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Fast evaluation of holonomic functions near and in regular singularities

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A holonomic function is an analytic function, which satisfies a linear differential equation $Lf = 0$ with polynomial coefficients. In particular, the elementary functions $\exp, \log, \sin$, etc. and many special functions like $\mathrm{erf}, \mathrm{Si}$, Bessel functions, etc. are holonomic functions.

In a previous paper, we have given an asymptotically fast algorithm to evaluate a holonomic function f at a non-singular point z' on the Riemann surface of f, up to any number of decimal digits while estimating the error. However, this algorithm becomes inefficient, when z' approaches a singularity of f.

In this paper, we obtain efficient algorithms for the evaluation of holonomic functions near and in singular points where the differential operator L is regular (or, slightly more generally, where L is *quasi-regular* — a concept to be introduced below).

1. Introduction

Let $\mathbb{K}$ be a subfield of $\mathbb{C}$. A *holonomic function* (over $\mathbb{K}$) is an analytic function f, which satisfies a linear differential equation

$$P_r(z)f^{(r)} + \cdots + P_0(z)f = 0, \tag{1.1}$$

where $P_0, \cdots, P_r$ are polynomials in $\mathbb{K}[z]$ with $P_r \neq 0$. The elementary functions exp, log, sin,... and many special functions like $\mathrm{erf}, \mathrm{Si}, \ldots$, Bessel functions, hypergeometric functions, etc. are holonomic. The class of holonomic functions also admits several interesting algebraic properties and has recently been the object of intensive study in computer algebra and mathematics (Stanley, 1980; Lipshitz, 1989; Zeilberger, 1990).

In (van der Hoeven, 1999), we have studied holonomic functions from the *exact numerical* point of view: requiring that all complex numbers z we compute with are *effective* (i.e. for any rational $\varepsilon > 0$ we can compute a "Gaussian rational" $\tilde{z} \in \mathbb{Q}[i]$ with $|\tilde{z} - z| \leqslant \varepsilon$), we were interested in algorithms to evaluate holonomic functions. Of course, we need be careful here, since f is actually defined on a Riemann surface $\mathcal{R}$ above $\mathbb{C}\backslash\Omega$, for some finite set Ω (since any element in Ω must be a zero of P_r).

More precisely, we select a base point ζ on $\mathcal{R}$, which projects on an effective $z \in \mathbb{C}\backslash\Omega$,

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and we give ourselves effective initial conditions[†] in ζ:

$$I(\zeta) = \begin{pmatrix} f(\zeta) \\ \vdots \\ \frac{1}{(r-1)!} f^{(r-1)}(\zeta) \end{pmatrix}.$$

Next, we consider a non-singular path $\zeta \rightsquigarrow \zeta'$ on $\mathcal{R}$, which is represented by a suitable effective broken line path $z \rightsquigarrow z'$ in $\mathbb{C} \backslash \Omega$, and the problem is to compute f in ζ'.

More generally, we may ask for the values of the first $r-1$ derivatives of f in ζ', i.e. to compute $I(\zeta')$ in terms of $I(\zeta)$. This linear relationship can be written

$$I(\zeta') = \Delta_{\zeta \rightsquigarrow \zeta'} I(\zeta), \tag{1.2}$$

where $\Delta_{\zeta \rightsquigarrow \zeta'}$ is a matrix which only depends on the homotopy class of the projection of $z \rightsquigarrow z'$ of $\zeta \rightsquigarrow \zeta'$ in $\mathbb{C} \backslash \Omega$. We will call $\Delta_{z \rightsquigarrow z'} = \Delta_{\zeta \rightsquigarrow \zeta'}$ the *transition matrix* along $z \rightsquigarrow z'$ or $\zeta \rightsquigarrow \zeta'$. These matrices satisfy the transitivity relation

$$\Delta_{z \rightsquigarrow z' \rightsquigarrow z''} = \Delta_{z' \rightsquigarrow z''} \Delta_{z \rightsquigarrow z'} \tag{1.3}$$

for the composition of paths. When $z \rightsquigarrow z' = z \circlearrowleft z$ is actually a loop around one of the singularities, then $\Delta_{z \circlearrowleft z}$ reduces to a monodromy matrix.

In section 2 we recall results from (Chudnovsky and Chudnovsky, 1990) and (van der Hoeven, 1999) about the efficient computation of transition matrices and the application to the evaluation of f. However, the algorithms we presented there have two main disadvantages:

- They suffer from numerical instability problems when ζ' approaches a singularity: the coefficients of the transition matrix $\Delta_{z \rightsquigarrow z'}$ grow as fast as the most violent solutions to (1.1) near the singularity.
- The algorithms do not allow us to compute the limit of f in a singularity, if such a limit exists.

In this paper, we will study both problems. Our approach is to generalize transition matrices in order to accommodate paths with endpoints in singularities or which pass through singularities. The main steps, which will be detailed below, are as follows: solve the equation (1.1) formally *in* the singularity; give analytical meanings to the solutions; use these solutions to prolongate I *into* the singularity.

Formal solutions. In section 3, we recall and refine some classical results about the formal resolution of (1.1) in singularities in terms of *transseries*. These are generalized series which recursively involve exponentials and logarithms. In this article, we assume the singularity at $z = 0$, and then it suffices to consider transseries which are obtained from the field of Laurent series in z^{-1}, from $\log z$, from monomials z^α and exponentials of polynomials in z^{-1}, by the ring operations and substitutions $z \mapsto \sqrt[p]{z}$.

Analytical meaning of transseries. Sometimes (and actually even rather often), the formal transseries solutions to (1.1) are all convergent and bounds for their coefficients

[†] We note a small difference with (van der Hoeven, 1999), where we did not divide each coefficient $f^{(i)}(\zeta)$ by $i!$. This difference is motivated by compatibility reasons with (van der Hoeven, 1997) in view of remark 3.2.

can be computed. We will mainly be concerned with this *convergent case* in this paper; in section 3.3.2 we will introduce the corresponding notion of quasi-regular differential operators. The *divergent case* requires Écalle's accelero-summation theory (Écalle, 1992; Écalle, 1993; Braaksma, 1991; Balser, 1994; Ramis, 1978; Ramis, 1980) and will be treated in a forthcoming paper.

Prolongation of I into singularities. In section 3.3.1, we explicitly introduce a special basis of r transseries solutions $f^{[0]}, \ldots, f^{[r-1]}$ to (1.1) — the basis of *canonical solutions*. Having fixed analytical meanings of $f^{[0]}, \ldots, f^{[r-1]}$, each actual analytic solution f to (1.1) can be expressed as a linear combination

$$f = \lambda_0 f^{[0]} + \cdots + \lambda_{r-1} f^{[r-1]}.$$

The column vector with entries $\lambda_0, \ldots, \lambda_{r-1}$ will now be considered as the prolongation of the initial conditions I *into* the singularity; we say that it is a *generalized value* of I. We notice that this prolongation depends on the way we associated an analytical meaning to $f^{[0]}, \ldots, f^{[r-1]}$; this is particularly important in the divergent case.

Singular transition matrices. In section 4, we introduce *singular transition matrices* which describe the linear dependencies between the generalized or ordinary values of I in singular or ordinary points, just as the usual transition matrices described the linear dependencies between the values of I in ordinary points.

In the convergent case, we show how to approximate singular transition matrices up to any desired precision; this enables us in particular to approximate the limit of a solution f to (1.1) in the singularity, if it exists. Modulo an interesting heuristic stated in section 5.1, we also obtain uniform complexity bounds for (singular) transition matrices along paths close to a given singularity, and whose entries are represented by floating point numbers.

As to the relation of our work with respect to previous work, the idea to "pass through" singularities in order to perform analytic continuations near singularities has been around for some time among the specialists of resummation theory. However, we think that it has never been made as explicit as in our paper. More generally, we feel a need of detailed papers about effective analytic continuation near singularities, with actual algorithms and results about the computational complexity. This paper is intended as a first step in this direction.

In remark 2.4, we will also point out that our algorithms are *exponentially faster* than classical numerical algorithms, such as the Runge-Kutta method. This is a general phenomenon; in a forthcoming paper, we plan to generalize our results to (regular) non linear differential equations. We also recall that our algorithms provide a totally effective error control.

As to the incorporation of numerical algorithms for computations with holonomic functions in computer algebra systems, it is important to have a zero-test for holonomic constants. In the last section, we propose such a test, which is based on a new heuristic. We also prove a uniform complexity result based on this heuristic for the evaluation of certain polynomial expressions involving holonomic functions near singularities.

2. Survey of the non-singular case

In (van der Hoeven, 1999), we studied the following questions (using the notations from the introduction):

Q1. How to guarantee the exactness of evaluation algorithms?

Q2. What is the asymptotic complexity of computing n digits of $f(\zeta')$?

Q3. How does the choice of the path $z \rightsquigarrow z'$ influence the complexity of effective analytic continuation? In particular, what happens if the path approaches a singularity?

We will briefly recall our results in what follows.

REMARK 2.1. During the refereeing of this paper, the author has been made aware of the paper (Chudnovsky and Chudnovsky, 1990), in which the questions **Q2** and **Q3** were studied before in a similar way as in (van der Hoeven, 1999).

REMARK 2.2. We stress that questions **Q1** and **Q2** should really be seen as independent. The first question amounts to the computation of certain bounds as a function of the path $z \rightsquigarrow z'$. These bound computations are *independent* from the required precision n in the second question.

In **Q2** and **Q3**, we are concerned with asymptotically fast algorithms (i.e. fast algorithms for large n). The techniques we will use there are very different from the bound computation techniques and from more classical techniques (such as the Runge-Kutta method).

2.1. Effective bounds

If $z \rightsquigarrow z' = z \to z'$ is a straightline path with z' close to z, then $f(\zeta')$ can be approximated by evaluating sufficiently many terms of the power series expansion

$$f(\zeta') = f_0 + f_1(z'-z) + f_2(z'-z)^2 + \cdots \tag{2.1}$$

of f in ζ. In order to obtain an exact numerical algorithm, we should therefore be able to estimate the committed error.

Now (1.1) implies that the coefficients f_k satisfy a linear recurrence relation with coefficients in $\mathbb{K}(k)$. This relation can be written in matrix form

$$F_{k+1} = A_k F_k,$$

for a certain q by q matrix with coefficients in $\mathbb{K}(k)$ and where the F_k are column vectors with entries $f_k, \ldots, f_{k+q-1}$. Actually, the matrices A_k tend to a constant matrix for $k \to \infty$, i.e. $A_k \in \mathbb{K}[[k^{-1}]]$. Let λ be the largest eigenvalue of the limit matrix A_∞. Estimating the product $A_k \cdots A_0$ for $k \to \infty$, we proved the following in section 2.2 of (van der Hoeven, 1999):

THEOREM 2.1. *There exists an algorithm, which given $\mu > \lambda$ computes a constant B such that $|f_k| \leqslant B\mu^k$ for all k.* □

In particular, this bound yields error estimations for the tails of the Taylor series expansion (2.1), since

$$|f(\zeta') - f_0 - \cdots - f_{k-1}(z'-z)^{k-1}| \leqslant \frac{B\tau^k}{1-\tau},$$

for $\tau = |z'-z|/\mu < 1$.

