}}$ and an integer $a > 0$. Let $\varepsilon \preccurlyeq 1$ be such that $\mathrm{l}_{\omega^{m-1}\cdot k} h = \mathrm{l}_{\omega^{m-1}\cdot a}(\mathrm{e}_{\omega^m} g) + \varepsilon$, then from the definition of l_{ω^m} it follows that

$$\begin{aligned} \mathrm{l}_{\omega^m} h &= k + \mathcal{T}_{\mathrm{l}_{\omega^m}}(\mathrm{l}_{\omega^{m-1}\cdot a}(\mathrm{e}_{\omega^m} g), \varepsilon) \\ &= g + (k-a) + \mathcal{R}_{\mathrm{l}_{\omega^m}}(\mathrm{l}_{\omega^{m-1}\cdot a}(\mathrm{e}_{\omega^m} g), \varepsilon). \end{aligned}$$

We claim that $\varepsilon = 0$. This follows from Fact 7.6.4, if we can show that

$$\operatorname{supp} g \succ \mathcal{R}_{\mathrm{l}_{\omega^m}}(\mathrm{l}_{\omega^{m-1}\cdot a}(\mathrm{e}_{\omega^m} g), \varepsilon).$$

Notice first that from $g \in \mathcal{F}_{m,\mathbb{T}}$ it follows that

$$\forall j \geqslant 0 : \mathrm{l}_{\omega^m} \|\operatorname{supp} g^{\downarrow}\| < g - j.$$

In particular, for $j = a + 1$ this implies

$$\|\operatorname{supp} g^{\downarrow}\| \prec \mathrm{l}_{\omega^{m-1}}(\mathrm{l}_{\omega^{m-1}\cdot a}(\mathrm{e}_{\omega^m} g)).$$

But from

$$\mathrm{l}_{\omega^{m-1}} \mathfrak{n} \prec \log \mathfrak{n} \prec \mathfrak{n} \cdot \log \mathfrak{n} \cdot \log_2 \mathfrak{n} \cdots$$

we obtain with $\mathfrak{n} = \mathrm{l}_{\omega^{m-1}\cdot a}(\mathrm{e}_{\omega^m} g)$ that

$$\operatorname{supp} g^{\downarrow} \succ \mathrm{l}'_{\omega^m}(\mathrm{l}_{\omega^{m-1}\cdot a}(\mathrm{e}_{\omega^m} g)) \succcurlyeq \mathcal{R}_{\mathrm{l}_{\omega^m}}(\mathrm{l}_{\omega^{m-1}\cdot a}(\mathrm{e}_{\omega^m} g), \varepsilon).$$

Hence the same holds for $\operatorname{supp} g$. Now $\varepsilon = 0$ implies $\mathrm{l}_{\omega^m} h = g + (k-a) \in \mathbb{T}$, thus $f \in \mathbb{T}$. From $\mathbf{T^m 2}$ for $\mathbb{T}$ it then follows that $\mathrm{e}_{\omega^m} f \in \mathfrak{M}$. This shows the lemma. □

Now we assume that for $m \leqslant i < n$ we have functions $\log, \ldots, \mathrm{l}_{\omega^i}$ such that the inductive assumption $\mathbf{(IA^{\geqslant m})_i}$ holds for the field $\mathbb{T}_{\mathrm{e}_{\omega^m}}$. We define a function $\mathrm{l}_{\omega^{i+1}}$ on $(\mathbb{T}_{\mathrm{e}_{\omega^m}})^+_\infty$ as follows. Let $f \in (\mathbb{T}_{\mathrm{e}_{\omega^m}})^+_\infty$ and $k \in \mathbb{N}$ such that $\mathfrak{d}_{\mathrm{l}_{\omega^i\cdot k} f} \in \operatorname{dom} \mathrm{l}_{\omega^{i+1}} \cap \mathfrak{M}$. This is possible by the inductive assumption $\mathbf{(IA^{\geqslant m})_i}$. Then there is a series $\varepsilon \in \mathbb{T}^{\downarrow}_{\mathrm{e}_{\omega^m}}$ such that $\mathrm{l}_{\omega^i\cdot k} f = \mathfrak{d}_{\mathrm{l}_{\omega^i\cdot k} f} + \varepsilon$. We let

$$\mathrm{l}_{\omega^{i+1}} f := k + \mathcal{T}_{\mathrm{l}_{\omega^{i+1}}}(\mathfrak{d}_{\mathrm{l}_{\omega^i\cdot k} f}, \varepsilon).$$

Lemma 8.4.6 *Assume that for the integers $m \leqslant i < n$ the inductive assumption* $\mathbf{(IA^{\geqslant m})_i}$ *holds for $\langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^i} \rangle$. Let the function $\mathrm{l}_{\omega^{i+1}}$ be defined as above. Then the inductive assumption* $\mathbf{(IA^{\geqslant m})_{i+1}}$ *holds for $\langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^{i+1}} \rangle$.*

Proof: From the definition of $\mathrm{l}_{\omega^{i+1}}$ it follows that for every $k \in \mathbb{N}$ and every $f \in (\mathbb{T}_{\mathrm{e}_{\omega^m}})^+_\infty$ the leading monomial of $\mathrm{l}_{\omega^{i+1} \cdot k} f$ is an element of $\mathfrak{M}$. Hence $(\mathbf{IA}_{\mathbf{1}}^{\geqslant \boldsymbol{m}})_{\boldsymbol{i+1}}$ and $(\mathbf{IA}_{\mathbf{3}}^{\geqslant \boldsymbol{m}})_{\boldsymbol{i+1}}$ hold, since $\mathbb{T}$ is of strength $i+1$. We only need to show that $\mathbb{T}_{\mathrm{e}_{\omega^m}}$ is of strength $i+1$, and to do this it suffices to show $\mathbf{T}^{\boldsymbol{i+1}}\mathbf{2}$. Let $f \in \operatorname{dom} \mathrm{e}_{\omega^{i+1}}$ with $\operatorname{supp} f^{\downarrow} \prec_{\mathrm{l}_{\omega^i}} \mathrm{e}_{\omega^{i+1}} f$. Then there is a series $h \in \mathbb{T}_{\mathrm{e}_{\omega^m}}$ with $f = \mathrm{l}_{\omega^{i+1}} h$. Choose $k \in \mathbb{N}$ large enough such that $\mathfrak{a} = \mathfrak{d}_{\mathrm{l}_{\omega^i \cdot k} h} \in \operatorname{dom} \mathrm{l}_{\omega^{i+1}} \cap \mathfrak{M}$ and $\mathrm{l}_{\omega^{i+1}} \mathfrak{a} \in \mathfrak{M}$. Then for $\varepsilon \preccurlyeq 1$ with $\mathrm{l}_{\omega^i \cdot k} h = \mathfrak{a} + \varepsilon$ we have

$$f = \mathrm{l}_{\omega^{i+1}} \mathfrak{a} + k + \mathcal{R}_{\mathrm{l}_{\omega^{i+1}}}(\mathfrak{a}, \varepsilon).$$

Hence $f^{\downarrow} = \mathcal{R}_{\mathrm{l}_{\omega^{i+1}}}(\mathfrak{a}, \varepsilon)$. Fact 7.6.4 implies $\varepsilon = 0$. Thus $f = \mathrm{l}_{\omega^{i+1}} \mathfrak{a} + k \in \mathbb{T}$. Applying $\mathbf{T}^{\boldsymbol{i+1}}\mathbf{2}$ for $\mathbb{T}$ finishes the proof. □

We summarize the results in the following

Proposition 8.4.7 *Let n, m be integers with $n > 0$ and $0 \leqslant m \leqslant n$. Let $\mathbb{T}$ be of strength n and $\mathbb{T}_{\mathrm{e}_{\omega^m}}$ be defined as above. Then there are functions l_{ω^i} for $i \leqslant n$ such that*

$$\langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^i} \rangle$$

is of strength i. Moreover, we have the following chain:

$$\begin{array}{ccccccc} \langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log \rangle & \hookrightarrow \cdots \hookrightarrow & \langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^{m-1}} \rangle & & & & \\ & & \big\downarrow & & & & \\ & & \langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^m} \rangle & \hookrightarrow \cdots \hookrightarrow & \langle \mathbb{T}_{\mathrm{e}_{\omega^m}}, \log, \ldots, \mathrm{l}_{\omega^n} \rangle. & & \end{array}$$

In particular, the field $\mathbb{T}_{\mathrm{e}_{\omega^m}}$ is of strength n.

Chapter 9

Exponential closures of positive strength

We can now apply the tools developed so far in order to construct a field of generalized power series with exponential and logarithmic functions of positive strength. These functions will be total on the set of positive and infinite elements. We show some properties of such fields.

9.1 Properties of extended fields

Throughout this section we fix a field $\mathbb{T} = C[[\mathfrak{M}]]$ which is of strength $n > 0$. By Proposition 8.4.7, the e_{ω^i}-extension of $\mathbb{T}$ is of strength n for every $i \leqslant n$. Starting from $\mathbb{T}$, we can thus generate ever larger transseries fields. Before we use these extensions to construct e_{ω^n}-closed fields, let us study some properties of e_{ω^n}-extensions.

Recall that for all $f \in \mathbb{T}_\infty^+$ we have defined a truncation $t_f \trianglelefteq f$ which defines a new monomial $\mathrm{e}_{\omega^n} t_f \in \mathfrak{N}_{n,\mathbb{T}}$, if $\pi_{n,\mathbb{T}} f = \infty$. This does not mean, however, that the function e_{ω^n} is defined for f in $\mathbb{T}_{\mathrm{e}_{\omega^n}}$. But — as the next lemma shows — using finitely many e_{ω^i}-extensions with $i \leqslant n$ is a strong enough tool to eventually ensure $f \in \operatorname{dom} \mathrm{e}_{\omega^n}$ in a field $\hat{\mathbb{T}} \supset \mathbb{T}$.

Lemma 9.1.1 *If $\mathbb{T}$ is of strength $n > 0$, then for all $f \in \mathbb{T}_\infty^+$:*

(1) If $\pi_{n,\mathbb{T}} f = \infty$, then $\pi_{n,\mathbb{T}_{\mathrm{e}_{\omega^n}}} f < \infty$.

(2) If $\pi_{n,\mathbb{T}} f < \infty$ and $\pi_{n-1,\mathbb{T}_{\mathrm{e}_{\omega^{n-1}}}}(\mathrm{e}_{\omega^n}(f - \pi_{n,\mathbb{T}} f)) = 0$, then

$$\pi_{n,\mathbb{T}_{\mathrm{e}_{\omega^{n-1}}}} f = \max(0, \pi_{n,\mathbb{T}} f - 1).$$

(3) If $\pi_{1,\mathbb{T}} f < \infty$, then $\pi_{1,\mathbb{T}_{\exp}} f = \max(0, \pi_{1,\mathbb{T}} f - 1)$.

Proof: (1) Let $f \in \mathbb{T}_\infty^+$ with $\pi_{n,\mathbb{T}} f = \infty$. Let $g \trianglelefteq f$ be the maximal truncation with $g \in \mathcal{F}_{n,\mathbb{T}}$, and let $\varepsilon \in \mathbb{T}^{\downarrow}$ with $f = g + \varepsilon$. Then $g \in \operatorname{dom} \mathrm{e}_{\omega^n}$ in $\mathbb{T}_{\mathrm{e}_{\omega^n}}$ and

$$\mathrm{l}_{\omega^n} \|\varepsilon\| \not< g - \mathbb{N}.$$

The assumption $\pi_{n,\mathbb{T}} f = \infty$ implies that $(g-j,\varepsilon)$ is not an e_{ω^n}-Taylor couple for $j \geqslant 0$. Assume for a contradiction that $\pi_{n,\mathbb{T}_{\mathrm{e}_{\omega^n}}} f = \infty$. Then from $g - j \in \operatorname{dom} \mathrm{e}_{\omega^n}$ in $\mathbb{T}_{\mathrm{e}_{\omega^n}}$ it follows that then for all $j \geqslant 0$ the sequence

$$(\mathrm{e}^{(i)}_{\omega^n}(g-j) \cdot \varepsilon^i)_{0 \leqslant i}$$

is not a Noetherian family. Lemma 7.3.3 implies $1 \prec \mathrm{e}'_{\omega^n}(g-j) \cdot \varepsilon$ for all $j \geqslant 0$. Hence

$$\forall j \geqslant 0 : \|\varepsilon\| < \mathrm{e}'_{\omega^n}(g-j).$$

From $\mathrm{e}'_{\omega^n}(h-1) \prec \mathrm{e}_{\omega^n} h$ for all series h, we now obtain $\|\varepsilon\| \prec \mathrm{e}_{\omega^n}(g-j)$ for all $j \geqslant 0$. Hence $\mathrm{l}_{\omega^n} \|\varepsilon\| < g - j$ which implies $\mathrm{l}_{\omega^n} \|\varepsilon\| < g - \mathbb{N}$. This contradiction shows (1).

The assumptions imply $\mathrm{e}_{\omega^n}(f - \pi_{n,\mathbb{T}} f) \in \mathbb{T}$ and

$$\mathrm{e}_{\omega^{n-1}}(\mathrm{e}_{\omega^n}(f - \pi_{n,\mathbb{T}} f)) = \mathrm{e}_{\omega^n}(f - (\pi_{n,\mathbb{T}} f - 1)) \in \mathbb{T}_{\mathrm{e}_{\omega^{n-1}}}.$$

This proves (2). The part (3) follows from (2) using the fact that for all $h \in \mathbb{T}^+_\infty$ we have $\pi_{0,\mathbb{T}_{\exp}} h = 0$. □

We will eventually consider chains of extensions. More generally, we will have families of totally ordered, multiplicative groups $(\mathfrak{M}_i)_{i \in I}$ which are totally ordered by set-inclusion such that for all $i \in I$ the fields $C[[\mathfrak{M}_i]]$ are of a common strength. Recall from Proposition 2.3.9 that if this common strength is 0, then $C[[\bigcup_I \mathfrak{M}_i]]$ is a transseries field. We generalize this proposition to fields of positive strengths.

Proposition 9.1.2 *Let α be an ordinal and $(\mathfrak{M}_i)_{i<\alpha}$ a sequence of totally ordered, multiplicative groups such that for some $n > 0$*

- *$\forall i < j < \alpha : \mathfrak{M}_i$ is a subgroup of $\mathfrak{M}_j$,*
- *$\forall i < \alpha :$ the field $\mathbb{T}_i = C[[\mathfrak{M}_i]]$ is of strength n,*
- *$\forall i < j < \alpha : \mathbb{T}_i$ is an e_{ω^n}-l_{ω^n}-substructure of $\mathbb{T}_j$.*

Let $\mathbb{T}_{<\alpha} = \bigcup_{i<\alpha} \mathbb{T}_i$ and $\mathbb{T}_\alpha = C[[\bigcup_{i<\alpha} \mathfrak{M}_i]]$. Then there is a function $\mathrm{l}_{\omega^n} : \mathbb{T}^+_\alpha \to \mathbb{T}_\alpha$ such that:

(1) $\mathbb{T}_\alpha$ is of strength n.

(2) If there is a cofinal set $J \subseteq \alpha$ with $\forall j \in J : \mathfrak{M}_{j+1} = (\mathfrak{M}_j)_{\mathrm{e}_{\omega^n}}$, then for all $f \in (\mathbb{T}_{<\alpha})^+_\infty$: $\pi_{n,\mathbb{T}_{<\alpha}} f < \infty$.

Proof: (1) First, we have to define a function l_{ω^n} on $\mathbb{T}^+_\alpha$. Let $f \in \mathbb{T}^+_\alpha$, then there is an $i < \alpha$ such that the leading term τ of f is a series in $\mathbb{T}_i$. Let R be the remainder of f, i.e. $f = \tau + R$. Then we let $\mathrm{l}_{\omega^n}\tau$ as in $\mathbb{T}_i$, and (τ, R) is a l_{ω^n}-Taylor couple in $\mathbb{T}_\alpha$. We let

$$\mathrm{l}_{\omega^n} f := \mathcal{T}_{\mathrm{l}_{\omega^n}}(\tau, R).$$

This definition is coherent by Chapter 6. This shows $\mathbf{T^n 1}$ and $\mathbf{T^n 3}$. Now let $f \in \operatorname{dom} \mathrm{e}_{\omega^n}$ in $\mathbb{T}_\alpha$ and $h \in \mathbb{T}_\alpha$ such that $f = \mathrm{l}_{\omega^n} h$. Suppose that $\operatorname{supp} f^{\downarrow} \prec_{\mathrm{l}_{\omega^{n-1}}} \mathrm{e}_{\omega^n} f$. Then for $\mathfrak{m} = \mathfrak{d}_h$ there is an integer $k \in \mathbb{N}$ such that

$$\mathfrak{n} := \mathfrak{d}_{\mathrm{l}_{\omega^{n-1}.k} h} = \mathfrak{d}_{\mathrm{l}_{\omega^{n-1}.k} \mathfrak{m}} \in \operatorname{dom} \mathrm{l}_{\omega^n} \cap \mathfrak{M}_i$$

for some $i < \alpha$. Then for $\mathrm{l}_{\omega^{n-1}\cdot k}h = \mathfrak{n} + \varepsilon$ we have $\mathrm{l}_{\omega^n}h = \mathcal{T}_{\mathrm{l}_{\omega^n}}(\mathfrak{n}, \varepsilon)$, and by Lemma 7.4.7 we have

$$\operatorname{supp} \mathrm{l}_{\omega^n}\mathfrak{n} \succ \mathcal{R}_{\mathrm{l}_{\omega^n}}(\mathfrak{n}, \varepsilon).$$

From Fact 7.6.4 it follows that $\operatorname{supp} f^{\downarrow} \prec_{\mathrm{l}_{\omega^{n-1}}} \mathrm{e}_{\omega^n} f$ implies $\varepsilon = 0$. Hence $f = \mathrm{l}_{\omega^n}\mathfrak{n} + k \in \mathbb{T}_i$. Condition $\mathbf{T^n2}$ for $\mathbb{T}_i$ then shows $\mathrm{e}_{\omega^n} f \in \mathfrak{M}_i \subseteq \mathfrak{M}_\alpha$, hence $\mathbf{T^n2}$. Condition $\mathbf{T^n4}$ follows from the same condition for $\mathbb{T}$. This shows (1).

(2) Let $f \in (\mathbb{T}_{<\alpha})^+_\infty$, then $f \in \mathbb{T}_i$ for some $i < \alpha$. Pick $j \in J$ with $j > i$. Then by Lemma 9.1.1, we have $\pi_{n,\mathbb{T}_{j+1}} f < \infty$. □

Finally, we consider questions concerning the size of the support of series in the e_{ω^n}-extension of $\mathbb{T}$. Again, our present results will generalize results about the support in exp-extensions $\mathbb{T}_{\exp}$.

Proposition 9.1.3 *Let $\mathbb{T} = C[[\mathfrak{M}]]$ be of strength n, and let κ_1, κ_2 be cardinal numbers such that C and $\mathfrak{M}$ have cofinal cardinality $< \kappa_1$ and $< \kappa_2$, respectively. Then for every $f \in \mathbb{T}_{\mathrm{e}_{\omega^n}}$ the support of f has cardinality $< \max(\kappa_1, \kappa_2)$.*

Proof: Let $(\mathfrak{a}_\beta)_{\beta<\tau}$ be well-ordered in $\mathfrak{M}_{\mathrm{e}_{\omega^n}}$. Then the lemma follows, if we can show that $\tau < \max(\kappa_1, \kappa_2)$. For each $\beta < \tau$ there are $\mathfrak{m}_\beta \in \mathfrak{M}$ and $\mathfrak{n}_\beta \in \mathfrak{N}_{n,\mathbb{T}}$ such that $\mathfrak{a}_\beta = \mathfrak{m}_\beta \cdot \mathfrak{n}_\beta$. The sequence $(\log \mathfrak{a}_\beta)_{\beta<\tau}$ is strictly decreasing in $\mathbb{T}_{\mathrm{e}_{\omega^n}}$. From the hypothesis about $\mathbb{T}$, we obtain that the support of each $\log \mathfrak{m}_\beta$ has cardinality $< \max(\kappa_1, \kappa_2)$. Since for every $\mathfrak{n} \in \mathfrak{N}_{n,\mathbb{T}}$ the support of $\log \mathfrak{n}$ is countable, we have $|\operatorname{supp} \log \mathfrak{a}_\beta| < \max(\kappa_1, \kappa_2)$. We apply Lemma 1.8.5 and conclude $|\tau| < \max(\kappa_1, \kappa_2)$. □

Example 9.1.4 Let $C = \mathbb{R}$, i.e. $\kappa_1 = \aleph_1$. Take the monomial group $\mathfrak{L}_n$ from Section 7.6. We apply Lemma 2.4.3 and obtain that every series $f \in \mathbb{R}[[\mathfrak{L}_n]]$ has countable support. Moreover, we see that applying the extension process (of strength $\leqslant n$) countably many times does not affect the size of the support of the series. It remains countable.

