Constructing reductions for creative telescoping

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The class of *reduction-based* algorithms was introduced recently as a new approach towards *creative telescoping*. Starting with Hermite reduction of rational functions, various reductions have been introduced for increasingly large classes of holonomic functions. In this paper we show how to construct reductions for general holonomic functions, in the purely differential setting.

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1. Introduction

Let $\mathbb{K}$ be an effective field of characteristic zero and $\phi \in \mathbb{K}[x]$ a non-zero polynomial. Consider the system of differential equations

$$\phi y' = A y, \tag{1}$$

where $A \in \mathbb{K}[x]^{r \times r}$ is an $r \times r$ matrix with entries in $\mathbb{K}[x]$ and y is a column vector of r unknown functions. Notice that any system of differential equations $y' = B y$ with $B \in \mathbb{K}(x)$ can be rewritten in this form by taking ϕ to be a multiple of all denominators.

Let y be a formal solution to (1) and consider the $\mathbb{K}[x, \phi^{-1}]$-module $\mathbb{M}$ of linear combinations $\lambda y = \lambda_{1,1} y_1 + \cdots + \lambda_{1,r} y_r$ where $\lambda \in \mathbb{K}[x, \phi^{-1}]^{1 \times r}$ is a row vector. Then $\mathbb{M}$ has the natural structure of a D-module for the derivation ∂ defined by $(\lambda y)' = (\lambda' + \phi^{-1} \lambda A) y$. A $\mathbb{K}$-linear mapping $[\cdot]: \mathbb{M} \to \mathbb{M}$ is said to be a *reduction* if $[f] - f \in \operatorname{Im} \partial$ for all $f \in \mathbb{M}$. Such a reduction is said to be *confined* if its image is a finite dimensional subspace of $\mathbb{M}$ and *normal* if $[f'] = 0$ for all $f \in \mathbb{M}$.

In this paper, we will show how to construct confined reductions. Such reductions are interesting for their application to creative telescoping [13, 7], as we briefly recall in section 2. The first reduction of this kind is Hermite reduction [2, 4], in which case $A = 0$. The existence of normal confined reductions has been shown in increasingly general cases [12, 6] and most noticeably for Fuchsian differential equations [5]. We refer to [4, 9] for more details and the application to creative telescoping.

Our construction of confined reductions proceeds in two stages. In section 4, we first focus on the $\mathbb{K}[x]$-submodule $\mathbb{M}^\sharp$ of $\mathbb{M}$ of linear combinations λy with $\lambda \in \mathbb{K}[x]^{1 \times r}$. We will construct a $\mathbb{K}$-linear *head reduction* $\lceil \cdot \rceil: \mathbb{M}^\sharp \to \mathbb{M}^\sharp$ such that $\lceil f \rceil - f \in \operatorname{Im} \partial$ and $\deg \lceil f \rceil$ is bounded from above for all $f \in \mathbb{M}^\sharp$. Here we understand that $\deg(\lambda y) := \deg \lambda := \max(\deg \lambda_{1,1}, \ldots, \deg \lambda_{1,r})$ for all $\lambda \in \mathbb{K}[x]^{1 \times r}$. The construction uses a variant of Gaussian elimination that will be described in section 3.

The head reduction may also be regarded as a way to reduce the valuation of f in x^{-1}, at the point at infinity. In section 5 we turn to tail reductions, with the aim to reduce the valuation of f at all other points in $\mathbb{K}$ and its algebraic closure $\hat{\mathbb{K}}$. This is essentially similar to head reduction *via* a change of variables, while allowing ourselves to work in algebraic extensions of $\mathbb{K}$.

In the last section 7, we show how to glue the head reduction and the tail reductions at each of the roots of ϕ together into a global confined reduction on $\mathbb{M}$. Using straightforward linear algebra and suitable valuation bounds, one can further turn this reduction into a normal one, as will be shown in detail in section 7.2.

The valuation bounds that are required in section 7.2 are proved in section 6. In this section we also prove degree and valuation bounds for so called head and tail choppers. The existence of head and tail choppers whose size is polynomial in the size of the original equation makes it possible to derive polynomial bounds for the complexity of creative telescoping: this follows from polynomial bounds for the dimension of $\mathrm{im}\,[\cdot]$ and for the reduction of an elements in $\mathbb{M}$. We intend to work out the details in a forthcoming paper.

2. Creative telescoping

Let $\mathbb{k}$ be an effective subfield of $\mathbb{C}$ and let $\partial_x = \partial / \partial x$ and $\partial_u = \partial / \partial u$ denote the partial derivations with respect to x and u. Consider a system of differential equations

$$\begin{cases} \phi\,\partial_x y &= A\,y \\ \phi\,\partial_u y &= B\,y, \end{cases} \tag{2}$$

where $\phi \in \mathbb{k}[x, u]$ is non-zero and $A, B \in \mathbb{k}[x, u]^{r \times r}$ are such that

$$\partial_x(\phi^{-1} B) + \phi^{-2} A\,B = \partial_u(\phi^{-1} A) + \phi^{-2} B\,A.$$

Setting $\mathbb{K} = \mathbb{k}(u)$, the first part of (2) then becomes of the form (1). Notice that any bivariate holonomic function is an entry of a solution to a system of the form (2).

Let y be a complex analytic solution of the above system of equations and let $\mathbb{M}$ be the $\mathbb{K}[x, \phi^{-1}]$-module generated by the entries of y. Notice that $\mathbb{M}$ is stable under both ∂_x and ∂_u. For any $f = \lambda\,y \in \mathbb{M}$ with $\lambda \in \mathbb{K}[x, \phi^{-1}]^{1 \times r}$ and any non-singular contour $\mathscr{C}$ in $\mathbb{C}$ between two points $\alpha, \beta \in \mathbb{k}$, we may consider the integral

$$F(u) \;=\; \int_{\mathscr{C}} f(x, u)\,\mathrm{d}x,$$

which defines a function in the single variable u. It is natural to ask under which conditions F is a holonomic function and how to compute a differential operator $L \in \mathbb{K}[\partial_u]$ with $L F = 0$.

The idea of *creative telescoping* is to compute a differential operator $K \in \mathbb{K}[\partial_u]$ and a function $\xi = \partial_x \chi$ with $\chi \in \mathbb{M}$ such that

$$K f(x, u) \;=\; \xi(x, u). \tag{3}$$

Integrating over $\mathscr{C}$, we then obtain

$$K F(u) \;=\; \int_{\mathscr{C}} \frac{\partial \chi}{\partial x}(x, u)\,\mathrm{d}x \;=\; \chi(\beta, u) - \chi(\alpha, u).$$

If the contour $\mathscr{C}$ has the property that $\chi(\beta) = \chi(\alpha)$ for all $\chi \in \mathbb{M}$ (where the equality is allowed to hold at the limit if necessary), then $L = K$ yields the desired annihilator with $L F = 0$. In general, we need to multiply K on the left with an annihilator of $\chi(\beta, u) - \chi(\alpha, u)$.

Now assume that we have a computable confined reduction $[\cdot]\colon \mathbb{M} \to \mathbb{M}$. Then the functions in the sequence $[f], [\partial_u f], [\partial_u^2 f], \ldots$ can all be computed and they belong to a finite dimensional $\mathbb{K}$-vector space V. Using linear algebra, that means that we can compute a relation

$$K_0\,[f] + \cdots + K_s\,[\partial_u^s f] = [K_0\,f + \cdots + K_s\,\partial_u^s f] = 0 \tag{4}$$

with $K_0, \ldots, K_s \in \mathbb{K}$. Taking

$$\begin{aligned} K &= K_0 + \cdots + K_s \partial_u^s \\ \xi &= (K f) - [K f] \in \partial_x \mathbb{M}, \end{aligned}$$

we thus obtain (3). If the relation (4) has minimal order s and the reduction $[\cdot]$ is normal, then it can be shown [9] that there exist no relations of the form (3) of order lower than s.

3. Row swept forms

Let $U \in \mathbb{K}^{r \times r}$ be a matrix and denote the i-th row of U by $U_{i,\cdot}$. Assuming that $U_{i,\cdot} \neq 0$, its *leading index* ℓ_i is the smallest index j with $U_{i,j} \neq 0$. We say that U is in *row swept form* if there exists a $k \in \{0, \ldots, r\}$ such that $U_{1,\cdot} \neq 0, \ldots, U_{k,\cdot} \neq 0, U_{k+1,\cdot} = \cdots = U_{r,\cdot} = 0$ and $U_{i',\ell_i} = 0$ for all $i < i' \leqslant k$. Notice that U has rank k in this case.