REMARK 2.3. In (van der Hoeven, 1999), we applied the theorem to the case when z is non-singular for (1.1), i.e. $P_r(z) \neq 0$. In that case, all solutions to (1.1) have convergence radius at least λ^{-1}. More precisely, λ^{-1} coincides with the convergence radius of $1/P_r$ in z.

2.2. Fast evaluation of truncated power series expansions

Assuming that $\mathbb{K}$ is an algebraic number field and $z, z' \in \mathbb{K}$, we will show now how to compute $f_0 + f_1(z'-z) + \cdots + f_k(z'-z)^k$ in an asymptotically efficient way. We first introduce the vectors

$$\begin{aligned} \Phi_k &= F_k z^k; \\ \Sigma_{k;l} &= F_k z^k + F_{k+1} z^{k+1} + \cdots + F_{k+l-1} z^{k+l-1}, \end{aligned}$$

for $k \in \mathbb{N}, l \geqslant 1$. We claim that for all k and $l \geqslant 1$, there exist matrices $M_{k;l}$ and $N_{k;l}$, such that

$$\begin{aligned} \Sigma_{k;l} &= M_{k;l} \Phi_k; \\ \Phi_{k+l} &= N_{k;l} \Phi_k. \end{aligned}$$

This is clearly so for $l = 1$, by taking $M_{k;1} = Id$ and $N_{k;1} = z A_k$. Assume $l > 0$ and decompose $l = l_1 + l_2$ with $l_1 = \lfloor \frac{l}{2} \rfloor$. Then we take

$$\begin{aligned} M_{k;l} &= M_{k;l_1} + M_{k+l_1;l_2} \, N_{k;l_1}; \\ N_{k;l} &= N_{k+l_1;l_2} \, N_{k;l_1}. \end{aligned}$$

These recursion formula yield an efficient divide and conquer algorithm to compute $M_{0;k}$ for large k (if such a matrix has fractional entries, then we put all its entries on common denominator and no gcd computations are performed in order to simplify this denominator). Denoting by $M(n)$ the time required to multiply two n-digit numbers, we proved the following complexity bound for this algorithm in section 3.2 of (van der Hoeven, 1999).

THEOREM 2.2. *Assume that $\mathbb{K}$ is an algebraic number field. Then the matrix $M_{0,k}$ can be computed in time $O(M(k \log^2 k))$.* □

Using the bounds from the previous section, this yields an efficient algorithm to evaluate $f(\zeta')$ up to any desired precision. The iterated derivatives $f', f'', \ldots$ of f can also be evaluated efficiently in ζ', because these derivatives are also holonomic.

2.3. General transition matrices

The approximation problem for transition matrices between two close points z and z' clearly reduces to the evaluation problem of p linearly independent solutions to (1.1) and its first $r-1$ iterated derivatives in z'. Using the bounds from section 2.1, we can do this up till any desired precision. If $\mathbb{K}$ is an algebraic number field, and $z, z' \in \mathbb{K}$, we can even use the asymptotically efficient algorithm from above.

To compute the transition matrices along general paths, we approximate the path by a broken line path and use the transitivity relation (1.3). In order to choose the broken line path in an optimal way, it is important to estimate the complexity of the computation of transition matrices as a function of the path. Denote by $D(\zeta, \rho)$ the compact disk with center ζ and radius ρ and by $\rho(z)$ the distance of z to the closest singularity. Assuming that $\mathbb{K}$ is an algebraic number field, we also denote by $\mathtt{size}(z)$ the memory space needed to store a number $z \in \mathbb{K}$. In section 4.1 of (van der Hoeven, 1999), we proved the following for straightline paths $z \to z'$ with z, z' above $\mathbb{K}$:

THEOREM 2.3. *Assume that*

(a) U *is an open domain on which* $|f|$ *is bounded.*
(b) $\mathbb{K}$ *is an algebraic number field.*
(c) $z \to z'$ *is the straightline path between two points* $z, z' \in \mathbb{K}$.
(d) We have $D(z, |z' - z|) \subseteq U$.

Denote $s = \mathtt{size}(z) + \mathtt{size}(z')$ *and* $\tau = \frac{\rho(z)}{|z'-z|}$. *Then* $f(z')$ *can be evaluated up to precision* 2^{-n} *in time*

$$O(M(n(s + \log n) \log n \log^{-1} \tau)),$$

uniformly in z *and* z', *provided that* $\log \log \tau = O(n)$.

We have also shown that an arbitrary broken line path $z \rightsquigarrow z'$ can be suitably approximated by a broken line path with vertices in $\mathbb{K}$ (but which depends on the required precision), in order to obtain an efficient approximation algorithm for $\Delta_{z \rightsquigarrow z'}$:

THEOREM 2.4. *Assume that* $\mathbb{K}$ *is an algebraic number field. Then* n *digits of* $\Delta_{z \rightsquigarrow z'}$ *(resp.* $f(z')$*) can be computed in time* $O(M(n \log^2 n \log \log n))$. □

REMARK 2.4. We stress that the above complexity is far better than the complexities achieved by classical numerical methods. For instance, the Runge-Kutta method needs a time $O(n)$ to get a result with a precision of $O(n^{-4})$. But a precision of $O(n^{-4})$ means that we only obtain $O(\log n)$ correct digits! In other words, in order to obtain n correct binary digits, Runge-Kutta's algorithm needs a time $O(2^{n/4})$. Therefore, Runge-Kutta's algorithm has an exponential complexity from our point of view. Nevertheless, this method remains superior for small precisions.

2.4. AN ALTERNATIVE ALGORITHM FOR BOUND COMPUTATIONS

One of the referees observed that I could have used Cauchy-Kovalevskaya's majorant method in order to obtain the bound from theorem 2.1 in (van der Hoeven, 1999). Indeed, I was not aware of this method at the time, but I rediscovered it since, and was actually planning to use it for a forthcoming paper on analytic continuation of solutions to non linear differential equations. For the sake of completeness, we apply it in this section to the case of linear differential equations in non-singular points. It would be interesting to know whether the technique can be generalized to the regular singular case which will be studied in the remainder of this paper.

So assume that $P_r(z) \neq 0$ and let λ^{-1} be the radius of convergence of $1/P_r$ in z. Given $\mu > \lambda$, we will show how to compute a B, such that $|f_k| \leqslant B\mu^k$ for all k. Now observe that (1.1) is equivalent to

$$f^{(r)}(z) = -\frac{P_{r-1}(z)}{P_r(z)} f^{(r-1)} - \cdots - \frac{P_0(z)}{P_r(z)} f. \tag{2.2}$$

We first compute bounds for the coefficients of the rational fractions $-P_i/P_r$, of the form

$$\left|\left(-\frac{P_i(z)}{P_r(z)}\right)_k\right| \leqslant M_i \nu^k,$$

where $\lambda < \nu < \mu$ (say $\nu = (\lambda + \mu)/2$). Then let

$$N = \left\lceil \frac{1}{\nu} \max_{i \in \{0, \ldots, r-1\}} \sqrt[r-i]{r M_i} \right\rceil,$$

so that

$$\begin{aligned}
\left|\left(-\frac{P_i(z)}{P_r(z)}\right)_k\right| &\leqslant \left(\frac{M_i}{1 - \nu z}\right)_k \leqslant \frac{(N\nu)^{r-i}}{r} \left(\frac{1}{1 - \nu z}\right)_k \\
&\leqslant \frac{(N + r - 1) \cdots (N + i)}{r} \left[\left(\frac{\nu}{1 - \nu z}\right)^{r-i}\right]_k
\end{aligned}$$

for all $i \in \{0, \ldots, r-1\}$ and k. In other words, the equation

$$g^{(r)}(z) = \frac{N + r - 1}{r} \left(\frac{\nu}{1 - \nu z}\right)^1 g^{(r-1)} + \cdots + \frac{(N + r - 1) \cdots N}{r} \left(\frac{\nu}{1 - \nu z}\right)^r g \tag{2.3}$$

is a “majorant” of the original equation (2.2). Furthermore,

$$g = A \left(\frac{1}{1 - \nu z}\right)^N \tag{2.4}$$

is a simple solution to (2.3) for each A. Take

$$A = \max_{i \in \{0, \ldots, r-1\}} \frac{|f_i|}{\left[\left(\frac{1}{1 - \nu z}\right)^N\right]_i} = \max_{i \in \{0, \ldots, r-1\}} \frac{|f_i|}{\nu^i \binom{N+i}{i}}.$$

Using the majorant technique, we now observe that

$$|f_k| \leqslant g_k = A \binom{N + k}{k} \nu^k \tag{2.5}$$

for all k, since (2.3) is a majorant of (2.2) and (2.5) holds for all $k < r$. Now g_k/μ^k is maximal for $k \approx N\nu/(\mu - \nu)$, whence

$$B = A \binom{\left\lceil \frac{\mu}{\mu - \nu} N \right\rceil}{N} \left(\frac{\nu}{\mu}\right)^{\left\lfloor \frac{\nu}{\mu - \nu} N \right\rfloor}$$

has the required property that $|f_k| \leqslant B \mu^k$ for all k.

Remark 2.5. It is possible to choose the bounds in a slightly sharper way in the above method. However, this leads to more complicated formulas and we do not expect the gain to be worth it in general. Therefore, we have preferred the above computationally “simple” method, which should be easier to implement and is expected to suffer less from overhead.

3. Formal solutions in a singularity

Consider the linear differential operator

$$L = L_r \delta^r + \cdots + L_0,$$

where $L_0, \ldots, L_r \in \mathbb{K}[z]$ are polynomials in z and δ denotes the derivation $z\frac{\partial}{\partial z}$. The interest of using δ instead of $\frac{\partial}{\partial z}$ is that δ preserves the valuation (when the valuation is non zero). We will study the singular behaviour of the solutions to

$$Lf = 0, \tag{3.1}$$

near $z = 0$; clearly, the study near other singularities is similar, modulo a translation. Throughout this section, we assume that at least one of the L_i is not divisible by z.