9.2 Iterating extensions and the closure of admissible fields

We have now all necessary tools to construct a field of generalized power series which has functions e_{ω^n} and l_{ω^n} for some $n > 0$ which both are total on the set of positive and infinite elements. As we will see, we can even extend the functions such that they are total on the set of positive elements.

Section 7.6 provides a transseries field of strength n. Let in the following $\mathbb{T}$ be such a field and let $f \in \mathbb{T}^+_\infty$. Suppose that $f \notin \operatorname{dom} \mathrm{e}_{\omega^n}$ in $\mathbb{T}$. Proposition 9.1.1 now suggests how to construct a field $\hat{\mathbb{T}} \supseteq \mathbb{T}$ such that f is in the domain of the function e_{ω^n} in $\hat{\mathbb{T}}$. First, we check whether $\pi_{n,\mathbb{T}} f = \infty$ or $< \infty$. If $\pi_{n,\mathbb{T}} f = \infty$, then we apply an e_{ω^n}-extension to $\mathbb{T}$, and we obtain an integer $k \in \mathbb{N}$ such that $f - k \in \operatorname{dom} \mathrm{e}_{\omega^n}$ in $\mathbb{T}_{\mathrm{e}_{\omega^n}}$. More generally, if there is an $i \leqslant n$ such that $\pi_{i,\mathbb{T}} f = \infty$, then we apply the e_{ω^i}-extension process to $\mathbb{T}$. We thus find an extension $\tilde{\mathbb{T}} \supseteq \mathbb{T}$ such that $\pi_{i,\tilde{\mathbb{T}}} f < \infty$ for all $i \leqslant n$. Now Proposition 9.1.1 shows how to choose the extensions to reduce all $\pi_{i,\hat{\mathbb{T}}} f$. Hence after a finite number of extensions, we will obtain a field $\hat{\mathbb{T}} \supseteq \mathbb{T}$ such that f is in the domain of e_{ω^n} in that field.

9.2.1 Cofinal partitions

Throughout this section, let λ be a limit ordinal. Recall that the set λ is totally ordered by $<=\in$. Two totally ordered sets $S = (S, \leqslant), P = (P, \leqslant)$ are **isomorphic**, in symbols $S \cong P$, iff there is a surjective and strictly increasing mapping $\varphi : S \to P$. Note that φ is bijective and that $\cong$ is an equivalence relation. We remark that for limit ordinals λ_1, λ_2 we have $\lambda_1 \cong \lambda_2$ if and only if $\lambda_1 = \lambda_2$. A set $S \subseteq \lambda$ is **cofinal** in λ iff there is no $\alpha < \lambda$ such that $S \subseteq \alpha$. We will use the following lemma.

Lemma 9.2.1 *Let $S \subseteq \lambda$ be totally ordered by $<=\in$. If $S \cong \lambda$, then S is cofinal in λ.*

Proof: Let $\lambda' \leqslant \lambda$ be the smallest limit ordinal such that S is cofinal in λ'. If $S \cong \lambda'$, then $\lambda' \cong S \cong \lambda$, hence $\lambda' = \lambda$.

Now assume that $S \ncong \lambda'$. Let $\varphi : S \to \lambda$ be bijective and increasing and $\psi : S \to \lambda'$ injective and increasing. Then $\vartheta = \psi \circ \varphi_{-1} : \lambda \to \lambda'$ is strictly increasing. We claim that $\vartheta_{i+1}(\lambda \setminus \lambda') < \vartheta_i(\lambda \setminus \lambda')$ for all $i \geqslant 0$. From $\operatorname{range} \vartheta \subseteq \lambda'$ we obtain $\vartheta(\lambda \setminus \lambda') < \lambda \setminus \lambda'$. Now assume that $\vartheta_i(\lambda \setminus \lambda') < \vartheta_{i-1}(\lambda \setminus \lambda')$. Let $\alpha \in \vartheta_{i+1}(\lambda \setminus \lambda')$. Then $\alpha = \vartheta(\beta)$ for some $\beta \in \vartheta_i(\lambda \setminus \lambda') < \vartheta_{i-1}(\lambda \setminus \lambda')$. Since ϑ is strictly increasing, this implies $\alpha = \vartheta(\beta) < \vartheta_i(\lambda \setminus \lambda')$, thus the claim.

If $\lambda' < \lambda$, then for $\alpha \in \lambda \setminus \lambda'$ the sequence $(\vartheta_i(\alpha))_{0 \leqslant i}$ is strictly decreasing. This contradiction shows $\lambda = \lambda'$. □

For an ordinal α and $n \in \mathbb{N}$, a function $p : \alpha \to n + 1 = \{0, \ldots, n\}$ is called a **partition of $\boldsymbol{\alpha}$ of strength $\boldsymbol{n}$**, or simply a **partition**, if n is clear from the context. For the limit ordinal λ and a partition $p : \lambda \to n + 1$ we define sets

$$S_{p,i} := p_{-1}(i) \subseteq \lambda \qquad (0 \leqslant i \leqslant n).$$

We say that p is a **cofinal partition** in λ iff $S_{p,i} \cong \lambda$ for all $i \leqslant n$. By Lemma 9.2.1 the sets $S_{p,i}$ are cofinal in λ, if p is cofinal in λ.

Example 9.2.2 Let us give examples in ω and ω^2. We write a partition $p : \lambda \to n + 1$ as a sequence $p = (p_0, p_1, \cdots)$. Let $n = 2$. First let $\lambda = \omega$.

$$\begin{aligned}
p_a &= (1, 2, 0, 2, 0, 2, \cdots) \\
p_b &= (1, 0, 1, 0, 1, 0, \cdots) \\
p_c &= (0, 1, 2, 0, 1, 2, \cdots) \\
p_d &= (1, 2, 0, 1, 2, 0, 0, 1, 2, 0, 0, 0, 1, 2, 0, 0, 0, 0, 1, 2, \cdots).
\end{aligned}$$

Then the partitions p_a, p_b are not cofinal in ω, and p_c, p_d are cofinal in ω. Let $\lambda = \omega^2$.

$$\begin{array}{rcl} p_e & = & (0,0,\dots,1,1,\dots,2,2,\dots,2,2,\dots,\cdots) \\ p_f & = & (0,1,2,0,1,2,\dots,0,1,0,1,\dots,0,1,0,1,\dots,\cdots) \\ p_g & = & (0,1,2,0,1,2,\dots,0,1,2,0,1,2,\dots,\cdots) \\ p_h & = & (0,1,0,1,\dots,0,2,0,2,\dots,0,1,2,0,1,2,\cdots). \end{array}$$

The partitions p_e, p_f are not cofinal in ω^2, but p_g and p_h are cofinal in ω^2.

Remark 9.2.3 Let $p : \lambda \to n+1$ be a partition. For $\alpha < \lambda$ we let $p\restriction_\alpha : \alpha \to n+1$ be the restriction of p to α. Then $p\restriction_\alpha$ is a partition of α of strength n. We say that a partition p of λ is **strongly cofinal** in λ iff for all limit ordinals $\lambda' \leqslant \lambda$ the partition $p\restriction_{\lambda'}$ is cofinal in λ'. Note that the partition p_h of Example 9.2.2 shows that not every cofinal partition is strongly cofinal.

Every ordinal α is either a limit ordinal, or there are λ and $n \in \omega$ such that $\alpha = \lambda + n$, where in the following we allow $\lambda = 0$. (Suppose that this is not the case, then let α be the smallest ordinal which is neither a limit ordinal nor of the form $\lambda + n$. Then α must be a successor ordinal, and $\alpha = \beta + 1$. But then $\beta < \alpha$ has the same property, contradicting the minimality of α.)

We let $\lfloor \alpha \rfloor$ be the integer such that $\alpha = \lambda + \lfloor \alpha \rfloor$ for some limit ordinal λ. A partition p is **regular** of strength n iff $\lfloor \alpha \rfloor \equiv p_\alpha \pmod{n+1}$. Hence the regular partition on ω is $p = (0,1,\dots,n,0,1,\dots,n,\cdots)$. Note that regular partitions are uniquely determined and strongly cofinal.

9.2.2 Closures

Let $\mathbb{T} = C[[\mathfrak{M}]]$ be of strength n. Fix α, an ordinal, and p, a partition of strength n of α. We define a sequence $(\mathbb{T}^{\langle p,\beta\rangle})_{\beta \leqslant \alpha}$ of fields of generalized power series over monomial groups $\mathfrak{M}^{\langle p,\beta\rangle}$ with the field of constants C. Moreover, each $\mathbb{T}^{\langle p,\beta\rangle}$ will be of strength n. We let

$$\begin{array}{rlrll} \mathfrak{M}^{\langle p,0\rangle} & := \mathfrak{M}_{\mathrm{e}_{\omega^{p(0)}}} & \mathbb{T}^{\langle p,0\rangle} & := C[[\mathfrak{M}^{\langle p,0\rangle}]] & \\ \mathfrak{M}^{\langle p,\beta+1\rangle} & := \mathfrak{M}^{\langle p,\beta\rangle}_{\mathrm{e}_{\omega^{p(\beta+1)}}} & \mathbb{T}^{\langle p,\beta+1\rangle} & := C[[\mathfrak{M}^{\langle p,\beta+1\rangle}]] & (\beta < \alpha) \\ \mathfrak{M}^{\langle p,\lambda\rangle} & := \displaystyle\bigcup_{\beta<\lambda} \mathfrak{M}^{\langle p,\beta\rangle} & \mathbb{T}^{\langle p,\lambda\rangle} & := C[[\mathfrak{M}^{\langle p,\lambda\rangle}]] & (\lambda \leqslant \alpha,\ \text{limit ordinal}) \end{array}$$

In other words, the field $\mathbb{T}^{\langle p,\beta+1\rangle}$ is the $\mathrm{e}_{\omega^{p(\beta+1)}}$-extension of $\mathbb{T}^{\langle p,\beta\rangle}$. If $\mathbb{T}^{\langle p,\beta\rangle}$ is of strength n, then so is $\mathbb{T}^{\langle p,\beta+1\rangle}$. By that we mean that the field $\mathbb{T}^{\langle p,\beta+1\rangle}$ is equipped with functions $\log,\dots,\mathrm{l}_{\omega^n}$ such that $\langle \mathbb{T}^{\langle p,\beta+1\rangle}, \log,\dots,\mathrm{l}_{\omega^n}\rangle$ is of strength n as done in Section 8.4. I.e. each step $\mathbb{T}^{\langle p,\beta\rangle} \to \mathbb{T}^{\langle p,\beta+1\rangle}$ requires intermediate steps

$$\begin{array}{ccccc} \mathbb{T}^{\langle p,\beta\rangle} & & & & \mathbb{T}^{\langle p,\beta+1\rangle} \\ \downarrow & & & & \| \\ \mathbb{T}^{\langle p,\beta+1\rangle} & \hookrightarrow \cdots \hookrightarrow & \langle \mathbb{T}^{\langle p,\beta+1\rangle}, \log,\dots,\mathrm{l}_{\omega^j}\rangle & \hookrightarrow \cdots \hookrightarrow & \langle \mathbb{T}^{\langle p,\beta+1\rangle}\mathrm{e}_{\omega^{i+1}}, \log,\dots,\mathrm{l}_{\omega^n}\rangle \end{array}$$

Similarly, we identify $\mathbb{T}^{\langle p,\lambda\rangle}$ with its structure of strength n for limit ordinals λ. We use Lemma 9.1.2 to show that $\mathbb{T}^{\langle p,\lambda\rangle}$ is of strength n. Thus $\mathbb{T}^{\langle p,\beta\rangle}$ will be an e_{ω^n}-l_{ω^n}-substructure of $\mathbb{T}^{\langle p,\gamma\rangle}$ for all $\beta < \gamma \leqslant \alpha$. Therefore, since $\mathbb{T}$ is of strength n, so are all $\mathbb{T}^{\langle p,\beta\rangle}$ for $\beta \leqslant \alpha$.

We define a set $\mathbb{T}^{\langle p,<\lambda\rangle}$ for every limit ordinal $\lambda \leqslant \alpha$ by

$$\mathbb{T}^{\langle p,<\lambda\rangle} := \bigcup_{\beta<\lambda} \mathbb{T}^{\langle p,\beta\rangle}.$$

Note that $\mathbb{T}^{\langle p,<\lambda\rangle}$ is a proper subset of $\mathbb{T}^{\langle p,\lambda\rangle}$ for every limit ordinal $\lambda \leqslant \alpha$. If p is cofinal in λ, then we call $\mathbb{T}^{\langle p,<\lambda\rangle}$ a **closure** of $\mathbb{T}$. A closure is **regular** iff p is regular. Since regular partitions are unique, we denote the regular closure of $\mathbb{T}$ by $\mathbb{T}^{\mathrm{reg}}_{<\lambda}$, and call it the **regular $\mathbf{e}_{\omega^n}$-closure** of length λ.

Proposition 9.2.4 *Let $\mathbb{T} = C[[\mathfrak{M}]]$ be of strength $n > 0$. Let p be a cofinal partition of strength n of the ordinal α . Then*

(1) The field $\mathbb{T}^{\langle p,\alpha\rangle}$ is of strength n.
(2) If λ is a limit ordinal, then functions e_{ω^n} and l_{ω^n} are total on $(\mathbb{T}^{\langle p,<\lambda\rangle})^+_\infty$.
(3) Suppose that C and $\mathfrak{M}$ have cofinal cardinality $< \kappa_1$ and $< \kappa_2$, respectively, for some cardinals κ_1, κ_2. If $\alpha < \max(\kappa_1, \kappa_2)$, then for all $\beta \leqslant \alpha$:

$$|\operatorname{supp} f| < \max(\kappa_1, \kappa_2)$$

for all $f \in \mathbb{T}^{\langle p,\beta\rangle}$.

Proof: (1) follows from the above considerations. We show (2). Let $f \in (\mathbb{T}^{\langle p,<\lambda\rangle})^+_\infty$. Then there is an ordinal $\beta < \lambda$ such that $f \in (\mathbb{T}^{\langle p,\beta\rangle})^+_\infty$. Since λ is a limit ordinal, we have $\beta + \omega \leqslant \lambda$. Since p is a cofinal partition, the sets

$$S_{p,i} \cap (\lambda \setminus \beta)$$

are cofinal as well for all $i \leqslant n$. Thus there is a sequence $(\gamma_j)_{j\in\mathbb{N}}$ such that

- $\beta < \gamma_0 < \gamma_1 < \cdots < \lambda$;
- for all $0 \leqslant j$ and $i \leqslant n$ we have $\emptyset \neq (\gamma_{j+1} \setminus \gamma_j) \cap S_{p,i}$.

Let $\hat{\mathbb{T}}_j = \mathbb{T}^{\langle p,\gamma_j\rangle}$. From part (1) of Proposition 9.1.1 it follows that we have $\pi_{0,\hat{\mathbb{T}}_1} f = 0$ and

$$k_i := \pi_{i,\hat{\mathbb{T}}_1} f < \infty \qquad (0 < i \leqslant n).$$

Let $K_i = k_1 + \cdots + k_i$ for all $i \leqslant n$. Then inductively applying part (2) of Proposition 9.1.1 shows that

$$\pi_{i,\hat{\mathbb{T}}_{K_i}} f = 0.$$

Hence $f \in \operatorname{dom} \mathrm{e}_{\omega^n}$ in $\hat{\mathbb{T}}_{K_n} \subseteq \mathbb{T}^{\langle p,<\lambda\rangle}$. The function l_{ω^n} is defined as in $\mathbb{T}^{\langle p,\lambda\rangle}$.

(3) follows from Proposition 9.1.3. This proves the proposition. □

Remark 9.2.5 From Proposition 9.2.4 it now follows that the regular e_{ω^n}-closure of $\mathbb{T}$ has total exponential and logarithmic functions of strength $i \leqslant n$ on the set of positive and infinite elements. Moreover, if for all $f \in \mathbb{T}$ the support of f is countable (as it is the case in the admissible field $\mathbb{R}[[\mathfrak{L}_n]]$) and λ is a countable limit ordinal, then all series in $\mathbb{T}^{\mathrm{reg}}_{<\lambda}$ have countable supports.

We can now state

Theorem 9.2.6 *For all $n \geqslant 0$ there are fields of generalized power series $\mathcal{K}_n$ such that there are functions* $\exp, \ldots, \mathrm{e}_{\omega^n}$ *and* $\log, \ldots, \mathrm{l}_{\omega^n}$ *which are totally defined on $(\mathcal{K}_n)^+_\infty$ with $\forall i < n$: $\forall f \in (\mathcal{K}_n)^+_\infty$:*

$$\begin{aligned} \mathrm{e}_{\omega^i} \circ \mathrm{e}_{\omega^{i+1}} f &= \mathrm{e}_{\omega^{i+1}}(f+1) \\ \mathrm{l}_{\omega^{i+1}} \circ \mathrm{l}_{\omega^i} f &= \mathrm{l}_{\omega^{i+1}} f - 1 \end{aligned}$$

Proof: Let $C = \mathbb{R}$ and $\mathfrak{M} = \mathfrak{L}_n$. Then the regular e_{ω^n}-closure $\mathcal{K}_n$ of $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ (of length ω) has the above properties by Proposition 9.2.4. This shows the theorem. □

Remark 9.2.7 Note that we can extend Theorem 9.2.6 to all positive elements of $\mathcal{K}_n$. Recall that $\mathbb{R}$ is an exponential field of strength n such that e_{ω^n} and l_{ω^n} are totally defined and analytic on $\mathbb{R}^+$. Then for each $0 < f \in \mathcal{K}_n^\updownarrow$ there is a positive real r_f and an infinitesimal ε_f such that $f = r_f + \varepsilon_f$. Then $\mathrm{e}_{\omega^n} f = \mathcal{T}_{\mathrm{e}_{\omega^n}}(r_f, \varepsilon_f)$ and $\mathrm{l}_{\omega^n} f = \mathcal{T}_{\mathrm{l}_{\omega^n}}(r_f, \varepsilon_f)$ are defined.

9.3 Generalizing structural properties

In this section, we generalize results about transseries fields to fields of higher strength. We work along the same lines as in the case of extensions of strength 0.

9.3.1 Tree-representations in fields of positive strength

We have seen how to define a tree-representation of a series of a transfinite exponential extension $\hat{\mathbb{T}}$ of a transseries field $\mathbb{T}$ (of strength 0). Recall that this only involves exp-extensions.

Let $\mathbb{T} = C[[\mathfrak{M}]]$ be of strength $n \geqslant 0$. Let α be an ordinal and p a partition of α of strength $\leqslant n$, i.e.

$$p : \alpha \longrightarrow n + 1 = \{0, \ldots, n\}.$$

Recall that $\mathbb{T}^{\langle p,\alpha\rangle}$ denotes the field of strength n which results from the extension process determined by p:

$$\mathbb{T} \longrightarrow \mathbb{T}^{\langle p,0\rangle} = \mathbb{T}_{\mathrm{e}_{\omega^{p(0)}}} \longrightarrow \mathbb{T}^{\langle p,1\rangle} = \mathbb{T}_{\mathrm{e}_{\omega^{p(1)}}} \longrightarrow \cdots.$$

For every term $c\mathfrak{m} \in C\mathfrak{M}^{\langle p,\alpha\rangle}$ and every series $f \in \mathbb{T}^{\langle p,\alpha\rangle}$, we have already defined maximal and minimal tree-representations. Moreover, we have introduced relative and relative-minimal tree-representations with respect to $\mathbb{T}$ of terms and series, if $p(\beta) = 0$ for all $\beta < \alpha$.