An invertible matrix $S \in \mathbb{K}^{r \times r}$ such that $S U$ is in row swept form will be called a *row sweaper* for U. We may compute such a matrix S using the routine **RowSweaper** below, which is really a variant of Gaussian elimination. Whenever we apply this routine to a matrix U such that the truncated matrix $\tilde{U}$ with rows $U_{1,\cdot}, \ldots, U_{k,\cdot}, 0, \ldots, 0$ is in row swept form, we notice that these first k rows are left invariant by the row sweaping process. In other words, the returned row sweaper S is of the form $S = \begin{pmatrix} \mathrm{Id}_k & 0 \\ * & * \end{pmatrix}$. If, in addition, the matrix U has rank k, then S is of the form $S = \begin{pmatrix} \mathrm{Id}_k & 0 \\ * & \mathrm{Id}_{r-k} \end{pmatrix}$.

Algorithm RowSweaper(U)

$S := \mathrm{Id}_r$, $R := U$
for i **from** 1 **to** r **do**
  if $R_{i',j} = 0$ for all $i' \geqslant i$ and j **then return** S
  Let $i' > i$ be minimal such that $R_{i',j} \neq 0$ for some j
  Swap the i-th and i'-th rows of S and R
  $v := R_{i,\ell_i}^{-1}$
  for i' **from** $i+1$ **to** r **do**
    $S_{i',\cdot} := S_{i',\cdot} - v R_{i',\ell_i} S_{i,\cdot}$, $R_{i',\cdot} := R_{i',\cdot} - v R_{i',\ell_i} R_{i,\cdot}$
return S

4. Head reduction

4.1. Head choppers

Let $i \in \mathbb{K}$ and $T \in \phi\, \mathbb{K}(i)[x, x^{-1}]^{r \times r}$. We may regard T as a Laurent polynomial with matrix coefficients $T_k \in \mathbb{K}(i)^{r \times r}$:

$$T = \sum_{k \in \mathbb{Z}} T_k x^k. \tag{5}$$

If $T \neq 0$, then we denote $\deg T = \max \{k \in \mathbb{Z} : T_k \neq 0\}$ and $\operatorname{val} T = \min \{k \in \mathbb{Z} : T_k \neq 0\}$. For any $\delta \in \mathbb{Z}$, we also denote $(\Xi^\delta T)(x, i) = x^\delta T(x, i + \delta)$. Setting

$$U = \Upsilon(T) := \phi^{-1} T A + T' + i x^{-1} T, \tag{6}$$

the equation (1) implies

$$(C x^i T y)' = C x^i U y, \tag{7}$$

for any constant matrix $C \in \mathbb{K}(i)^{r\times r}$. The matrix U can also be regarded as a Laurent polynomial with matrix coefficients $U_k \in \mathbb{K}(i)^{r\times r}$. We say that T is a *head chopper* for (1) if $U_{\deg U}$ is an invertible matrix.

PROPOSITION 1. *For all $\delta \in \mathbb{Z}$, we have*

$$\Upsilon(\Xi^\delta T) = \Xi^\delta \Upsilon(T).$$

Proof. Setting $U = \Upsilon(T)$, $\tilde{T} = \Xi^\delta T$ and $\tilde{U} = \Upsilon(\tilde{T})$, we have

$$\begin{aligned}\tilde{U}(x,i) &= \phi^{-1} x^\delta T(x,i+\delta) A + x^\delta T'(x,i+\delta) + \delta x^{\delta-1} T(x,i+\delta) + i x^{\delta-1} T(x,i+\delta)\\ &= x^\delta (\phi^{-1} T(x,i+\delta) A + T'(x,i+\delta) + (i+\delta) x^{-1} T(x,i+\delta))\\ &= x^\delta U(x,i+\delta).\end{aligned}$$

In other words, $\tilde{U} = \Xi^\delta U$. □

PROPOSITION 2. *Assume that $\delta \in \mathbb{Z}$ and that $P \in \mathbb{K}(i)^{r\times r}$ is invertible. Then*

a) T is a head chopper for (1) if and only if $\Xi^\delta T$ is a head chopper for (1).

b) T is a head chopper for (1) if and only if PT is a head chopper for (1).

Proof. Assume that T is a head chopper for (1). Setting $\tilde{T} = \Xi^\delta T$ and $\tilde{U} = \Upsilon(\tilde{T})$, we have $\tilde{U} = \Xi^\delta U$ and $\tilde{U}_{\deg \tilde{U}}(i) = U_{\deg U}(i+\delta)$ is invertible. Similarly, setting $\hat{T} = PT$ and $\hat{U} = \Upsilon(\hat{T})$, we have $\hat{U} = PU$, whence $\hat{U}_{\deg \hat{U}} = PU_{\deg U}$ is invertible. The opposite directions follow by taking $-\delta$ and P^{-1} in the roles of δ and P. □

4.2. Head annihilators

Notice that the equations (5–7) and Proposition 1 generalize to the case when $T \in \phi\, \mathbb{K}(i)[x, x^{-1}]^{n\times r}$ for some arbitrary n. Notice also that $\deg U \leqslant \deg T + \sigma$, where $\sigma := \max(\deg A - \deg \phi, -1)$. Given $d \in \mathbb{Z}$ and $e \in \mathbb{N}$, let

$$\begin{aligned}M_d &= \{T \in \phi\, \mathbb{K}(i)[x, x^{-1}]^{1\times r} : \deg T \leqslant d\}\\ M_{d,e} &= \{T \in M_d : \deg \Upsilon(T) \leqslant d + \sigma - e\}.\end{aligned}$$

It is easy to see that both M_d and $M_{d,e}$ are $\mathbb{K}(i)[\Xi^{-1}]$-modules.

Now consider a matrix $T \in \phi\, \mathbb{K}(i)[x, x^{-1}]^{r\times r}$ with rows $T_{1,\cdot}, \ldots, T_{r,\cdot} \in M_{d,e}$ ordered by increasing degree $\deg T_{1,\cdot} \leqslant \cdots \leqslant \deg T_{n,\cdot}$. Let $U = \Upsilon(T)$, let $N = \mathrm{N}(T)$ be the matrix with rows $\Xi^{-\deg T_{1,\cdot}} T_{1,\cdot}, \ldots, \Xi^{-\deg T_{r,\cdot}} T_{r,\cdot}$, and let k be maximal such that $\deg T_{k,\cdot} < d$. We say that T is a (d,e)-*head annihilator* for (1) if the following conditions are satisfied:

HA1. The rows of T form a basis for the $\mathbb{K}(i)[\Xi^{-1}]$-module $M_{d,e}$;

HA2. The matrix N_0 is invertible;

HA3. The first k rows of $U_{d+\sigma-e}$ are $\mathbb{K}(i)$-linearly independent.

The matrix $\phi\, x^{d-\deg \phi}\, \mathrm{Id}_r$ is obviously a $(d, 0)$-head annihilator. If $k = r$, then we notice that **HA3** implies that T is a head chopper for (1). The following proposition is also easily checked:

PROPOSITION 3. *For any $\delta \in \mathbb{Z}$, we have*

$$\begin{aligned}M_{d+\delta} &= \Xi^\delta M_d\\ M_{d+\delta,e} &= \Xi^\delta M_{d,e}.\end{aligned}$$

Moreover, T is a (d,e)-head annihilator if and only if $\Xi^\delta T$ is a $(d+\delta,e)$-head annihilator. □

PROPOSITION 4. *Let T be a (d,e)-head annihilator for (1). Let $U=\Upsilon(T)$ and k be as in* **HA1**–**HA3** *and denote $k^*=\operatorname{rank}(U_{d+\sigma-e})$. Then there exists an invertible matrix $J\in\mathbb{K}(i)^{r\times r}$ of the form $J=\begin{pmatrix}\mathrm{Id}_k & 0\\ * & *\end{pmatrix}$ such that the last $r-k^*$ rows of $JU_{d+\sigma-e}$ vanish and such that JT is a (d,e)-head annihilator for (1).*

Proof. Let $J=\begin{pmatrix}\mathrm{Id}_k & 0\\ V & W\end{pmatrix}$ be the row sweeper for $U_{d+\sigma-e}$ as computed by the algorithm **RowSweaper** from section 3. By construction, $\deg (JT)_{j,\cdot}=\deg T_{j,\cdot}$ for all $j\leqslant k$. We claim that $\deg (JT)_{j,\cdot}=\deg T_{j,\cdot}=d$ for all $j>k$. Indeed, if $\deg (JT)_{j,\cdot}<d$, then this would imply that $(JN_0)_{j,\cdot}=0$, which contradicts **HA2**. From our claim, it follows that $\deg (JT)_{1,\cdot}\leqslant\cdots\leqslant\deg (JT)_{n,\cdot}$ and k is maximal with the property that $\deg (JT)_{k,\cdot}<d$. Since the first k rows of U and $JU=\Upsilon(JT)$ coincide, the first k rows of $(JU)_{d+\sigma-e}$ are $\mathbb{K}(i)$-linearly independent. This shows that **HA3** is satisfied for JT. As to **HA2**, let $\tilde{J}\in\mathbb{K}(i)^{r\times r}$ be the invertible matrix with $\tilde{J}(i)=J(i-d)=\begin{pmatrix}\mathrm{Id}_k & 0\\ V(i-d) & W(i-d)\end{pmatrix}$. Then we notice that $\mathrm{N}(JT)=\tilde{J}\,\mathrm{N}(T)$, whence $\mathrm{N}(JT)_0=\tilde{J}N_0$ is invertible. The rows of JT clearly form a basis for $M_{d,e}$, since J is invertible. □