In general, the homogeneous differential equation (3.1) does not necessarily have r linearly independent power series solutions. But it is a well known fact [(Fabry, 1885; Poincaré, 1886; Birkhoff, 1909; Birkhoff, 1913; Ince, 1926; Turrittin, 1963; Wasow, 1967)] that a complete basis of *transseries* solutions (i.e. generalized series which involve logarithms and exponentials in a recursive way) can always be found and computed [(van Hoeij, 1997; van Hoeij, 1996; Della Dora *et al.*, 1982)]. More precisely, for some finite algebraic extension $\hat{\mathbb{K}}$ of $\mathbb{K}$, there exists a basis of cardinal r of formal solutions f of the form

$$f \in \hat{\mathbb{K}}[\log z]_t[[\sqrt[p]{z}]] z^\alpha e^{P(1/\sqrt[p]{z})}. \tag{3.2}$$

Here $\alpha \in \hat{\mathbb{K}}$, P is a polynomial with coefficients in $\hat{\mathbb{K}}$ and no constant term, and $\hat{\mathbb{K}}[\log z]_t$ stands for the set of polynomials in $\log z$ over $\hat{\mathbb{K}}$ of degrees strictly less than t. We call $e^{P(1/\sqrt[p]{z})}$ the *purely exponential part* of f.

There are several algorithms to compute all triples (p, α, P) for which solutions of the form (3.2), with $f \asymp (\log z)^i z^\alpha e^{P(1/\sqrt[p]{z})}$ (for some i), exist (Della Dora *et al.*, 1982; van Hoeij, 1997; van Hoeij, 1996; van der Hoeven, 1997). Let us call such a triple (p, α, P) *admissible*†, if there are no other such triples of the form $(pq, \alpha - \frac{\beta}{pq}, P \circ z^q)$ with $q, \beta \in \mathbb{N}^*$. In order to find all solutions to (3.1), it then suffices to solve this equation in $\hat{\mathbb{K}}[\log z][[\sqrt[p]{z}]] z^\alpha e^{P(1/\sqrt[p]{z})}$ for all admissible triples (p, α, P).

In this section we shall concentrate on how to find these solutions for a fixed admissible triple (p, α, P). In section 3.1 we first show that it suffices to consider the case when $p = 1, \alpha = 0$ and $P = 0$. This reduces the general problem to finding all solutions of the form

$$f = f_0 + \cdots + \frac{1}{(t-1)!} f_{t-1} \log^{t-1} z, \tag{3.3}$$

to (3.1), where $f_0, \ldots, f_{t-1}$ are power series in z. In section 3.2, we establish recurrence relations for the coefficients of these power series. We conclude in section 3.3.

† The definition of admissible triples may seem a bit technical. It is motivated by the observations that

$$\hat{\mathbb{K}}[\log z]_t[[\sqrt[p]{z}]] z^{\alpha-\beta} e^{P(1/\sqrt[p]{z})} \subseteq \hat{\mathbb{K}}[\log z]_t[[\sqrt[p]{z}]] z^\alpha e^{P(1/\sqrt[p]{z})}$$

and

$$\hat{\mathbb{K}}[\log z]_t[[\sqrt[pq]{z}]] z^\alpha e^{P(1/\sqrt[pq]{z})} \subseteq \hat{\mathbb{K}}[\log z]_t[[\sqrt[p]{z}]] z^\alpha e^{P(1/\sqrt[p]{z})}$$

for all $\beta, q \in \mathbb{N}^*$.

3.1. Reduction to the case $p=1, \alpha=0$ and $P=0$

Consider the problem of finding the solutions (3.2) to $Lf=0$, for fixed p, α and P. We will first reduce this problem to the case when $p=1$. Given a linear differential operator $L = L_r \delta^r + \cdots + L_0$ with coefficients in $\mathbb{K}(z)$, there exists a unique linear differential operator $L_{\circ z^p}$ such that

$$L_{\circ z^p}(f \circ z^p) = (Lf) \circ z^p,$$

for all series $f \in \hat{\mathbb{K}}[\log z][[\sqrt[p]{z}]]$. The coefficients of $L_{\circ z^p}$ are given explicitly by

$$L_{\circ z^p, i} = \frac{L_i \circ z^p}{p^i},$$

whence they belong to $K(z^p)$. Now solving the equation $Lf=0$ with

$$f \in \hat{\mathbb{K}}[\log z][[\sqrt[p]{z}]] z^\alpha e^{P(1/\sqrt[p]{z})}$$

is equivalent to solving the equation $L_{\circ z^p}(f \circ z^p) = 0$ with

$$f \circ z^p \in \hat{\mathbb{K}}[\log z][[z]] z^{p\alpha} e^{P(1/z)}.$$

This reduces the general problem to the case when $p=1$.

In a similar fashion, the general case with $p=1$ reduces to the case $p=1, \alpha=0$ and $P=0$: given a linear differential operator $L = L_r \delta^r + \cdots + L_0$ and a transseries φ (below, we will actually take $\varphi = z^\alpha e^{P(z^{-1})}$), there exists a unique linear differential operator $L_{\times\varphi} = L_{\times\varphi,r} \delta^r + \cdots + L_{\times\varphi,0}$, such that

$$L_{\times\varphi} f = L(\varphi f)$$

for all f. We call $L_{\times\varphi}$ a *multiplicative conjugate* of L. Its coefficients are given explicitly by

$$L_{\times\varphi,i} = \sum_{j=i}^{r} \binom{j}{i} L_j \delta^{j-i} \varphi.$$

Now letting $\varphi = z^\alpha e^{P(z^{-1})}$, we observe that the coefficients of $L_{\times\varphi}$ are rational functions in $\hat{\mathbb{K}}(z)$ multiplied by φ. Since solving $Lf=0$ for $f \in \hat{\mathbb{K}}[\log z][[z]] z^\alpha e^{P(z^{-1})}$ is equivalent to solving $L_{\times\varphi}(h/\varphi) = 0$ for $f/\varphi \in \hat{\mathbb{K}}[\log z][[z]]$, we reduced our initial problem to the case when $p=1, \alpha=0$ and $P=0$.

3.2. Recurrence relations

In this section, we will give recurrence relations for the coefficients of the f_i from (3.3). Given a linear differential operator L, we will denote by μ_L the polynomial

$$\mu_L(k) = L_{r,0} k^r + \cdots + L_{0,0},$$

in k, where $L_{i,0}$ stands for the constant term of L_i. We also denote by L' the "derivative" of L:

$$L' = r L_r \delta^{r-1} + \cdots + 2 L_2 \delta + L_1.$$

Notice that $\mu_{L'} = (\mu_L)'$.

3.2.1. Extraction of coefficients

The action of L on $f = f_0 + \cdots + \frac{1}{(t-1)!} f_{t-1} \log^{t-1} z$ is expressed conveniently using the successive "derivatives" of L:

$$\begin{aligned} Lf &= Lf_0 + \cdots + \frac{1}{(t-1)!}(Lf_{t-1}) \log^{t-1} z \\ &\vdots \\ &+ \frac{1}{i!}\left(L^{(i)} f_i + \cdots + \frac{1}{(t-1-i)!}(L^{(i)} f_{t-1}) \log^{t-1-i} z\right) \\ &\vdots \\ &+ \frac{1}{(t-1)!} L^{(t-1)} f_{t-1}. \end{aligned}$$

Hence, the equation $Lf = 0$ yields the equations

$$\begin{array}{l} Lf_{t-1} = 0; \\ Lf_{t-2} + L' f_{t-1} = 0; \\ \vdots \\ Lf_0 + L' f_1 + \frac{1}{2} L'' f_2 + \cdots + \frac{1}{(t-1)!} L^{(t-1)} f_{t-1} = 0 \end{array} \tag{3.4}$$

for the f_i.

Let us now extract the k-th Taylor coefficients of these relations using the rules

$$(\delta g)_k = k g_k; \tag{3.5}$$

$$(z g)_k = g_{k-1}. \tag{3.6}$$

These rules imply

$$\begin{array}{lcl} (Lg)_k &=& Q_0(k) g_k + \cdots + Q_q(k) g_{k-q}; \\ (L' g)_k &=& Q_0'(k) g_k + \cdots + Q_q'(k) g_{k-q}; \\ & \vdots & \\ (L^{(t-1)} g)_k &=& Q_0^{(t-1)}(k) g_k + \cdots + Q_q^{(t-1)}(k) g_{k-q}, \end{array} \tag{3.7}$$

for certain polynomials $Q_0, \ldots, Q_q \in \mathbb{K}[k]$ with $Q_0 = \mu_L$. Of course, we understand that the k-th coefficient of a power series vanishes, whenever $k \neq \mathbb{N}$.

3.2.2. The generic case

Combination of (3.4) and (3.7) yields

$$\sum_{l=0}^{q} \sum_{j=i}^{t-1} \frac{Q_l^{(j-i)}(k)}{(j-i)!} f_{j,k-l} = 0, \tag{3.8}$$

for $0 \leqslant i \leqslant t-1$. For "generic" k, we have $Q_0(k) = \mu_L(k) \neq 0$. Then the relations (3.8) become

$$f_{i,k} = \frac{-1}{Q_0(k)} \left(\sum_{j=i+1}^{t-1} \frac{Q_0^{(j-i)}(k)}{(j-i)!} f_{j,k} + \sum_{l=1}^{q} \sum_{j=i}^{t-1} \frac{Q_l^{(j-i)}(k)}{(j-i)!} f_{j,k-l} \right). \tag{3.9}$$

Taking successive values $t-1,\ldots,0$ for i, we can interpret (3.9) as recurrence relations for $f_{t-1,k},\ldots,f_{0,k}$ in terms of previous coefficients $f_{j,k-l}$ with $l>0$. Denoting by F_k the column vector with tq entries

$$f_{0,k},\ldots,f_{0,k-q+1},\ldots,f_{t-1,k},\ldots,f_{t-1,k-q+1},$$

these relations can also be written as a matrix relation

$$F_k = A_k F_{k-1}, \tag{3.10}$$

where the entries of A_k are rational functions in $\mathbb{K}(k)$.

3.2.3. The degenerate case

Assume now that we are in the degenerate case where k is a zero of μ_L of multiplicity $\nu_k>0$. Then the system of equations (3.8) becomes overdetermined and does not necessarily admit a solution. The degenerate case corresponds to the situation when higher powers of $\log z$ are needed in order to express the solutions to $Lf=0$.

Nevertheless, we will now show that, if $f_{i,k-l}=0$ for all $i\geqslant t-\nu_k$ and $l>0$, then the system of equations (3.8) again admits a natural solution F_k of the form (3.10). The condition that $f_{i,k-l}=0$ for all $i\geqslant t-\nu_k$ and $l>0$ corresponds to assuming that t was taken sufficiently large; indeed, in section 3.3.1, we will show how to choose such a t, so that all solutions in $\hat{\mathbb{K}}[\log z][[z]]$ to $Lf=0$ are actually in $\mathbb{K}[\log z]_t[[z]]$.