However, if extensions of positive strength are involved in the definition of $\mathbb{T}^{\langle p,\alpha\rangle}$, then these tree-representations do not suffice anymore. Moreover, some of the properties do not generalize to the new setting. Let us explain this with an example.

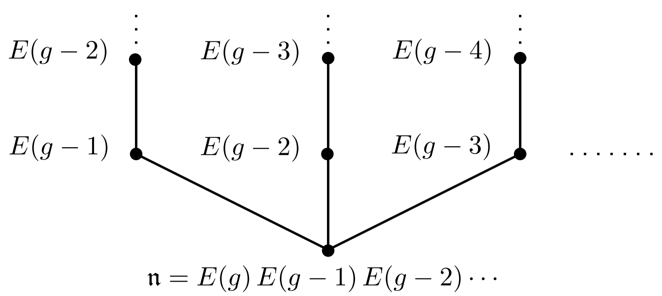
Example 9.3.1 Let $\mathbb{T} = C[[\mathfrak{M}]]$ be of strength 1 and $\alpha = 1$. We choose $p(0) = 1$ as partition. Note that in this case $\mathbb{T}^{\langle p,\alpha\rangle} = \mathbb{T}_E$. From $g \in \mathcal{F}_{1,\mathbb{T}}$, we obtain

$$\mathfrak{n} := E(g)\, E(g-1)\, E(g-2) \cdots \in \mathfrak{N}_{1,\mathbb{T}}.$$

The root of the maximal tree-representation $T_{\mathfrak{n},\max}$ is labeled by $\mathfrak{n}$, and the labels of the successors of the root are $E(g-1), E(g-2), \ldots$. Now we remark that for every $i \geqslant 1$, the maximal tree-representation of $E(g-i)$ has only one successor, which is labeled by $E(g-i-1)$. Hence, the paths in $T_{\mathfrak{n},\max}$ are of the form

$$[\mathfrak{n}, E(g-i), E(g-i-1), E(g-i-2), \ldots],$$

where $i \geqslant 1$. Thus the minimal tree-representation of $\mathfrak{n}$ is the tree of height 1 such that the leaves are labeled by monomials $E(g-i) \in \mathfrak{N}_{1,\mathbb{T}}$.



Note that no label of $T_{\mathfrak{n},\max}$ is an element of $C\mathfrak{M}$. Moreover, this labeled tree does not provide any information about the series g. As for the relative and relative-minimal tree-representations, they do not even exist.

On the other hand, once we know that a monomial is of the form $E(h) \in \mathfrak{N}_{1,\mathbb{T}}$, the maximal tree-representation of $E(h)$ consists only of the admissible path $[E(h), E(h-1), E(h-2), \ldots]$. The series $h \in \mathbb{T}$ admits maximal, minimal and relative-minimal tree-representations.

Hence, the notion of tree-representation for transseries fields will usually not be enough, and we have to extend this notion as follows.

Definition 9.3.2 *Let $\mathbb{T} = C[[\mathfrak{M}]]$ be of strength n and $p : \alpha \to n+1$ a partition for some ordinal number α. Let $t \in C\mathfrak{M}^{\langle p,\alpha\rangle}$. Then a* **tree-representation of strength n** *of t* **with respect to $\mathbb{T}$** *is a labeled tree $T = (T, l)$ such that $l(\mathsf{r}(T)) = t$ and such that for the labeling $l : T \to C\mathfrak{M}_\alpha$ we have*

TRn1. $\forall \mathsf{n} \in T \setminus \mathrm{leaf}(T)$*: if* $l(\mathsf{n})$ *is of the form* $\mathrm{l}_\beta(\mathrm{e}_{\omega^m} g)$*, then there exists a bijection* $\varphi : \mathrm{term}\, g \to \mathrm{succ}(\mathsf{n})$ *with*

(i) $\forall s, t \in \mathrm{term}\, g : s \succ t \Leftrightarrow l(\varphi(s)) \succ l(\varphi(t))$ *and*

(ii) $\forall t \in \mathrm{term}\, g : l(\varphi(t)) = t$.

TRn2. $\forall \mathsf{n} \in T \setminus \mathrm{leaf}(T)$*: if* $l(\mathsf{n})$ *is not of the form* $\mathrm{l}_\beta(\mathrm{e}_{\omega^m} g)$*, then there exists a bijection* $\varphi : \mathrm{term}\, \log \mathfrak{d}_{l(\mathsf{n})} \to \mathrm{succ}(\mathsf{n})$ *with*

(i) $\forall s, t \in \mathrm{term}\, \log \mathfrak{d}_{l(\mathsf{n})} : s \succ t \Leftrightarrow l(\varphi(s)) \succ l(\varphi(t))$ *and*

(ii) $\forall t \in \mathrm{term}\, \log \mathfrak{d}_{l(\mathsf{n})} : l(\varphi(t)) = t$.

We say that $T = (T, l)$ *represents* $l(\mathsf{r}(T)) = t$*. If for all* $\mathsf{n} \in T$ *we have* $l(\mathsf{n}) \in C\mathfrak{M} \Rightarrow \mathsf{n} \in \mathrm{leaf}(T)$*, then we say that* T *is a* **relative** *tree-representation of strength* n *with respect to* $\mathbb{T}$*.*

Remark 9.3.3 We extend the definition of tree-representations to series $f \in \mathbb{T}^{\langle p,\alpha\rangle}$ as follows. If $f \in C\mathfrak{M}^{\langle p,\alpha\rangle}$, then (T, l) is a tree-representation of the series f iff it is a tree-representation of the term f. If $1 < |\mathrm{term}\, f|$, then a labeled tree $T = (T, l)$ is a tree-representation of f iff the restriction of l to every child K_t (where $t \in \mathrm{term}\, f$) is a tree-representation of t and $l(\mathsf{r}(T)) = f$.

Remark 9.3.4 In the case $n = 0$, this is exactly the definition of tree-representations in transfinite exponential extensions, since the condition **TRn1** never applies in this case. Moreover, we remark that this definition is upwards-compatible: if $n_1 < n_2$ and T is a tree-representation of strength n_1, then T is a tree-representation of strength n_2.

Example 9.3.5 Let $f \in \mathbb{T}^{\langle p,\alpha\rangle}$. If $f \in C\mathfrak{M}$, then T_f is the labeled tree with only one element, the root, which is labeled with f. If $f \in \mathbb{T} \setminus C\mathfrak{M}$, then T_f is the tree of height 1 such that the label of $\mathsf{r}(T_f)$ is f and such that there is a bijection between the leaves of T_f and $\mathrm{term}\, f$. Note that this coincides with the construction of T_f from Example 3.2.12 in Chapter 3.

Now assume that for all $\beta < \alpha$ and all series from $\mathbb{T}^{\langle p,\beta\rangle}$ there is a representation as a tree w.r.t. $\mathbb{T}$. In particular, terms from $C\mathfrak{M}^{\langle p,\beta\rangle}$ have a representation w.r.t. $\mathbb{T}$. If $f \in \mathbb{T}^{\langle p,\beta\rangle}$ for some $\beta < \alpha$, then we let T_f be a tree-representation defined in $\mathbb{T}^{\langle p,\beta\rangle}$.

If

$$f \in \mathbb{T}^{\langle p,\alpha\rangle} \setminus \bigcup_{\beta<\alpha} \mathbb{T}^{\langle p,\beta\rangle},$$

then we define $\mathrm{pre}(T_f)$ to be the tree of height 1 such that the label of $\mathsf{r}(T_f)$ is f and such that there is a bijection between the leaves of T_f and $\mathrm{term}\, f$.

If α is a limit ordinal, then for all $t \in \mathrm{term}\, f$ we already have a representation T_t w.r.t. $\mathbb{T}$. If α is a successor ordinal $\beta + 1$, then we have to define T_t for those $t \in \mathrm{term}\, f$ such that $\mathfrak{d}_t \notin \mathfrak{M}^{\langle p,\beta\rangle}$. Fix $t \in \mathrm{term}\, f$. We distinguish three cases.

First, assume that $p(\alpha) = 0$. Then $\mathfrak{M}^{\langle p,\alpha\rangle} = \mathbb{T}^{\langle p,\beta\rangle}_{\exp}$. Hence for every $c\mathfrak{m} \in C\mathfrak{M}^{\langle p,\alpha\rangle}$ there is a purely infinite series $\log \mathfrak{m} \in (\mathbb{T}^{\langle p,\beta\rangle})^{\uparrow}$, which already admits a representation $T_{\log \mathfrak{m}}$. The root of $T_{\log \mathfrak{m}}$ is labeled with $\log \mathfrak{m}$, and the successors of the root are labeled with elements from $\mathrm{term}\, \log \mathfrak{m}$. We let T_t be the labeled tree which we obtain if we replace the root of $T_{\log \mathfrak{d}_t}$ by t.

Second, assume that $0 < m = p(\alpha)$ and that $\mathfrak{d}_t = \mathrm{l}_\beta(\mathrm{e}_{\omega^m} g)$ for some $\beta \leqslant \omega^m$ and $g \in \mathcal{F}_{m,\mathbb{T}^{\langle p,\beta\rangle}}$. Then $g \in \mathbb{T}^{\langle p,\beta\rangle}$ admits a representation as a labeled tree T_g w.r.t. $\mathbb{T}$ already. We let T_t be the labeled tree which results from replacing the root of T_g by t.

Finally, assume that $0 < m = p(\alpha)$ and that $\mathfrak{d}_t$ is not of the form $\mathrm{l}_\beta(\mathrm{e}_{\omega^m} g)$. Then for all $\mathfrak{n} \in \operatorname{supp} \operatorname{term} \mathfrak{d}_t$ we have either $\mathfrak{n} \in \mathfrak{M}^{\langle p,\gamma\rangle}$ with $\gamma < \alpha$ or $\mathfrak{n} = \mathrm{l}_\beta(\mathrm{e}_\omega g)$ for some $\beta \leqslant \omega^m$ and $g \in \mathcal{F}_{m,\mathbb{T}^{\langle p,\beta\rangle}}$. Hence for all $s \in \operatorname{term} \log \mathfrak{d}_t$, there are already labeled trees T_s w.r.t. $\mathbb{T}$. As in the exponential case, we let $\operatorname{pre}(T_t)$ be the tree of height 1, such that the root is labeled with t and such that there is a bijection between the set of leaves and the $\operatorname{term} \log \mathfrak{d}_t$. We let

$$T_t := \operatorname{pre}(T_t)[T_s]_{s \in \operatorname{term} \log \mathfrak{d}_t}.$$

We let T_f be the representation that we obtain by substituting T_t into the leaf of $\operatorname{pre}(T_f)$ which is labeled with t, i.e.

$$T_f := \operatorname{pre}(T_f)[T_t]_{t \in \operatorname{term} f}.$$

In fact, the labeled tree T_f is a relative tree-representation of strength n of the series $f \in \mathbb{T}^{\langle p,\alpha\rangle}$.

Tree representations in purely exponential extensions (i.e. in the case $n = 0$) have several properties shown in Chapter 3. Some of them generalize to the case of positive strength. Large parts of the proofs are similar to the case $n = 0$, and we will not repeat those parts. We only give the information needed to extend the proofs to the situation $n > 0$.

9.3.2 Properties of tree-representations

The following properties are formulated for the tree-representations of series f. Notice that f can be a term or monomial.

Proposition 9.3.6 *Each series $f \in \mathbb{T}^{\langle p,\alpha\rangle}$ admits a unique relative tree-representation $T_{f,\mathbb{T}}$ of strength $n \geqslant 0$ w.r.t. $\mathbb{T}$.*

Proof: Assume that there are monomials which admit two different relative tree-representations T, T'. Let $\mathfrak{m} \in \mathfrak{M}^{\langle p,\gamma\rangle}$ with this property such that γ is minimal. We only need to consider the case where $\mathfrak{m} = \mathrm{l}_\beta(\mathrm{e}_{\omega^n} g)$. Then the relative tree-representation of g is unique, since $g \in \mathbb{T}^{\langle p,\nu\rangle}$ for some $\nu < \gamma$. The roots of T and T' are labeled with $\mathfrak{m}$. But the labelings of T and T' restricted to $T \setminus \mathsf{r}(T)$ and $T' \setminus \mathsf{r}(T')$ are the labeling of $T_g \setminus \mathsf{r}(T_g)$, hence identical. But then $T = T'$. Contradiction. The rest follows as in Proposition 3.2.13. □

Proposition 9.3.7 *A relative tree-representation of strength $n \geqslant 0$ does not contain infinite chains for $\leqslant$, the ordering in the underlying tree of $T_{f,\mathrm{rel}}$.*

Proof: We extend the proof of Proposition 3.2.14 by showing that if $l(\mathsf{n}_i) \in \mathfrak{M}^{\langle p,\beta_i\rangle}$ (where β_i is minimal) and $l(\mathsf{n}_i) = \mathrm{l}_\beta(\mathrm{e}_{\omega^m} g)$, then $l(\mathsf{n}_{i+1}) \in \mathfrak{M}^{\langle p,\beta_{i+1}\rangle}$ (where β_{i+1} is minimal) and $\beta_{i+1} < \beta_i$. But $l(\mathsf{n}_{i+1}) \in \operatorname{supp} g$ and $g \in \mathbb{T}^{\langle p,\beta_{i+1}\rangle}$ for some $\beta_{i+1} < \beta_i$ imply this property. □

Remark 9.3.8 Proposition 3.2.15 does not generalize. For instance, if $g = x^2 + x$, then it is not possible to decide whether g contributes to a monomial of the form $\exp(x^2 + x)$ or to some $\mathrm{l}_\beta(\mathrm{e}_\omega(x^2 + x))$. Nonetheless, the closure properties can be generalized.

9.3.3 Minimal and maximal tree-representations of higher strength

We now introduce maximal and minimal tree-representations for extensions of positive strength. Throughout this section, we fix a field $\mathbb{T}$ of strength n, an ordinal $\alpha > 0$ and a partition $p : \alpha \to n+1$. Recall that for fixed terms or series in $\mathbb{T}^{\langle p,\alpha\rangle}$ there are already maximal and minimal tree-representations as defined in Chapter 3. We had also defined relative and relative-minimal tree-representations for transfinite exponential extensions, but these were with respect to the starting field.

As mentioned above, these representations may be not sufficient to express all information about the given object. For objects from $\mathbb{T}^{\langle p,\alpha\rangle}$, maximal and minimal representations will have to take in account the partition p.

Let $t \in C\mathfrak{M}^{\langle p,\alpha\rangle}$ and $T_{t,\mathrm{rel}}$ the relative tree-representation of t w.r.t. $\mathbb{T}$ with labeling l. For every leaf n, the term $l(\mathsf{n})$ is an element of $C\mathfrak{M}$ and admits therefore a maximal tree-representation $T_{l(\mathsf{n}),\max}$. We let

$$T_{t,\max} := T_{t,\mathrm{rel}}[T_{l(\mathsf{n}),\max}]_{\mathsf{n}\in\mathrm{leaf}(T_{t,\mathrm{rel}})}$$

be the **maximal tree-representation** of t **with respect to** $\mathbb{T}$. Similarly, we define the maximal tree-representation w.r.t. $\mathbb{T}$ of series from $\mathbb{T}^{\langle p,\alpha\rangle}$.

Remark 9.3.9 If $p(\beta) = 0$ for all $\beta < \alpha$, then we have $\mathbb{T}^{\langle p,\alpha\rangle} = \mathbb{T}_\alpha$, a transfinite exponential extension. For terms and series in $\mathbb{T}_\alpha$, we have already defined maximal tree-representations. We remark, though, that this is coherent with the above definition: both, the maximal tree-representation and the maximal tree-representation *w.r.t.* $\mathbb{T}$ yield the same tree. Since there is no danger of confusing the two representations, we use the same symbol for them.

We define the **minimal tree-representation** of t w.r.t. $\mathbb{T}$ as in Chapter 3, i.e. as the sub-tree (T, l) of the maximal tree-representation of t w.r.t. $\mathbb{T}$ such that a node n of T is a leaf if and only if there is an admissible path P in $T_{t,\max}$ such that

$$\begin{aligned} \mathsf{n}_{P,n} &= \mathsf{n} \\ t_{P,n+i} &= \log_i \mathfrak{m}_{P,n} \end{aligned}$$

for some $n \in \mathbb{N}$ and all $i \geqslant 0$. One shows similarly as Proposition 3.2.9 that every term (or series) has a unique minimal tree-representation $T_{t,\min}$ w.r.t. $\mathbb{T}$. Remark 9.3.9 remains also true for minimal tree-representations w.r.t. $\mathbb{T}$.

9.3.4 Closure properties

Let $\kappa > \aleph_0$ be a cardinal number. Proposition 9.2.4 implies that if the support of all series from $\mathbb{T}$ have cardinality $< \kappa$ and if $|\alpha| < \kappa$, then series from $\mathbb{T}^{\langle p,\alpha\rangle}$ also have supports of cardinality less than κ. The situation changes in general for $\alpha \geqslant \kappa$.

We will now assume that we add the κ-support condition to the definition of generalized power series, that is, we only allow generalized power series such that the support has a cardinality smaller than κ. We will show that in this case the extension process

$$\mathbb{T} \longrightarrow \mathbb{T}^{\langle p,0\rangle} \longrightarrow \mathbb{T}^{\langle p,1\rangle} \longrightarrow \cdots \longrightarrow \mathbb{T}^{\langle p,\beta\rangle} \longrightarrow \cdots$$

is stabilizing.

Proposition 9.3.10 *Assume that generalized power series have κ-support. Let p be a partition of the class of ordinals of strength $\leqslant n$. Then there exists a unique ordinal λ such that*

1. $\forall \alpha < \lambda : \mathbb{T}^{\langle p,\alpha\rangle} \subsetneq \mathbb{T}^{\langle p,\lambda\rangle}$,
2. $\forall \alpha \geqslant \lambda : \mathbb{T}^{\langle p,\alpha\rangle} = \mathbb{T}^{\langle p,\lambda\rangle}$.

Proof: We use the same techniques as in the proof of Proposition 3.3.1, but we have to make the following adaptations. Every relative tree-representation $T = T_{f,\mathbb{T}}$ admits a function

$$\iota : T \setminus (\mathsf{r}(T) \cup \operatorname{leaf}(T)) \longrightarrow \{0, \ldots, n\} \times \omega^n$$

which is defined by

$$\iota(t) := \begin{cases} (m, \beta) & \text{if} \quad l(t) = \mathrm{l}_\beta(\mathrm{e}_{\omega^m} g) \\ (0,0) & \text{else.} \end{cases}$$

Then we change condition $\boldsymbol{\mathcal{T}3}$ into

$\boldsymbol{\mathcal{T}3'}$**.** the inner nodes are labeled with elements from $(n+1) \times \omega^n \times C$

and we consider the class $\boldsymbol{\mathcal{T}}$ of labeled trees T such that $\boldsymbol{\mathcal{T}1}$, $\boldsymbol{\mathcal{T}2}$, $\boldsymbol{\mathcal{T}3'}$ and $\boldsymbol{\mathcal{T}4}$ hold. Paths in a tree T in the class $\boldsymbol{\mathcal{T}}$ are represented by tuples from the set

$$((n+1) \times \omega^n \times C)^\star \times C\mathfrak{M}.$$

This shows that $\boldsymbol{\mathcal{T}}$ is a set. Consistently replacing $\mathbb{T}_\alpha$ by $\mathbb{T}^{\langle p,\alpha\rangle}$ shows the existence and uniqueness of λ. □

9.3.5 Strong cofinal partitions

Let $\mathbb{T}$ be of strength n and $p : \lambda \to n+1$ a cofinal partition of the limit ordinal λ. We have shown that $\mathbb{T}^{\langle p,<\lambda\rangle}$ is a field such that e_{ω^n} is total on the set of positive and infinite series.

