Computation of the monodromy of generalized polylogarithms

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Generalized polylogarithms

Polylogarithms

$s = (s_1, s_2, \ldots, s_k)$: multiindex of positive integers.

$$\mathrm{Li}_s(z) = \sum_{n_1 > n_2 > \cdots > n_k > 0} \frac{z^{n_1}}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}.$$

Converges for $|\zeta| < 1$.

Generalized ζ-function

At $z = 1$, we have convergence if $s_1 > 1$:

$$\zeta_s = \mathrm{Li}_s(1) = \sum_{n_1 > n_2 > \cdots > n_k > 0} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}},$$

In particular, $\zeta(p) = \zeta_{(p)}$ for all $p > 1$.

Iterated integrals

Definition of Li_w for $w \in X^* = \{x_0, x_1\}^*$

$$\begin{aligned} L_{x_0^n}(z) &= \frac{1}{n!}\log^n(z); \\ L_{x_0 w}(z) &= \int_0^z L_w(t)\frac{dt}{t}; \\ L_{x_1 w}(z) &= \int_0^z L_w(t)\frac{dt}{1-t}. \end{aligned}$$

Convergence at 0 if $w \in X^* x_1$.
Convergence at 1 if $w \in x_0 X^*$.

$$\mathrm{Li}_{(s_1,\ldots,s_k)} = \mathrm{Li}_{x_0^{s_1-1}x_1 x_0^{s_2-1}x_1 \cdots x_0^{s_k-1}x_1}.$$

Combinatorial encoding

$$L(z) = \sum_{w\in X^*} L_w(z) w$$

Differential equation for L:

$$\frac{d}{dz}L(z) = \left(\frac{x_0}{z} + \frac{x_1}{1-z}\right) L(z).$$

Indeed:

$$L(z) = 1 + \sum_{u\in X^*} L_{x_0 u}(z) x_0 u + \sum v \in X^* L_{x_1 v}(z) x_1 v,$$

Main results

1. $L(z)$ is a Lie exponential

2. Asymptotic expansions of L

3. Monodromy of L around 1

4. Fast computations

5. Linear independence of the L_w

$L(z)$ is a Lie exponential

Shuffle product

$$\begin{aligned}\forall w \in X^*, \quad & 1 \shuffle w &= w \shuffle 1 = w,\\ \forall u, v \in X^*, \quad & xu \shuffle yv &= x(u \shuffle yv) + y(xu \shuffle v).\end{aligned}$$

Example: $x_0 x_1 \shuffle x_1 = x_1 x_0 x_1 + 2 x_0 x_1^2$.

$x_0 \quad x_1 \qquad x_0 \qquad x_1 \qquad x_0 \quad x_1$

$x_1 \qquad\qquad x_1 \qquad\qquad x_1$

$R\langle X\rangle$: non commutative polynomials in X over ring R.
Shuffle algebra $\mathrm{Sh}_{\mathbb{Q}}\langle X\rangle$: $R\langle X\rangle$ with $\shuffle$ (extended linearly).

Lyndon words

Let $<$ be the lexicographical ordering on X^*.
A Lyndon word is a non empty word $l \in X^*$, such that

$$\forall u, v \in X^+,\ l = uv \Rightarrow l < v$$

Lyndon words up till length 5:

$$\{x_0, x_0^4 x_1, x_0^3 x_1, x_0^3 x_1^2, x_0^2 x_1, x_0^2 x_1 x_0 x_1, x_0^2 x_1^2,\\ x_0^2 x_1^3, x_0 x_1, x_0 x_1 x_0 x_1^2, x_0 x_1^2, x_0 x_1^3, x_0 x_1^4, x_1\}.$$

Radford: The Lyndon words form a transcendence basis for $\mathrm{Sh}_{\mathbb{Q}}\langle X\rangle$.