PROPOSITION 5. *Let T be a (d,e)-head annihilator for (1). Let $U=\Upsilon(T)$, let $k^*=\operatorname{rank}(U_{d+\sigma-e})$, and assume that the last $r-k^*$ rows of $U_{d+\sigma-e}$ vanish. Let T^* be the matrix with rows $(\Xi^{-1}T)_{1,\cdot},\ldots,(\Xi^{-1}T)_{k^*,\cdot},T_{k^*+1,\cdot},\ldots,T_{r,\cdot}$. Then T^* is a $(d,e+1)$-head annihilator for (1).*

Proof. We have $\deg T^*_{j,\cdot}=\deg T_{j,\cdot}-1<d$ for all $j\leqslant k^*$ and $\deg T^*_{j,\cdot}=\deg T_{j,\cdot}=d$ for all $j>k^*$. In particular, we have $\deg T^*_{1,\cdot}\leqslant\cdots\leqslant\deg T^*_{n,\cdot}$ and k^* is maximal with the property that $\deg T^*_{k^*,\cdot}<d$. Setting $U^*=\Upsilon(T^*)$, we also observe that $U^*_{j,\cdot}=\Xi^{-1}(U_{j,\cdot})$ for all $j\leqslant k^*$. Since $\operatorname{rank}(U_{d+\sigma-e})=k^*$ and the last $r-k^*$ rows of $U_{d+\sigma-e}$ vanish, the first k^* rows of both $U_{d+\sigma-e}$ and $U^*_{d+\sigma-e-1}$ are $\mathbb{K}(i)$-linearly independent. In other words, **HA3** is satisfied for T^*. As to **HA2**, we observe that $\mathrm{N}(T^*)=\mathrm{N}(T)$, whence $\mathrm{N}(T^*)_0=N_0$ is invertible.

Let us finally show that T^* forms a basis for the $\mathbb{K}(i)[\Xi^{-1}]$-module $M_{d,e+1}$. So let $R\in M_{d,e+1}$. Then $R\in M_{d,e}$, so $R=\Lambda(T)$ for some row matrix $\Lambda=\Lambda_0+\Lambda_1\Xi^{-1}+\cdots\in\mathbb{K}(i)[\Xi^{-1}]^{1\times r}$. Setting $S=\Upsilon(\Lambda(T))$, we have $\deg S\leqslant d+\sigma-e-1$, whence $S_{d+\sigma-e}=\Lambda_0U_{d+\sigma-e}=0$. Since the first k^* rows of $U_{d+\sigma-e}$ are $\mathbb{K}(i)$-linearly independent and the last $r-k^*$ rows of $U_{d+\sigma-e}$ vanish, we get $(\Lambda_0)_{1,j}=0$ for all $j\leqslant k^*$. Let $\tilde{\Lambda}$ be the row vector with $\tilde{\Lambda}_{1,j}=\Lambda_{1,j}\Xi$ for $j\leqslant k^*$ and $\tilde{\Lambda}_{1,j}=\Lambda_{1,j}$ for $j>k^*$. By what precedes, we have $\tilde{\Lambda}\in\mathbb{K}(i)[\Xi^{-1}]^{1\times r}$ and $R=\Lambda_{1,1}(T_{1,\cdot})+\cdots+\Lambda_{1,r}(T_{r,\cdot})$. Now we have $\Lambda_{1,j}(T_{j,\cdot})=\Lambda_{1,j}(\Xi^{-1}(T^*_{j,\cdot}))=\tilde{\Lambda}_{1,j}(T^*_{j,\cdot})$ for $j\leqslant k^*$ and $\Lambda_{1,j}(T_{j,\cdot})=\tilde{\Lambda}_{1,j}(T^*_{j,\cdot})$ for $j>k^*$. In other words, $R=\tilde{\Lambda}(T^*)$, as desired. □

4.3. Computing head choppers

Propositions 4 and 5 allow us to compute (d,e)-head annihilators for (1) with arbitrarily large e. Assuming that we have $k=r$ in **HA3** for sufficiently large e, this yields the following algorithm for the computation of a head chopper for (1):

Algorithm HeadChopper(ϕ, A)

$T:=\phi\,\mathrm{Id}_r, U:=\Upsilon(T)$
repeat
    if $U_{\deg U}$ is invertible **then return** T
    $J:=$ **RowSweaper**$(U_{\deg U})$
    $(T,U):=(JT,JU)$
    $k^*:=\operatorname{rank}(U_{\deg U})$, $\Delta:=\begin{pmatrix}\mathrm{Id}_{k^*}\Xi^{-1} & 0\\ 0 & \mathrm{Id}_{r-k^*}\end{pmatrix}$
    $(T,U):=(\Delta T,\Delta U)$

PROPOSITION 6. *Let $d=\deg\phi$. Consider the value of T at the beginning of the loop and after e iterations. Then T is a (d,e)-head annihilator.*

Proof. We first observe that $U = \Upsilon(T)$ throughout the algorithm. Let us now prove the proposition by induction over e. The proposition clearly holds for $e = 0$. Assuming that the proposition holds for a given e, let us show that it again holds at the next iteration. Consider the values of T and U at the beginning of the loop and after e iterations. Let k be maximal such that $\deg T_{k,\cdot} < d$. From the induction hypothesis, it follows that the first k rows of $U_{\deg U}$ are $\mathbb{K}(i)$-linearly independent, whence the matrix J is of the form $J = \begin{pmatrix} \mathrm{Id}_k & 0 \\ * & * \end{pmatrix}$. Now Proposition 4 implies that $J\,T$ is still a (d, e)-head annihilator. Since the last $r - k^*$ rows of $(J\,U)_{\deg(JU)}$ vanish, Proposition 5 also implies that $\Delta(JT)$ is a $(d, e+1)$-head annihilator. This completes the induction. Notice also that $k^* \geqslant k$ is maximal with the property that $\deg(\Delta(JT))_{k^*,\cdot} < d$. □

Proposition 7. *If the algorithm* **HeadChopper** *does not terminate, then there exists a non zero row matrix $R \in \phi\,\mathbb{K}(i)[[x^{-1}]]^{1\times r}$ with $\Upsilon(R) = 0$. In particular, $(R\,y)' = 0$.*

Proof. Assume that **HeadChopper** does not terminate. Let T_e be the value of T at the beginning of the main loop after e iterations. Also let J_e and Δ_e be the values of J and Δ as computed during the $(e+1)$-th iteration.

Let k_e be maximal such that $\deg T_{k_e,\cdot} < d := \deg \phi$. Using the observation made at the end of the above proof, we have $k_0 \leqslant k_1 \leqslant \cdots$, so there exist an index $e_0 \in \mathbb{N}$ and $k_\infty < r$ with $k_e = k_\infty$ for all $e \geqslant e_0$. Furthermore,

$$J_e = \begin{pmatrix} \mathrm{Id}_{k_e} & 0 \\ * & * \end{pmatrix}, \qquad \Delta_e = \begin{pmatrix} \mathrm{Id}_{k_{e+1}}\,\Xi^{-1} & 0 \\ 0 & \mathrm{Id}_{r-k_{e+1}} \end{pmatrix},$$

and

$$T_{e+1} \;=\; \Delta_e(J_e\,T_e).$$

Moreover, for $e \geqslant e_0$, the row sweaper J_e is even of the form

$$J_e \;=\; \begin{pmatrix} \mathrm{Id}_{k_\infty} & 0 \\ * & \mathrm{Id}_{r-k_\infty} \end{pmatrix}.$$

By induction on $e \in \mathbb{N}$, we observe that $T_e \in \phi\;\mathbb{K}(i)[x^{-1}]^{r\times r}$. For $e \geqslant e_0$, we also have $\deg\,(\phi^{-1}\,T_e)_{j,\cdot} \leqslant e_0 - e$ for all $j \leqslant k_\infty$, again by induction. Consequently, $\deg\,(\phi^{-1}\,T_{e+1} - \phi^{-1}\,T_e) \leqslant e_0 - e$ for all $e_0 - e$, which means that the sequence $\phi^{-1}\,T_e$ converges to a limit $\phi^{-1}\,T_\infty$ in $\mathbb{K}(i)[[x^{-1}]]^{r\times r}$. By construction, the first k_∞ rows of T_∞ are zero, its last $r - k_\infty$ rows have rank $r - k_\infty$, and $\Upsilon(T_\infty) = 0$. We conclude by taking R to be the last row of T_∞. □

Theorem 8. *The algorithm* **HeadChopper** *terminates and returns a head chopper for* (1).