So assume that k is a zero of μ_L of multiplicity $\nu_k>0$ and assume that $f_{i,k-l}=0$ for all $i\geqslant t-\nu_k$ and $l>0$. Then the equations (3.8) trivially hold for $t-\nu_k\leqslant i<t$, independently of the values of $f_{t-1,k},\ldots,f_{t-\nu_k,k}$. For $0\leqslant i<t-\nu_k$, we obtain the relations

$$f_{i+\nu_k,k} = \frac{-(\nu_k!)}{Q_0^{(\nu_k)}(k)}\left(\sum_{j=i+\nu_k+1}^{t-1}\frac{Q_0^{(j-i)}(k)}{(j-i)!}f_{j,k} + \sum_{l=1}^{q}\sum_{j=i}^{t-1}\frac{Q_l^{(j-i)}(k)}{(j-i)!}f_{j,k-l}\right). \tag{3.11}$$

Taking successive values $t-\nu_k-1,\ldots,0$ for i, we can again interpret (3.11) as recurrence relations for $f_{t-1,k},\ldots,f_{\nu_k,k}$ in terms of previous coefficients $f_{j,k-l}$ with $l>0$.

Finally, since the equations (3.8) do not involve $f_{\nu_k-1,k},\ldots,f_{0,k}$, these coefficients can be chosen arbitrarily. For our purpose in section 3.3.1 of finding "canonical" solutions to $Lf=0$, it is convenient to take $f_{\nu_k-1,k}=\cdots=f_{0,k}=0$. Then the recurrence relations (3.11) can again be rewritten as a matrix relation

$$F_k = A_k F_{k-1}, \tag{3.12}$$

where the entries of A_k are rational functions in $\mathbb{K}(k)$, and where the rows which correspond to the entries $f_{\nu_k-1,k},\ldots,f_{0,k}$ of F_k are taken to be zero.

Remark 3.1. Of course, all solutions to $Lf=0$ of the form (3.3) do not necessarily satisfy the recurrence relation (3.12), but the "canonical" solutions that we will construct now do satisfy it (for all but one k, which corresponds to the initial condition).

3.3. Solving $Lf = 0$ in the ring of logarithmic power series

3.3.1. A canonical basis of solutions

Assume that $(f_{d,n} \log^d z + \cdots + f_{0,n})z^n + o(z^n)$ is a solution to $Lf = 0$ in $\hat{\mathbb{K}}[\log z][[z]]$, with $f_{d,n} \neq 0$. In section 3.2.2 we have shown that the $f_{i,k}$ are linear combinations of the entries of F_{k-1} for non-singular k. Therefore, the dominant exponent n of f in z must be a zero of μ_L. Let ν_n be the multiplicity of this zero. In section 3.2.3 we have shown that the $f_{i,n}$ are linear combinations of the entries of F_{k-1} for $i \geqslant \nu_n$. Therefore, we also must have $d < \nu_n$.

Conversely, let us show how to construct a solution $f = f^{[n,d]}$ to $Lf = 0$ of the form

$$f^{[n,d]} = (\log z)^d z^n + \sum_{k>n} \sum_{i=0}^{t-1} \frac{f_{i,k}}{i!} (\log z)^i z^k,$$

for each couple (n, d) with $d < \nu_n$. Let

$$t = d + 1 + \nu_{n+1} + \nu_{n+2} + \cdots \leqslant r \tag{3.13}$$

and take F_n to be the column vector, whose only non zero entry corresponds to $f_{d,n} = d!$. We take $F_k = 0$ for all $k < n$ and $F_k = A_k F_{k-1}$ for $k > n$, with the notations from sections 3.2.2 and 3.2.3.

We have to show that our choice of t is indeed sufficiently large, such that the condition from section 3.2.3, that $f_{i,k-l} = 0$ for all $i \geqslant t - \nu_k$ and $l > 0$, holds for all k. Actually, by induction over k, we observe that $f_{i,k-l} = 0$ for all $i \geqslant t - \nu_k - \nu_{k+1} - \cdots$ and $l > 0$. Notice also that, by construction, the coefficients of all F_k are actually in $\mathbb{K}$.

We claim that the $f^{[n,d]}$ with $d < \nu_n$ form a basis for the space of solutions to (3.1) in $\hat{\mathbb{K}}[\log z][[z]]$. Let f be such a solution to (3.1) and consider it as a generalized series in z and $(\log z)^{-1}$. If $f = 0$ then we have nothing to prove. Otherwise, we may write $f = c_1 (\log z)^{d_1} z^{n_1} + o((\log z)^{d_1} z^{n_1})$ and we already observed that $0 \leqslant d_1 < \nu_{d_1}$. Hence, $\tilde{f} = f - c_1 f^{[n_1,d_1]}$ is again a solution to (3.1), which is either zero, or it has an asymptotically smaller dominant term. Repeating the argument, we find an expression for f as a finite linear combination of the $f^{[n,d]}$, since there are only a finite number of $f^{[n,d]}$. We have proved:

Theorem 3.1. *The equation $Lf = 0$ for f in $\hat{\mathbb{K}}[\log z][[z]]$ has a basis of solutions in $\mathbb{K}[\log z]_t[[z]]$, with t as in (3.13). In particular, the $f^{[n,d]}$ with $d < \nu_n$ form such a basis.*

We will call the basis formed by the $f^{[n,d]}$ the *canonical basis* of solutions to $Lf = 0$ in $\hat{\mathbb{K}}[\log z][[z]]$. More generally, for each admissible triple (p, α, P), we construct a basis of *canonical solutions* $f^{[n,d]}_{(p,\alpha,P)}$ of the form (3.2) to (3.1): we first reduce to the case when $p = 1, \alpha = 0$ and $P = 0$ as described in section 3.1, we next take the canonical solutions to the obtained equation, which finally yield the desired canonical solutions when translating back. The collection of all these canonical solutions for the different admissible triples forms a basis of r formal solutions to (3.2) — the basis of canonical solutions.

Remark 3.2. The basis of canonical solutions coincides with the basis constructed in sections 4.5 and 4.8.3 of (van der Hoeven, 1997). The first explicit constructions of fundamental systems of solutions occur in (Frobenius, 1873; Ince, 1926). Computationally simpler fundamental systems of solutions were first given in (van Hoeij, 1997) and (van der Hoeven, 1997). In all cases, the dominant monomials of the basis elements are pairwise distinct, so that they differ only by triangular linear transformations. However, our "canonical bases" are the most intrinsic ones from the asymptotic point of view. Indeed, in (van der Hoeven, 1997) we prove some characteristic properties of such bases, which justify our terminology.

3.3.2. Convergence and fast evaluation of the canonical solutions

Let $f = f^{[n,d]}$ be one of the canonical solutions and adopt the notations from section 3.2. We have shown that the recurrence relation satisfied by the coefficients of f can be expressed by matrix relations

$$F_k = A_k F_{k-1},$$

with an initial condition at $k = n$. Now looking at the recurrence relations (3.8), we observe that A_k tends to a finite limit when $k \to \infty$, if and only if $\deg Q_0 \geqslant \deg Q_i$, for all i. This is again equivalent to the condition that the constant term $L_{r,0}$ of L_r does not vanish. If this is the case, the operator L is said to be *regular*. The following generalization of theorem 2.1 is the effective version of a well-known theorem in (Frobenius, 1873).

Theorem 3.2. *Assume that L is regular. Then there exists an algorithm which computes $B, \mu > 0$, such that each entry of F_k is bounded by $B\mu^k$ for all k.*

Proof. For all but a finite number of "singular" k, the coefficients of A_k are given by rational functions in $\mathbb{K}(k)$. Furthermore, A_k tends to a finite limit A_∞ for $k \to \infty$, since L is regular. Therefore, the algorithm from section 2.1 (i.e. the algorithm **B** from section 2.2 in (van der Hoeven, 1999)) is easily adapted to our slightly more general situation: with the notations from (van der Hoeven, 1999) (where N_k plays the role of A_k), it suffices to require k_0 to be larger than any singular value of k. □

Remark 3.3. With the notations from (van der Hoeven, 1999), it may be necessary to take k_0 quite large. In practice, it is therefore preferable not to estimate the product $N_{k_0-1} \cdots N_0$ by mere evaluation. Instead, we recommend to treat apart those N_k (with $0 \leqslant k < k_0$) for which the estimation $||E_k|| \leqslant \varepsilon$ already holds, and the remaining ones (for which $|\mu_L(k)|$ is small or zero).

The binary splitting method from section 2.2 can also again be used to evaluate

$$F_0 + F_1 z + \cdots + F_k z^k = (Id + (A_1 z) + \cdots + (A_k z) \cdots (A_1 z)) F_0,$$

for sufficiently small values of z in $\mathbb{K}$. The bitwise complexity of this algorithm is the same as before, again due to the fact that the entries of A_k are rational functions in k with exceptional values in the finite number of poles. In particular, analogues of theorems 2.2 and 2.3 hold. Combined with theorem 3.2, this yields

Theorem 3.3. *Assume that L is regular and let μ be as in theorem 3.2. Assume that $\mathbb{K}$ is an algebraic number field and let $z \in \mathbb{K}$ be such that $|z| \leqslant \lambda/\mu$, for fixed $\lambda < 1$.*

Then there exists an algorithm which simultaneously computes $f_0(z), \ldots, f_{t-1}(z)$ up to n decimal digits in time $O(M(n \log n(\log n + \mathtt{size}(z))))$, uniformly in z. □

More generally, for each admissible triple (p, α, P) reduction to the case $p = 1, \alpha = 0$ and $P = 0$ yields an equation

$$(L_r^{(p,\alpha,P)} \delta^r + \cdots + L_0^{(p,\alpha,P)}) f = 0,$$

which has to be solved in $\hat{\mathbb{K}}[\log z][[z]]$. If $L_{r,0}^{(p,\alpha,P)} \neq 0$ for all admissible triples (p, α, P), then we will say that L is *quasi-regular*.

Proposition 3.1. *L is quasi-regular, if and only if the canonical solutions to $Lf = 0$ all have the same purely exponential part.*

Proof. The operator L is regular if and only if μ_L admits r zeros in $\mathbb{C}$, when counted with multiplicities. Furthermore, such a zero λ has multiplicity ν, if and only if there exist exactly ν canonical basis elements with dominant terms of the form $z^\lambda \log^j z$, namely $z^\lambda, \ldots, z^\lambda \log^{\nu-1} z$ (indeed, this follows from theorem 3.1 for a multiplicative conjugation $L_{\times z^\alpha}$ of L, for which $(1, 0, 0)$ is an admissible triple; notice that $\mu_L(\lambda) = \mu_{L_{\times z^\alpha}}(\lambda - \alpha)$). Hence, L is regular if and only if there exist r canonical basis elements with purely exponential part 1.