Lie exponentials

$R\langle\langle X\rangle\rangle$: non commutative series in X over ring R.
$\mathcal{L}ie_R\langle\langle X\rangle\rangle \subset R\langle\langle X\rangle\rangle$: Lie series generated from X by [].
$(S|u) = S_u$: coefficient of u in $S \in R\langle\langle X\rangle\rangle$.
$S \in R\langle\langle X\rangle\rangle$ Lie exponential if

1. There exists a Lie series $L \in \mathcal{L}ie_R\langle\langle X\rangle\rangle$ with $S = e^L$.
2. $\forall u, v \in X^*$, $(S|u \,\text{ш}\, v) = (S|u)(S|v)$.
3. $\Delta(S) = S \otimes S$, where Δ denotes the usual coproduct $\Delta : R\langle\langle X\rangle\rangle \to R\langle\langle X\rangle\rangle \otimes R\langle\langle X\rangle\rangle$, which is defined on letters $x \in X$ by $\Delta(x) = x \otimes 1 + 1 \otimes x$.

$L(z)$ is a Lie exponential

Suffices to prove this for a particular z.
Observation: $L(\varepsilon) = e^{x_0 \log \varepsilon} + O(\sqrt{\varepsilon})$.
Indeed, $L_w(\varepsilon) = O(\varepsilon^n \log^{|w|_1} \varepsilon)$, by induction.
Consequence: $L(x)$ “is a Lie exponential at the limit”.

Rigorous proof

Claim: let $T(z) = \Delta L(z) - L(z) \otimes L(z)$. Then

$$\begin{aligned} T'(z) &= \big(\Delta V(z)\big)\ T(z), \\ \lim_{\varepsilon \to 0+} T(\varepsilon) &= 0, \end{aligned}$$

where $V(z) = (\frac{x_0}{z} + \frac{x_1}{1-z})$.
Consequence: $T(z) = 0$ for all z by induction and integration.

Formula for T'

Clearly $\Delta(L') = (\Delta L)'$, whence

$$\begin{aligned} T' &= \Delta(L') - (L' \otimes L + L \otimes L') \\ &= \Delta(VL) - (VL \otimes L + L \otimes VL) \\ &= \Delta(V)\Delta(L) - (V \otimes 1 + 1 \otimes V)(L \otimes L) \\ &= \Delta(V)\Delta(L) - (\Delta V)(L \otimes L) \\ &= \Delta(V)\ [\Delta(L) - (L \otimes L)] \\ &= \Delta(V)\ T. \end{aligned}$$

Limit condition

$L(\varepsilon) = e^{x_0 \log \varepsilon} + O(\sqrt{\varepsilon})$ for $\varepsilon \to 0^+$ and
$e^{x_0 \log \varepsilon}$ is a Lie exponential. Therefore,

$$\begin{aligned} T(\varepsilon) &= \Delta L(\varepsilon) - L(\varepsilon) \otimes L(\varepsilon), \\ &= \Delta(O(\sqrt{\varepsilon})) - O(\sqrt{\varepsilon}) \otimes e^{x_0 \log \varepsilon} \\ &\quad - e^{x_0 \log \varepsilon} \otimes O(\sqrt{\varepsilon}) - O(\sqrt{\varepsilon}) \otimes O(\sqrt{\varepsilon}). \end{aligned}$$

Asymptotic expansions of L at 1

The claim

$L_w(1-\varepsilon)$ is a convergent series in $C[\log\varepsilon]_{|w|}[[\varepsilon]]$.
In view of the shuffle relation

$$L_{v \,\text{Ш}\, w}(z) = L_v(z) L_w(z).$$

we only need to treat the case when w is a Lyndon word.

Proof by induction on $|w|$

For x_0 and x_1:

$$\begin{aligned} L_{x_0}(1-\varepsilon) &= -\varepsilon - \tfrac{1}{2}\varepsilon^2 - \tfrac{1}{3}\varepsilon^3 - \cdots; \\ L_{x_1}(1-\varepsilon) &= \log\varepsilon. \end{aligned}$$

For longer Lyndon words $w = x_0 v \in x_0 X^* x_1$:

$$\begin{aligned} L_w(1-\varepsilon) &= \int_0^{1-\varepsilon} \frac{L_v(z)}{z} dz \\ &= \zeta_w - \int_{1-\varepsilon}^{1} \frac{L_v(z)}{z} dz \\ &= \zeta_w - \int_0^{\varepsilon} \frac{L_v(1-z)}{1-z} dz. \end{aligned}$$

Induction hypothesis $\Rightarrow$ convergent expansion for $L_v(1-z)$.
Integrate convergent asymptotic expansion of $\frac{L_v(1-z)}{1-z}$.

