Undecidability versus undecidability

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Prague, 10-8-1998

Main problems

Analytic continuation

Given an algebraic differential equation with complete initial conditions, how to (efficiently) compute the value of the solution in a point up to any desired precision ?

Asymptotic behaviour

Given an algebraic differential equation in x, compute all possible asymptotic behaviours of solutions for $x \to \infty$.

The linear case

The equation

$$L_r(z)f^{(r)}(z) + \cdots + L_0(z)f(z) = 0,$$

with $L_i(z) \in \mathbb{Q}[i][z]$ and $L_0(z) \neq 0$.
Only singularities: zeros of $L_r(z)$.

Analytic continuation

Let $z \rightsquigarrow z'$ be a nonsingular path with $z, z' \in \mathbb{Q}[i]$.
Transition matrix $M_{z \rightsquigarrow z'}$:

$$\begin{pmatrix} f(z') \\ \vdots \\ f^{r-1}(z') \end{pmatrix} = M_{z \rightsquigarrow z'} \begin{pmatrix} f(z) \\ \vdots \\ f^{r-1}(z) \end{pmatrix}.$$

Theorem: $M_{z \rightsquigarrow z'}$ can be approximated up to n decimal digits in time $O(n \log^3 n \log \log n)$.

Questions

What if $z \rightsquigarrow z'$ passes through singularities?
Complexity if $z \rightsquigarrow z'$ approaches a singularity.

Asymptotic expansions in 0

Basis of formal generalized series solutions:

$$f(z) = (f_0 + \cdots + f_{r-1} \log^{r-1} z) z^\alpha e^P,$$

where

$$\begin{cases} p \in \mathbb{N}^*; \\ \alpha \in \mathbb{C}; \\ f_0, \ldots, f_{r-1} \in \mathbb{C}[[\sqrt[p]{z}]]; \\ P \in \mathbb{C}[[\sqrt[p]{z^{-1}}]]. \end{cases}$$

Regular singular case

If L is a regular singular ($\Rightarrow$ the f_i are convergent), then the analytic continuation result generalizes.

General case

Treated via resummation or accelero-summation (Écalle).

Example of resummation

$$\tilde{f} = \sum_{n \geqslant 1} f_n z^n = \sum_{n \geqslant 1} (-1)^n (n-1)! z^n.$$

Apply the formal Borel transform

$$\hat{f} = \tilde{\mathcal{B}} \tilde{f} = \sum_{n \geqslant 0} \frac{f_{n+1}}{n!} \zeta^n = \frac{1}{1+\zeta}.$$

Apply the analytical Laplace transform

$$f(z) = (\mathcal{L}\hat{f})(z) = \int_0^\infty \hat{f}(\zeta) e^{-\zeta/z} d\zeta = \int_0^\infty \frac{e^{-\zeta/z}}{1+\zeta} d\zeta.$$

f defined for all z with $\Re z > 0$.

The differential equation

Both $\tilde{f}$ and f satisfy

$$z^2 f' + f = z.$$

Transseries

Examples

$$
\begin{aligned}
f &= e^{e^x} + 2e^{e^x - \log^2 x} + 6e^{e^x - 2\log^2 x} + \cdots;\\
g &= 1 + x^{-1}e^x + x^{-2}e^x + \cdots + e^{-x} + x^{-1}e^{-x} + \cdots + \cdots;\\
h &= e^{e^x + x^{-1}e^x + \cdots} + x^{-1}e^{e^x + x^{-1}e^x + \cdots} + \cdots.
\end{aligned}
$$

Advantages

– Transseries describe violent singularities.

– Rich algebraic structure.

– Analysis through accelero-summation.

Drawback

They only describe strongly monotonic asymptotic behaviour.

Exp-log functions

Constructed from $\mathbb{Q}$ and x by $+, -, \times, /, \exp$ and log.

Assume a zero test for exp-log constants.

There exists an asymptotic expansion program for exp-log functions.

Exp-log systems

Given an asymptotical exp-log system in several variables, like

$$\begin{aligned} e^{x+e^y} - e^{y+e^x} &= xy + \log^y x; \\ 1 &\prec\!\!\prec x; \\ x &\preccurlyeq\!\!\prec e^y. \end{aligned}$$

There exists a "desingularization" algorithm:

The solution set S in $\mathbb{T}$ is written

$$S = S_1 \amalg \cdots \amalg S_n.$$

Each S_i determined by a formula depending on

- A finite number of infinitesimal transseries parameters.
- A finite number of real parameters.
- Exp-log constraints on the real parameters.

$\longrightarrow$ Generalization to more general transseries systems.

Differential equations

Given an algebraic differential equation.

There exists a generic resolution algorithm in $\mathbb{T}$:

The solution set S in $\mathbb{T}$ is written

$$S = S_1 \amalg \cdots \amalg S_n.$$

Each S_i determined by a formula depending on

- A finite number of infinitesimal transseries parameters.
- A finite number of real parameters.
- First order exp-log constraints on the real parameters.

Intermediate value theorem

Let P be a differential polynomial over $\mathbb{T}$.

Assume $P(f) < 0$ and $P(g) > 0$ for $f < g \in \mathbb{T}$.

Then there exists a $h \in \mathbb{T}$ with $f < h < g$ and $P(h) = 0$.

Consequence:

$$f'' f^{1998} + e^{e^x} f f' f''' - \Gamma(\log^{10} x) = e^{\log^8 x}$$

admits a solution in $\mathbb{T}$.

Undecidable? I can decide

Grigoriev and Singer

The problem whether a given system of algebraic differential equations admits a power series solution (with generalized exponents) is undecidable.