Proposition 9.3.11 *Let $p, q : \lambda \to n+1$ be strong cofinal partitions. Then $\mathbb{T}^{\langle p,<\lambda\rangle} = \mathbb{T}^{\langle q,<\lambda\rangle}$.*

Proof: We show that for all limit ordinals $\lambda' < \lambda$ we have $\mathfrak{M}^{\langle p,\lambda'\rangle} = \mathfrak{M}^{\langle q,\lambda'\rangle}$. This shows the proposition. Let $\lambda' < \lambda$ be the smallest limit ordinal such that $\mathfrak{M}^{\langle p,\lambda'\rangle} \neq \mathfrak{M}^{\langle q,\lambda'\rangle}$.

For every $\mathfrak{m} \in \mathfrak{M}^{\langle p,\lambda'\rangle}$ there is a limit ordinal κ and an integer $n \in \mathbb{N}$ such that $\mathfrak{m} \in \mathfrak{M}^{\langle p,\kappa+n\rangle}$. Since q is strongly cofinal, there are integers $0 < i_1 < \cdots < i_n$ such that $q(\kappa + i_j) = p(\kappa + j)$ for $1 \leqslant j \leqslant n$. But then

$$\mathfrak{M}^{\langle p,\kappa+i_n\rangle} \supseteq \mathfrak{M}^{\langle q,\kappa+n\rangle}.$$

This shows $\mathfrak{M}^{\langle p,\lambda'\rangle} \supseteq \mathfrak{M}^{\langle q,\lambda'\rangle}$. By symmetry, we obtain the equality, which contradicts the minimality of λ'. □

9.4 Concluding remarks

Transseries fields of positive strength share many properties with usual transseries fields. We have shown how to extend such fields, and we have shown how to construct a field which is closed under an exponential function of positive strength.

Furthermore, we have seen that we can indeed carry out this process arbitrarily often. Introducing a restriction on the size of the supports has also the same stabilizing effect in fields of positive strength that it already had in transseries fields. And finally, transseries of positive strength admit a tree-representation, which describes the series in a canonical way.

Of course, all this generates again many questions which we have to leave unanswered in this thesis. One might for instance be interested in the effects that occur when we replace exponential extensions by nested extensions. In particular, can the results about derivations and compositions be generalized to transfinite extensions including nested extensions? The same question can of course be posed for extension processes which do not allow nested extensions, but exponential extensions of positive strength. Or what can one say about the set of derivations in given transseries fields in general, can they be classified?

Let us point out that we provide for at least some of these questions tools. Especially, the relative tree-representations of fields of positive strength allow to generalize the results about derivations and compositions. Already, the definition of such functions were in the purely exponential case defined using purely structural properties of tree-representations: the existence of paths and tree-embeddings.

Even though we do not give answers to any these questions here, we hope that we have at least made the point that the end of this thesis is by far not the end of the story of transseries.

Glossary

General notations

$\alpha, \beta, \gamma, \ldots$	ordinal numbers
$\kappa, \kappa_1, \kappa_2, \cdots$	cardinal numbers
$k, m, n, \ldots$	integer numbers
$a, b, K, \ldots$	tuples of integers, words over $\mathbb{Z}$
$\lvert\alpha\rvert$	cardinality of the ordinal α
$\lvert k\rvert$	$k_1 + \cdots + k_i$ for $k \in \mathbb{Z}^i$, 29
$k!$	$k_1! \cdots k_i!$ for $k \in \mathbb{N}^i$, 29
$f^p x^q = (f(x^q))^p$	for a function f
$\phi_n x$	n-th iteration of the function ϕ, $n \in \mathbb{Z}$
f^k	$f^{k_1} \cdots f^{k_i}$ for a function f and $k \in \mathbb{Z}^i$, 12
$\bar{f}^k$	$f_1^{k_1} \cdots f_i^{k_i}$ for $\bar{f} = (f_1, \ldots, f_i)$ and $k \in \mathbb{Z}^i$
$\bar{f}^{(k)}$	$f_1^{(k_1)} \cdots f_i^{(k_i)}$ for $\bar{f} = (f_1, \ldots, f_i)$ and $k \in \mathbb{N}^i$
$f(A_1, \ldots, A_i)$	the set $\{f(a_1, \ldots, a_i) \mid a_i \in A_i\}$ for n-ary functions f and sets $A_1, \ldots, A_i$
$\mathcal{R}(A_1, \ldots, A_i)$	$\mathcal{R} \cap A_1 \times \cdots A_i$ for n-ary relations $\mathcal{R}$ and sets $A_1, \ldots, A_i$
$f_x = f(x)$	index convention for series f, 7
$f_{i,x} = (f_i)_x$	double index convention for series, 7
a_k	$a_{k_1} \cdots a_{k_i}$ for a sequence $a = (a_i)_{0 \leqslant i}$ and $k \in \mathbb{N}^i$
a^k	$a_0^{k_1} \cdots a_{i-1}^{k_i}$ for a sequence $a = (a_i)_{0 \leqslant i}$ and $k \in \mathbb{N}^i$
$\coprod_{i \in I} X_i, \coprod X_i$	disjoint union of the sets $(X_i)_{i \in I}$, 10
K^*	the set $K \setminus \{0_K\}$ for a field K, 5
$T(i, n)$	the set $\{k \in \mathbb{N}^i \mid \lvert k\rvert = n\}$ for $i, n \in \mathbb{N}$, 29
$T^*(i, n)$	the set $\{k \in (\mathbb{N}^+)^i \mid \lvert k\rvert = n\}$ for $i, n \in \mathbb{N}$, 29
$\widehat{Z}$	the tuple $(Z_1, \ldots, Z_{i-1})$, if $Z = (Z_1, \ldots, Z_i)$,

Orders

$(P, \leqslant_P)$	a partial order; also denoted by P, if $\leqslant_P$ is clear from the context, 1

$\leqslant = \leqslant_P$	the ordering on P, 1
$\geqslant$	the inverse ordering of $\leqslant$, 1
$\lvert a \rvert$	absolute value of the element a in a total order, 5
$a \perp_{\leqslant} b, a \perp b$	a and b are incomparable in the ordering $\leqslant$, 2
(G)	final segment generated by the set G, 2
$P^\star$	set of words over P, 3
$P^\sharp$	set of non-empty words over P, 3
$P^\diamond$	set of commutative words over P, 4
$P^\dagger$	set of non-empty commutative words over P, 4
$\prod w = w_1 \cdots w_n$	product notation for the word $w = [w_1, \ldots, w_n]$, 5
$\prec_R, \preccurlyeq_R, \asymp_R$	asymptotical relations with respect to the ring R, 6

Series

$\operatorname{supp} f$	support of a function f, 7
$\operatorname{term} f$	set of terms of a function f, 7
$C[[P]]$	set of functions $f : P \to C$ with Noetherian support, 10
$\mathfrak{M} = (\mathfrak{M}, \succcurlyeq)$	order with (semi)group structure, set of monomials, 7
$\mathbb{S} = C[[\mathfrak{M}]]$	set of generalized power series over $\mathfrak{M}$ with coefficients in C, 7
$\mathfrak{M}^\uparrow, \mathfrak{M}^\downarrow$	sets of infinite or infinitesimal monomials, 8
$\mathfrak{M}^{\updownarrow} = \mathfrak{M}^\uparrow \coprod \{1\}$	set of infinite monomials, 8
$\mathfrak{M}^{\downarrow\!\!\!\!-} = \mathfrak{M}^\downarrow \coprod \{1\}$	set of infinitesimal monomials, 8
$\mathbb{S}^\uparrow, \mathbb{S}^\downarrow$	sets of purely infinite and purely infinitesimal series respectively, 8
$\mathbb{S}^{\updownarrow}, \mathbb{S}^{\downarrow\!\!\!\!-}$	sets $\mathbb{S}^\uparrow + C$ and $\mathbb{S}^\downarrow$ of infinite and infinitesimal series respectively, 8
$\mathbb{S}_\infty, \mathbb{S}_\infty^+$	the set of infinite and positive, infinite series respectively, 19 and 19
$\sum F, \prod F$	sum and product of the Noetherian family $F = (f_a)_{a \in A}$, 11 and **??**
$f = \sum_{\mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}$	sum convention for functions $f : P \to C$, 7
$f = f^\uparrow + f^= + f^\downarrow$	representation ofthe series f as sum of its purely infinite, purely infinitesimal and constant parts, 18
$f^{\updownarrow} = f^\uparrow + f^=$	infinite part of the series$f$, 18
$f^{\downarrow\!\!\!\!-} = f^\downarrow + f^=$	infinitesimal part of theseries f, 18
$f = c_f \mathfrak{d}_f (1 + \delta_f)$	canonical representation of the series f, 18
$f = c_f \mathfrak{d}_f + R_f$	canonical representation of the series f, 18
$c_f, \mathfrak{d}_f$	leading coefficient and leading monomial of the series f, 18
$R_f = f - c_f \mathfrak{d}_f$	remainder of the series f, 18
$g \trianglelefteq f, g \triangleleft f$	g is a truncation (resp. proper truncation) of f, 21

Transseries fields

$\mathbb{T}$	a transseries field, 30
∂	a derivation, 75
$f' = \partial f$	alternative notation for the derivation of a series f, 75

$F^{[i]}$	the i-th pseudo-derivation operator of F, **??**
$\mathcal{R}_F(x,\varepsilon)$	restricted Taylor series of F in x, **??**
$\mathcal{T}_F(x,\varepsilon)$	Taylor series of F in x, **??**

Trees

$T=(T,\leqslant)$	tree T with ordering $\leqslant$, 51
$\mathsf{n},\mathsf{t},\mathsf{s},\ldots$	nodes of a tree, 51
$\mathsf{r}=\mathsf{r}(T)$	root of T, 51
$\mathsf{h}(\mathsf{n})$	height of the node n, 51
$\mathrm{path}(T)$	set of paths in T, 52
$\mathrm{leaf}(T)$	leaves of T, 51
$\mathrm{succ}(\mathsf{n})$	set of successors of the node n, 51
$\mathrm{pred}(\mathsf{n})$	set of predecessors of the node n, 51
K_{n}	child generated by n, i.e. $\{\mathsf{t}\in T \mid \mathsf{n}\leqslant\mathsf{t}\}$, 52
$\mathsf{n}_{P,i}$	node of height i in the path P, 52
$T_{f,\max}, T_{f,\min}$	maximal and minimal tree-representation of f, 55 and 57
$T_{f,\mathbb{T}}, T_{f,\mathrm{rm},\mathbb{T}}$	relative and relative-minimal tree-representation of f with respect to $\mathbb{T}$, 59 and 62
$\succcurlyeq$	ordering on the set of path, 83

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Appendix A

Exponential fields of positive strength

C is an **exponential field of strength 0** iff it is an exponential field. For $n > 0$, the field C is an **exponential field of strength n** iff there are functions $\exp, \dots, \mathrm{e}_{\omega^n}$ such that

E1. C is an exponential field of strength $n-1$ for the functions $\exp, \dots, \mathrm{e}_{\omega^{n-1}}$,

E2. $\exists c_n \in C : \forall c_n \leqslant x < y$:

(i) $x \in \operatorname{dom} \mathrm{e}_{\omega^n}$,

(ii) $x + 1 < \mathrm{e}_{\omega^n} x$ and $\mathrm{e}_{\omega^n} x < \mathrm{e}_{\omega^n} y$,

(iii) $\mathrm{e}_{\omega^n} x \in \operatorname{dom} \mathrm{e}_{\omega^{n-1}}$ and $\mathrm{e}_{\omega^{n-1}} \circ \mathrm{e}_{\omega^n} x = \mathrm{e}_{\omega^n}(x+1)$.

By (ii), the function e_{ω^n} is unbounded in C. Since e_{ω^n} is strictly increasing on $(c, \infty) \subseteq C$, its inverse function l_{ω^n} is uniquely defined on $(\mathrm{e}_{\omega^n} c, \infty) \subseteq C$. Moreover, the function $\mathrm{l}_{\omega^{n-1}}$ is defined on $(\mathrm{e}_{\omega^n}(c+1), \infty)$ and satisfies

$$\mathrm{l}_{\omega^n} \circ \mathrm{l}_{\omega^{n-1}} x = \mathrm{l}_{\omega^n} x - 1.$$

If C is an exponential field of strength n, then e_{ω^n} and l_{ω^n} are called the **exponential** and **logarithmic functions of strength n**. Note that exponential and logarithmic functions are of strength 0. Exponential and logarithmic functions $E = \mathrm{e}_\omega$ and $L = \mathrm{l}_\omega$ of strength 1 are also called **super-exponential** and **super-logarithmic** functions, respectively.

Remark A.0.1 Let $C = \mathbb{R}$. Recall that the exponential function exp is ultimately faster than every polynomial function over $\mathbb{R}$, i.e. for every $i \in \mathbb{N}$ there is a real number d_i such that

$$\forall d_i < x : \quad x^i < e^x.$$

The same holds for exponential functions of positive strength. We show inductively that for all $i, n \in \mathbb{N}$ there are $d_{n,i} \in \mathbb{R}$ (for a fixed set of functions $\exp, \mathrm{e}_\omega, \mathrm{e}_{\omega^2}, \dots$) with

$$\forall d_{n,i} < x : \quad x^i < \mathrm{e}_{\omega^n} x.$$

The initial case $n = 0$ is clear. Now suppose that we have shown that $d_{n,i}$ exists for a fixed n and all i. Let c_n and c_{n+1} as in **E2**. Then applying e_{ω^n} yields

$$\forall x > \max(c_n, c_{n+1}) : \quad x + 2 < \mathrm{e}_{\omega^n}(x+1) < \mathrm{e}_{\omega^{n+1}}(x+1).$$

Hence for all $x > \max(d_{n,i}, c_n + 1, c_{n+1} + 1)$ we have

$$x^i < \mathrm{e}_{\omega^n} x < \mathrm{e}_{\omega^{n+1}} x.$$

This finishes the induction.

Example A.0.2 Let $C = \mathbb{R}$ and $\exp x = \sum_{0 \leqslant i} \frac{1}{i!} x^i$. Then $\langle \mathbb{R}, \exp \rangle$ is an exponential field of strength 0. Suppose that for some integer $n \geqslant 0$ we have functions $\exp, \ldots, \mathrm{e}_{\omega^n}$ defined on $[0, \infty)$ such that $\langle \mathbb{R}, \exp, \ldots, \mathrm{e}_{\omega^n} \rangle$ is an exponential field of strength n. Let $c_n \in \mathbb{R}^+$ be such that for all real numbers $x \geqslant c_n$ we have $\mathrm{e}_{\omega^n} x > x + 1$.

Let $f : [0, 1] \to \mathbb{R}$ be strictly increasing with $f(0) = c_n + 1$ and $f(1) = \mathrm{e}_{\omega^n}(c_n + 1)$. We define a function $\mathrm{e}_{\omega^{n+1}}$ on $[0, \infty)$ as follows. Let $x \geqslant 0$ and $n_x \in \mathbb{N}$ such that $r_x = x - n_x \in [0, 1)$. Then we let

$$\mathrm{e}_{\omega^{n+1}} x := \mathrm{e}_{\omega^n \cdot n_x} f(r_x).$$

For all $x \geqslant 0$ we have $n_{x+1} = n_x + 1$, which shows the functional equation for e_{ω^n} and $\mathrm{e}_{\omega^{n+1}}$. Since e_{ω^n} and f are strictly increasing, so is $\mathrm{e}_{\omega^{n+1}}$. We have to show that there is a $c_{n+1} \geqslant 0$ such that $\mathrm{e}_{\omega^{n+1}} x > x + 1$ for all $x \geqslant c_{n+1}$. From $\mathrm{e}_{\omega^{n+1}} 0 = c_n + 1$ we obtain $\mathrm{e}_{\omega^{n+1}} 1 = \mathrm{e}_{\omega^n}(c_n + 1) > c_n + 2$. Let us assume that for all integers $k > 0$

$$\mathrm{e}_{\omega^{n+1}} k > c_n + 1 + k.$$

Then $\mathrm{e}_{\omega^{n+1}}(k + 1) > \mathrm{e}_{\omega^n}(c_n + 1 + k) > c_n + 2 + k$. Choosing c_n large enough, we may let $c_{n+1} = c_n + 1$. This shows the existence of c_{n+1}.

The function $\mathrm{e}_{\omega^{n+1}}$ defined as in Example A.0.2 is continuous. We remark that it is possible to construct C^0- and even C^k-solutions (for $k \in \mathbb{N} \cup \{\infty\}$). Results by Écalle (see [Éca92]) imply that there are always quasi-analytic functions e_{ω^n}. In [Kne50], Kneser has constructed an analytic super-exponential function. Using his result, we prove:

Proposition A.0.3 *For all $n \geqslant 0$, there are analytic exponential functions e_{ω^i} of strength $i \leqslant n$ such that $\langle \mathbb{R}, \exp, \ldots, \mathrm{e}_{\omega^n} \rangle$ is an exponential field of strength n.*

Proof: The case $n = 0$ is clear; and the case $n = 1$ has been shown by Kneser. In fact, Kneser's proof can be applied to e_{ω^n} and $\mathrm{e}_{\omega^{n+1}}$, if there is an analytic expansion of e_{ω^n} into $\mathbb{C}$ such that there is a $z \in \mathbb{C}$ and an open neighbourhood $\mathcal{U}$ of z such that e_{ω^n} is holomorph on $\mathcal{U}_z$ and such that z is a fixed point of e_{ω^n}.

First remark that for all 1-periodic, analytic functions $g : \mathbb{R} \to \mathbb{R}$ the function

$$\mathrm{e}^g_{\omega^n} x := \mathrm{e}_{\omega^n}(x + g(x))$$

again defines an analytic exponential function of strength n. In particular, for constant functions $g(x) = -b \in \mathbb{R}$, we obtain that

$$\mathrm{e}^{-b}_{\omega^n} x := \mathrm{e}_{\omega^n}(x - b)$$

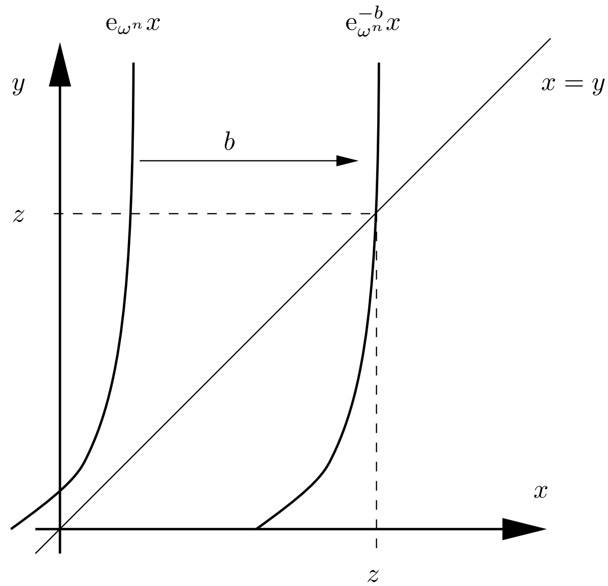


Figure A.1: Translation of $\mathrm{e}_{\omega^n} x$ by b.

is the translation of e_{ω^n} in direction 1 by b. Since $\lim_{x \to \infty} \mathrm{e}_{\omega^n} x = \infty$, there must be a real number b such that $\mathrm{e}_{\omega^n}^{-b}$ admits a fixed point $z \in \mathbb{R}$ (i.e. the graph of $\mathrm{e}_{\omega^n}^{-b}$ eventually cuts the line $x = y$, see Figure A.1). Hence there is an open neighbourhood $\mathcal{U}_z \subset \mathbb{C}$ such that $\mathrm{e}_{\omega^n}^{-b}$ extends holomorphically to $\mathcal{U}_z$. Now apply Kneser's proof. □

In [Bos86], Boshernitzan considers super-exponential functions E on $\mathbb{R}$ in order to construct Hardy-fields containing functions of ultimately faster growth than $\exp_i$ ($i \in \mathbb{N}$). For C^1-functions E he shows that E' is ultimately bounded by E^3, i.e. there is some $x_2 \in \mathbb{R}$ such that

$$\forall x_2 < x : \qquad E'x < E^3 x.$$

In the following, we will strengthen this bound and generalize the result to exponential functions of arbitrary positive strength. The case of strength 1 will be treated separately since the proof is simpler here.