Now assume that L is general and let f be a canonical basis element with purely exponential part $e^{P(1/\sqrt[p]{z})}$. After a multiplicative conjugation with $e^{P(1/\sqrt[p]{z})}$ and a substitution $z \mapsto z^p$, we obtain an operator $\tilde{L}$, which is regular, if and only if L is quasi-regular. Now each canonical solution to $Lf = 0$ with purely exponential part $e^{P(1/\sqrt[p]{z})}$ corresponds to a unique canonical solution to $\tilde{L}\tilde{f} = 0$ with purely exponential part 1. Hence, L is quasi-regular, if and only if there exist r canonical basis elements with purely exponential part $e^{P(1/\sqrt[p]{z})}$. □

3.4. A worked example

Consider the equation

$$z^3 \delta^3 g + (3z^2 - 2z^3)\delta^2 g + (3z - 7z^2)\delta g + (1 - 5z + 3z^2 - z^4) g = 0. \tag{3.14}$$

It can be shown that $(1, 0, \frac{1}{z})$ is the only admissible triple associated to this equation. Hence, there exists a fundamental system of solutions to (3.14) in $\mathbb{Q}[[z]][\log z] e^{1/z}$. The multiplicative conjugation

$$g = f e^{1/z},$$

transforms the equation (3.14) into

$$\delta^3 f - 2\delta^2 f - zf = 0. \tag{3.15}$$

In a first stage, let us search for power series solutions of this equation. Using the rules (3.5) and (3.6), we obtain the following recurrence relation for the coefficients of f:

$$f_n = \frac{1}{n^2(n-2)} f_{n-1}. \tag{3.16}$$

Taking $f_0 = f_1 = 0$ and $f_2 = 1$, this recurrence relation enables us to compute $f_3, f_4, \ldots$. However, if $f_1 \neq 0$, then the recurrence relation will not be valid for $n = 2$, and we will need to introduce logarithms into the solution.

In general, we therefore have to seek for solutions of the form

$$f = f_0 + f_1 \log z + \tfrac{1}{2} f_2 \log^2 z,$$

with $f_0, f_1, f_2 \in C[[z]]$. For $n \in \mathbb{N} \backslash \{0, 2\}$, the recurrence relation (3.10), which replaces (3.16), becomes

$$\begin{pmatrix} f_{0,n} \\ f_{1,n} \\ f_{2,n} \end{pmatrix} = \begin{pmatrix} \frac{1}{n^2(n-2)} & \frac{-3n+4}{n^3(n-2)^2} & \frac{6n^2-16n+12}{n^4(n-2)^3} \\ 0 & \frac{1}{n^2(n-2)} & \frac{-3n+4}{n^3(n-2)^2} \\ 0 & 0 & \frac{1}{n^2(n-2)} \end{pmatrix} \begin{pmatrix} f_{0,n-1} \\ f_{1,n-1} \\ f_{2,n-1} \end{pmatrix}. \tag{3.17}$$

In the degenerate case when $n = 2$, the equation (3.12) becomes

$$\begin{pmatrix} f_{0,n} \\ f_{1,n} \\ f_{2,n} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{3n^2-4n} & \frac{3-2n}{(3n^3-4n)^2} & 0 \\ 0 & \frac{1}{3n^2-4n} & 0 \end{pmatrix} \begin{pmatrix} f_{0,n-1} \\ f_{1,n-1} \\ f_{2,n-1} \end{pmatrix}, \tag{3.18}$$

where we assume that $f_{2,n-1} = 0$. Now the canonical basis of solutions is given by

$$\begin{aligned} f^{[0,1]} &= \log z + (-\log z - 1) z + (-\tfrac{1}{4} \log^2 z - \tfrac{3}{16} \log z) z^2 + \\ &\quad (-\tfrac{1}{36} \log^2 z + \tfrac{11}{432} \log z - \tfrac{1}{48}) z^3 + \cdots \\ f^{[0,0]} &= 1 - z - \tfrac{1}{4} z^2 \log z + (-\tfrac{1}{36} \log z + \tfrac{5}{108}) z^3 + \cdots \\ f^{[2,0]} &= z^2 + \tfrac{1}{9} z^3 + \tfrac{1}{288} z^4 + \tfrac{1}{21600} z^5 + \cdots. \end{aligned}$$

The corresponding initial conditions are $(f_{0,0}, f_{1,0}, f_{2,0}) = (0, 1, 0)$, $(f_{0,0}, f_{1,0}, f_{2,0}) = (1, 0, 0)$, resp. $(f_{0,2}, f_{1,2}, f_{2,2}) = (1, 0, 0)$. The basis elements $f^{[0,1]}$, $f^{[0,0]}$ and $f^{[2,0]}$ may be evaluated fast up till any precision using the dichotomic algorithm from section 2.2.

4. Singular transition matrices

4.1. Extended Riemann surfaces

In what follows, we would like to consider initial conditions in singularities. Therefore, it is convenient to extend the Riemann surface of f with points above the singularities. In this section, we give an abstract construction of this extension in the case of a "Riemann surface with only isolated singularities".

In the sequel, a *Riemann surface above* $\mathbb{C}$ is a non empty connected separated topological space $\mathcal{R}$, together with a continuous projection $\pi : \mathcal{R} \to \mathbb{C}$, such that for each point $\zeta \in \mathcal{R}$, there exists a neighbourhood U of ζ, such that π restricted to U is a homeomorphism and maps U onto an open disk in $\mathbb{C}$. A *broken line path* on $\mathcal{R}$ is a continuous mapping $\varphi : [0, 1] \to \mathcal{R}$, such that $\pi \circ \varphi$ is a broken line path in $\mathbb{C}$. The endpoint $\varphi(1)$ is uniquely determined by $\varphi(0)$ and $\pi \circ \varphi$. Therefore, given a base point $\zeta \in \mathcal{R}$, other points $\zeta' \in \mathcal{R}$ are conveniently represented by broken line paths $\pi(\zeta) = z_0 \to \cdots \to z_n = \pi(\zeta')$ on $\mathbb{C}$.

Fix a base point ζ on $\mathcal{R}$, where all broken line paths start. An *open-ended broken line path* on $\mathcal{R}$ is a continuous mapping $\varphi : [0, 1[\to \mathcal{R}$, such that $\pi \circ \varphi$ is the restriction to $[0, 1[$ of a broken line path ψ on $\mathbb{C}$. We say that $\mathcal{R}$ has a *singularity* at "the end of φ", if ψ cannot be lifted back into a path on $\mathcal{R}$ starting at ζ. We say that $\mathcal{R}$ has an *isolated* *singularity* at "the end of φ", if there exists an open disk $D \subseteq \mathbb{C}$ with center $\psi(1)$, such that for all sufficiently small $\varepsilon > 0$ and any broken line path ξ in $D \backslash \{\psi(1)\}$ starting at $\psi(1-\varepsilon)$, the path $t \mapsto \psi(t(1-\varepsilon))$ composed with ξ can be lifted back to $\mathcal{R}$. We say that $\mathcal{R}$ *has only isolated singularities*, if this is the case for all open-ended broken line paths on $\mathcal{R}$.

Assume that $\mathcal{R}$ has only isolated singularities and let ζ be a base point for $\mathcal{R}$. Consider the set $\mathcal{P}$ of open-ended line paths φ on $\mathcal{R}$, such that $\pi(\varphi(1-t)) = a + t$ for all $t \in]0, \varepsilon]$ and some $a \in \mathbb{C}$ and $\varepsilon > 0$. Two paths φ, ψ in $\mathcal{P}$ are said to be equivalent, if there exists an $\varepsilon > 0$, such that $\varphi(1-t) = \psi(1-t)$ for all $0 < t < \varepsilon$. Then the set $\hat{\mathcal{R}} = \mathcal{P}/\sim$ of paths in $\mathcal{P}$ modulo equivalence is called the *singular extension* of $\mathcal{R}$. The non-singular open-ended line paths φ can be extended into broken line paths $\hat{\varphi}$ and correspond to the usual points $\hat{\varphi}(1)$ in $\mathcal{R}$ (conversely, each point on $\mathcal{R}$ can be represented by a broken line path φ; without loss of generality, we may choose $\varphi \in \mathcal{P}$, by deforming it in the neighbourhood of $\psi(1)$). The singular open-ended line paths $\varphi \in \mathcal{P}$ correspond to new, singular, points in $\hat{\mathcal{R}} \backslash \mathcal{R}$. We have a natural projection $\hat{\pi} : \hat{\mathcal{R}} \to \mathbb{C}$, which extends π: if φ is a broken line path in $\mathcal{P}$, such that $\pi \circ \varphi$ is the restriction of ψ to $[0,1[$, then we take $\hat{\pi}(\overline{\varphi}) = \psi(1)$, where $\overline{\varphi}$ denotes the equivalence class of φ.

Let $\hat{\zeta}$ be a point on $\hat{\mathcal{R}}$, which is represented by $\varphi : [0,1[\to \mathcal{R}$. For all $\varepsilon > 0$ sufficiently small, we have $\pi(\varphi(1-\varepsilon)) = \hat{\pi}(\hat{\zeta}) + \varepsilon$. For such ε, we set $\hat{\zeta} + \varepsilon = \varphi(1-\varepsilon)$. Since $\mathcal{R}$ has only isolated singularities, for all sufficiently small $\varepsilon > 0$ and $\alpha \in \mathbb{R}$, the path in $\mathbb{C}$, which starts at $\hat{\pi}(\hat{\zeta}) + \varepsilon$ and which turns counterclockwise around $\pi(\hat{\zeta})$ by an angle α, can be lifted back into a path $\varphi_{\varepsilon,\alpha}$ on $\mathcal{R}$, which starts at $\hat{\zeta} + \varepsilon$. We define $\hat{\zeta} + \varepsilon e^{i\alpha} = \varphi_{\varepsilon,\alpha}(1)$. Let $n \in \mathbb{N}^* \cup \{\infty\}$ be smallest, such that $\hat{\zeta} + \varepsilon e^{i(\alpha + 2n\pi)} = \hat{\zeta} + \varepsilon e^{i\alpha}$ for all sufficiently small $\varepsilon > 0$ and $\alpha \in \mathbb{R}$. If $n = 1$, then we say that $\hat{\zeta}$ is a *removable singularity*. If $1 < n < \infty$, then $\hat{\zeta}$ is an *algebraic singularity*, if $n = \infty$, $\hat{\zeta}$ is said to be a *logarithmic singularity*.