Expansion of $L_{x_1 x_0^2}(1-\varepsilon)$

Using the program `xtaylor`, we find

$$w = x_0 x_1^2 - x_1 \text{ш}\, x_0 x_1 + \frac{1}{2} x_0 \text{ш}\, x_1^{\text{ш}\, 2}.$$

We have

$$\begin{aligned}
L_{x_0}(1-\varepsilon) &= -\varepsilon - \tfrac{1}{2}\varepsilon^2 - O(\varepsilon^{5/2}); \\
L_{x_1}(1-\varepsilon) &= \log\varepsilon; \\
L_{x_0x_1}(1-\varepsilon) &= \zeta_{x_0x_1} - \int_0^{\varepsilon} \frac{\log z}{1-z} dz \\
&= \zeta_{x_0x_1} - (\log\varepsilon - 1)\varepsilon - (\tfrac{1}{2}\log\varepsilon - \tfrac{1}{4})\varepsilon^2 - O(\varepsilon^{5/2}); \\
L_{x_0x_1^2}(1-\varepsilon) &= \zeta_{x_0x_1^2} - \int_0^{\varepsilon} \frac{\log^2 z}{1-z} dz \\
&= \zeta_{x_0x_1^2} - (\log^2\varepsilon - 2\log\varepsilon + 2)\varepsilon - (\tfrac{1}{2}\log^2\varepsilon - \\
&\quad \tfrac{1}{2}\varepsilon + \tfrac{1}{4})\varepsilon^2 - O(\varepsilon^{5/2}).
\end{aligned}$$

Hence

$$\begin{aligned}
L_{x_1^2x_0}(1-\varepsilon) &= \zeta_{x_0x_1}\log\varepsilon + \zeta_{x_0x_1^2} \\
&- (\tfrac{5}{2}\log^2\varepsilon - 3\log\varepsilon + 2)\varepsilon \\
&- (\tfrac{5}{4}\log^2\varepsilon - \tfrac{3}{4}\log\varepsilon + \tfrac{1}{4})\varepsilon^2 \\
&- O(\varepsilon^{5/2}).
\end{aligned}$$

Monodromy of L

Chen series

γ: differentiable path $[0,1] \to \mathbb{C}\backslash\{0,1\}$ between a and b.

S_γ: evaluation in b of solution to

$$\begin{aligned} \frac{d}{dz}S(z) &= \left(\frac{x_0}{z} + \frac{x_1}{1-z}\right) S(z); \\ S(a) &= 1. \end{aligned}$$

$S_\gamma \in \mathbb{C}\langle\langle X \rangle\rangle$: Chen series.

Properties

S_γ is a Lie exponential, determined by the homotopy class of γ.

$$\begin{aligned} S_{\gamma_1\gamma_2} &= S_{\gamma_2} S_{\gamma_1}; \\ S_{\gamma^{-1}} &= S_\gamma^{-1}. \end{aligned}$$

Also

$$L(z) = S_{z_0 \rightsquigarrow z} L(z_0).$$

Estimations for Chen series around 0 and 1

$\gamma_0(R)$: circular path around 0 of radius R.

$\gamma_1(R)$: circular path around 1 of radius R.

By induction over $|w|$, assuming $R < 1/2$:

$$(S_{\gamma_0(R)}|w) \leqslant \frac{1}{|w|!}(2\pi)^{|w|}(2R)^{|w|_{x_1}}.$$

For $\varepsilon \to 0^+$, this estimate yields

$$\begin{aligned} S_{\gamma_0(\varepsilon)} &= e^{2i\pi x_0} + O(\varepsilon), \\ S_{\gamma_1(\varepsilon)} &= e^{-2i\pi x_1} + O(\varepsilon). \end{aligned}$$

Monodromy

$\mathcal{M}_0 L$: analytic continuation of L around 0.