Proof. We already observed that $U = \Upsilon(T)$ throughout the algorithm. If the algorithm terminates, then it follows that T is indeed a head chopper for (1). Assume for contradiction that the algorithm does not terminate and let $R \in \phi\,\mathbb{K}(i)[[x^{-1}]]^{1\times r}$ be such that $\Upsilon(R) = 0$. Let $y \in \mathbb{L}^{r\times r}$ be a fundamental system of solutions to the equation (1), where $\mathbb{L}$ is some differential field extension of $\mathbb{K}(i)((x^{-1}))$ with constant field $\mathbb{K}(i)$. From $\Upsilon(R) = 0$ we deduce that $(R\,y)' = 0$, whence $R\,y \in \mathbb{K}(i)^r$. More generally, $\Upsilon(\Xi^{-j}\,R) = 0$ whence $((\Xi^{-j}\,R)\;y)' = 0$ and $(\Xi^{-j}\,R)\;y \in \mathbb{K}(i)^r$ for all $j \in \mathbb{N}$. Since the space $\mathbb{K}(i)^r$ has dimension r over $\mathbb{K}(i)$, it follows that there exists a polynomial $\Lambda \in \mathbb{K}(i)[\Xi^{-1}]$ of degree at most r in Ξ^{-1} such that $\Lambda(R)\;y = 0$ and $\Lambda(R) \neq 0$. Since y is a fundamental system of solutions, we have $\det y \neq 0$. This contradicts the existence of an element $\Lambda(R) \in \mathbb{L}^r \setminus \{0\}$ with $\Lambda(R)\,y = 0$. □

4.4. Head reduction

Let T be a head chopper for (1). Replacing T by $\Xi^{\mathrm{val}\,T}T$ if necessary, we may assume without loss of generality that $T \in \phi\;\mathbb{K}(i)[x]^{r\times r}$ and $U = \Upsilon(T) \in \mathbb{K}(i)[x]^{r\times r}$. Let $\tau = \deg U$. Writing $T = N\,/\,D$ with $N \in \phi\,\mathbb{K}[i][x]^{r\times r}$ and $D \in \mathbb{K}[i]$, let $\mathcal{I}$ to be the set of *exceptional indices* $i \in \mathbb{K}$ for which $D(i) = 0$ or $(\det U_\tau)(i) = 0$. For any $d \in \mathbb{Z}$, let

$$\Lambda_d \;=\; \{\lambda \in \mathbb{K}[x]^{1\times r} \colon \forall i > d, i - \tau \notin \mathcal{I} \Rightarrow \lambda_i = 0\}.$$

If $d \geqslant \tau$ and $i = d - \tau \notin \mathcal{I}$, then the matrix $U_\tau(i) \in \mathbb{K}^{r \times r}$ is invertible. We define the $\mathbb{K}$-linear mapping $\pi_d: \Lambda_d \to \Lambda_{d-1}$ by

$$\pi_d(\lambda) \;=\; \lambda - (\lambda_d U_\tau^{-1}(i))\, x^i\, U(i).$$

We indeed have $\pi_d(\lambda) \in \Lambda_{d-1}$, since $(\lambda_d\, U_\tau^{-1}(i))\; x^i\; U(i) = \lambda_d\, x^d + O(x^{d-1})$. The mapping π_d also induces a mapping $\Lambda_d\, y \to \Lambda_{d-1}\, y$; $\lambda\, y \mapsto \pi_d(\lambda)\; y$ that we will still denote by π_d. Setting $c = \lambda_d U_\tau^{-1}(i)$, the relation (7) yields

$$(\lambda - \pi_d(\lambda))\, y = c\, x^i\, U(i)\, y = (c\, x^i\, T(i)\, y)'.$$

This shows that the mapping π_d is a reduction. If $d \geqslant \tau$ and $i = d - \tau \in \mathcal{I}$, then we have $\Lambda_d = \Lambda_{d-1}$ and the identity map $\pi_d: \Lambda_d\, y \to \Lambda_{d-1}\, y$ is clearly a reduction as well.

Since compositions of reductions are again reductions, we also obtain a reduction $\pi_\tau \circ \cdots \circ \pi_d$: $\Lambda_d\; y \to \Lambda_{\tau-1}\; y$ for each d. Now let $\lceil \cdot \rceil: \mathbb{K}[x]^{1 \times r}\, y \to \mathbb{K}[x]^{1 \times r}\, y$ be the unique mapping with $\lceil \lambda\, y \rceil = (\pi_\tau \circ \cdots \circ \pi_d)(\lambda\, y)$ for all $d \geqslant \tau$ and $\lambda \in \Lambda_d$. Then $\lceil \cdot \rceil$ is clearly a reduction as well and it has a finite dimensional image $\operatorname{im} \lceil \cdot \rceil \subseteq \Lambda_{\tau-1}$. For any $\lambda \in \mathbb{K}[x]^{1 \times r}$, we call $\lceil \lambda\, y \rceil$ the *head reduction* of $\lambda\, y$. The following straightforward algorithm allows us to compute head reductions:

Algorithm HeadReduce($\boldsymbol{\lambda}$)

repeat
  if $\lambda_{i+\tau} = 0$ for all $i \in \mathbb{N} \setminus \mathcal{I}$ **then return** λ
  Let $i \in \mathbb{N} \setminus \mathcal{I}$ be maximal with $\lambda_{i+\tau} \neq 0$
  $c := \lambda_{i+\tau} U_\tau^{-1}(i)$
  $\lambda := \lambda - c\, x^i\, U(i)$

PROPOSITION 9. *The routine* **HeadReduce** *terminates and is correct.* □

Remark 10. It is straightforward to adapt **HeadReduce** so that it also returns the certificate $\kappa \in \phi\, \mathbb{K}[x]^{1 \times r}$ with $\lambda\, y - (\kappa\, y)' \in \Lambda_{\tau-1}\, y$. Indeed, it suffices to start with $\kappa := 0$ and accumulate $\kappa := \kappa + c\, x^i\, T(i)$ at the end of the main loop.

Remark 11. The algorithm **HeadReduce** is not very efficient. The successive values of c can be computed more efficiently in a relaxed manner [10].

Remark 12. The algorithm **HeadReduce** also works for matrices $\lambda \in \mathbb{K}[x]^{n \times r}$ with an arbitrary number of rows n. This allows for the simultaneous head reduction of several elements in $\mathbb{K}[x]^{1 \times r}\, y$, something that might be interesting for the application to creative telescoping.

5. TAIL REDUCTION

5.1. Tail choppers

Head reduction essentially allows us to reduce the valuation in x^{-1} of elements in $\mathbb{M}$ *via* the subtraction of elements in $\partial \mathbb{M}$. Tail reduction aims at reducing the valuation in $x - \alpha$ in a similar way for any α in the algebraic closure $\hat{\mathbb{K}}$ of $\mathbb{K}$. More precisely, let $i \in \mathbb{K}$, $\alpha \in \hat{\mathbb{K}}$ and $T \in \phi\, \hat{\mathbb{K}}(i)[x, (x - \alpha)^{-1}]^{r \times r}$. We may regard T as a Laurent polynomial in $x - \alpha$ with matrix coefficients $T_k \in \hat{\mathbb{K}}(i)^{r \times r}$:

$$T \;=\; \sum_{k \in \mathbb{Z}} T_k\, (x - \alpha)^k. \tag{8}$$

If $T \neq 0$, then we denote its valuation in $x - \alpha$ by $\operatorname{val}_\alpha T = \min \{k \in \mathbb{Z}: T_k \neq 0\}$. Setting

$$U \;=\; \Upsilon_\alpha(T) \;:=\; \phi^{-1}\, T A + T' + i\, (x - \alpha)^{-1}\, T, \tag{9}$$

the equation (1) implies

$$(C\,(x-\alpha)^i\,T y)' \;=\; C\,(x-\alpha)^i\,U y, \tag{10}$$

for any matrix $C \in \hat{\mathbb{K}}(i)^{r\times r}$. The matrix U can also be regarded as a Laurent polynomial with matrix coefficients $U_k \in \hat{\mathbb{K}}(i)^{r\times r}$. We say that T is a *tail chopper at* α for (1) if $U_{\mathrm{val}_\alpha U}$ is an invertible matrix. In fact, it suffices to consider tail choppers at the origin:

LEMMA 13. *Let $T \in \phi\, \hat{\mathbb{K}}(i)[x, (x-\alpha)^{-1}]^{r\times r}$, where $\alpha \in \hat{\mathbb{K}}$. Define $\tilde{T}(x, i) = T(x+\alpha, i)$, $\tilde{\phi}(x) = \phi(x+\alpha)$ and $\tilde{A}(x) = A(x+\alpha)$. Then T is a tail chopper at α for (1) if and only if $\tilde{T}$ is a tail chopper at 0 for $\tilde{\phi}\,\tilde{y}' = \tilde{A}\,\tilde{y}$.*

Proof. Setting $\tilde{U} = \Upsilon_0(\tilde{T})$, we have $\tilde{U}(x) = U(x+\alpha)$. Consequently, $\mathrm{val}_\alpha \tilde{U} = \mathrm{val}_0 U$ and $\tilde{U}_{\mathrm{val}_\alpha \tilde{U}} = U_{\mathrm{val}_0 U}$. □

There is also a direct link between head choppers and tail choppers at 0 *via* the change of variables $x \leftrightsquigarrow x^{-1}$.