4.2. Singular transition matrices

Throughout this section, we denote by $f^{[0]}, \ldots, f^{[r-1]}$ the r linearly independent canonical formal solutions to (3.1) in the neighbourhood of 0. We have shown in the previous section that we may write

$$f^{[i]}(z) = [f_0^{[i]}(\sqrt[p_i]{z}) + \cdots + \frac{1}{(t_i - 1)!} f_{t_i - 1}^{[i]}(\sqrt[p_i]{z}) \log^{t_i - 1} z] z^{\alpha_i} e^{P_i(1/\sqrt[p_i]{z})}, \tag{4.1}$$

for certain p_i, α_i, P_i, t_i and power series $f_0^{[i]}, \ldots, f_{t_i-1}^{[i]} \in \hat{\mathbb{K}}[[\sqrt[p_i]{z}]]$.

4.2.1. Singular transition matrices in the convergent case

Let $\mathcal{R}$ denote the Riemann surface† of the solutions to (3.1). We assume that we have fixed a determination of $\log(z - \omega)$ on $\mathcal{R}$ for each $\omega \in \Omega$. Since the only singularities of solutions to the equation (3.1) are above the zeros of P_r (recall that P_r is the leading coefficient in (1.1), which is obtained from (3.1) by rewriting L as a linear differential operator in $\frac{\partial}{\partial z}$), $\mathcal{R}$ has only isolated singularities. Therefore, we may extend the Riemann

† This surface is obtained by considering the universal covering surface $\mathcal{U}$ above $\mathbb{C} \backslash \Omega$. The solutions to (3.1) are clearly defined on $\mathcal{U}$. We define $\zeta_1, \zeta_2 \in \mathcal{U}$ to be equivalent, if their projections on $\mathbb{C}$ coincide and if all solutions to (3.1) take the same values in ζ_1 and ζ_2 (i.e. the monodromy matrix between ζ_1 and ζ_2 is the identity). Now we take $\mathcal{R} = \mathcal{U}/\sim$.

surface $\mathcal{R}$ into $\hat{\mathcal{R}}$, as described in the previous section. Now assume that we have a singular point ω on $\hat{\mathcal{R}}$ above 0 and assume that we are in the convergent case, i.e. L is quasi-regular in 0. Then the series $f_j^{[i]}$ actually converge on a small neighbourhood

$$B_\rho(\omega) = \{\omega + \varepsilon e^{i\alpha} | 0 < \varepsilon < \rho, \alpha \in \mathbb{R}\} \subseteq \mathcal{R}$$

of ω, whence the expansions (4.1) yield genuine functions on $B_\rho(\omega)$, which can be extended to $\mathcal{R}$ by analytic continuation.

Now each formal solution $f = a_0 f^{[0]} + \cdots + a_{r-1} f^{[r-1]}$ to (3.1) can be represented by a column vector V with entries $a_0, \ldots, a_{r-1}$. If W is the column vector with entries $f(\zeta), \ldots, \frac{1}{(r-1)!} f^{(r-1)}(\zeta)$ for $\zeta = \omega + \varepsilon e^{i\alpha} \in B_\rho(\omega)$, then W depends linearly on V. We wish to see this dependency as a generalization of (1.2). For this purpose, we encode the projection of the straightline path $\omega \to \zeta$ by $0_\alpha \to z$. Writing $I(\omega)$ and $I(\zeta)$ for V resp. W, the matrix $\Delta_{\omega \to \zeta} = \Delta_{0_\alpha \to z}$ such that

$$I(\zeta) = \Delta_{\omega \to \zeta} I(\omega),$$

is called the *singular transition matrix* along the path $\omega \to \zeta$ or $0_\alpha \to z$.

4.2.2. Singular transition matrices versus transition matrices

Singular transition matrices generalize usual transition matrices: if 0 was actually a non-singular point, then the canonical solutions $f^{[0]}, \ldots, f^{[r-1]}$ are precisely the unique power series solutions whose asymptotic expansions are given by $f^{[i]}(z) = z^i + O(z^r)$, and the entries $a_0, \ldots, a_{r-1}$ indeed coincide with $f(\zeta), \ldots, \frac{1}{(r-1)!} f^{(r-1)}(\zeta)$. The transitivity relation for usual transition matrices generalizes to

$$\Delta_{0_\alpha \to z \rightsquigarrow z'} = \Delta_{z \rightsquigarrow z'} \Delta_{0_\alpha \to z} \tag{4.2}$$

for concatenations of straightline paths $0_\alpha \to z$ on $\hat{\mathcal{R}}$ with arbitrary paths $z \rightsquigarrow z'$ on $\mathcal{R}$, such that $0_\alpha \to z \rightsquigarrow z'$ is homotopic to $0_\beta \to z'$ for some β. Actually, this relation yields a way to extend the definition of $\Delta_{0_\alpha \to z}$ to more general paths $0_\alpha \to z \rightsquigarrow z'$ on $\hat{\mathcal{R}}$.

The functions associated to the formal solutions being linearly independent, the matrices $\Delta_{0_\alpha \rightsquigarrow z}$ are necessarily invertible. It follows that (4.2) also yields a new way to compute transition matrices between points *near* the singularity:

$$\Delta_{z \rightsquigarrow {}_\alpha 0_\alpha \rightsquigarrow z'} = \Delta_{0_\alpha \rightsquigarrow z'} \Delta_{0_\alpha \rightsquigarrow z}^{inv},$$

where $0_\alpha \rightsquigarrow z$ is the inverse path of $z \rightsquigarrow {}_\alpha 0$ and the path $z \rightsquigarrow {}_\alpha 0_\alpha \rightsquigarrow z'$ is homotopic to a path which does not pas through the singularity. It turns out that this way of computing transition matrices can be more efficient than the way described in (van der Hoeven, 1999), when we are close to a singularity.

4.2.3. Singular transition matrices in the divergent case

Of course, it may also happen that one of the $f_j^{[i]}$ diverges: for instance, the divergent series $\sum_k (-1)^k k! z^k$ is formally holonomic. Using the process of resummation and, more generally, multisummation, it is nevertheless generally possible to associate genuine functions to the $f^{[i]}$, which are first defined on a certain sector of $\mathcal{L}_\rho$ and then continued analytically. This means that we can again define singular transition matrices, which now *depend* on the multisummation process used. The actual numerical approximation of such matrices up to any precision will be treated in a forthcoming paper.

4.2.4. Boundary value problems and renormalization

Singular transition matrices are not only useful for computing limits of solutions to (1.1) in singularities. Actually, the knowledge of the singular transition matrix between two possibly singular points can be used to solve more general boundary value problems. As an example, assume that f is a solution of a second order equation like (1.1) with the boundary conditions $f(a) = A$ and $f(b) = B$ for given A and B, where a and b are potentially singular and L is quasi-regular in a and b. Then $f'(a)$ and $f'(b)$ can be computed directly from the transition matrix between a and b, in general.

The singular transition matrices can also be used to renormalize logarithmically divergent integrals of differential equations. For example, assume that all solutions to (3.1) near the origin are in $\mathbb{K}[\log z]_r[[z]]$. Then the mapping

$$\begin{array}{rcl} \rho : \mathbb{K}[\log z]_r[[z]] & \longrightarrow & \mathbb{K}; \\ \sum_{i<r,j} f_{i,j} (\log z)^i z^j & \longmapsto & f_{0,0} \end{array}$$

is a ring homomorphism, called the *renormalization mapping*. Convergent power series are mapped to their values in 0 by this mapping. But ρ also associates values to functions with logarithmic singularities in an additively and multiplicatively coherent way. The singular transition matrices may be used to compute the renormalizations of holonomic functions near regular singular points.

4.2.5. Monodromy using singular transition matrices

Another interesting application of singular transition matrices is in the computation of the monodromy of the differential equation around the singularity in 0. Indeed, let $\varepsilon \circlearrowleft \varepsilon$ be the path starting at a small ε and turning counterclockwise around 0. Then

$$\Delta_{\varepsilon \circlearrowleft \varepsilon} = \Delta_{\varepsilon \to_0 0_{2\pi} \to \varepsilon} = \Delta_{0_0 \to \varepsilon} \Delta_{0_0 \to_{2\pi} 0} \Delta_{\varepsilon \to_0 0}.$$

Here the matrix $\Delta_{0_0 \to_{2\pi} 0}$ is obtained formally as follows. When substituting

$$\begin{array}{rcl} \log z & \longrightarrow & \log z + 2\pi i; \\ z^c & \longrightarrow & e^{2\pi i c} z^c \quad (\forall c), \end{array}$$

in the formal canonical solutions to $Lf = 0$ in 0, we obtain a new basis of formal solutions. In particular, these solutions can be written as linear combinations of the canonical solutions; this linear relationship is expressed by the matrix $\Delta_{0_0 \to_{2\pi} 0}$.

4.2.6. Approximation of singular transition matrices

Assume that $\mathbb{K}$ is an algebraic number field and that L is quasi-regular. In section 3.3.2, we have shown how to efficiently evaluate the $f_j^{[i]}$ up to any desired precision, given a fixed, sufficiently small z. Substituting $\log z$ and the $e^{P_i(1/\sqrt[p_i]{z})}$ by their values in (4.1), this also yields an efficient method to evaluate the $f^{[i]}(z)$ up to any desired precision.

We claim the method from section 3.3.2 can also be used to evaluate the derivatives of the $f^{[i]}(z)$ efficiently and up to any desired precision. In order to see this, we first notice that it suffices to be able to evaluate the derivatives of the $f_j^{[i]}$ as is seen by differentiating (4.1). Now the coefficients F_k of the series

$$I(z) = F_0 + F_1 z + F_2 z^2 + \cdots$$

from section 3.3.2 are related by matrix relations

$$F_k = A_k F_{k-1},$$

where the coefficients of A_k are rational functions in k for all but a finite number of k. The same holds for the coefficients $(k+1)F_{k+1}$ of the series

$$I'(z) = F_1 + 2F_2 z + 3F_3 z^2 + \cdots,$$

which are related by

$$(k+1)F_{k+1} = \frac{k+1}{k} A_{k+1}(kF_k),$$

as well as for the coefficients of the higher derivatives of I. Hence the same method can be used to evaluate these series, which proves our claim. It follows

THEOREM 4.1. *Let $\mathbb{K}$ be an algebraic number field and assume that L is quasi-regular.*

a. There exists an algorithm which computes a radius ρ, such that the expansions (4.1) converge for $|z| \leqslant \rho$.
b. Let $z \in \mathbb{K}$ and α be given with $|z|$ sufficiently small. Then there exists an algorithm to approximate $\Delta_{0_\alpha \rightsquigarrow z}$ up till n decimal digits in time $O(M(n \log^2 n))$. □

4.3. An application to the Riemann zeta function

4.3.1. Generalized polylogarithms and the zeta function

Let $r \geqslant 1$ be an integer. The polylogarithm

$$\mathrm{Li}_r(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^r}$$

is a holonomic function with a singularity in 1. Nevertheless, for $r > 1$, the limit of Li_r in 1 exists and

$$\zeta(r) = \mathrm{Li}_r(1),$$

where ζ is the Riemann zeta function.