$\mathcal{M}_1 L$: analytic continuation of L around 1.

$$\mathcal{M}_i L(t) = S_{\gamma_i(t)} L(t) \quad \text{for } i = 0, 1.$$

We will show how to compute Lie exponentials $M_0, M_1 \in \mathbb{C}\langle X \rangle$, which do not depend on t, such that $\forall t \in]0,1[$

$$\mathcal{M}_i L(t) = L(t) M_i, \quad \text{for } i = 0, 1.$$

Monodromy around 0

$$\begin{aligned}\mathcal{M}_0 L(t) &= S_{\varepsilon \rightsquigarrow t} S_{\gamma_0(\varepsilon)} S_{t \rightsquigarrow \varepsilon} L(t), \\ &= L(t)\ L^{-1}(\varepsilon) S_{\gamma_0(\varepsilon)} L(\varepsilon).\end{aligned}$$

This yields

$$M_0 = \lim_{\varepsilon \to 0+} L^{-1}(\varepsilon) S_{\gamma_0(\varepsilon)} L(\varepsilon) = e^{2i\pi x_0}.$$

Indeed, for $\varepsilon \to 0$, we have commutation of

$$\begin{aligned}S_{\gamma_0(\varepsilon)} &= e^{2i\pi x_0} + O(\varepsilon) \quad \text{and} \\ L(\varepsilon) &= e^{(\log \varepsilon) x_0} + O(\sqrt{\varepsilon}).\end{aligned}$$

Monodromy around 1

$$\begin{aligned}\mathcal{M}_1 L(t) &= S_{1-\varepsilon \rightsquigarrow t} S_{\gamma_1(\varepsilon)} S_{t \rightsquigarrow 1-\varepsilon} L(t), \\ &= L(t)\ L^{-1}(1-\varepsilon) S_{\gamma_1(\varepsilon)} L(1-\varepsilon),\end{aligned}$$

whence

$$M_1 = \lim_{\varepsilon \to 0+} L^{-1}(1-\varepsilon) e^{-2i\pi x_1} L(1-\varepsilon).$$

By abstract nonsense, this limit exists.
Algorithm: using that $L(1-\varepsilon) \in \mathbb{R}[\log \varepsilon][[\varepsilon]]\langle\langle X \rangle\rangle$.

Efficient algoritm

Factorization of a Lie exponential S

$$S = \sum_{w \in X^*} (S|w)\ w \prod_{l \in \mathcal{L}yndon(X)}^{\searrow} e^{(S|P^*(l))P(l)}.$$

Example at order 4

$$\begin{aligned} L(z) &= e^{L_{x_1}(z)x_1} e^{L_{x_0x_1x_1}(z)[[x_0,x_1],x_1]} e^{L_{x_0x_1}(z)[x_0,x_1]} \\ &\times e^{L_{x_0x_0x_1}(z)[x_0,[x_0,x_1]]} e^{L_{x_0}(z)x_0}. \end{aligned}$$

Formula for M_1

$$\begin{aligned} M_1 &= Z^{-1} e^{-2i\pi x_1} Z \\ Z &= \prod_{l \notin \{x_0,x_1\}}^{\searrow} e^{\zeta_{P^*(l)} P(l)} \end{aligned}$$

Here $P^*(l) \in x_0 \mathbb{Z}\langle X \rangle x_1$ for $l \notin \{x_0, x_1\}$.

Bracket form

$l \in \mathcal{L}yndon(X)$: Lyndon word.

$l = uv$ with v as long as possible and $u, v \in \mathcal{L}yndon(X)$.

bracket form $P(l)$ of l:

$$\begin{aligned} P(l) &= [P(u), P(v)] \\ P(x) &= x \quad \text{for each letter } x \in X, \end{aligned}$$

$B_1 = \{P(l);\ l \in \mathcal{L}yndon(X)\}$: basis for the free Lie algebra.