LEMMA 14. *Let $T \in \phi\, \hat{\mathbb{K}}(i)[x, x^{-1}]^{r\times r}$. Setting $\tilde{x} = x^{-1}$, we define $\tilde{\phi}(\tilde{x}) = -x^2\,\phi(x)$, $\tilde{A}(\tilde{x}) = A(x)$ and $\tilde{T}(\tilde{x}, i) = T(x, -i)$. Then T is a tail chopper at 0 for (1) if and only if $\tilde{T}$ is a head chopper for $\tilde{\phi}\,\tilde{y}' = \tilde{A}\,\tilde{y}$.*

Proof. Setting $\tilde{U} = \Upsilon(\tilde{T})$, we have

$$\begin{aligned}
\tilde{U}(\tilde{x}, -i) &= \tilde{\phi}(\tilde{x})^{-1}\,\tilde{T}(\tilde{x}, -i)\,\tilde{A}(\tilde{x}) + \frac{\partial \tilde{T}}{\partial \tilde{x}}(\tilde{x}, -i) - i\,\tilde{x}^{-1}\,\tilde{T}(\tilde{x}, -i)\\
&= -x^2\,\phi(x)^{-1}\,T(x, i)\,A(x) - x^2\,T'(x, i) - i\,x\,T(x, i)\\
&= -x^2\,(\phi(x)^{-1}\,T(x, i)\,A(x) + T'(x, i) + i\,x^{-1}\,T(x, i))\\
&= -x^2\,U(x, i).
\end{aligned}$$

Consequently, $\deg \tilde{U} = \mathrm{val}_0 U + 2$ and $\tilde{U}_{\deg \tilde{U}}(-i) = U_{\mathrm{val}_0 U}(i)$. □

Finally, the matrix $\phi\,\mathrm{Id}_r$ is a tail chopper at almost all points α:

LEMMA 15. *Let $\alpha \in \hat{\mathbb{K}}$ be such that $\phi(\tilde{\alpha}) \neq 0$. Then $\phi\,\mathrm{Id}_r$ is a tail chopper for (1) at α.*

Proof. If $\phi(\tilde{\alpha}) \neq 0$ and $T = \phi\,\mathrm{Id}_r$, then (9) becomes $U = i\,(x-\alpha)^{-1}\,\phi\,\mathrm{Id}_r + O((x-\alpha)^0)$ for $x \to \alpha$. In particular, $\mathrm{val}_\alpha U = -1$ and $U_{\mathrm{val}_\alpha(U)} = i\,\phi(\alpha)\,\mathrm{Id}_r$ is invertible in $\hat{\mathbb{K}}(i)^{r\times r}$. □

5.2. Computing tail choppers

Now consider a monic square-free polynomial $\psi \in \mathbb{K}[x]$ and assume that we wish to compute a tail chopper for (1) at a root α of ψ in $\hat{\mathbb{K}}$. First of all, we have to decide how to conduct computations in $\hat{\mathbb{K}}$. If ψ is irreducible, then we may simply work in the field $\mathbb{L} = \mathbb{K}[x]/(\psi)$ instead of $\hat{\mathbb{K}}$ and take α to be the residue class of x, so that α becomes a generic formal root of ψ. In general, factoring ψ over $\mathbb{K}$ may be hard, so we cannot assume ψ to be irreducible. Instead, we rely on the well known technique of *dynamic evaluation* [8].

For convenience of the reader, let us recall that dynamic evaluation amounts to performing all computations as if ψ were irreducible and $\mathbb{L} = \mathbb{K}[x]/(\psi)$ were a field with an algorithm for division. Whenever we wish to divide by a non-zero element $a \bmod \psi$ (with $a \in \mathbb{K}[x]$) that is not invertible, then $\gcd(a, \psi)$ provides us with a non trivial factor of ψ. In that case, we launch an exception and redo all computations with $\gcd(a, \psi)$ or $\psi/\gcd(a, \psi)$ in the role of ψ.

So let $\alpha \in \mathbb{L}$ be a formal root of ψ and define $\tilde{x} = (x - \alpha)^{-1}$, $\tilde{y}(\tilde{x}) = y(x)$, $\tilde{\phi}(\tilde{x}) = -(x - \alpha)^2 \phi(x)$ and $\tilde{A}(\tilde{x}) = A(x)$. Let $\tilde{T}(\tilde{x}, i)$ be a head chopper for the equation $\tilde{\phi}\, \tilde{y}' = \tilde{A}\, \tilde{y}$, as computed using the algorithm from section 4.3. Then $T(x, i) = \tilde{T}(\tilde{x}, -i)$ is a tail chopper at α by Lemmas 13 and 14.

5.3. Tail reduction

Let T be a tail chopper for (1) at $\alpha \in \hat{\mathbb{K}}$. Let $\tilde{x} = (x - \alpha)^{-1}$, $\tilde{y}(\tilde{x}) = y(x)$, $\tilde{\phi}(\tilde{x}) = -(x - \alpha)^2 \phi(x)$ and $\tilde{A}(\tilde{x}) = A(x)$ be as above, so that $\tilde{T}(\tilde{x}, i) = T(x, -i)$, is a head chopper for the equation $\tilde{\phi}\, \tilde{y}' = \tilde{A}\, \tilde{y}$. In particular, rewriting linear combinations $\lambda\, y$ with $\lambda \in \hat{\mathbb{K}}[(x - \alpha)^{-1}]^{1 \times r}$ as linear combinations $\tilde{\lambda}\, \tilde{y}$ with $\tilde{\lambda} \in \hat{\mathbb{K}}[\tilde{x}]^{1 \times r}$, we may head reduce $\tilde{\lambda}\, \tilde{y}$ as described in section 4.4. Let $\tilde{\mu} \in \hat{\mathbb{K}}[\tilde{x}]^{1 \times r}$ be such that $\tilde{\mu}\, \tilde{y} = \lceil \tilde{\lambda}\, \tilde{y} \rceil$. Then we may rewrite $\tilde{\mu}\, \tilde{y}$ as an element $\mu\, y$ of $\hat{\mathbb{K}}[(x - \alpha)^{-1}]^{1 \times r} y$. We call $\mu\, y$ the *tail reduction* of $\lambda\, y$ at α and write $\mu\, y = \lfloor \lambda\, y \rfloor_\alpha$.

Let $\tilde{\mathcal{I}}$ be the finite set of exceptional indices for the above head reduction and $\mathcal{I} = -\tilde{\mathcal{I}}$. Setting $U = \Upsilon_\alpha(T)$ and $\tau = \mathrm{val}_\alpha U$, it can be checked that the following algorithm computes the tail reduction at α:

Algorithm TailReduce($\boldsymbol{\lambda}$)

repeat
  if $\lambda_{i+\tau} = 0$ for all $i \in (-\mathbb{N}) \setminus \mathcal{I}$ **then return** λ
  Let $i \in (-\mathbb{N}) \setminus \mathcal{I}$ be minimal with $\lambda_{i+\tau} \neq 0$
  $c := \lambda_{i+\tau} U_\tau^{-1}(i)$
  $\lambda := \lambda - c\, (x - \alpha)^i\, U(i)$

6. Degree and valuation bounds

6.1. Cyclic vectors

In the particular case when

$$\phi = -L_r, \qquad A = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ L_0 & L_1 & \cdots & L_{r-1} \end{pmatrix}, \qquad y = \begin{pmatrix} f \\ \vdots \\ f^{(r-1)} \end{pmatrix} \tag{11}$$

for some operator $L \in \mathbb{K}[x][\partial]$ of order r, the system (1) is equivalent to

$$L f \;=\; 0. \tag{12}$$

Given a general system (1), there always exists an element $f \in \mathbb{K}(x)\, y$ such that $f, \ldots, f^{(r-1)}$ are $\mathbb{K}(x)$-linearly independent. Such an element f is called a *cyclic vector* and, with respect to the basis $f, \ldots, f^{(r-1)}$ of $\mathbb{K}(x)\, y$, the equation (1) transforms into an equation of the form (12). For efficient algorithms to compute cyclic vectors, we refer to [3].

In the remainder of this section, we focus on systems (1) that are equivalent to (12), with ϕ, A and y as in (11).