Proposition A.0.4 *Let $\varepsilon > 0$ be a real number and E be a C^1-super-exponential function on $\mathbb{R}$. Then there is a real number $x_\varepsilon \in \operatorname{dom} E$ such that*

$$\forall x_\varepsilon < x : \qquad E'x < E^{1+\varepsilon} x. \tag{A.1}$$

Proof: First, we remark that we only need to show the proposition for $\varepsilon = 1/m$ with $m \in \mathbb{N}^+$. For the rest of the proof, we fix m. Furthermore, we may assume $[0, 1) \subseteq \operatorname{dom} E$ and that $E'r > 0$

for all $r \in [0,1)$ (by a translation of E if necessary). Let C be such that $|\log E'r| \leqslant C$ for all $r \in [0,1)$. Let $a = E(0)$.

Claim 1: $\forall x \in \mathbb{R} : \forall k \in \mathbb{N} : x + \cdots + \exp_k x < \exp_{k+1} x$.

We show the claim inductively. The case $k = 0$ follows from $1 + x \leqslant e^x$. As for the inductive step, we remark that for all $y \in \mathbb{R}$ we have $0 < 1 + (y-1)^2$, thus $2y < e^y$ for all $y > 0$. Applying this to $y = \exp_{k+1} x > 0$ yields

$$\sum_{i=0}^{k+1} \exp_i x < 2 \exp_{k+1} x < \exp_{k+2} x,$$

whence the claim 1.

Claim 2: Fix $D \in \mathbb{R}$. Then there is an integer $N_D \in \mathbb{N}$ such that

$$\forall n \geqslant N_D : \forall y \in [a, e^a] : \quad m \cdot (y + \cdots + \exp_n y) + D < \exp_{n+1} y.$$

For large enough n we have $D < m \cdot (y + \cdots + \exp_n y)$. Thus claim 2 holds if we can show that there is an N_D such that for all $n \geqslant N_D$ and all $y \in [a, e^a]$ the inequality

$$2m \cdot (y + \cdots + \exp_n y) < \exp_{n+1} y \tag{A.2}$$

holds. By claim 1, we have for all n and y

$$2m \cdot (y + \cdots + \exp_n y) < 4m \cdot \exp_n y.$$

Let $z_m \in \mathbb{R}$ be such that $4m \cdot z < e^z$ for all $z \geqslant z_m$. Then there is an integer $N_D \in \mathbb{N}$ such that $z_m < \exp_n y$ for all $y \in [a, e^a]$ and all $n \geqslant N_D$. Thus

$$2m \cdot (y + \cdots + \exp_n y) < 4m \cdot \exp_n y < \exp_{n+1} y.$$

This shows inequality (A.2) and therefore claim 2.

Let for the rest of the proof $D = m \cdot C$ and N_D as in claim 2. Then for all $x \geqslant N_D + 1$ there are $n \geqslant N_D + 1$ and $r \in [0,1)$ such that $x = n + r$. From the functional equation $Ex = \exp E(x-1)$ we obtain

$$\begin{aligned} E'x &= Ex \cdot E'(x-1) \\ &= Ex \cdots E(1+r) \cdot E'r \\ &= \exp(Er) \cdots \exp_n(Er) \cdot E'r. \end{aligned}$$

Let $y = Er$, then $y \in [a, e^a]$ and

$$\log E'x = y + \cdots + \exp_{n-1} y + \log E'r \leqslant y + \cdots + \exp_{n-1} y + C.$$

Since $n - 1 \geqslant N_D$, claim 2 now implies

$$m \log E'x \leqslant m \cdot (y + \cdots + \exp_{n-1} y) + D < \exp_n y = Ex,$$

hence $E'x < E(x+1)^{\frac{1}{m}}$. Multiplying with $E(x+1)$ yields

$$E'(x+1) < E(x+1)^{1+\frac{1}{m}}.$$

This proves the proposition for $x_{\frac{1}{m}} = N_D + 2$. □

Remark A.0.5 Fix $0 < \varepsilon$. Let x_ε be minimal such that for all $x > x_\varepsilon$ the inequality (A.1) holds on (x_ε, ∞). Then the function $\varepsilon \mapsto x_\varepsilon$ is decreasing. In other words, the stronger we want inequality (A.1) to be, the bigger we have to choose x_ε. The same remains true for the general case.

Proposition A.0.6 *Let* e_{ω^n} *be a* C^1*-exponential function of strength* n *on* $\mathbb{R}$*. For each* $\varepsilon > 0$ *there is an* $x_\varepsilon \in \operatorname{dom} \mathrm{e}_{\omega^n}$ *such that*

$$\forall x_\varepsilon < x : \qquad \mathrm{e}'_{\omega^n} x < \mathrm{e}^{1+\varepsilon}_{\omega^n} x.$$

Proof: Let $\Psi = \mathrm{e}_{\omega^n}$ and c_n the constant from **E2** for Ψ. Let ψ be the exponential function of strength $n-1$ such that for all $c_n < x \in \mathbb{R}$ with $\Psi x \in \operatorname{dom} \psi$ the functional equation

$$\psi(\Psi x) = \Psi(x+1) \tag{A.3}$$

holds. The constant from **E2** for ψ is denoted by c_{n-1}. By applying a translation $\Psi(x+b)$ if necessary, we may assume that Ψ satisfies the following conditions.

Ψ1. $[0,1] \subseteq \operatorname{dom} \Psi$,
Ψ2. $c_n \leqslant 0$,
Ψ3. $1 < c_{n-1} < \Psi(0)$.

Note that by **Ψ3** we have $\Psi x \in \operatorname{dom} \psi$ for all $x \geqslant 0$. Hence equation (A.3) holds for all $x \geqslant 0$. Let $a = \Psi(0)$. Then $\psi(a) = \Psi(1)$, and for all $y \in [a, \psi(a)]$ we have $c_{n-1} < y$. Condition **E2** for ψ implies $y < \psi(y)$, and using a simple induction shows $y + i < \psi_i(y)$ for all i. Whence

$$\forall y \in [a, \psi(a)] : \qquad \lim_{k\to\infty} \psi_k(y) = +\infty \tag{A.4}$$

Claim 1: Let $D \in \mathbb{R}$. Then there is an integer $N_D \in \mathbb{N}$ such that

$$\forall k > N_D : \forall y \in [a, \psi(a)) : \quad D < \sum_{i=1}^{k-1} \log \psi_i(y).$$

To show claim 1, we remark first that from (A.4) it follows that

$$\forall y \in [a, \psi(a)] : \qquad \lim_{k\to\infty} \log \psi_k(y) = +\infty.$$

Since ψ is continuous on $[a, \psi(a)]$, there is an integer N_D such that for all $k > N_D$ and all $y \in [a, \psi(a)]$ the inequality $D < \log \psi_{k-1}(y)$ holds. But then the same inequality is true on $[a, \psi(a))$. From **Ψ3** we now obtain $0 < \log \psi_i(y)$. Thus

$$\forall k > N_D : \forall y \in [a, \psi(a)) : \qquad D < \log \psi(y) + \cdots + \log \psi_{k-1}(y).$$

This shows claim 1.

Claim 2: $\exists N_a : \forall k > N_a : \forall y \in [a, \psi(a)) : \quad \sum_{i=1}^{k-1} \log \psi_i(y) < \log \psi_k(y).$

Recall from Remark A.0.1 that $d_{n-1,3}$ is the real number with

$$\forall x > d_{n-1,3} : \quad x^3 < \psi(x).$$

Choose $M \in \mathbb{N}$ such that $d_{n-1,3} < \psi_M(y)$ for all $y \in [a, \psi(a))$. Such an integer exists for the same reason as N_D exists in claim 1. We assume that M is sufficiently large such that $c_{n-1} < \psi_M(y)$ for all $y \in [a, \psi(a))$. Consequently, $\log \psi_M(y) < \log \psi_{M+1}(y)$. We show by induction that

$$\forall i \geqslant 0 : \quad \sum_{j=1}^{i} \log \psi_{M+j}(y) + \sum_{j=0}^{i} \log \psi_{M+j}(y) < \log \psi_{M+i+1}(y)$$

whenever $y \in [a, \psi(a))$. For $i = 0$ there is nothing to show. Assume that the inequality holds up to i. Then

$$\sum_{j=1}^{i+1} \log \psi_{M+j}(y) + \sum_{j=0}^{i+1} \log \psi_{M+j}(y) < 3 \log \psi_{M+i+1}(y). \tag{A.5}$$

From $d_{n-1,3} < \psi_{M+i+1}(y)$ we obtain $\psi^3_{M+i+1}(y) < \psi_{M+i+2}(y)$. Applying this to inequality (A.5) yields the inductive step. Now choose i large enough such that

$$\sum_{j=0}^{M-1} \log \psi_j(y) < \sum_{j=1}^{i+1} \log \psi_{M+j}(y).$$

Then claim 2 holds for $N_a = M + i + 1$.

Claim 3: Let $K \in \mathbb{N}^+$. Then there is an integer $N_K \in \mathbb{N}$ such that

$$\forall k > N_K : \forall y \in [a, \psi(a)) : \quad K \cdot \sum_{i=0}^{k-1} \log \psi_i(y) < \log \psi_k(y).$$

Let again $d_{n-1,2K} \in \mathbb{R}$ be such that $x^{2K} < \psi(x)$ for all $x > d_{n-1,2K}$. Let $N_{2K} \in \mathbb{N}$ be such that for all $k > N_{2K}$ and all $y \in [a, \psi(a))$ the inequality $d_{n-1,2K} < \psi_{k-1}(y)$ holds. Then

$$\forall k > N_{2K} : \forall y \in [a, \psi(a)) : \quad \psi^{2K}_{k-1}(y) < \psi_k(y). \tag{A.6}$$

On the other hand, by claim 2 we have for $N_K > \max(N_a, N_{2K})$

$$\forall k > N_K : \forall y \in [a, \psi(a)) : \quad \log \psi(y) + \cdots \log \psi_{k-2}(y) < \log \psi_{k-1}(y). \tag{A.7}$$

Adding $\log \psi_{k-1}(y)$ and multiplying by K, we see that (A.7) is equivalent to

$$\forall k > N_K : \forall y \in [a, \psi(a)) : \quad K \cdot \sum_{i=0}^{k-1} \log \psi_i(y) < 2K \cdot \log \psi_{k-1}(y).$$

Invoking (A.6) proves claim 3.

We now show the proposition. First, we remark that we can restrict ourselves to $\varepsilon = \frac{1}{m}$ where $m \in \mathbb{N}^+$. Fix m. The function Ψ' is continuous on $[a, \psi(a)]$, hence there is a real number C with

$$\forall y \in [a, \psi(a)) : \quad |\log \Psi'(y)| \leqslant C.$$

Let $D = m \cdot C$ and $N = \max(N_{m+1}, N_D)$ (where N_D and N_{m+1} are the integers from claim 1 and claim 3 respectively). Then since $\log y > 0$:

$$\forall k > N : \forall y \in [a, \psi(a)) : \quad (m+1) \cdot \sum_{i=0}^{k-1} \log \psi_i(y) + D < \sum_{i=1}^{k} \log \psi_i(y) < \sum_{i=0}^{k} \log \psi_i(y).$$

Hence

$$\forall k > N : \forall y \in [a, \psi(a)) : \quad \sum_{i=0}^{k-1} \log \psi_i(y) + C < \frac{1}{m} \log \psi_k(y),$$

which is equivalent to

$$\forall k > N : \forall y \in [a, \psi(a)) : \quad \sum_{i=0}^{k} \log \psi_i(y) + C < (1+\varepsilon) \log \psi_k(y). \tag{A.8}$$

For $x > N$, we have $k > N$ and $r \in [0, 1)$ such that $x = k + r$. Then (A.8) and $y = \Psi(r)$ imply

$$\forall x > N : \quad \log((\Psi x) \cdots \psi_k(\Psi x) \cdot \Psi' r) < (1+\varepsilon) \log \psi_k \Psi(r) = \log(\Psi x)^{1+\varepsilon}.$$

Hence

$$\Psi' x = \Psi x \cdots \psi_k(\Psi x) \cdot \Psi' r < (\Psi x)^{1+\varepsilon}.$$

The proposition holds thus for $x_\varepsilon = N$. $\square$

Appendix B

Introduction (English Version)

In this thesis, we present the construction of fields with functions which are faster than every iterated exponential function. This introduction will describe what we mean by "construction", "faster than" and "exponential function". By doing this, we hope to give the reader a good idea of what he can expect from this thesis, and we hope to provide a motivation for the presented work. Moreover, this introduction will serve as a guide to help the reader through the different parts of the thesis.

We start by explaining some basic concepts and by presenting the main results. We go on to summarize what is known about super-exponential functions. The third part of this introduction will motivate the given construction. Then we will come to the "road map" of the thesis: we give a short summary of each of the forthcoming chapters, thus equipping a possible reader with an orientation guide. This will be of particular interest since some chapters are rather technical, and there is a real danger of losing the overview when working through the unavoidable details. Finally, we list some of the notations used.

B.1 Main results

The main objective of the presented thesis is to study the possibility of the existence of fast-growing functions on fields of generalized power series.

For a totally ordered field C and a totally ordered multiplicative group $\mathfrak{M}$, a function

$$f : \mathfrak{M} \longrightarrow C$$

is a generalized power series, if the set of $\mathfrak{m} \in \mathfrak{M}$ with $f(\mathfrak{m}) \neq 0$ (called the support of f) is well-ordered in $\mathfrak{M}$. For fixed $C, \mathfrak{M}$ the set $\mathbb{S} = C[[\mathfrak{M}]]$ of generalized power series $f : \mathfrak{M} \to C$ admits a multiplication and an addition which provide $\mathbb{S}$ with a field structure. Thus every polynomial $P \in \mathbb{S}[X]$ with coefficients in $\mathbb{S}$ corresponds canonically to a function $f_P : \mathbb{S} \to \mathbb{S}$.

Moreover, since C and $\mathfrak{M}$ are totally ordered, it is possible to introduce a total ordering on $\mathbb{S}$. Hence, there is a natural interpretation of "growth" in $\mathbb{S}$. Indeed, for two polynomials $P, Q \in \mathbb{S}[X]$, we say that P is faster than Q, if there is some $s \in \mathbb{S}$ such that

$$|f_Q(t)| < |f_P(t)|$$

for all series $t > s$. Classical results about generalized power series fields imply that distinct polynomials can be compared in this sense. In analogy with functions over the real line, the question arises whether there are functions on $\mathbb{S}$ or at least on some interval $(f, +\infty)$ which are faster than every polynomial in

$\mathbb{S}[X]$. Taking this analogy further, one might even be interested in the existence of exponential functions on $\mathbb{S}$.[1]

The field $\mathbb{S} = C[[\mathfrak{M}]]$ has *a priori* no reason to provide more structure than the one described above. So, admitting exponential and logarithmic functions requires extra assumptions on the basic objects C and $\mathfrak{M}$. Without listing these assumptions here, let us remark that so called transseries fields yield the right setting for introducing exponential and logarithmic functions.[2] Transseries fields will be denoted by $\mathbb{T}$ rather than by $\mathbb{S}$.

An important feature of transseries fields $\mathbb{T}$ is that on the one hand the logarithm is totally defined on the set of positive series, but that on the other hand the exponential function is not total. To overcome this problem, Dahn introduced a process which extends $\mathbb{T}$ to a transseries field $\mathbb{T}_{\exp}$, thus building towers of transseries fields

$$\begin{array}{c} \vdots \\ \cup \\ \mathbb{T}_{\exp,\exp} \\ \cup \\ \mathbb{T}_{\exp} \\ \cup \\ \mathbb{T} \end{array}$$

such that the logarithm and the exponential function can be totally defined on the positive subset of their union. We will show how to continue this construction beyond the union of this tower, thus constructing transseries fields $\mathbb{T}_\alpha$ for every ordinal number α. Moreover, the ordering on the extended field is such that for sufficiently large series f, the exponential of f is larger than every f^i ($i \in \mathbb{N}$).

Again, in view of the field of real numbers, it is natural to ask whether fields of generalized power series possess more structural properties than the field structure or – as in the case of transseries fields – logarithmic functions. In particular, can we introduce infinite sums, derivations and compositions in such fields?

As for the exponential function, it is necessary to give those notions a meaning for generalized power series. Let $\mathbb{S} = C[[\mathfrak{M}]]$ and $\mathcal{F} = (f_i)_{i \in I} \in \mathbb{S}^I$. Any notion of infinite sums should coincide with the field operations, if I is a finite set. This is satisfied if $\mathcal{F}$ is a Noetherian family, i.e. if the union of the supports of all f_i is well-ordered in $\mathfrak{M}$ and if for every $\mathfrak{m} \in \mathfrak{M}$, there are only finitely many $i \in I$ such that $f_i(\mathfrak{m}) \neq 0$. If this is the case, then we let $\sum \mathcal{F}$ be the series in $\mathbb{S}$ with $\sum \mathcal{F}(\mathfrak{m}) = \sum_I f_i(\mathfrak{m})$.

The canonical notion of a derivation ∂ on $\mathbb{S}$ should have the following properties:

- ∂ is constantly 0 on C
- for all $f, g \in \mathbb{S}$ we have $\partial(fg) = \partial(f) \cdot g + f \cdot \partial(g)$
- if $\mathcal{F} = (f_i)_{i \in I}$ is a Noetherian family, then so is $\partial(\mathcal{F}) = (\partial(f_i))_{i \in I}$ and $\partial(\sum \mathcal{F}) = \sum \partial(\mathcal{F})$.

Moreover, if $\mathbb{S}$ is a transseries field, then the condition

[1] One has, of course, to specify what an exponential function is. To serve the purpose of this section, namely to present the main results, we will assume that exponential functions F are non-constant functions with $F(x+y) = F(x) \cdot F(y)$ whenever both sides are defined.

[2] There are, however, approaches alternative to the one presented in this thesis. All these approaches are similar in that they have to overcome the same problem: the exponential and logarithmic functions are not simultaneously totally defined on the set of all positive series. This fact has been independently shown by S. and F.-V. Kuhlmann and S. Shelah [KKS97] and J. van der Hoeven [vdH97].

- if $0 < f$, then $\partial(f) = f \cdot \partial(\log f)$

should hold. We state our first result.

Result 1 *If ∂ is a derivation on $\mathbb{T}$, then for every ordinal number α there is a unique derivation ∂_α on $\mathbb{T}_\alpha$ which extends ∂.*

Similarly, one can define a notion of composition. Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be transseries fields, then a function $\vartriangle : \mathbb{T}_1 \to \mathbb{T}_2$ is a right-composition if the following conditions hold:

- $\vartriangle$ is injective and $\forall c \in C : \vartriangle(c) = c$
- $\vartriangle$ is multiplicative
- if $\mathcal{F} = (f_i)_{i \in I}$ is a Noetherian family (in $\mathbb{T}_1$), then so is $\vartriangle(\mathcal{F}) = (\vartriangle(f_i))_{i \in I}$ (in $\mathbb{T}_2$) and $\vartriangle(\sum \mathcal{F}) = \sum \vartriangle(\mathcal{F})$
- for all $f \in \operatorname{dom} \exp$ in $\mathbb{T}_1$: $\vartriangle(\exp f) = \exp \vartriangle(f)$.