More generally, one may consider generalized polylogarithms L_w, where w is a word on the alphabet $\{0,1\}$: we define

$$L_{0^r}(z) = \frac{1}{r!} \log^r z; \tag{4.3}$$

$$L_{0w}(z) = \int_0^z L_w(t) \frac{dt}{t}; \tag{4.4}$$

$$L_{1w}(z) = \int_0^z L_w(t) \frac{dt}{1-t}, \tag{4.5}$$

where (4.3) hold for all $r \in \mathbb{N}$, where (4.4) holds for all words w that contain at least one 1, and where (4.5) holds for all words w. The existence of the integrals is ensured by the fact that $L_w = O(\sqrt{z})$, for all words that contain at least one 1. The L_w indeed generalize the Li, since $\mathrm{Li}_r = L_{0^{r-1}1}$. The limit $\zeta(w) \equiv L_w(1)$ of L_w in 1 exists for words of the form $w = 0v1$.

4.3.2. Convergence of the polylogarithms at $z = 0$

Consider the two differential operators

$$\Omega_0 = \delta, \quad \Omega_1 = \frac{1-z}{z}\delta.$$

For each word $w = w_1 \cdots w_r$ of length r, we define the differential operator of order $r+1$ by

$$\Omega_{w_1 \cdots w_r} = z^{|w|_1} \delta \Omega_{w_r} \cdots \Omega_{w_1}.$$

Here $|w|_1$ denotes the number of ones in w. Since $\delta(z^{-1}\delta) = z^{-1}(\delta^2 - \delta)$, the constant factor $z^{|w|_1}$ in the above definition ensures that the operators Ω_w all have their coefficients in $\mathbb{Q}[z]$. The purpose of this section is to show that Ω_w is regular (whence *a fortiori* quasi-regular) for all w.

We must show that the constant coefficient $\Omega_{w,r+1,0}$ of $\Omega_{w,r+1}$ does not vanish for each word w of length r. This is clear for the empty word. Assume now that the assertion holds for a given word w. Then

$$\Omega_{0w} = \Omega_w \delta,$$

whence $\Omega_{0w,r+2} = \Omega_{w,r+1} \neq 0$, which proves our claim for the word $0w$. As to the word $1w$, we have

$$\Omega_{1w} = z\Omega_w \left(\frac{1-z}{z}\delta\right).$$

Again, we get $\Omega_{1w,r+2} = \Omega_{w,r+1} \neq 0$, as desired.

4.3.3. Convergence of the polylogarithms at $z = 1$

In order to study the polylogarithms at $z = 1$, we introduce operators $\hat{\Omega}_w$, which coincide with the Ω_w up to a constant factor, by setting $\hat{\delta} = (1-z)d/dz$,

$$\hat{\Omega}_0 = \frac{z}{1-z}\hat{\delta}, \quad \hat{\Omega}_1 = \hat{\delta}$$

and

$$\hat{\Omega}_{w_1 \cdots w_r} = (1-z)^{|w|_0} \hat{\delta} \hat{\Omega}_{w_r} \cdots \hat{\Omega}_{w_1}.$$

In a similar way as above, one proves that $\hat{\Omega}_w$ is regular at $z = 1$ for each w.

4.3.4. Fast computation of the $\zeta(w)$

From the previous two sections, we conclude that the Ω_w are regular in both 0 and 1 for all w. Therefore, the "doubly singular" transition matrices $\Delta_{w,0_0 \to \frac{1}{2} \to_{-\pi} 1}$ associated to the equations $\Omega_w f = 0$ can be approximated efficiently by theorem 4.1. In particular, we infer:

THEOREM 4.2. *For each word $w \in \{0,1\}^*$ there exists an algorithm to approximate $\zeta(w)$ up till n decimal digits in time $O(M(n \log^2 n))$.* □

REMARK 4.1. In the particular case of polylogarithms, it is possible to get a more explicit formula for the $\zeta(w)$, by using Chen series (Chen, 1971; Minh and Petitot, 1998; Minh *et al.*, 1998) and exploiting the symmetry of the problem with respect to the transformation $z \leftrightarrow 1 - z$. We will just state the result. Denote $L(z) = \sum_w L_w(z) w$,

$$\hat{L}(z) = \sum_{w_1 \cdots w_r} L_{(1-w_r)\cdots(1-w_1)} w$$

and $Z = \sum_w \zeta_w w$. Then

$$Z = \hat{L}(\tfrac{1}{2}) L(\tfrac{1}{2}).$$

The advantage of this formula is that it improves the dependence of the constant factor of the approximation algorithm on w.

5. Practical computations with holonomic functions

In (van der Hoeven, 1999) and this paper, we have shown how to evaluate holonomic functions efficiently, even near and in singularities, in the case of quasi-regular operators. However, from the computer algebra point of view, several questions were not answered: how to test whether a holonomic constant is zero and, more generally, how to compute a floating point approximation of a holonomic constant in an efficient way? Indeed, the second problem is more general, since the computation of the exponent of a floating point approximation of a constant includes a zero test.

In their full generality, these problems are extremely hard. Nevertheless, in section 5.1, we propose a heuristic which enables us to give solutions to these problems, which we expect to be satisfactory in practice. In section 5.2, we go more deeply into the problem of computing floating point approximations of holonomic constants, by studying polynomial expressions involving holonomic functions near singularities. We will state a uniform complexity result, which is again based on our heuristic.

5.1. The holonomic constants problem

5.1.1. Exp-log constants

Actually, the problem of giving an effective zero test for holonomic constants is already very hard in the case of so called *exp-log constants*, which are constructed from the rationals using the field operations, exponentiation and the logarithm. The best actual result is an effective zero test for exp-log constants under the hypothesis that a difficult number theoretical conjecture (namely Schanuel's conjecture) holds (Richardson, 1997). Nevertheless, no information at all is provided about the efficiency of such a zero test.

Nevertheless, from a practical point of view, it is often a good idea to evaluate the exp-log constant c we want to test for zero up till a certain number of digits, which depends on the size of the exp-log constant, and check whether the result vanishes. The only known straightforward counterexamples in which this strategy fails are constructed by exploiting large cancellations like

$$e^{10^{-10^{10}}} - 1 \approx 0.$$

Nevertheless, this problem can be avoided by restricting the argument x of any subexpression of the form $\exp x$ of c to be bounded by $1/N \leqslant |x| \leqslant N$ for some given constant N. Denoting by $\mathfrak{E}_N$ the set of such exp-log expressions, we conjectured in our PhD (van der Hoeven, 1997) that an exp-log constant $c \in \mathfrak{E}_N$ is either zero, or can be proven to be non zero by evaluating $e^{O(\mathtt{size}(c))}$ digits, where $\mathtt{size}(c)$ denotes the size of c as an expression[†]. Whether this conjecture holds or not, it does provide a reliable heuristic zero-test for exp-log constants and the heuristic remains even reliable when we replace the exponential bound on the number of digits by a smaller one.

5.1.2. Holonomic constants

We will now propose a generalization of the above to the case of holonomic constants. Let $\mathbb{K}$ denote the field of algebraic numbers and let $\mathfrak{F}$ be the class of holonomic functions f over $\mathbb{K}$, with initial conditions in $\mathbb{K}$ at a non-singular point z in $\mathbb{K}$. We consider f as being defined on an open disk with center z. We encode functions $f \in \mathfrak{F}$ by the equations they satisfy and their initial conditions; choosing dense representations, f again has a natural size $\mathtt{size}(f)$. Now consider the class $\mathfrak{H}$ of constant expressions formed from $\mathbb{K}$ by using the field operations, and applying holonomic functions in $\mathfrak{F}$.

In order to state our heuristic, we associate to each constant in $\mathfrak{H}$ its size as an expression in a non conventional way. The size of an integer is the number of its binary digits and the size of $\sqrt{-1}$ is 1. If $c, c' \in \mathfrak{H}$ and $*$ is a field operation, then we take $\mathtt{size}(c * c') = \mathtt{size}(c) + \mathtt{size}(c') + 1$. Given a holonomic function $f \in \mathfrak{F}$ with its initial conditions in z and a constant $c \in \mathfrak{H}$, we finally define

$$\mathtt{size}(f(c)) = \mathtt{size}(f) + \mathtt{size}(c) + \max\left(0, \left\lceil \log \sup_{|u-z| \leqslant |c-z|} |f(u)| \right\rceil\right).$$

Notice that all constants in $\mathbb{K}$ indeed have a size, since the algebraic functions over $\mathbb{Q}$ are holonomic.

An easy structural induction on general expressions $c \in \mathfrak{H}$ shows that

$$|c| \leqslant e^{\mathtt{size}(c)},$$

for all $c \in \mathfrak{H}$. Our heuristic states that we also have some similar lower bound for $|c|$, if $c \neq 0$:

H. For each $c \in \mathfrak{H}$, we have either $c = 0$ or $e^{-H(\mathtt{size}(c))} \leqslant |c| \leqslant e^{\mathtt{size}(c)}$.

The choice of the function H, which is assumed to be positive and non decreasing, is left open for the moment. We conjecture that the heuristic holds for sufficiently large H, such as $H(n) = e^n$ and probably even $H(n) = n^2$. In practice, it will be most convenient to take $H(n) = C$ or $H(n) = Cn$, although the heuristic is false as a mathematical statement for $H(n) = O(n)$. Nevertheless, $H(n) = n$ will rarely fail on practical examples. From a complexity point of view it is also interesting to study the case $H(n) = n \log^C n$.

[†] One has to be careful at this point, since the size function is defined for expressions and not for numbers. For instance, $1 + 1$ is equal to 2 as a number, but not as an expression.

5.1.3. Properties of the size function

It is interesting to study some of the properties of the way we associate sizes to holonomic constants. We first notice that for $x \in \mathfrak{H} \cap \mathbb{R}$, we have

$$\mathtt{size}(e^x) = \mathtt{size}(\exp) + \mathtt{size}(x) + \lceil |x| \rceil.$$

In order to obtain the size for arbitrary $z \in \mathfrak{H}$, we may rewrite the imaginary part $\Im z = 2\pi n + r$ with $n \in \mathbb{Z}$ and $|r| \leqslant \pi$. Then $\mathtt{size}(n) = O(\log(|\Im z| + 1))$ and $\mathtt{size}(r) = O(\mathtt{size}(z))$, since $\log(|\Im z| + 1) = O(\mathtt{size}(z))$. Writing $e^z = e^{\Re z + r\sqrt{-1}}$, we infer

$$\mathtt{size}(e^z) \leqslant |\Re z| + O(\mathtt{size}(z)). \tag{5.1}$$

This formula establishes the link with section 5.1.1: for real x, the smallest exp-log expression in $\mathfrak{E}_N$ which represents e^x also has size $O(|x|)$. As to logarithms, any $z \in \mathfrak{H}^*$ can be rewritten as a product $z = 2^n e^{m\pi\sqrt{-1}/16} z'$ for integers n and $0 \leqslant m < 32$, where $\mathtt{size}(n) = O(|\log |z||) = O(\mathtt{size}(z))$ and $|z' - 1| \leqslant \frac{1}{2}$ with $\mathtt{size}(z') = O(\mathtt{size}(z))$. Using $\log z = n \log 2 + m\pi\sqrt{-1}/16 + \log z'$, it follows that

$$\mathtt{size}(\log z) = O(\mathtt{size}(z)). \tag{5.2}$$

We plan to come back on more properties of the size function in a forthcoming paper.