6.2. Formal transseries solutions

Let us start with a quick review of some well known results about the asymptotic behaviour of solutions to (12) when $x \to \infty$. We define

$$\begin{aligned} \mathbb{S} &= \hat{\mathbb{K}}((x^{-\mathbb{Q}}))[\log x] \\ \mathfrak{S} &= x^{-\mathbb{Q}} \log^{\mathbb{N}} x \end{aligned}$$

to be the set of polynomials in $\log x$ whose coefficients are Puiseux series in x^{-1}, together with the corresponding set of *monomials*. The set $\mathfrak{S}$ is asymptotically ordered by

$$x^{\alpha}\log^{i}x\preccurlyeq x^{\beta}\log^{j}x\iff\alpha>\beta\vee(\alpha=\beta\wedge i\leqslant\beta)$$

and elements f of $\mathbb{S}$ can be written as series $f=\sum_{\mathfrak{m}\in\mathfrak{S}}f_{\mathfrak{m}}\mathfrak{m}$ with $f_{\mathfrak{m}}\in\hat{\mathbb{K}}$. We call $\operatorname{supp}f:=\{\mathfrak{m}\in\mathfrak{S}\colon f_{\mathfrak{m}}\neq 0\}$ the *support* of f. If $f\neq 0$, then the maximal element $\mathfrak{d}_f$ of the support is called the *dominant monomial* of f.

We may also regard elements f of $\mathbb{S}$ as series $\sum_{\alpha\in\mathbb{Q}}f_{\alpha}x^{-\alpha}$ in x^{-1} with coefficients in $\hat{\mathbb{K}}[\log x]$. If $f\neq 0$, then we denote by $v(f)=\operatorname{val}_{\infty}f=\max\{\alpha\in\mathbb{Q}\colon f_{\alpha}\neq 0\}$ the corresponding valuation of f in x^{-1}. Notice that we have $\mathfrak{d}_f=x^{-v(f)}\log^{i}x$ for some $i\in\mathbb{N}$.

Let $\mathbb{Q}^{>}=\{a\in\mathbb{Q}\colon a>0\}$. We write $\hat{\mathbb{K}}[x^{\mathbb{Q}^{>}}]$ for the set of finite $\hat{\mathbb{K}}$-linear combinations of elements of $x^{\mathbb{Q}^{>}}$. The sets $\hat{\mathbb{K}}[x^{\mathbb{Q}^{\geqslant}}]$ and $\hat{\mathbb{K}}[x^{\mathbb{Q}}]$ are defined likewise. Consider the formal exponential monomial group

$$\mathfrak{E}\;=\;x^{\hat{\mathbb{K}}}\mathrm{e}^{\hat{\mathbb{K}}[x^{\mathbb{Q}^{>}}]}.$$

It is well known [1, 11] that the equation (12) admits a basis $h_1,\ldots,h_r$ of formal solutions of the form

$$h_i\;=\;\varphi_i\,\mathfrak{e}_i,$$

with $\varphi_i\in\mathbb{S}$, $\mathfrak{e}_i\in\mathfrak{E}$ and such that the monomials $\mathfrak{d}_{\varphi_1}\mathfrak{e}_1,\ldots,\mathfrak{d}_{\varphi_r}\mathfrak{e}_r$ are pairwise distinct. We will write

$$\mathfrak{H}_L\;=\;\{\mathfrak{d}_{\varphi_1}\mathfrak{e}_1,\ldots,\mathfrak{d}_{\varphi_r}\mathfrak{e}_r\}.$$

Notice that this result generalizes to the case when $L\in\mathbb{K}[x^{\mathbb{Q}}][\partial]$ *via* a straightforward change of variables $x\rightsquigarrow x^{\alpha}$ with $\alpha\in\mathbb{Q}^{>}$.

6.3. Action of linear differential operators on transseries

Let us now consider a linear differential operator $L\in\hat{\mathbb{K}}[x^{\mathbb{Q}}][\partial]$. Such an operator can also be expanded with respect to x^{-1}; we denote by $v(L)$ the valuation of L in x^{-1} and by $D_L\in\hat{\mathbb{K}}[\partial]$ the coefficient of $x^{-v(L)}$ in this expansion.

From the valuative point of view it is more convenient to work with linear differential operators $L\in\hat{\mathbb{K}}[x^{\mathbb{Q}}][\theta]$, where $\theta=x\,\partial_x$ is the Euler derivation. Such operators can be expanded with respect to x^{-1} in a similar way and $v(L)$ and D_L are defined as above.

For $L\in\hat{\mathbb{K}}[x^{\mathbb{Q}}][\partial]$, let L_{θ} be the corresponding operator in $\hat{\mathbb{K}}[x^{\mathbb{Q}}][\theta]$. If L has order r, then

$$v(L)\;\leqslant\;v(L_{\theta})\;\leqslant\;v(L)+r.$$

For $L\in\hat{\mathbb{K}}[x^{\mathbb{Q}}][\theta]$, let L_{∂} be the corresponding operator in $\hat{\mathbb{K}}[x^{\mathbb{Q}}][\theta]$. If L has order r, then

$$v(L)-r\;\leqslant\;v(L_{\partial})\;\leqslant\;v(L).$$

Given $\mathfrak{e}\in\mathfrak{E}$, we notice that its logarithmic Euler derivative $\sigma:=\theta\,\mathfrak{e}\,/\,\mathfrak{e}$ belongs to $\hat{\mathbb{K}}[x^{\mathbb{Q}^{\geqslant}}]$. Let $L_{\ltimes\mathfrak{e}}\in\hat{\mathbb{K}}[x^{\mathbb{Q}}][\theta]$ be the operator obtained from L by substituting $\theta+\sigma$ for θ. Then

$$L_{\ltimes\mathfrak{e}}(f)\;=\;\mathfrak{e}^{-1}L(f\,\mathfrak{e})$$

for all $f\in\mathbb{S}$. If L has order r, then

$$v(L)+r\,v(\sigma)\;\leqslant\;v(L_{\ltimes\mathfrak{e}})\;\leqslant\;v(L)-r\,v(\sigma).$$

We call $L_{\ltimes\mathfrak{e}}$ the *twist* of L by $\mathfrak{e}$.

Now consider an operator $L\in\hat{\mathbb{K}}[x^{\mathbb{Q}}][\partial]$ and $f\in\mathbb{S}$. If $D_L(\mathfrak{d}_f)=0$, then it can be shown that $\mathfrak{d}_f\in\mathfrak{H}_L$. Otherwise, $L f\sim D_L(\mathfrak{d}_f)\,x^{v(L)}$, whence $v(L f)=v(L)+v(f)$. In other words,

$$\mathfrak{d}_f\notin\mathfrak{H}_L \implies v(L f)=v(L)+v(f).$$

More generally, if $f\in\mathbb{S}$ and $\mathfrak{e}\in\mathfrak{E}$, then

$$\mathfrak{d}_f\,\mathfrak{e}\notin\mathfrak{H}_L \implies v(\mathfrak{e}^{-1}L(f\,\mathfrak{e}))=v(L_{\ltimes\mathfrak{e}})+v(f). \tag{13}$$

Let

$$\hbar \;=\; -\min\{v(\theta\,\mathfrak{e}/\mathfrak{e})\colon \mathfrak{m}\,\mathfrak{e}\in\mathfrak{H}_L, \mathfrak{m}\in\mathfrak{S}, \mathfrak{e}\in\mathfrak{E}\setminus\{1\}\}.$$

If $L\in\hat{\mathbb{K}}[x][\theta]$, then it can be shown using the Newton polygon method that $\hbar\leqslant\deg L-\operatorname{val}L$.

6.4. Degree and valuation bounds for head and tail choppers

Let $\sigma=-\min(v(A)-v(\phi),1)$ and notice that $v(\Upsilon(T))\geqslant v(T)-\sigma$ for all $T\in\mathbb{K}(i)[x,x^{-1}]^{1\times r}$.

THEOREM 16. *For any $T\in\mathbb{K}(i)[x,x^{-1}]^{1\times r}$ and $U=\Upsilon(T)$, we have*

$$v(T)-\sigma \;\leqslant\; v(U) \;\leqslant\; v(T)+2\,r\,\hbar+r+1.$$

Proof. Let ξ be a transcendental constant over $\mathbb{K}$ with $\partial_x\,\xi=0$ and let $\mathbb{L}=\mathbb{K}(\xi)$. Then $T(\xi)\in\mathbb{L}[x,x^{-1}]^{1\times r}$ satisfies $v(T(\xi))=v(T)$, $v(U(\xi))=v(U)$, and

$$(x^{\xi}\,T(\xi)\,y)' \;=\; x^{\xi}\,U(\xi)\,y.$$

We may rewrite $T(\xi)\,y=K f$ for the linear differential operator $K=T(\xi)_{1,1}+\cdots+T(\xi)_{1,r}\,\partial^{r-1}\in\mathbb{L}[x,x^{-1}][\partial]$. Notice that $v(K)=v(T(\xi))=v(T)$. Similarly, we may rewrite $U(\xi)\,y=H f$ for some operator $H\in\mathbb{L}[x,x^{-1}][\partial]$ with $v(H)=v(U(\xi))=v(U)$.