Result 2 *If $\vartriangle : \mathbb{T}_1 \to \mathbb{T}_2$ is a right-composition, then for every ordinal number α there is a unique right-composition $\vartriangle_\alpha : \mathbb{T}_{1,\alpha} \to \mathbb{T}_{2,\alpha}$ which extends $\vartriangle$.*

An immediate question arising from the above is whether there is a link between derivations and right-compositions. In particular, can Taylor-series developments be generalized to transseries fields? This question is not only interesting in its own right. If we want to study structural properties of transseries fields, in particular the existence of super-exponential functions, then we have to answer this question affirmatively.

The first step is to extend the notion of right-compositions to compositions in general. Fix transseries fields $\mathbb{T}_i$ $(i = 1, 2, 3)$ with derivations ∂^1, ∂^2 on $\mathbb{T}_1$ and $\mathbb{T}_2$ respectively.[3] A partially defined function $\circ : \mathbb{T}_1 \times \mathbb{T}_3 \to \mathbb{T}_2$ is a compatible composition if it satisfies the following conditions:

- $\mathbb{T}_3 \subseteq \mathbb{T}_2$, and the restriction of ∂^2 to $\mathbb{T}_3$ is a derivation
- for every series $g \in \mathbb{T}_3$ with $C < g$, the function $\vartriangle_g : \mathbb{T}_1 \to \mathbb{T}_2$ with $\vartriangle_g(f) = f \circ g$ is a right-composition
- for every $\mathfrak{m} \in \mathfrak{M}_1$ larger than 1, the function $\mathfrak{m} \circ \cdot : \{f \in \mathbb{T}_3 \,|\, C < f\} \to \mathbb{T}_2$ is strictly increasing
- $\circ$ satisfies the chain rule for compositions, i.e. for all $f \in \mathbb{T}_1$ and all $g \in \mathbb{T}_3$ with $g \in \operatorname{dom}(f \circ \cdot)$, we have $g \in \operatorname{dom}(f' \circ \cdot)$ and $(f \circ g)' = (f' \circ g) \cdot g'$
- let $f \in \mathbb{T}_1$, $g \in \mathbb{T}_3$ and $(\varepsilon_i)_{i \in I}$ be a Noetherian family in $\mathbb{T}_2$ such that

  $$\forall i \in I : \forall \mathfrak{m} \in \operatorname{supp} f : C < \left| \frac{\mathfrak{m} \circ g}{\mathfrak{m}' \circ g \cdot \varepsilon_i} \right|,$$

  then $g + \sum_I \varepsilon_i \in \operatorname{dom} f \circ \cdot$, $(f^{(n)} \circ g \cdot \varepsilon_i)_{0 \leqslant n, i \in I^n}$ is a Noetherian family and

  $$f \circ (g + \textstyle\sum_I \varepsilon_i) = \displaystyle\sum_{0 \leqslant n} \frac{1}{n!} f^{(n)} \circ g \cdot \sum_{i \in I^n} \varepsilon_i,$$

  where $\varepsilon_i = \varepsilon_{i_1} \cdots \varepsilon_{i_n}$ for $i = (i_1, \ldots, i_n) \in I^n$.

Result 3 *If $\circ : \mathbb{T}_1 \times \mathbb{T}_3 \to \mathbb{T}_2$ is a compatible composition, then for every ordinal number α there is a unique compatible composition $\circ_\alpha : \mathbb{T}_{1,\alpha} \times \mathbb{T}_3 \to \mathbb{T}_{2,\alpha}$ which extends $\circ$.*

[3] We abbreviate both $\partial^1(f)$ and $\partial^2(f)$ by f' for better readability. The n-th derivation of f will in both cases be denoted by $f^{(n)}$.

A natural follow-up question to the above results about transseries fields concerns the existence of generalized power series fields admitting not only exponential functions, but also functions with faster growth than every iteration of the exponential function. For instance, a function E satisfying the functional equation $E(x+1) = \exp \circ E(x)$ has this property. Note that once we have a function E, a function $\mathcal{E}$ with $\mathcal{E}(x+1) = E \circ \mathcal{E}(x)$ will also be faster than every $\exp_i$. We let e_{ω^i} be an exponential function of strength $i \geqslant 0$, if $\mathrm{e}_{\omega^0} = \exp$ and

$$\begin{array}{rcl} \mathrm{e}_{\omega^1}(x+1) & = & \mathrm{e}_{\omega^0} \circ \mathrm{e}_{\omega^1}(x) \\ \mathrm{e}_{\omega^2}(x+1) & = & \mathrm{e}_{\omega^1} \circ \mathrm{e}_{\omega^2}(x) \\ \mathrm{e}_{\omega^3}(x+1) & = & \mathrm{e}_{\omega^2} \circ \mathrm{e}_{\omega^3}(x) \\ & \vdots & \end{array}$$

Exponential functions of strength 1 are also called super-exponential functions. As there is no reason for generalized power series fields to admit an exponential function, transseries fields do not necessarily have exponential functions of positive strength. However, one can again choose a set of additional conditions which provide the right framework for the definition of exponential functions of strength $n \geqslant 0$. Those fields will be called transseries fields of strength n.

Result 4 *For all $n \in \mathbb{N}$, there are transseries fields of strength n.*

Generalizing Dahn's exp-extension process, we introduce e_{ω^n}-extensions which extend transseries fields $\mathbb{T}$ of strength n to fields $\mathbb{T}_{\mathrm{e}_{\omega^n}}$ which are again of strength n. Using these extensions we show

Result 5 *Let $n \geqslant 0$. There are fields $\mathcal{K}_n$ of generalized power series with exponential functions of strength n such that both e_{ω^n} and the inverse function l_{ω^n} are total on $\{f \in \mathcal{K}_n \,|\, C < f\}$.*

B.2 Super-exponential functions – a short history

Super-exponential functions and related problems have already been studied occasionally. In this section we highlight some results; by no means, however, do we claim completeness.

In contrast to our construction, super-exponential functions have either been used to construct other classes of functions (in particular, fractional iterates of some given function) and therefore rather been a tool, or else the main attention has been given to super-exponential functions (or at least germs of such functions) over the real line. To our knowledge, exponential functions of strength higher than 1 have not yet been investigated.

First steps towards super-exponential functions can be traced back to the 19th century, when P. du Bois-Raymond showed that there is no limit to growth for real functions. More precisely, let $f_1 \prec f_2 \prec \cdots$ be functions[4] defined on some interval $(a, \infty) \subseteq \mathbb{R}$. Then there exists a function $F : (a, \infty) \to \mathbb{R}$ such that $f_i \prec F$ for all i. G. H. Hardy [Har10] gives two proofs of this fact. Applied to the set of functions $\exp_i = \exp \circ \cdots \circ \exp$ (the i-fold iteration of the exponential function), this fact yields the existence of a function F which is faster than every iteration of exp.

The mere existence of such a function F, though, says nothing about the actual behaviour of F. In order to at least give a restriction on the growth of such functions, we have introduced the notion of super-exponential functions as solutions to the functional equation

$$\exp E(x) = E(x+1). \tag{B.1}$$

[4]The symbol $\prec$ denotes Hardy's relation of domination between real functions which is defined by $f \prec g$ iff $\lim\limits_{x \to \infty} f(x)/g(x) = 0$.

Now, if E is a solution to equation (B.1) and g a 1-periodic function, then $E^*(x) = E(x + g(x))$ is also a solution. Hence, super-exponential functions are far from being uniquely defined by the functional equation which determines their growth.

The next significant progress in the study of solutions E was made by H. Kneser in the 1940s. In [Kne50], he constructs an analytic super-exponential function by using a complex fixed point of e^x and conformal transformations. In fact, he uses his solution to define an analytic half-iterate of the exponential function, i.e. an analytic function φ with

$$\varphi \circ \varphi(x) = e^x.$$

More generally, he defines a set of analytic functions $\exp_r$ (where $r \in \mathbb{R}$), called the fractional iterates of exp, with the properties

$$\begin{aligned} \exp_1(x) &= e^x \\ \exp_{r+s}(x) &= \exp_r \circ \exp_s(x) \end{aligned}$$

for all $r, s \in \mathbb{R}$. By constructing E and its unique inverse function L, he obtains the desired functions by letting $\exp_r(x) = E(L(x) + r)$.[5]

Kneser's article led to more study of fractional iterates, where e^x was occasionally replaced by other functions. Most notably, work by G. Szekeres and K. W. Morris [Sze58],[Sze62],[SM62] considers functions of exponential growth, that is, functions f such that

$$\exp_{k-1}(x) \prec f_k \prec \exp_{k+1}(x)$$

for all $k \in \mathbb{N}^+$. Examples are e^x and $e^x - 1$. Fractional iterates[6] of f can now be constructed by solving the functional equation

$$B(f(x)) = B(x) + 1,$$

which is also called the Abel equation, and then letting $f_r(x) = B_{-1}(B(x) + r)$. Note that for $f(x) = e^x$, a solution B is a super-logarithmic function. An interesting result of Szekeres concerns the uniqueness of B for a large class of functions f. As mentioned above, super-exponential and -logarithmic functions are far from being uniquely determined. The situation changes, if we consider functions f which are real analytic for $x \geqslant 0$, which satisfy $x < f(x)$ and $0 < f'(x)$ for $x > 0$ and which allow a development

$$f(x) = x + ax^2 + \cdots \qquad \text{where } a > 0.$$

This is the case for $f(x) = e^x - 1$ and $a = 1/2$. Then there is only one function b such that

$$\lim_{x \to 0^+} x^2 b(x) = \frac{1}{a}$$

with $b = B'$ for a solution B of the Abel equation. In other words, B is uniquely determined up to a constant.

Finally, super-exponential functions appear in M. Boshernitzan's work about trans-exponential functions. In [Bos86], he considers solutions E of the functional equation

$$h(E(x)) = E(x + 1),$$

[5] Even though his article was published in 1950, Kneser mentions that the question of the existence of analytic functions $\exp_{\frac{1}{2}}$ was discussed during the 1941 meeting of the German Mathematical Association and that these discussions were sparked by a need for a "reasonable" solution in industry. Considering the time and place as well as the political situation in Germany, one might with hindsight wonder what interest German industrials had in analytic half-iterates of e^x or what caused Kneser to mention this fact at all.

[6] i.e. a set of functions f_r such that $f_1 = f$ and $f_{r+s} = f_r \circ f_s$ for all $r, s \in \mathbb{R}$

where $h(x)$ is either e^x or $e^x - 1$. Due to their growth properties, he calls solutions E trans-exponentials, and he shows that there are germs of solutions which belong to Hardy fields. As an intermediate result, he shows that for C^1-solutions E, the inequality $E' \prec E^3$ yields. We will come back to this observation later.

B.3 Motivations

Having given a short review on the historical developments concerning super-exponential functions, we will now devote a few remarks to our motivations for our construction. As mentioned above, super-exponential functions have until now mainly served as a tool to obtain sets of fractional iterates. There are, however, other reasons to pursue the construction. We will mention two of them here.

The first motivation lies in the model-theoretic study of the field of real numbers. Let $\bar{\mathbb{R}}$ denote the reals with their field structure and $\mathcal{L}$ the language of ordered rings. A well known observation by A. Tarski [Tar51] states that every definable subset of $\bar{\mathbb{R}}$ is the finite union of intervals in $\mathbb{R} \cup \{\pm\infty\}$. In other words, definable sets can already be expressed using the relation $\leqslant$ and parameters from $\mathbb{R} \cup \{\pm\infty\}$. Tarski's result led to the question of how to add functions to $\bar{\mathbb{R}}$ (and likewise function symbols to $\mathcal{L}$) without losing this property for definable sets (see for instance [vdD84]). More precisely, if $\mathcal{F}$ is a set of functions over the reals and $\mathcal{L}_{\mathcal{F}}$ is the language of ordered rings augmented by a functional symbol for every function in $\mathcal{F}$, is every definable subset of $\langle \bar{\mathbb{R}}, \mathcal{F} \rangle$ then a finite union of intervals?

For $\mathcal{F}$ with this property, one says that $\langle \bar{\mathbb{R}}, \mathcal{F} \rangle$ is o-minimal. During the 1980s and 1990s, o-minimal structures have been intensively studied.[7] Furthermore, o-minimal structures admit many interesting topological properties (cell-decomposition, stratification, triangularisation, etc.), which have been studied in great detail by L. van den Dries [vdD98].

More important to us are the sets $\mathcal{F}$, which can be added to $\bar{\mathbb{R}}$ such that the resulting structure remains o-minimal. We are especially interested in possible growth properties of definable functions. A first important result states that one can add restricted analytic functions to $\bar{\mathbb{R}}$ (see [vdD86]). Here, the growth of definable functions is ultimately polynomially bounded. In his paper [Wil96], A. Wilkie then shows that one can add the exponential function to $\bar{\mathbb{R}}$ and still retain the o-minimality property. This result has been generalized [vdDMM94], [Res93] to a great extend, but one always obtains structures with exponential bounds for all definable functions. Hence, a natural question is whether or not there are o-minimal structures $\langle \bar{\mathbb{R}}, \mathcal{F} \rangle$ with definable functions which are not bound by some $\exp_k$. Certainly, $\langle \bar{\mathbb{R}}, E \rangle$ is a candidate.

In view of J.P. Ressayre's proof of Wilkie's theorem it is interesting to have a non-archimedean model of $\mathrm{Th}(\bar{\mathbb{R}}, \exp, E)$. We do not know whether our construction really contributes to a concluding solution to this question, but recent results by Ressayre [Res99] suggest that our model is at least a tool to gain more insight into the behaviour of super-exponential functions in non-standard models. Moreover, once the o-minimality of the reals with super-exponential functions is shown, the question of the limits of growth of definable functions arises anew. Hence, it makes sense to treat the construction for arbitrary strength rather than just for strength 1.[8]

Our second motivation is J. van der Hoeven's programme to construct a field of transseries in which every algebraic, functional or differential equation with parameters in this field has a solution within the field itself, if it admits solutions at all. In this context, adding a super-exponential function (or exponential

[7] For introductory articles see [PS86], [KPS86].

[8] In this context, we remark that an affirmative answer to the question of o-minimality of the structure $\langle \bar{\mathbb{R}}, E \rangle$ also addresses the question of leveled o-minimal structures studied by D. Marker and Ch. Miller [MM97].

functions of arbitrary strength for that matter) can be seen as closing the field under solutions of the functional equation $E(x+1) = \exp E(x)$.

Let us in connection with this mention that Section 2.5 about nested transmonomials and -series is also part of this programme. In fact, we do not need nested objects for the construction of E or L (and their higher-strength versions), but they yield solutions to functional equations. Work by van der Hoeven is currently still in progress, and we hope that our present work is a helpful contribution towards the conclusion of his programme.

B.4 The structure of the thesis

We now outline the structure of the thesis chapter by chapter.

Chapter 1: The first chapter introduces the very basics. Although it is not our aim to make the thesis a completely self-contained exposition, we start by recalling some well-known concepts and results.

We begin with the definition of an ordering being a binary anti-symmetric, reflexive and transitive relation over some set P. In connection with orders, we introduce the notions of comparability, total orders, anti-chains, decreasing chains and well-founded orders. We repeat that these objects are mathematical folklore, and that we do not claim any originality in introducing them. The same is true for the generalization of the concept of well-ordered sets in total orderings to general orderings: an order is Noetherian if it has no strictly decreasing chains and no infinite anti-chains.

The theory of Noetherian orders is well-studied, and we give some equivalent formulations, which we will freely use throughout the rest of the thesis. Next, we introduce words over a given set P, where we make a distinction between commutative and non-commutative words, $P^\diamond$ and $P^\star$ respectively. Moreover, if there is an ordering $\leqslant$ defined on P, then we introduce orderings $\leqslant_{P^\diamond}$ and $\leqslant_{P^\star}$ on the sets $P^\diamond$ and $P^\star$ respectively. We recall Higman's result that if $(P, \leqslant)$ is Noetherian, then so are $(P^\diamond, \leqslant_{P^\diamond})$ and $(P^\star, \leqslant_{P^\star})$.

After a short reminder of what an archimedean field is and how to generalize this notion to modules, we finally introduce the main object of our study, the generalized power series. In fact, at this stage, we define the set $\mathbb{S} = C[[\mathfrak{M}]]$ of generalized power series over $\mathfrak{M}$ with coefficients in C rather generally by allowing $\mathfrak{M}$ to be any ordered semi-group and C a ring. Then $f \in C[[\mathfrak{M}]]$ if $f : \mathfrak{M} \to C$ is a function with Noetherian support in $\mathfrak{M}$. In general, however, we will let $\mathfrak{M}$ be an ordered, multiplicative abelian group. At this point it is important to introduce a whole set of notations. We start with sub-sets of $\mathfrak{M}$. Let $\succcurlyeq$ be the ordering of $\mathfrak{M}$, then

$$\begin{aligned} \mathfrak{M}^{\uparrow} &= \{\mathfrak{m} \in \mathfrak{M} \mid \mathfrak{m} \succ 1\}, \\ \mathfrak{M}^{\updownarrow} &= \{\mathfrak{m} \in \mathfrak{M} \mid \mathfrak{m} \succcurlyeq 1\}, \\ \mathfrak{M}^{\downarrow} &= \{\mathfrak{m} \in \mathfrak{M} \mid 1 \succ \mathfrak{m}\}, \\ \mathfrak{M}^{\underline{\downarrow}} &= \{\mathfrak{m} \in \mathfrak{M} \mid 1 \succcurlyeq \mathfrak{m}\}. \end{aligned}$$

Moreover, we let $\mathbb{S}^{\uparrow} = C[[\mathfrak{M}^{\uparrow}]]$, and define the sets $\mathbb{S}^{\updownarrow}, \mathbb{S}^{\downarrow}, \mathbb{S}^{\underline{\downarrow}}$ accordingly. We also use the arrow-notation as an operator on the set of series by letting $f^{\uparrow} \in \mathbb{S}$ with

$$f^{\uparrow}(\mathfrak{m}) = \begin{cases} f(\mathfrak{m}) & \text{if } \mathfrak{m} \in \mathfrak{M}^{\uparrow}, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we can define series $f^{\updownarrow}, f^{\downarrow}, f^{\underline{\downarrow}}$, which are elements of $\mathbb{S}^{\updownarrow}, \mathbb{S}^{\downarrow}, \mathbb{S}^{\underline{\downarrow}}$, respectively. We write $f_{\mathfrak{m}}$ instead of $f(\mathfrak{m})$ to express the idea that f should be seen as a series (hence the name) rather than as a function, thus $f_{\mathfrak{m}}$ acting as the coefficient belonging to the monomial $\mathfrak{m}$. Using this convention, we write $f = \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}$.

We introduce an addition and a multiplication on $\mathbb{S}$ by letting

$$\begin{aligned} f+g &= \sum_{\mathfrak{m}\in\mathfrak{M}} (f_\mathfrak{m}+g_\mathfrak{m})\mathfrak{m}, \\ f\cdot g &= \sum_{\mathfrak{m}\in\mathfrak{M}} (\sum_{\mathfrak{a}\mathfrak{b}=\mathfrak{m}} f_\mathfrak{a} g_\mathfrak{b})\mathfrak{m}. \end{aligned}$$

These operations equip $\mathbb{S}$ with a natural ring structure. There are also canonical embeddings of both C and $\mathfrak{M}$ into $\mathbb{S}$. Moreover, it is shown that $\mathbb{S}$ is a field if and only C is a field.

In order to show the latter property, it is necessary to introduce a notion of addition which extends the sum of finitely many series. Sure enough, one may not find a reasonable expression $f_1+f_2+\cdots$ for every arbitrarily given sequence $(f_1, f_2, \dots)$ of series in $\mathbb{S}$. However, if the sequence $F=(f_i)_{i\in I}\in\mathbb{S}^I$ is such that $\bigcup_{i\in I}\operatorname{supp} f_i$ is Noetherian in $\mathfrak{M}$ and that for all $\mathfrak{m}\in\mathfrak{M}$ there are only finitely many $i\in I$ such that $\mathfrak{m}\in\operatorname{supp} f_i$, then we may let

$$\sum F = \textstyle\sum_I f_i = \displaystyle\sum_{\mathfrak{m}\in\mathfrak{M}}\sum_{i\in I} f_{i,\mathfrak{m}}\mathfrak{m}.$$

Sequences F with the above properties are called Noetherian families, and it is shown that Noetherian families admit good algebraic properties.