Bases for the free Lie algebra

$$w = l_1^{\alpha_1} l_2^{\alpha_2} \ldots l_k^{\alpha_k}, \quad l_1 > l_2 > \cdots > l_k, \quad k \geqslant 0$$

unique factorization of $w \in X^*$ as a $\searrow$ product of Lyndon words.

$$\begin{aligned} P(w) &= P(l_1)^{\alpha_1} P(l_2)^{\alpha_2} \ldots P(l_k)^{\alpha_k}, \\ P^*(w) &= C P^*(l_1)^{ш\, \alpha_1}\ ш \ldots ш\ P^*(l_k)^{ш\, \alpha_k}, \\ &\quad \text{where } C = (\alpha_1! \alpha_2! \ldots \alpha_k!)^{-1} \\ P^*(l) &= x P^*(w), \quad \forall l \in \mathcal{L}yndon(X), \\ &\quad \text{where } l = xw,\ x \in X,\ w \in X^*. \end{aligned}$$

$B = \{P(w);\ w \in X^*\}$: Poincaré–Birkoff–Witt basis.

$B^* = \{P^*(w);\ w \in X^*\}$: dual basis.

l	$P(l)$	$P^*(l)$
x_0	x_0	x_0
x_1	x_1	x_1
x_0x_1	$[x_0, x_1]$	x_0x_1
$x_0{}^2x_1$	$[x_0, [x_0, x_1]]$	$x_0{}^2x_1$
$x_0x_1{}^2$	$[[x_0, x_1], x_1]$	$x_0x_1{}^2$
$x_0{}^3x_1$	$[x_0, [x_0, [x_0, x_1]]]$	$x_0{}^3x_1$
$x_0{}^2x_1{}^2$	$[x_0, [[x_0, x_1], x_1]]$	$x_0{}^2x_1{}^2$
$x_0x_1{}^3$	$[[[x_0, x_1], x_1], x_1]$	$x_0x_1{}^3$
$x_0{}^4x_1$	$[x_0, [x_0, [x_0, [x_0, x_1]]]]$	$x_0{}^4x_1$
$x_0{}^3x_1{}^2$	$[x_0, [x_0, [[x_0, x_1], x_1]]]$	$x_0{}^3x_1{}^2$
$x_0{}^2x_1x_0x_1$	$[[x_0, [x_0, x_1]], [x_0, x_1]]$	$2x_0{}^3x_1{}^2 + x_0{}^2x_1x_0x_1$
$x_0{}^2x_1{}^3$	$[x_0, [[[x_0, x_1], x_1], x_1]]$	$x_0{}^2x_1{}^3$
$x_0x_1x_0x_1{}^2$	$[[x_0, x_1], [[x_0, x_1], x_1]]$	$3x_0{}^2x_1{}^3 + x_0x_1x_0x_1{}^2$
$x_0x_1{}^4$	$[[[[x_0, x_1], x_1], x_1], x_1]$	$x_0x_1{}^4$
$x_0{}^5x_1$	$[x_0, [x_0, [x_0, [x_0, [x_0, x_1]]]]]$	$x_0{}^5x_1$
$x_0{}^4x_1{}^2$	$[x_0, [x_0, [x_0, [[x_0, x_1], x_1]]]]$	$x_0{}^4x_1{}^2$
$x_0{}^3x_1x_0x_1$	$[x_0, [[x_0, [x_0, x_1]], [x_0, x_1]]]$	$2x_0{}^4x_1{}^2 + x_0{}^3x_1x_0x_1$
$x_0{}^3x_1{}^3$	$[x_0, [x_0, [[[x_0, x_1], x_1], x_1]]]$	$x_0{}^3x_1{}^3$
$x_0{}^2x_1x_0x_1{}^2$	$[x_0, [[x_0, x_1], [[x_0, x_1], x_1]]]$	$3x_0{}^3x_1{}^3 + x_0{}^2x_1x_0x_1{}^2$
$x_0{}^2x_1{}^2x_0x_1$	$[[x_0, [[x_0, x_1], x_1]], [x_0, x_1]]$	$6x_0{}^3x_1{}^3 + 3x_0{}^2x_1x_0x_1{}^2 + x_0{}^2x_1{}^2x_0x_1$
$x_0{}^2x_1{}^4$	$[x_0, [[[[x_0, x_1], x_1], x_1], x_1]]$	$x_0{}^2x_1{}^4$
$x_0x_1x_0x_1{}^3$	$[[x_0, x_1], [[[x_0, x_1], x_1], x_1]]$	$4x_0{}^2x_1{}^4 + x_0x_1x_0x_1{}^3$
$x_0x_1{}^5$	$[[[[[x_0, x_1], x_1], x_1], x_1], x_1]$	$x_0x_1{}^5$