Let Ω, $\Theta\in\mathbb{L}[x,x^{-1}][\theta]$ be the operators given by $\Omega=(\theta+\xi)\,K_\theta$ and $\Theta=H_\theta$, so that $v(\Omega)\leqslant v(T)+r$ and $v(U)\leqslant v(\Theta)$. By construction,

$$\begin{aligned}\Omega f &= \theta\,K_\theta f+\xi\,K_\theta f\\ &= x^{-\xi}\,\theta(x^{\xi}\,K_\theta f)\\ &= x^{-\xi}\,\theta(x^{\xi}\,T(\xi)\,y)\\ &= x\,U(\xi)\,y\\ &= x\,\Theta\,f.\end{aligned}$$

Since K has order at most $r-1$, there exists a monomial $\mathfrak{m}\,\mathfrak{e}\in\mathfrak{H}_L\setminus\mathfrak{H}_K$ with $\mathfrak{m}\in\mathfrak{S}$ and $\mathfrak{e}\in\mathfrak{E}\setminus\{1\}$. Now the solutions of $\Omega\,g=0$ are spanned by the solutions to $K_\theta\,g=0$ and any particular solution to the inhomogeneous equation $K_\theta g=x^{-\xi}$. In [11, Chapter 7] it is shown that the latter equation admits a so called distinguished solution $g\in x^{-\xi}\,\mathbb{L}((x^{-\mathbb{Q}}))[\log x]$ with $v(x^{\xi}\,g)=-v(K_\theta)$ and $\mathfrak{d}_{x^\xi g}\,x^{-\xi}\notin\mathfrak{H}_K$. Since ξ is transcendental over $\mathbb{K}$, it follows that $\mathfrak{m}\,\mathfrak{e}\in\mathfrak{H}_L\setminus\mathfrak{H}_\Omega$. Let $\varphi\in\mathbb{S}$ be a solution to $L\,(\varphi\,\mathfrak{e})=0$ with $\mathfrak{d}_\varphi=\mathfrak{m}$. Then $f=\varphi\,\mathfrak{e}$ satisfies

$$v(x\,\mathfrak{e}^{-1}\,\Theta\,f) \;=\; v(x\,\Theta_{\ltimes\mathfrak{e}}\,\varphi) \;\geqslant\; v(\Theta_{\ltimes\mathfrak{e}})+v(\mathfrak{m})-1,$$

whereas

$$v(\mathfrak{e}^{-1}\,\Omega\,f) \;=\; v(\Omega_{\ltimes\mathfrak{e}}\,\varphi) \;=\; v(\Omega_{\ltimes\mathfrak{e}})+v(\mathfrak{m}).$$

(The last equality follows from (13) and the fact that $\mathfrak{m}\,\mathfrak{e}\in\mathfrak{H}_L\setminus\mathfrak{H}_\Omega$.) Now $\Omega\,f=x\,\Theta\,f$ implies

$$v(\Omega)+r\,\hbar\geqslant v(\Omega)-r\,v\!\left(\tfrac{\theta\,\mathfrak{e}}{\mathfrak{e}}\right)\geqslant v(\Omega_{\ltimes\mathfrak{e}})\geqslant v(\Theta_{\ltimes\mathfrak{e}})-1\geqslant v(\Theta)+r\,v\!\left(\tfrac{\theta\,\mathfrak{e}}{\mathfrak{e}}\right)-1\geqslant v(\Theta)-r\,\hbar-1,$$

whence

$$v(U) \;\leqslant\; v(\Theta) \;\leqslant\; v(\Omega)+2\,r\,\hbar+1 \;\leqslant\; v(T)+2\,r\,\hbar+r+1.$$

We already noticed before that $v(U) \geqslant v(T) - \sigma$. □

We notice that our definition of σ coincides with the one from section 4.2, since the degree in x coincides with the opposite of the valuation in x^{-1}. Theorem 16 immediately yields a bound for the number e of iterations that are necessary in **HeadChopper** in order to obtain a head chopper. More precisely:

COROLLARY 17. *There exists a head chopper $T \in M_{0,e}$ for (1) with $e \leqslant r + 1 + 2\, r\, \hbar + \sigma$.*

Proof. Let $\tilde{T}$ be the head chopper as computed by **HeadChopper** and let R be its last row. Then $v(R) = v(\phi)$ and $\tilde{T} \in M_{\deg \phi, e}$ where $e = v(\Upsilon(R)) - v(\phi) + \sigma$. Now the theorem implies $v(\Upsilon(R)) \leqslant v(R) + r + 1 + 2\, r\, \hbar$, whence $e \leqslant r + 1 + 2\, r\, \hbar + \sigma$. We conclude that $T = \Xi^{-\deg \phi}(\tilde{T})$ is a head chopper in $M_{0,e}$. □

Let $\sigma_\infty = \sigma$, $\hbar_\infty = \hbar$, and let $e_\infty \in \mathbb{N}$ be the smallest number for which there exists a head chopper T with

$$\mathrm{val}_\infty \Upsilon(T) = \mathrm{val}_\infty T - \sigma_\infty + e_\infty. \tag{14}$$

We will call e_∞ the *defect* of (1) at infinity. Collary 17 shows that $e_\infty \leqslant r + 1 + 2\, r\, \hbar_\infty + \sigma_\infty$. In a similar way, given $\alpha \in \hat{\mathbb{K}}$, let $\sigma_\alpha = -\min(\mathrm{val}_\alpha A - \mathrm{val}_\alpha \phi, -1)$ and let $e_\alpha \in \mathbb{N}$ be the smallest number for which there exists a tail chopper T with

$$\mathrm{val}_\alpha \Upsilon_\alpha(T) = \mathrm{val}_\infty T - \sigma_\alpha + e_\alpha. \tag{15}$$

We call e_α the *defect* of (1) at α. Defining $\hbar_\alpha$ in a similar way as h_∞, but at the point $x = \alpha$, one has $e_\alpha \leqslant r + 1 + 2\, r\, \hbar_\alpha + \sigma_\alpha$.

6.5. Valuation bounds for differentiation on $\mathbb{M}$

Let $R \in \mathbb{K}((x^{-1}))^{1 \times r}$ and $S = \phi^{-1} R\, A + R'$, so that $(R\, y)' = S\, y$. Let $\mathfrak{H}_L^\sharp := \mathfrak{H}_L\, \{1, \ldots, x^r\}$. The proof technique from Theorem 16 can also be used for studying $v(S)$ as a function of $v(R)$:

THEOREM 18. *For all $R \in \mathbb{K}((x^{-1}))^{1 \times r}$ with $\mathfrak{d}_R^{-1} \notin \mathfrak{H}_L^\sharp$ and $S = \phi^{-1} R\, A + R'$, we have*

$$v(R) - \sigma \;\leqslant\; v(S) \;\leqslant\; v(R) + 2\, r\, \hbar + r + 1.$$

Proof. Let $i = -v(R)$ and rewrite $R = x^i \tilde{R}$ and $S = x^i \tilde{S}$ with $v(\tilde{R}) = 0$. Then we have

$$(x^i \tilde{R}\, y)' \;=\; x^i \tilde{S}\, y.$$

We may rewrite $\tilde{R}\, y = K f$ and $\tilde{S} = H f$ for some $K, H \in \mathbb{L}[x, x^{-1}][\partial]$ with $v(K) = v(\tilde{R}) = 0$ and $v(H) = v(\tilde{S})$. Let $\Omega, \Theta \in \mathbb{L}[x, x^{-1}][\theta]$ be the operators given by $\Omega = (\theta + i)\, K_\theta$ and $\Theta = H_\theta$, so that $0 \leqslant v(\Omega) = v(K_\theta) \leqslant r$ and $v(\tilde{S}) \leqslant v(\Theta)$. In a similar way as in the proof of Theorem 16, we deduce that

$$\Omega\, f \;=\; x\, \Theta\, f.$$

Since K has order at most $r - 1$, there exists a monomial $\mathfrak{m}\, \mathfrak{e} \in \mathfrak{H}_L \setminus \mathfrak{H}_K$ with $\mathfrak{m} \in \mathfrak{G}$ and $\mathfrak{e} \in \mathfrak{E} \setminus \{1\}$. The distinguised solution to the equation $K_\theta\, g = x^{-i}$ with $\mathfrak{H}_\Omega = \mathfrak{H}_K \cup \{\mathfrak{d}_g\}$ has valuation $v(g) = -i - v(K_\theta)$, so that $-i - r \leqslant v(g) \leqslant -i$. Since $x^{-i} = \mathfrak{d}_R^{-1} \notin \mathfrak{H}_L^\sharp$, it follows that $\mathfrak{d}_g \notin \mathfrak{H}_L$, whence $\mathfrak{m}\, \mathfrak{e} \in \mathfrak{H}_L \setminus \mathfrak{H}_\Omega$. In a similar way as in the proof of Theorem 16, we obtain

$$v(\Omega) \;\geqslant\; v(\Theta) - 2\, r\, \hbar - 1,$$

whence

$$v(S) - v(R) \;=\; v(\tilde{S}) - v(\tilde{R}) \;=\; v(H) - v(K) \;\leqslant\; v(\Theta) - v(\Omega) + r \;\leqslant\; 2\, r\, \hbar + r + 1.$$

The inequality $v(S) \geqslant v(R) - \sigma$ in the other direction is straightforward. □

Corollary 19. *We can compute an integer $\nu_\infty \in \mathbb{Z}$ such that for all $R \in \mathbb{K}((x^{-1}))^{1 \times r}$ and $S = \phi^{-1} R A + R'$ with $\operatorname{val}_\infty R \leqslant \nu_\infty$, we have*

$$\operatorname{val}_\infty S \ \leqslant\ \operatorname{val}_\infty R - \sigma_\infty + e_\infty.$$

Proof. Let T be a head chopper for (1) such that $U = \Upsilon(T)$ satisfies $v(U) = v(T) - \sigma_\infty + e_\infty = 0$. Let ζ be sufficiently small such that $x^i \notin \mathfrak{H}_L^\sharp$ and $U_{\deg U}(i)$ is defined and invertible for all $i \leqslant \zeta$. We take $\nu_\infty = \zeta - 2 r \hbar - r - 1$.