The summation of Noetherian families can be seen in the more general context of strong algebras. Without going into details here, we only mention that generalized power series fields $C[[\mathfrak{M}]]$ are strong C-algebras with respect to the above summations $\sum_I$. A key property, which will be used throughout the construction process in this thesis, is the following. Let $C[[\mathfrak{M}]], C[[\mathfrak{N}]]$ be rings of generalized power series. Let $\varphi:\mathfrak{M}\longrightarrow C[[\mathfrak{N}]]$ be a mapping such that the image of every Noetherian set in $\mathfrak{M}$ is a Noetherian family in $C[[\mathfrak{N}]]$. Then φ extends to a unique mapping $\hat{\varphi}: C[[\mathfrak{M}]]\longrightarrow C[[\mathfrak{N}]]$ such that for any Noetherian family $(f_i)_{i\in I}$ in $C[[\mathfrak{M}]]$ we have

$$\textstyle\sum_I \hat{\varphi}(f_i) = \hat{\varphi}(\sum_I f_i).$$

Moreover, if φ preserves multiplication, then so does $\hat{\varphi}$. Also, if for $\mathfrak{m},\mathfrak{n}\in\mathfrak{M}$, the mapping φ satisfies $\varphi(\mathfrak{m}\mathfrak{n}) = \mathfrak{m}\cdot\varphi(\mathfrak{n})+\varphi(\mathfrak{m})\cdot\mathfrak{n}$, then $\hat{\varphi}$ inherits this property as well, i.e. for all $f,g\in C[[\mathfrak{M}]]$ we have $\hat{\varphi}(fg) = f\cdot\hat{\varphi}(g)+\hat{\varphi}(f)\cdot g$.

The rest of the first chapter is devoted to generalized power series fields $C[[\mathfrak{M}]]$ where both C and $\mathfrak{M}$ are totally ordered. One effect is that we have several canonical ways of representing series $f\in\mathbb{S}$. First, we notice that since now $\mathfrak{M}=\mathfrak{M}^{\uparrow}\ \cup\ \{1\}\ \cup\ \mathfrak{M}^{\downarrow}$, there is a unique constant $f^{=}=f_1\in C$ such that

$$\begin{aligned} f &= f^{\uparrow}+f^{=}+f^{\downarrow} \\ &= f^{\uparrow}+f^{\updownarrow} \\ &= f^{\updownarrow}+f^{\downarrow}. \end{aligned}$$

Moreover, the support of f is well-ordered in $(\mathfrak{M},\succcurlyeq)$ and admits thus a minimal element, called the leading monomial of f, denoted by $\mathfrak{d}_f$. The value which f takes in $\mathfrak{d}_f$ is c_f, the leading coefficient. Let $\tau_f = c_f\mathfrak{d}_f$ be the leading term of f, then there are series R_f, δ_f with

$$\begin{aligned} f &= \tau_f + R_f \\ &= \tau_f(1+\delta_f). \end{aligned}$$

Also, the total orderings of C and $\mathfrak{M}$ induce a total ordering on $\mathbb{S}$ defined by

$$\begin{aligned} 0<f &\quad\Leftrightarrow\quad 0<c_f \\ g<f &\quad\Leftrightarrow\quad 0<f-g. \end{aligned}$$

We finish the first chapter with some general considerations about truncations and the behaviour of supports for sequences of series. For a given $f \in \mathbb{S}$, a series g is a truncation of f if the support of g is an initial segment of the support of f and if the two series coincide on the support of g. In other terms, there is a monomial $\mathfrak{m}_g$ such that

$$g = \sum_{\mathfrak{m} \succ \mathfrak{m}_g} f_{\mathfrak{m}} \mathfrak{m}.$$

We will sometimes use truncations and their properties in proofs. Similarly, we introduce cofinal cardinalities as a tool. For a total order $P = (P, \leqslant)$ we say that P has cofinal cardinality $< \kappa$ (where κ is a cardinal number), if every well-ordered set in P has cardinality less than κ. The real numbers with their natural ordering have, for instance, cofinal cardinality $< \aleph_1$. However, we show that if C and $\mathfrak{M}$ have cofinal cardinalities $< \kappa_1$ and $< \kappa_2$ respectively, then for every strictly decreasing sequence $(f_\alpha)_{\alpha<\tau}$ in $\mathbb{S}$ we must have $|\tau| < \max(\kappa_1, \kappa_2)$.

Chapter 2: Fields of generalized power series – up to this point – provide little structure. However, by demanding some well-chosen properties, we single out classes of generalized power series fields which have at least logarithmic and exponential functions. To this end, we start the second chapter by fixing the conditions of a function to be called an exponential function.

In fact, a function exp which is partially defined on a totally ordered field C, is an exponential function if it is strictly increasing, if $a + 1 \leqslant \exp a$ for all $a \in C$ in the domain of exp, and if

$$\exp(a + b) = (\exp a)(\exp b),$$

whenever both sides are defined. The field C is called an exp-log field in this case.

If C is an exp-log field such that $C = \operatorname{dom} \exp$, then one can define a function exp on $C[[\mathfrak{M}]]^{\mapsdown}$ by

$$\exp f = \exp(f^{=}) \cdot e(f^{\downarrow}),$$

where $e(x) = \sum_{\mathbb{N}} \frac{1}{n!} x^n$. The range of exp is the set $\mathbb{S}^{\mapsdown,+}$ of positive non-infinite series. Hence every $\mathbb{S}$ provides a basic exp-log field structure. The inverse function of exp is denoted by log and satisfies

$$\log f = \log c_f + l(f^{\downarrow})$$

for all $0 < f \in \mathbb{S}^{\mapsdown}$, where $l(x) = \sum_{1 \leqslant n} \frac{(-1)^{n+1}}{n} x^n$. Additional properties are, however, required in order for a field $\mathbb{S}$ to allow a logarithm to be defined on the set of all positive elements. A field $C[[\mathfrak{M}]]$ is said to be a transseries field, if C is an exp-log field with $C = \operatorname{dom} \exp$ and if log extends partially to $\mathbb{T} = C[[\mathfrak{M}]]$ such that

- **T1.** $\operatorname{dom} \log = \mathbb{T}^+$
- **T2.** $\log \mathfrak{M} \subseteq \mathbb{T}^{\uparrow}$
- **T3.** $\log(1 + f) = l(f)$, for all $f \in \mathbb{T}^{\downarrow}$
- **T4.** for every sequence $(\mathfrak{m}_i)_{0 \leqslant i} \subseteq \mathfrak{M}$ such that $\mathfrak{m}_{i+1} \in \operatorname{supp} \log \mathfrak{m}_i$ for all $0 \leqslant i$, there is an integer $n_0 \in \mathbb{N}$ such that

$$\forall n_0 \leqslant n : \forall \mathfrak{n} \in \operatorname{supp} \log \mathfrak{m}_n : \mathfrak{n} \succcurlyeq \mathfrak{m}_{n+1} \ \wedge \ (\log \mathfrak{m}_n)_{\mathfrak{m}_{n+1}} = \pm 1.$$

Conditions **T1** – **T3** allow Dahn's extension process [Dah84], whereas condition **T4** is essential for nested extensions. Both the exponential and the nested extensions are the focus of this chapter.

In order to distinguish transseries fields from usual generalized power series fields, we will use $\mathbb{T}$ instead of $\mathbb{S}$. Elements of transseries fields are called transseries. A simple example of a transseries field is $\mathbb{L} = \mathbb{R}[[\log^{\mathbb{Z}^\star} x]]$, where

$$\log^{\mathbb{Z}^\star} x = \{\log^a x = x^{a_0} \log^{a_1} x \cdots \log_n^{a_n} x \mid a \in \mathbb{Z}^\star\}.$$

Now, since there are no transseries fields $\mathbb{T}$ such that both exp and log are total on $\mathbb{T}$ and $\mathbb{T}^+$ respectively, there is a need to enlarge $\mathbb{T}$. This is where Dahn's extension process comes into play. One lets $\mathbb{T}_{\exp} = C[[\exp \mathbb{T}^{\uparrow}]]$, the exponential extension of $\mathbb{T}$. For an ordinal number α one defines the field $\mathbb{T}_\alpha = C[[\mathfrak{M}_\alpha]]$ by

$$\mathbb{T}_\alpha = \begin{cases} \mathbb{T} & \text{if } \alpha = 0 \\ \mathbb{T}_{\beta,\exp} & \text{if } \alpha = \beta + 1 \\ C[[\bigcup\limits_{\beta<\alpha} \mathfrak{M}_\beta]] & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Fields of the form $\mathbb{T}_\alpha$ are also called transfinite exponential extensions of $\mathbb{T}$. There are two different ways to obtain fields of generalized power series fields for which exp and log are total. First, if λ is a limit ordinal, then $\bigcup_{\alpha<\lambda} \mathbb{T}_\alpha$ has this property; but it is no longer of the form $C[[\mathfrak{N}]]$. The second possibility is to only allow transseries such that the cardinality of the support does not exceed a fixed cardinal number. In the latter case, the extension process is stabilizing. One reason why we treat the case of exponential extensions in great detail is that many of the underlying principles will re-occur in a similar shape in the construction of exponential extensions of positive strength. In fact, the plan we have to follow is to

- extend the monomial group to a set $\hat{\mathfrak{M}} \supseteq \mathfrak{M}$,
- define a multiplicative group structure on $\hat{\mathfrak{M}}$,
- define an ordering on $\hat{\mathfrak{M}}$ which is compatible with the multiplication,
- define a logarithm on $\hat{\mathfrak{M}}$ and $\hat{\mathbb{T}} = C[[\hat{\mathfrak{M}}]]$ such that $\hat{\mathbb{T}}$ is a transseries field.

Let us mention a general result about transseries fields. Suppose that C and $\mathfrak{M}$ have cofinal cardinality $< \kappa_1$ and $< \kappa_2$ respectively. Then we show that

$$|\operatorname{supp} f| < \max(\kappa_1, \kappa_2)$$

for all series $f \in \mathbb{T}_{\exp}$.

The second part of the chapter demonstrates the possibility of introducing nested monomial expressions. By that we mean transmonomials like

$$e^{x^2 + e^{\log_2^2 x + e^{\log_4^2 x + e^{\cdot^{\cdot^{\cdot}}}}}}. \tag{B.2}$$

The expression (B.2) provides a canonical solution to the functional equation

$$f(x) = \exp(x^2 + f(\log_2 x)).$$

Expressions of this kind also occur naturally in the characterization of intervals of transseries. For more on this see [vdH97].

Monomials like (B.2) have *a priori* no reason to belong to a given transseries field. One can easily check that it is not in $\mathbb{L}$, for instance, nor in any transfinite exponential extension $\mathbb{L}_\alpha$. We provide a tool for extending transseries fields $\mathbb{T}$ by nested monomials, thus giving us a means to close such fields under functional equations which lead to such expressions. More precisely, for sequences $\varphi = (\varphi_0, \varphi_1, \dots)$ and $\sigma = (\sigma_0, \sigma_1, \dots) \in \{-1, +1\}^{\mathbb{N}}$ with

- $\forall i \geqslant 0 : \varphi_i \in \mathbb{T}^{\uparrow} \wedge 0 < \varphi_{i+1}$,
- $\forall i \geqslant 0 : \forall \mathfrak{m} \in \operatorname{supp} \varphi_i : \exists j > i : \forall \psi \in \mathbb{T}^{\uparrow}$:

$$\operatorname{supp} \varphi_j \succ \psi \Rightarrow \mathfrak{m} \succ \sigma_i \, e^{\varphi_{i+1} + \sigma_{i+1} e^{\cdot^{\cdot^{\cdot \sigma_{j-1} e^{\varphi_j + \psi}}}}},$$

we show how to construct an transseries field $\mathbb{T}_{\text{nest}}$ containing $\mathbb{T}$ and the expression

$$e^{\varphi_0+\sigma_0 e^{\varphi_1+\sigma_1 e^{\cdot^{\cdot^{\cdot}}}}}.$$

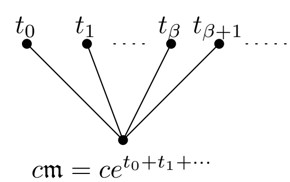
Chapter 3: This and the next two chapters pursue a structural study of transseries fields. This study is motivated by the necessity for a Taylor-series-like development in the construction of super-exponential functions. More precisely, we need notions of derivations and compositions for both transseries fields and their transfinite exponential extensions. Chapter 3 prepares the ground by introducing the representation of terms and series as trees.

We start by a general review of trees. Historically, there are many different approaches to this topic, depending on the purpose of the particular problem at hand. It seems therefore reasonable to define exactly the set of objects (tree, node, height, path) which we will need. Also, we define labeled trees and embeddings between trees. We show properties which are relevant to our later work. Moreover, we introduce a generalization of labeled trees, the labeled structures. Readers familiar with those objects and their properties may of course skip the technicalities of these sections.

We then apply the thus developed toolbox to series in fields $\mathbb{T} = C[[\mathfrak{M}]]$. The idea is indeed quite simple. Firstly, let $c\mathfrak{m} \in C\mathfrak{M}$. Then $\log \mathfrak{m}$ is a series in $\mathbb{T}$. Hence, we have the set term $\log \mathfrak{m} = \{t_\beta \mid \beta < \alpha\}$ for some ordinal α such that

$$t_0 \succ t_1 \succ \cdots \succ t_\beta \succ t_{\beta+1} \succ \cdots$$

We represent $c\mathfrak{m}$ as a tree with a root which is labeled with $c\mathfrak{m}$ such that from this root a branch leads to a leaf labeled by t_β for every $\beta < \alpha$. A tree thus constructed is unique for $c\mathfrak{m}$.



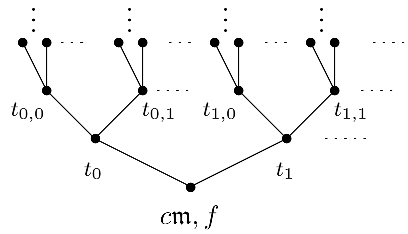
Clearly, to every t_β we can define a similar tree, and what is more, we can insert that tree into the leaf of the first tree labeled by t_β. Inductively continuing this process, we obtain the representation of the term $c\mathfrak{m}$ as a labeled tree of infinite height. Let us call this tree $T_{c\mathfrak{m},\max}$.

Secondly, it is possible to extend the concept from mere terms to series f. Note that the first step in the construction of the tree $T_{c\mathfrak{m},\max}$ can be imitated. We replace the term $c\mathfrak{m}$ by f as the label of the root, and we replace the set of terms in $\log \mathfrak{m}$ by the set of terms of f. Then we continue as above.[9]

The representation thus obtained will be called the maximal tree-representation of the given term or series. In fact, we first formalize the notion of a tree-representation, and then show the existence and uniqueness of maximal tree-representations. From the maximal tree-representation we derive several other tree-representations of terms or series. First, we observe that – by the properties of transseries fields – a path P in $T_{t,\max}$ either admits an integer i such that the label $t_{P,i}$ of the node of height i in P is log-confluent of order 0, or for every $i \in \mathbb{N}$ there is some $j \geqslant i$ such that

$$\text{term} \log t_{P,j} \setminus \{t_{P,j+1}\} \neq \emptyset.$$

[9]We remark that in practice, if the given series is already a term, then we leave out the first step.



In the former case, we say that P is convergent. It is shown that $T_{t,\max}$ is completely determined by the set of its convergent paths. The minimal tree-representation of the term t, symbolized by $T_{t,\min}$ is the sub-tree of $T_{t,\max}$ such that a node is a leaf if and only if its label is log-confluent at order 0. One defines $T_{f,\min}$ of series f in a similar way.

The tree-representations $T_{t,\max}$ and $T_{t,\min}$ exist uniquely for all terms (or series) in transseries fields $\mathbb{T}$. For terms from transfinite exponential extensions $\hat{\mathbb{T}} = \mathbb{T}_\alpha$ we define two more tree-representations with respect to the field $\mathbb{T}$. The relative tree-representation $T_{t,\mathbb{T}}$ of t with respect to $\mathbb{T}$ is the sub-tree of $T_{t,\max}$ where a node is a leaf if and only if its label is an element from $C\mathfrak{M}$. The relative-minimal tree-representation w.r.t. $\mathbb{T}$ is the sub-tree of $T_{t,\max}$ such that a node is a leaf if and only if the label of the node is log-confluent at order 0 and an element from $C\mathfrak{M}$. The latter representation will be symbolized by $T_{t,\mathrm{rm},\mathbb{T}}$. We remark that all these trees are uniquely determined. Moreover, we show further properties and give an application of the use of these trees.

Chapter 4: We turn to derivations and the possibility of the existence of derivations for transfinite exponential extensions. We assume that there is a derivation ∂ on $\mathbb{T}$. Recall from above that by that we mean that ∂ is a strongly linear mapping which sends elements from C to 0, which satisfies the functional equation

$$\partial(fg) = \partial(f) \cdot g + f \cdot \partial(g)$$

for all series f, g and such that $\partial(f) = f \cdot \partial(\log f)$ for all $0 < f$. We fix an ordinal number $\alpha > 0$ and show that there is at most one derivation $\partial_\alpha : \mathbb{T}_\alpha \to \mathbb{T}_\alpha$ which extends the given derivation ∂.

There are two ways of defining ∂_α, and we mention here only the definition which uses a transfinite induction. Under the assumption that there are already derivations ∂_β for all $\beta < \alpha$, we define a function

$$\varphi : \mathfrak{M}_\alpha \longrightarrow \mathbb{T}_\alpha$$

using the fact that for every $\mathfrak{m} \in \mathfrak{M}_\alpha$, the series $\log \mathfrak{m}$ is contained in a field $\mathbb{T}_\beta$ with $\beta < \alpha$. We let

$$\varphi(\mathfrak{m}) := \mathfrak{m} \cdot \partial_\beta(\log \mathfrak{m}).$$

This definition does not depend on the choice of the ordinal β. Moreover, we show that if φ is a Noetherian mapping, then its unique strongly linear extension $\hat{\varphi} : \mathbb{T}_\alpha \to \mathbb{T}_\alpha$ is the derivation ∂_α. The problem thus reduced, it remains to show that φ is Noetherian. We invoke the Noetherian-like property concerning paths by associating in a canonical way a path to every element of $\operatorname{supp} \partial(\mathfrak{m})$ for some $\mathfrak{m} \in \mathfrak{M}_\alpha$. The claimed Noetherianity can then be shown. We thus extend derivations from a given transseries field to all transfinite exponential extensions.

Chapter 5: Similar to derivations, we can introduce a notion of compositions between transseries fields as done in the introduction. In fact, one shows that for every transseries field $\mathbb{T}$ there are right-compositions $\vartriangle : \mathbb{L} \to \mathbb{T}$ which are defined as follows. Let $g \in \mathbb{T}^+_\infty$ (the set of positive elements such that

$\mathfrak{d}_g \in \mathfrak{M}^{\uparrow}$). Then we can replace x by g. More precisely, we show that the mapping $\varphi : \log^{\mathbb{Z}^\star} x \longrightarrow \mathbb{T}$ defined by

$$\mathfrak{m} = \log^a x \longmapsto \mathfrak{m} \circ g = g^{a_0} \log^{a_1} g \cdots \log_n^{a_n} g$$

is Noetherian and extends thus uniquely to $\hat{\varphi} : \mathbb{L} \longrightarrow \mathbb{T}$ with $f \circ g = \sum_{\mathfrak{m}} f_{\mathfrak{m}} \, \mathfrak{m} \circ g$.