Idea

Consider the system Σ:

$$\begin{aligned} y'x &= \beta y; \\ \beta' &= 0; \\ z'yx + z''x^2 &= x + y. \end{aligned}$$

Σ admits a solution iff $n^{-1} \in \mathbb{N}$.

But...

The system admits always a solution if we allow logarithms.

Compare: resolution polynomial equations through radicals.

Undecidable? II can decide

Simple trigonometric systems

Language: $(\mathbb{Q}, +, -, \times)$ and $\varphi_{a,b}(x) = a \sin(x + b)$,
for each variable x and $a, b \in \mathbb{Q}$.
Equations in this language are undecidable.

Game

You give a simple trigonometric system and $\varepsilon > 0$.
I may change each $\varphi_{a,b}$ in the equations into $\varphi_{a',b'}$,
with $|a' - a| < \varepsilon$ and $|b' - b| < \varepsilon$.
Equations can be solved with probability 1.

Conclusions

– Ill posed problems are undecidable.

– The undecidable instances of an undecidable problem in real analysis are "singular".

Decidable? Please show me...

Sine-exp bombs

$$\sin(10^{10^{10^{10}}}) > 0?$$

Limsup

Determine

$$\limsup_{x \to \infty} \frac{\sin(10^{10^{10}} x) - e^{\cos x}}{3 + \sin^4 x}.$$

Idea: rewrite $\sin(10^{10^{10}} x)$ in terms of $\sin x$...

Algebraic systems

Random system of 100 polynomial equations in 100 unknowns.

Other classical example

Factor $11^{1000} + 2$.

Must we decide?

A Diophantine problem

Asymptotic expansion of

$$\exp\exp[(\sin(x) + \sin(e^\pi x) + e^{-x} - 2)x]?$$

Note

The expansion of $e^{e^{\lambda x}}$ (λ parameter) is

$$\begin{cases} e^{e^{\lambda x}}, & \text{if } \lambda > 0; \\ e, & \text{if } \lambda = 0; \\ 1 + e^{\lambda x} + \frac{1}{2}e^{2\lambda x} + \cdots, & \text{otherwise.} \end{cases}$$

Approach

Distinguish three cases depending on sign of

$$\sin(x) + \sin(e^\pi x) + e^{-x} - 2;$$

Substitute value for $x \longrightarrow$ we know in which case we are.

Finiteness of number of cases?

Reduction: weakly oscillatory $\longrightarrow$ strongly monotonic.

Decidable? Undecidable?

Constant problems

Test whether an exp-log constant vanishes.

Test whether an algebraic differential constant vanishes.

Richardson's theorem

If Schanuel's conjecture holds,

then there exists a zero test for exp-log constants.

Conjecture. **(Schanuel)** *Let* $\alpha_1, \cdots, \alpha_n \in \mathbb{C}$ *be* $\mathbb{Q}$*-linearly independent. Then*

$$\operatorname{tr} \deg_{\mathbb{Q}} \mathbb{Q}[\alpha_1, \cdots, \alpha_n, e^{\alpha_1}, \cdots, e^{\alpha_n}] \geqslant n.$$

Consequence

If Schanuel's conjecture holds, then many strongly monotonic asymptotical problems can be solved effectively.

Some conjectures

Exp-log conjecture

N: be some large number ($N > 10$).

$\mathfrak{E}_N$: the set of exp-log constants constructed as follows:

- $1 \in \mathfrak{E}_N$.
- $\mathfrak{E}_N$ stable under $+, -, \times, /, \log$.
- If $c \in \mathfrak{E}_N$ with $N^{-1} < c < N$, then $\exp c \in \mathfrak{E}_N$.

Conjecture: Let $c \in \mathfrak{E}_N$ be of size s as a tree.
Then $\exists K_N$, such that $c = 0$ iff $|c| < e^{-e^{K_N s}}$.
Fine tuning: $e^{-\varphi_N(s)}$ instead of $e^{-e^{K_N s}}$.

Power series analogue

Consider the following class $\mathfrak{S}$ of power series:

- $K \subseteq \mathfrak{S}$, $z \in \mathfrak{S}$.
- $\mathfrak{S}$ stable under $+, -, \times$.
- $\mathfrak{S}$ stable under $\operatorname{zinv}, \operatorname{zlog}, \exp$:
  $\operatorname{zinv}(z) = (1+z)^{-1}, \operatorname{zlog}(z) = \log(1+z)$.

Question: How many terms of $f \in \mathfrak{S} \setminus \{0\}$ can vanish as a function of the size f.

Algebraic differential analogue

Non standard size function for constant expressions:

- $\text{size}(1) = 1$.
- $\text{size}(x + y) = \text{size}(x - y) = \text{size}(xy) = \text{size}(x/y) =$
  $\text{size}(x) + \text{size}(y) + 1$.
- f: differential algebraic function near 0.
  $\text{size}(f)$: size of equation+ initial coefficients.

$$\text{size}(f(x)) = \text{size}(f) + \text{size}(x) + \max(0, \lceil \log \sup_{|u| \leqslant |z|} |f(u)| \rceil).$$

Conjecture: as above with non standard size function.

Final approach

– Find the right problem in the right context.

– Theoretically decidable cases?

– Theoretically undecidable cases are singular?

– Theoretically decidable cases modulo oracles?

– Decidability via a finite number of possible cases?

– Complexities of phenomena involved?

– Better complexities using plausible heuristics?

– Practically decidable cases?

– Decidability via a finite number of possible cases?