5.2. Floating point approximations

5.2.1. Introduction

Consider a holonomic function $f(z)$ in the neighbourhood of one of its singularities, say 0. Since the behaviour of f may become exponential, it is not a good idea to approximate $f(z)$ by numbers in $\mathbb{Q}[i]$ for small values of z. For instance, if $f(z) = e^{1/z}$, then the mere representation of $e^{1/10^{-n}}$ up to precision < 1 already necessitates a space of the order 10^n, while the number $z = 10^{-n}$ has size $O(n)$.

In order to store good approximations to numbers like $e^{1/10^{-n}}$ in $O(n)$ space we are therefore lead to the consideration of floating point representations. For us, a real floating point number will consist of a mantissa between 1 and 10 in $\mathbb{Q}$ (whose size depends on the required precision) and an exponent, which is an integer. The size of such a number is the sum of the sizes of the mantissa and the exponent. Also, the number 0 is represented by a special symbol of size one. Complex floating point numbers are represented via their real and imaginary parts.

By "computing a floating point approximation of precision n" of a real number $x \neq 0$, we shall mean the computation of a floating point number $y = M 10^E$, such that $|x - y| \leqslant 10^{E-n}$. Although floating point representation allow us to work efficiently with much larger numbers, a problem with the computation of floating point approximation is zero testing, because of its special representation. More generally, in order to compute a floating point approximation *efficiently*, one needs an efficient algorithm to find the approximate exponent; this may be very hard when subtracting two almost identical quantities, which leads to massive cancellations. In the case of holonomic constants, we will use the heuristic **H** from the previous section to treat this problem.

In the remainder of this section, we will be interested in the following problem: given a polynomial expression φ in holonomic functions $f_1, \ldots, f_n$ admitting a quasi-regular singularity at 0, how to evaluate P efficiently near 0? Now after a change of variable $z \mapsto z^p$, each of the f_i can be written

$$f_i(z) = (f_{i,0} + \cdots + f_{i,t_i-1} \log^{t_i-1}) z^{\alpha_i} e^{P_i(z^{-1})},$$

where the $f_{i,j}$ are convergent series in z, the α_i in $\hat{\mathbb{K}}$ and the P_i polynomials in z^{-1}. Substitution of these expressions in φ and expansion then shows that we may assume without loss of generality that φ is a linear combination of the f_i.

Now the functions $f_{i,j}$ and z^{α_i} have only a polynomial growth near 0, whence they can be approximated in a classical way. Our problem therefore reduces to the question how to efficiently compute a floating point approximation of a linear combination of large exponentials with small coefficients. We will now give such an algorithm, based in our heuristic **H**.

5.2.2. A fast heuristic approximation algorithm

In this section, we show how to evaluate sums of the form

$$u = c_0 e^{z_0} + \cdots + c_{r-1} e^{z_{r-1}} \tag{5.3}$$

in an efficient way, where the c_i and the z_i are small (but not "too small") holonomic constants. The idea of the algorithm is as follows. We first reorder the terms in (5.3), such that

$$\Re z_0 \geqslant \Re z_1 \geqslant \cdots \geqslant \Re z_{r-1}.$$

Usually, $\Re z_0 > \Re z_1$ and e^{z_0} is "huge w.r.t." e^{z_1}. If $c_0 \neq 0$, the heuristic **H** implies that c_0 is "reasonably large", while the numbers $c_1, \ldots, c_r$ are "reasonably small". Consequently, u is "almost equal" to $c_0 e^{z_0}$, of which a floating point approximation is easily obtained. If $c_0 = 0$, the term $c_0 e^{z_0}$ vanishes and we recursively evaluate the sum $c_1 e^{z_1} + \cdots + c_{r-1} e^{z_{r-1}}$. In general, $\Re z_0$ and $\Re z_1$ may be "almost equal", and the rôle of c_0 is replaced by constants of the form

$$c_0 + \cdots + c_{i-1} e^{z_{i-1} - z_0};$$

in this case $\Re z_0, \ldots, \Re z_{i-1}$ are "almost equal", while $\Re z_i$ is "quite smaller" than $\Re z_0$.

The notions of "almost equal" and "quite smaller" in the above discussion depend on the sizes of the c_i and the z_i, as well as the heuristic **H**. In order to make them more precise, we need some more notations. Let $E > 0$ be a constant such that for all z,

$$\mathtt{size}(e^z) \leqslant |\Re z| + E \mathtt{size}(z);$$

the existence of E follows from (5.1). We recursively define functions $H_0, H_1, \ldots$ by $H_0 = 0$ and

$$H_i(N) = H(H_{i-1}(N) + \cdots + H_0(N) + (N+2)(i+E)) + N + \log(2r).$$

For some particular choices of H we have the following asymptotic bounds:

- $H_r(N) = O(N)$, if $H(N) = O(N)$.
- $H_r(N) = O(N \log^{\alpha r} N)$, if $H(N) = O(N \log^{\alpha} N)$.
- $H_r(N) = O(N^{\alpha r})$, if $H(N) = O(N^{\alpha})$.
- $H_r(N) = \exp \overset{r \text{ times}}{\cdots} \exp O(N)$, if $H(N) = O(\exp N)$.

Now consider the following algorithm to evaluate sums of the form (5.3).

Algorithm F. The algorithm computes a floating point approximation up till n decimal digits of

$$u = c_0 e^{z_0} + \cdots + c_{r-1} e^{z_{r-1}},$$

where the c_i and z_i are constants in $\mathbb{C}$, whose sizes are bounded by N.

F0. [Trivial case] If $r = 0$, then return 0.

F1. [Reorder] Reorder indices such that $\Re z_0 \geqslant \cdots \geqslant \Re z_{r-1}$.

F2. [Determine gap] Let i be the minimal index such that

$$\Re(z_0 - z_i) > H_i(N).$$

Take $i = r$ if such an index does not exist.

F3. [Zero test] Test whether the constant

$$\lambda = c_0 + \cdots + c_{i-1} e^{z_{i-1} - z_0}$$

vanishes. If so, apply the algorithm recursively on the sum

$$c_i e^{z_i} + \cdots + c_{r-1} e^{z_{r-1}};$$

otherwise, proceed with the next step.

F4. [Return approximation] Compute an approximation of

$$\lambda' = c_0 + c_1 e^{z_1 - z_0} + \cdots + c_{r-1} e^{z_{r-1} - z_0}$$

in $\mathbb{Q}[i]$ with error $< e^{-H_i(N)} 10^{-n-2}$ and convert it to floating point format; this yields a floating point approximation of λ' up till $n+1$ decimal digits. Also compute a floating point approximation of e^{z_0} up till $n+1$ decimal digits. Return the product with the previous one.

5.2.3. Correctness proof and complexity analysis

Theorem 5.1. *Assume the heuristic* **H**. *Then the algorithm* **F** *is correct and its execution time is bounded by* $O(M((n + H_r(N)) \log^2(n + H_r(N))))$.

Proof. The algorithm clearly terminates. It suffices to prove the correctness in the case when $c_0 + \cdots + c_{i-1} e^{z_{i-1} - z_0}$ does not vanish. By the minimality of the index i, we have

$$\Re(z_0 - z_j) \leqslant H_i(N), \text{ for all } 0 < j < i.$$

In particular, it follows that the size of $c_0 + \cdots + c_{i-1} e^{z_{i-1} - z_0}$, when considered as a constant in $\mathfrak{H}$ is bounded by

$$\texttt{size}(\lambda) \leqslant H_{i-1}(N) + \cdots + H_0(N) + (N + 2)(i + E).$$

Therefore, **H** implies

$$|\lambda| \geqslant e^{\Re z_0 - H(H_{i-1}(N) + \cdots + H_0(N) + (N+2)(i+E))}. \tag{5.4}$$

On the other hand, we have

$$|c_i e^{z_i} + \cdots + c_{r-1} e^{z_{r-1}}| \leqslant r e^{\Re z_i + N}.$$

Since $\Re(z_0 - z_i) > H_i(N)$, it follows by our definition of $H_i(N)$ that

$$|c_i e^{z_i} + \cdots + c_{r-1} e^{z_{r-1}}| \leqslant \tfrac{1}{2} |c_0 e^{z_0} + \cdots + c_{i-1} e^{z_{i-1}}|.$$

In particular, we get upper and lower bounds for u

$$\frac{1}{2} |\lambda| e^{\Re z_0} \leqslant |u| \leqslant \frac{3}{2} |\lambda| e^{\Re z_0}$$

and similarly for λ'

$$\frac{1}{2} |\lambda| \leqslant |\lambda'| \leqslant \frac{3}{2} |\lambda|.$$

Because of the lower bound (5.4) for $|\lambda|$, it follows that an evaluation of λ' up to precision $< e^{-H_i(N)} 10^{-n-2}$ yields $n+1$ decimal digits of its floating point representation. Finally the multiplication of two floating point numbers with precisions of $n+1$ decimal digits indeed yields a floating point number with a precision of n decimal digits.

As to the time complexity bound: the zero test of $c_0 + \cdots + c_{i-1} e^{z_{i-1} - z_0}$ takes a time $O(M(H_i(N) \log^2 H_i(n)))$, while the evaluation of λ' up to precision $< e^{-H_i(N)} 10^{-n-2}$ takes a time $O(M((n + H_i(N)) \log^2 (n + H_i(N))))$. This completes the proof. □

Returning to our problem of the approximation of φ, we observe that $N = O(n)$ for small z. Therefore theorem 5.1 yields

THEOREM 5.2. *Assume that $\mathbb{K}$ is an algebraic number field, assume the heuristic* **H** *and assume that φ is a polynomial expression of holonomic functions $f_1, \ldots, f_n$ over $\mathbb{K}$ with quasi-regular singularities in* 0. *Then there exists a computable constant $\rho > 0$, such that for all $z = \varepsilon e^{i\alpha}$ above $\mathbb{K}$ with $\varepsilon < \rho$, the value of $\varphi(z)$ can be computed up till n decimal digits in time $O(M(H_r(s) \log^2 H_r(s)))$, where $s = n + \mathtt{size}(z)$. This bound is uniform in z.* □

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