Next, we consider extensions of right-compositions. More precisely, let $\mathbb{T}_1, \mathbb{T}_2$ be transseries fields with monomial groups $\mathfrak{M}, \mathfrak{N}$, respectively. We suppose the $\vartriangle : \mathbb{T}_1 \to \mathbb{T}_2$ is a right-composition, i.e. a multiplicative, strongly linear mapping such that for all $f \in \mathbb{T}_1$ we have

$$\begin{aligned} \vartriangle(f) = 0 \quad &\Rightarrow \quad f = 0, \\ f \in \operatorname{dom} \exp \quad &\Rightarrow \quad \vartriangle(\exp f) = \exp \vartriangle(f). \end{aligned}$$

Again, we show that there is at most one extension of $\vartriangle_\alpha$ to $\mathbb{T}_{1,\alpha}$. As for the derivation, we show its existence by using a transfinite induction. Assuming the existence of $\vartriangle_\beta$ for all $\beta < \alpha$, we define a mapping

$$\begin{aligned} \varphi : \mathfrak{M}_\alpha \quad &\longrightarrow \quad \mathbb{T}_{2,\alpha} \\ \mathfrak{m} \quad &\longmapsto \quad \exp \vartriangle_\beta(\log \mathfrak{m}) \end{aligned}$$

if $\log \mathfrak{m} \in \mathbb{T}_{1,\beta} \subseteq \mathbb{T}_{1,\alpha}$. We establish a link between right-compositions and tree-representations, and we can use this connection to show that φ is a Noetherian mapping. This implies that $\hat{\varphi} = \vartriangle_\alpha$. On the other hand, the correspondence will open a combinatorial way of defining extensions of right-compositions.

The third part of the chapter considers Taylor-series developments, which can be seen as the canonical link between derivations and compositions. Therefore, it is natural to ask whether we have something similar for those operators in transseries fields. In fact, we first formalize the concept by introducing the notion of compatible compositions. If $\mathbb{T}_i = C[[\mathfrak{M}_i]]$ ($i = 1, 2, 3$) are transseries fields, then we call a function $\circ : \mathbb{T}_1 \times \mathbb{T}_3 \to \mathbb{T}_2$ a compatible composition, if it satisfies a number of conditions. First, we assume that there are derivations on $\mathbb{T}_1$ and $\mathbb{T}_2$. Moreover, we demand that $\mathbb{T}_3 \subseteq \mathbb{T}_2$ and that the restriction of the derivation of $\mathbb{T}_2$ to $\mathbb{T}_3$ is an derivation. Secondly, if we fix a series $g \in (\mathbb{T}_3)^+_\infty$, then the function

$$\begin{aligned} \vartriangle_g : \mathbb{T}_1 \quad &\longrightarrow \quad \mathbb{T}_2 \\ f \quad &\longmapsto \quad f \circ g \end{aligned}$$

is a right-composition. The third point that we need is

$$\forall \mathfrak{m} \in \mathfrak{M}_1^{\uparrow} : \forall \mathfrak{n}_1, \mathfrak{n}_2 \in \mathfrak{M}_3^{\uparrow} : \quad \mathfrak{n}_1 \succ \mathfrak{n}_2 \Rightarrow \mathfrak{m} \circ \mathfrak{n}_1 \succ \mathfrak{m} \circ \mathfrak{n}_2.$$

Also, if $\circ$ is defined for $(f, g) \in \mathbb{T}_1 \times \mathbb{T}_3$, then $f' \circ g$ is defined as well and the chain rule $(f \circ g)' = f' \circ g \cdot g'$ holds. The last condition, that we demand, requires the most attention. Suppose that $\circ$ is defined for the couple $(f, g) \in \mathbb{T}_1 \times \mathbb{T}_3$. Fix a Noetherian family $(\varepsilon_i)_{i \in I}$ in $\mathbb{T}_2$. We can certainly not expect that $\circ$ is defined for the couple $(f, g + \sum_I \varepsilon_i)$. If, on the other hand, for all $i \in I$ we have

$$\forall \mathfrak{m} \in \operatorname{supp} f : \frac{\mathfrak{m} \circ g}{\mathfrak{m}' \circ g} \succ \varepsilon_i,$$

then we demand that $(f, g + \sum_I \varepsilon_i) \in \operatorname{dom} \circ$ and that $(f^{(n)} \circ g \cdot \varepsilon_i)_{0 \leqslant n, i \in I^n}$ is a Noetherian family with

$$f \circ (g + \textstyle\sum_I \varepsilon_i) = \displaystyle\sum_{0 \leqslant \mathbb{N}} f^{(n)} \circ g \cdot \sum_{i \in I^n} \varepsilon_i,$$

where $\varepsilon_i = \varepsilon_{i_1} \cdots \varepsilon_{i_n}$. Note that this is a very strong Taylor-property which allows a great freedom in the manipulation of series.

If we have a compatible composition $\circ : \mathbb{T}_1 \times \mathbb{T}_3 \to \mathbb{T}_2$, then for every fixed series $g \in (\mathbb{T}_3)^+_\infty$ the right-composition

$$\begin{array}{rcl} \Delta_g : \mathbb{T}_1 & \longrightarrow & \mathbb{T}_2 \\ f & \longmapsto & f \circ g \end{array}$$

extends uniquely to a right-composition

$$\Delta_{g,\alpha} : \mathbb{T}_{1,\alpha} \longrightarrow \mathbb{T}_{2,\alpha}.$$

It is therefore natural to ask whether the function

$$\begin{array}{rcl} \circ_\alpha : \mathbb{T}_{1,\alpha} \times (\mathbb{T}_3)^+_\infty & \longrightarrow & \mathbb{T}_{2,\alpha} \\ (f,g) & \longmapsto & \Delta_{g,\alpha}(f) \end{array}$$

is a compatible composition. In fact, it is the last condition, which is the hardest to prove. Using the facts about labeled structures as shown in Chapter 3, we can show this property, too.

Chapter 6: Until now, we were mainly concerned with functions of at most exponentially growth. From now on, our interest will be on functions which are faster. We start with some general considerations. The main question remains of how to define super-exponential and super-logarithmic functions in generalized power series fields. Since we will eventually broaden that question to exponential functions of arbitrary positive strength, we will first focus on properties originating from the defining functional equation. In fact, it is one of the purposes of chapter 5 to settle some technical questions once and for all.

Fix an transseries field $\mathbb{T}$ and assume that we have defined a function ϕ at least partially on that field. A special case would be $\phi = \log$, but actually we have $\phi = \mathrm{l}_{\omega^n}$ in mind. At any rate, we are interested to know how to define a function Φ on $\mathbb{T}$ such that the functional equation

$$\Phi\phi(g) = \Phi(g) - 1 \quad (g \in \mathbb{T})$$

holds whenever both sides make sense. We employ two ideas. First, if $g = f + \varepsilon$ such that the sequence

$$(\Phi(f), \Phi'(f) \cdot \varepsilon, \Phi''(f) \cdot \varepsilon^2, \dots)$$

is a Noetherian family, then we will let

$$\Phi(g) = \Phi(f + \varepsilon) = \Phi(f) + \Phi'(f) \cdot \varepsilon + \Phi''(f) \cdot \varepsilon^2 + \cdots .$$

In other words, we define $\Phi(g)$ using the Taylor-development of Φ in f. There are, however, some problems to consider.

- We already need a partial function Φ in order to let $f \in \operatorname{dom} \Phi$.
- What if there is no derivation defined on $\mathbb{T}$?
- What if there are $g = \hat{f} + \hat{\varepsilon}$ which also allow the Taylor-series development?

In fact, the first problem does not concern us here; we will simply work under the assumption that there is already a partially defined function Φ on $\mathbb{T}$. The second problem is in theory more serious. Even though we will always have a derivation in our applications, we address this problem because of its generality. In fact, we show a way to define $\Phi'(f), \Phi''(f), \dots$ by imitating a derivation. The third problem then disappears completely, for we then have only to apply the last point of the definition of a compatible composition.

So, let us dwell on the second point a little more. We will show that $\Phi'(f)$ is uniquely determined by the functional equation in transseries fields, if there is an derivation. But then, we can express $\Phi'(f)$ without even mentioning a derivation at all. The same principle will hold for higher derivations. For instance, if we want to determine the first derivative of a super-logarithmic function, we will find that

$$L'x = \frac{1}{x\,\log x\,\log_2 x\cdots} = \frac{1}{\exp(\log x + \log_2 x + \cdots)}.$$

We remark that in this example, we have $L'x \in \mathbb{L}_{\exp}$. The same will be true for $L''x, L'''x, \ldots$. In fact, derivatives of higher degrees can be recursively defined.

Now, there can be series g such that a decomposition into a sum $f + \varepsilon$ as above is not possible. To overcome those situations, we use the function ϕ to reduce the series g. Even though the above process may fail on g, there could be some $n \in \mathbb{N}$ such that $\phi_n(g)$ can be decomposed into such a sum enabling a definition of $\Phi(\phi_n(g))$ in the above sense. But then, we may let

$$\Phi(g) = \Phi(\phi_n(g)) + n.$$

Again, we have to tackle one problem. There could be more than one integer allowing that definition. In other words, we have to show that $\Phi(\phi_n(g)) + n = \Phi(\phi_m(g)) + m$, whenever both sides make sense. This property, which we will call the vertical coherence of the definition of Φ, will be shown.

Finally, we add some remarks about possible inverse functions ψ, Ψ of ϕ, Φ. It should be noticed that for these functions the functional equation $\psi(\Psi(f)) = \Psi(f+1)$ holds, whenever both sides are defined.

Chapter 7: We are now well-equipped to extend Dahn's construction to exponential functions of higher strength. To do so, we use the same method as in the exponential extension process, only, due to the amount of technical work involved, we have decided to split the process into two chapters. Actually, the difficulties arise from the fact that we want to treat all possible positive strength at a time. We could, of course, first construct fields with super-exponential and -logarithmic functions and then generalize the construction to general positive strength. But this would mean repeating the same definitions, lemmas and properties, thus lengthening the exposition unnecessarily.

First, we introduce some notations which will be helpful for keeping the formulas short. Also, we introduce the convention that if we speak of an exponential or logarithmic function of strength 0, then we mean the usual exponential and logarithmic functions. We fix a positive integer n. A totally ordered field C is an exponential field of strength n if there are functions $\exp, \ldots, \mathrm{e}_{\omega^{n-1}}$ such that C is an exponential field of strength $n-1$ for these functions and if there is a constant $c_n \in C$ and a function e_{ω^n} partially defined on C such that for all $c_n \leqslant x < y$ we have

- $x \in \operatorname{dom} \mathrm{e}_{\omega^n}$,
- $x + 1 < \mathrm{e}_{\omega^n} x$ and $\mathrm{e}_{\omega^n} x < \mathrm{e}_{\omega^n} y$,
- $\mathrm{e}_{\omega^n} x \in \operatorname{dom} \mathrm{e}_{\omega^{n-1}}$ and $\mathrm{e}_{\omega^{n-1}} \mathrm{e}_{\omega^n} x = \mathrm{e}_{\omega^n}(x+1)$.

We show that $\mathbb{R}$ is an exponential field of strength n and that – by generalizing Kneser's proof – we can also assume that e_{ω^n} is analytic. Moreover, we strengthen a result by Boshernitzan. Assume that e_{ω^n} is a C^1-function on $\mathbb{R}$ and $\varepsilon > 0$. Then there is a real number x_ε such that $\mathrm{e}'_{\omega^n} x < \mathrm{e}^{1+\varepsilon}_{\omega^n} x$ for all $x > x_\varepsilon$.

The next step is to define transseries fields of strength n. Recall that we had a specific composition result for transseries fields. This composition result can indeed be generalized to positive strength. Similarly, we need to extend the definition of log-confluence to a confluence property for logarithmic functions of any non-negative strength.[10] We then say that $\mathbb{T} = C[[\mathfrak{M}]]$ is of strength n if it is of strength

[10]The only difficulty in this approach of treating all positive strength at once is in keeping track of the dependencies.

$n-1$, if C is an exponential field of strength n and if there is a partially defined function l_{ω^n} with $\mathbb{T}_\infty^+ \subseteq \operatorname{dom} \mathrm{l}_{\omega^n}$ such that

- if $f, \mathrm{l}_{\omega^{n-1}} f \in \operatorname{dom} \mathrm{l}_{\omega^n}$, then $\mathrm{l}_{\omega^n} \circ \mathrm{l}_{\omega^{n-1}} f = \mathrm{l}_{\omega^n} f - 1$
- if $f \in \mathbb{T}_\infty^+$, then there is some $k \in \mathbb{N}$ such that f is $\mathrm{l}_{\omega^{n-1}}$-confluent at order k, such that

$$\mathrm{l}_{\omega^{n-1} \cdot k} f = \mathfrak{m} + \varepsilon$$

  with $\mathfrak{m} \in \operatorname{dom} \mathrm{l}_{\omega^n}$, and $\mathrm{l}'_{\omega^n} \mathfrak{m} \in \mathbb{T}$ and

$$\mathrm{l}_{\omega^n} f = k + \mathrm{l}_{\omega^n} \mathfrak{m} + \mathrm{l}'_{\omega^n} \mathfrak{m} \cdot \varepsilon + \frac{1}{2!} \mathrm{l}''_{\omega^n} \mathfrak{m} \cdot \varepsilon^2 + \cdots$$

- for all $f \in \operatorname{dom} \mathrm{e}_{\omega^n}$ with

$$\forall k \in \mathbb{N} : \forall \mathfrak{m} \in \operatorname{supp} f^{\downarrow} : 1 \prec \mathfrak{m} \cdot \mathrm{e}_{\omega^n}(f-k)$$

  we have $\mathrm{e}_{\omega^n} f \in \mathfrak{M}$
- $\mathbb{T}_\infty^+$ is l_{ω^n}-confluent.

This definition provides the right framework for both the generalized Dahn-process and the extension of the definition of transseries fields of strength $n-1$ to strength n.

The rest of this chapter consists of three different parts. First, we show some properties of transseries fields, which will have applications later. Secondly, we show that the partial composition result of strength n holds for transseries fields of strength n, thus making way for the definition of transseries fields of strength $n+1$. Finally, we give an example of an transseries field of strength n.

Chapter 8: This chapter extends Dahn's process to positive strength..

Fix integers $0 \leqslant i \leqslant n$ and assume that $\mathbb{T}$ is of strength n. Note that $\mathbb{T}$ is also of strength i, hence that there are functions e_{ω^i} and l_{ω^i}. Now, as in the exponential case, the function e_{ω^i} is not totally defined on the set $\mathbb{T}_\infty^+$. Again, we will define a field $\hat{\mathbb{T}} \supseteq \mathbb{T}$, which is of strength n and such that $\hat{\mathbb{T}}_\infty^+ \subseteq \operatorname{dom} \mathrm{e}_{\omega^i}$. Obtaining such a field is an iterative process.

In a first instance, we construct an extension $\mathbb{T}_{\mathrm{e}_{\omega^i}} = C[[\mathfrak{M}_{\mathrm{e}_{\omega^i}}]]$ of $\mathbb{T}$. In fact, we start by defining a set of new monomials, $\mathfrak{N}_{i,\mathbb{T}}$, as follows. Let $\mathcal{F}_{i,\mathbb{T}} \subseteq \mathbb{T}_\infty^+$ be the set of series f such that $f - k \notin \operatorname{dom} \mathrm{e}_{\omega^i}$ for all k. Since we want to add $\mathrm{e}_{\omega^i} f$ as a monomial, we let $\mathfrak{N}_{i,\mathbb{T}}$ be the multiplicative closure of the set

$$\{\mathrm{l}'_\alpha(\mathrm{e}_{\omega^n} f) \mid \alpha \leqslant \omega^i \ \wedge \ f \in \mathcal{F}_{i,\mathbb{T}}\}.$$

We then let $\mathfrak{M}_{\mathrm{e}_{\omega^i}} = \mathfrak{M} \cdot \mathfrak{N}_{i,\mathbb{T}}$. We now have to work through the following programme:

- define a multiplicative group structure on $\mathfrak{M}_{\mathrm{e}_{\omega^i}}$,
- define an ordering on $\mathfrak{M}_{\mathrm{e}_{\omega^i}}$ which is compatible with the multiplication,
- for $j = 0, \ldots, n$, define functions l_{ω^j} on $\mathfrak{M}_{\mathrm{e}_{\omega^i}}$ and $\mathbb{T}_{\mathrm{e}_{\omega^i}} = C[[\mathfrak{M}_{\mathrm{e}_{\omega^i}}]]$ such that

$$\langle \mathbb{T}_{\mathrm{e}_{\omega^i}}, \log, \ldots, \mathrm{l}_{\omega^j} \rangle$$

  is of strength j.

Note that the extension step $\mathbb{T} \to \mathbb{T}_{\mathrm{e}_{\omega^i}}$ does not only generate an transseries field of strength i, but that $\mathbb{T}_{\mathrm{e}_{\omega^i}}$ is even of strength n. Also, we remark that the extension step involves numerous intermediate steps, namely the construction of transseries fields $\langle \mathbb{T}_{\mathrm{e}_{\omega^i}}, \log, \ldots, \mathrm{l}_{\omega^j} \rangle$ of strength $j \leqslant n$.

Chapter 9: In the last chapter we show how to apply those processes in order to define exponential closures of positive strength

We use the extension process to construct the exponential closure of strength n. Let $\mathbb{T}$ be a transseries field[11] of strength n. Then we let

$$\begin{aligned} \mathbb{T}_0 &:= \mathbb{T} \\ \mathbb{T}_{m+1} &:= (\mathbb{T}_m)_{\exp, \mathrm{e}_\omega, \ldots, \mathrm{e}_{\omega^n}} \\ \mathbb{T}_{<\omega} &:= \bigcup_{m<\omega} T_m. \end{aligned}$$

In other words, we apply iteratively the extensions for $\exp, \ldots, \mathrm{e}_{\omega^n}$ and then take the inductive limit of the resulting chain of transseries fields. Now, the field $\mathbb{T}_{<\omega}$, although not a transseries field itself, is a field of generalized power series with functions e_{ω^n} and l_{ω^n} which are total on the set $\mathbb{T}^+_{<\omega,\infty}$.

Moreover, we show that we have a certain degree of freedom in the choice of the order of the iterative process. We also reconsider the behaviour of the supports of series under the extension processes. Finally, we generalize the results about tree-representations.

B.5 Remarks on the notations

We introduce various notations for objects defined in the thesis. There are, however, some general notations which will be employed throughout the following chapters.

Integer numbers will in general be denoted by k, m, n and occasionally by i, j, where the latter two usually stand for non-negative integers. For tuples of integers, we will often write a, b, K, L.

We reserve $\alpha, \beta, \gamma, \ldots$ for ordinal numbers. In particular, λ will generally stand for a limit ordinal. Cardinal numbers will be denoted by κ and indexed versions thereof.

Unless otherwise stated, I and J will stand for index sets and i, j for elements of I and J. It will be clear from the context whether i and j stand for integers or indices.

Let f and R be a n-ary function and relation respectively. For sets $A_1, \ldots, A_n$ we let

$$\begin{aligned} f(A_1, \ldots, A_n) &= \{f(a) \mid a \in A_1 \times \cdots \times A_n\} \\ R(A_1, \ldots, A_n) &= R \cap A_1 \times \cdots \times A_n. \end{aligned}$$

In other words, they denote the restriction of f and R to the sets $A_1, \ldots, A_n$. For integers n, we let f_n be the n-th iteration of the function f. Note that with this notation f_{-1} denotes the inverse function of f. Where an exponentiation is defined, $f^p x^q$ stands for the term $(f(x^q))^p$. All the above conventions are understood in cases where the terms are well-defined.

Finally, we remark that we use "iff" for definitions and "if and only if" for equivalent statements. Similarly, we use $t :\Leftrightarrow s$ to define the term t by s, and we use $t \Leftrightarrow s$ to express the equivalence of t and s.

Further notations are introduced when they occur in the text. We add a glossary and an index at the end of the thesis for further orientation.

[11] This always suggests the existence of appropriate functions l_{ω^i} for all $i \leqslant n$.