Table 1: Lyndon words, bracket forms and the dual basis

Some results

A classical result

$$\mathcal{M}_1 \operatorname{Li}_k(z) = \operatorname{Li}_k(z) - 2i\pi \frac{\log^{k-1}(z)}{(k-1)!}, \quad k > 0.$$

Computational results

$$\begin{aligned}
\mathcal{M}_1 L_{x_1} &= L_{x_1} - 2\pi i; \\
\mathcal{M}_1 L_{x_0 x_1^2} &= L_{x_0 x_1^2} - 2i\pi L_{x_0 x_1} - 2\pi^2 L_{x_0} + 2i\pi \zeta_{x_0 x_1}; \\
\mathcal{M}_1 L_{x_0 x_1 x_0 x_1^3} &= L_{x_0 x_1 x_0 x_1^3} - 2i\pi L_{x_0 x_1 x_0 x_1^2} + \\
&\quad 4\pi^2 L_{x_0^2 x_1^2} - \tfrac{8i\pi^3}{3} L_{x_0^2 x_1} - \pi^2 L^2_{x_0 x_1} + \\
&\quad \left(\tfrac{4i\pi^3}{3} L_{x_0} + 2i\pi \zeta_{x_0 x_1^2} + 2\pi^2 \zeta_{x_0 x_1}\right) L_{x_0 x_1} + \\
&\quad \left(-8i\pi \zeta_{x_0 x_1^3} - 4\pi^2 \zeta_{x_0 x_1^2} + \tfrac{4i\pi^3}{3} \zeta_{x_0 x_1}\right) L_{x_0} + \\
&\quad -4\pi^2 \zeta_{x_0^2 x_1^2} + \tfrac{8i\pi^3}{3} \zeta_{x_0^2 x_1} + 2i\pi \zeta_{x_0 x_1 x_0 x_1^2} - \pi^2 \zeta^2_{x_0 x_1}.
\end{aligned}$$

Linear independence

Given $n \geqslant 0$, assume that we have a $\mathbb{C}$-linear relation

$$(1) \qquad \sum_{|w| \leqslant n} \lambda_w L_w = 0, \quad \lambda_w \in \mathbb{C}$$

Induction on n: $\lambda_w = 0$ for all w. Rewrite (1) as

$$(2) \qquad \lambda_1 + \sum_{|u|<n} \lambda_{ux_0} L_{ux_0} + \sum_{|u|<n} \lambda_{ux_1} L_{ux_1} = 0.$$

Now

$$\begin{aligned} M_0 &= 1 + 2i\pi x_0 + \text{words of length} > 1 \\ M_1 &= 1 - 2i\pi x_1 + \text{words of length} > 1 \end{aligned}$$

Applying $(\mathcal{M}_0 - Id)$ and $(Id - \mathcal{M}_1)$ on (2),

$$\begin{aligned} 2i\pi \sum_{|u|=n-1} \lambda_{ux_0} L_u + \sum_{|u|<n-1} \mu_u L_u &= 0, \\ 2i\pi \sum_{|u|=n-1} \lambda_{ux_1} L_u + \sum_{|u|<n-1} \nu_u L_u &= 0, \end{aligned}$$

for certain coefficients μ_u and ν_u.
Induction hypothesis $\Rightarrow$ λ_{ux_0} and λ_{ux_1} vanish for $|u| = n-1$.
Consequently,

$$\sum_{|w| \leqslant n-1} \lambda_w L_w = 0,$$

whence $\lambda_w = 0$ for all w, again by the induction hypothesis.