Assume for contradiction that $v(R) \leqslant \nu_\infty$ and $v(S) > v(R) - \sigma_\infty + e_\infty$. Let $\tilde{R}$ and $\tilde{S}$ be such that $\tilde{S} y = \lceil S y \rceil$ and $(\tilde{R} y)' = \tilde{S} y$. By construction, $v(\tilde{R} - R) \geqslant v(S) + \sigma_\infty - e_\infty$, whence $v(\tilde{R}) = v(R)$, and $v(\tilde{S}) > \zeta$. But then $v(\tilde{S}) - v(\tilde{R}) = v(\tilde{S}) - v(R) > \zeta - \nu_\infty = 2 r \hbar + r + 1$. This contradicts Theorem 18, since $v(\tilde{R}) \leqslant \nu_\infty \leqslant \zeta$ implies $\mathfrak{d}_{\tilde{R}}^{-1} = x^{v(\tilde{R})} \notin \mathfrak{H}_L^\sharp$. □

Corollary 20. *Given $\alpha \in \hat{\mathbb{K}}$, we can compute an integer $\nu_\alpha \in \mathbb{Z}$ such that for all $R \in \hat{\mathbb{K}}((x - \alpha))^{1 \times r}$ and $S = \phi^{-1} R A + R'$ with $\operatorname{val}_\alpha R \leqslant \nu_\alpha$, we have*

$$\operatorname{val}_\alpha S \ \leqslant\ \operatorname{val}_\alpha R - \sigma_\alpha + e_\alpha.$$

Proof. Follows from the previous proposition *via* a change of variables. □

7. Global reduction

7.1. Gluing together the head and tail reductions

Let us now study how head and tail reductions can be glued together into a global confined reduction on $\mathbb{M} = \mathbb{K}[x, \phi^{-1}]^{1 \times r} y$. More generally, we consider the case when $\mathbb{M} = \mathbb{K}[x, \psi^{-1}]^{1 \times r} y$, where $\psi \in \mathbb{K}[x]$ is a monic square-free polynomial such that ϕ divides ψ^t for some $t \in \mathbb{N}$.

We assume that we have computed a head chopper for (1) and tail choppers T_{α_i} for (1) at each the roots $\alpha_1, \ldots, \alpha_\ell$ of ψ in $\hat{\mathbb{K}}$. In particular, we may compute the corresponding head and tail reductions. Given an element σ of the Galois group of $\hat{\mathbb{K}}$ over $\mathbb{K}$, we may also assume without loss of generality that the tail choppers were chosen such that $T_{\sigma(\alpha_i)} = \sigma(T_{\alpha_i})$ for all i.

Partial fraction decomposition yields $\hat{\mathbb{K}}$-linear mappings

$$\rho_{\alpha_i} \colon \hat{\mathbb{K}}[x, \psi^{-1}]^{1 \times r} \to (x - \alpha_i)^{-1} \hat{\mathbb{K}}[(x - \alpha_i)^{-1}]^{1 \times r}$$

and

$$\rho_\infty \colon \hat{\mathbb{K}}[x, \psi^{-1}]^{1 \times r} \to \hat{\mathbb{K}}[x]^{1 \times r}$$

with

$$\lambda \ = \ \rho_\infty(\lambda) + \rho_{\alpha_1}(\lambda) + \cdots + \rho_{\alpha_\ell}(\lambda),$$

for all $\lambda \in \hat{\mathbb{K}}[x, \psi^{-1}]^{1 \times r}$. This allows us to define a global reduction $[\lambda y]$ of λy by

$$[\lambda y] \ = \ \lceil \rho_\infty(\lambda) y \rceil + \lfloor \rho_{\alpha_1}(\lambda) y \rfloor_{\alpha_1} + \cdots + \lfloor \rho_{\alpha_\ell}(\lambda) y \rfloor_{\alpha_\ell}.$$

By our assumption that the tail choppers were chosen in a way that is compatible with the action of the Galois group, we have $[\lambda y] \in \mathbb{K}[x, \psi^{-1}]^{1 \times r} y$ whenever $\lambda \in \mathbb{K}[x, \psi^{-1}]^{1 \times r}$. Furthermore, the restriction of the reduction on $\hat{\mathbb{K}}[x, \psi^{-1}]^{1 \times r} y$ to $\mathbb{K}[x, \psi^{-1}]^{1 \times r} y$ is still a reduction.

Remark 21. It is plausible that computations in algebraic extensions can be avoided by combining tail choppers at conjugate roots under the action of the Galois group. However, we have not yet worked this idea out.

7.2. Normalizing the reduction

Given the confined reduction $[\cdot]: \mathbb{M} \to \mathbb{M}$ from section 7.1, let us show how to construct a normal confined reduction $[\![\cdot]\!]: \mathbb{M} \to \mathbb{M}$. For each $\sigma \in \{\infty, \alpha_1, \ldots, \alpha_k\}$, assume that we have computed constants ν_σ and τ_σ with the property that for all $\kappa y, \lambda y \in \mathbb{M}$ with $v_\sigma(\kappa) \leqslant \nu_\sigma$ and $(\kappa y)' = \lambda y$, we have $v_\sigma(\lambda) \leqslant v_\sigma(\kappa) + \tau_\sigma$.

Consider the finite dimensional $\mathbb{K}$-vector space $[\mathbb{M}] := \{[f]: f \in \mathbb{M}\}$ and let $\mu_\sigma := \min\{\mathrm{val}_\sigma \lambda: \lambda y \in [\mathbb{M}] \setminus \{0\}\}$ for all $\sigma \in \{\infty, \alpha_1, \ldots, \alpha_\ell\}$. Let Ω be the $\mathbb{K}$-subvector space of $\mathbb{M}$ of all $\lambda y \in \mathbb{M}$ with $\mathrm{val}_\sigma \lambda \geqslant \min(\mu_\sigma, \nu_\sigma) - \tau_\sigma$ for all $\sigma \in \{\infty, \alpha_1, \ldots, \alpha_\ell\}$. This space is finite dimensional and our assumptions imply that we cannot have $(\lambda y)' \in [\mathbb{M}]$ for $\lambda y \in \mathbb{M} \setminus \Omega$. In other words, for any $f \in \mathbb{M}$ with $f' \in [\mathbb{M}]$, we have $f \in \Omega$.

Now let $V := \partial \Omega \cap [\mathbb{M}]$ and let W be a supplement of V in $[\mathbb{M}]$ so that $[\mathbb{M}] = V \oplus W$. We may compute bases of V and W using straightforward linear algebra. The canonical $\mathbb{K}$-linear projections $\pi_V: [\mathbb{M}] \to V$ and $\pi_W: [\mathbb{M}] \to W$ with $\pi_V + \pi_W = \mathrm{Id}$ are also computable. We claim that we may take $[\![f]\!] := \pi_W([f])$ for every $f \in \mathbb{M}$.

PROPOSITION 22. *The mapping* $[\![\cdot]\!]: \mathbb{M} \to \mathbb{M}; f \mapsto \pi_W([f])$ *defines a computable normal confined reduction on* $\mathbb{M}$.

Proof. The mapping $[\![\cdot]\!]$ is clearly a computable confined reduction on $\mathbb{M}$. It remains to be shown that $[\![f']\!] = 0$ for all $f \in \mathbb{M}$. Now $[f'] - f' \in \partial \mathbb{M}$, so $[f'] \in \partial \mathbb{M}$ and there exists a $g \in \mathbb{M}$ with $g' = [f']$. Since $g' \in [\mathbb{M}]$, it follows that $g \in \Omega$ and $g' \in \partial \Omega \cap [\mathbb{M}] = V$. In other words, $[f'] = g' \in V$ and $[\![f']\!] = \pi_W([f']) = 0$. □

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