Complex transseries

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Successes of analyzable functions

- First order singular (but not too singular) systems.
- Real transseries and analyzable functions.

The quest of new successes

- Complex transseries and analyzable functions.
- Multivariate case and o-minimality.
- Partial differential eqations.

Real transseries

- Accelero-summation and well-behaved averages.
- Écalle's proof of Dulac's conjecture.
- Algorithm to solve any asymptotic algebraic differential equation

$$P(f)=0 \quad (f \prec \mathfrak{m}).$$

- Intermediate value theorem. Example:

$$P(f)=f^7+e^{e^x} f^3 f'''+\Gamma(\log\Gamma(x)+1)=0$$

- Extension to differential-difference equations

$$f(e^{\log^2 x})\, f''(x^2)\, f(q\,x)+e^{e^x} f(x)^2+f(x+1)+\log x=0.$$

Well-ordered power series

- Totally ordered constant field C.
- Monomial group $\mathfrak{M}$, with total ordering $\succcurlyeq$.
- [Hahn 1907] Set of well-ordered series

$$C[[\mathfrak{M}]] = \{f : \mathfrak{M} \rightarrow C \,|\, \operatorname{supp} f \text{ is well-ordered}\}$$

  forms a totally ordered field.
- $f = c_f\, \mathfrak{d}_f\, (1 + \delta_f)$
- $f \preccurlyeq g \Leftrightarrow \mathfrak{d}_f \preccurlyeq \mathfrak{d}_g$
- Canonical decomposition:

$$\begin{array}{ccccccc} f = & f^{\uparrow} & + & f^{=} & + & f^{\downarrow} \\ & \| & & \| & & \| \\ & \displaystyle\sum_{\mathfrak{m} \succ 1} f_{\mathfrak{m}}\, \mathfrak{m} & & f_1 & & \displaystyle\sum_{\mathfrak{m} \prec 1} f_{\mathfrak{m}}\, \mathfrak{m} \end{array}$$

Grid-based series

f grid-based $\Longleftrightarrow \exists\ \mathfrak{m}_1, \ldots, \mathfrak{m}_k \prec 1$ and $\mathfrak{n}$ with

$$\operatorname{supp} f \subseteq \{\mathfrak{m}_1, \ldots, \mathfrak{m}_k\}^* \,\mathfrak{n}.$$

$C [\![\mathfrak{M}]\!] \subseteq C[[\mathfrak{M}]]$: field of grid-based series.

Example

For $f = x^2 + x + 1 + x^{-1} + \cdots$, we have $\operatorname{supp} f \subseteq \{x^{-1}\}^* \,x^2$.

Construction of the field of real transseries

Logarithmic transseries

Start with monomial group

$$\mathfrak{L} = \mathfrak{E}_0 = \{x^{\alpha_0} (\log x)^{\alpha_1} (\log \log x)^{\alpha_2} \cdots (\log_l x)^{\alpha_l} \colon \alpha_0, \ldots, \alpha_l \in \mathbb{R}\}$$

and logarithm on $\mathbb{R}[[\mathfrak{L}]]^{+}_{*}$:

$$\log (c\, x^{\alpha_0} \cdots \log_l^{\alpha_l} x (1 + \delta)) = \\ \log c + \alpha_0 \log x + \cdots + \alpha_l \log_{l+1} x + \log (1 + \delta).$$

Inductive step

Assume $\mathfrak{E}_n$ given, with logarithm on $\mathbb{R}[[\mathfrak{E}_n]]_*^+$.

$$\mathfrak{E}_{n+1} = \exp \mathbb{R}[[\mathfrak{E}_n]]^{\uparrow},$$

with

$$\exp f^{\uparrow} \succcurlyeq \exp g^{\uparrow} \Leftrightarrow f \geqslant g.$$

Take

$$\log (c\, e^{f^{\uparrow}} (1+\delta)) = \log c + f^{\uparrow} + \log (1+\delta).$$

Inductive limit: $\mathbb{T} = C[[\mathfrak{E}_0 \cup \mathfrak{E}_1 \cup \cdots]]$.

Example

$e^{e^x(1+\frac{1}{x}+\frac{1}{x^2}+\cdots)} \in \mathfrak{E}_2$.

Series with complex coefficients

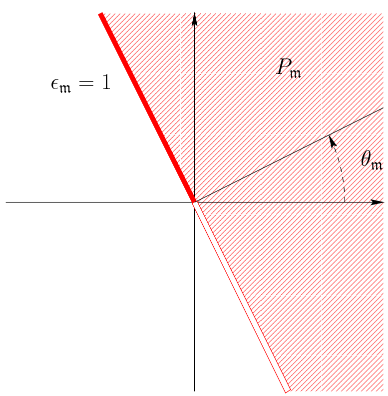
- For each $\mathfrak{m} \in \mathfrak{M}$, select a set of "positive constants"

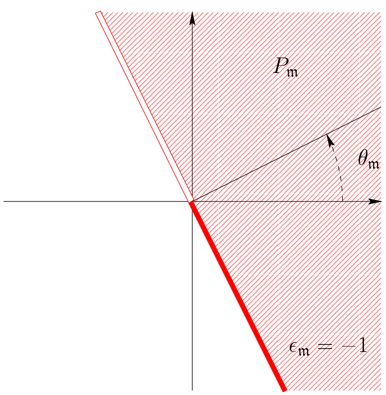
$$P_{\mathfrak{m}} = \{c \in \mathbb{C} | (\operatorname{Re}(c\, e^{-i\theta_{\mathfrak{m}}}) > 0) \vee (\operatorname{Re}(c\, e^{-i\theta_{\mathfrak{m}}}) = 0 \wedge \operatorname{Im}(\epsilon_{\mathfrak{m}}\, c\, e^{-i\theta_{\mathfrak{m}}}) > 0)\}.$$

- For $f \in \mathbb{C}[[\mathfrak{M}]]^{\neq}$, define $f > 0 \Longleftrightarrow c_f \in P_{\mathfrak{d}(f)}$.

$\longrightarrow$ $\mathbb{C}[[\mathfrak{M}]]$ is a totally ordered (strong) vector space.

$\longrightarrow$ $\exp \mathbb{C}[[\mathfrak{M}]]^{\uparrow}$ is a monomial group.





Construction of field of complex transseries

- Many possible choices of the $\theta_{\mathfrak{m}}$ and $\epsilon_{\mathfrak{m}}$:
  - $\mathfrak{L} \longrightarrow \mathfrak{L}_{\boldsymbol{\theta},\boldsymbol{\epsilon}}$
  - $\mathfrak{E}_n \longrightarrow \mathfrak{E}_{n,\boldsymbol{\theta},\boldsymbol{\epsilon}}$
- Under the assumption that for all $i \geqslant i_0$ we have
  - $\mathfrak{d}(\log_{i+1} \mathfrak{m}) = \log \mathfrak{d}(\log_i \mathfrak{m})$.
  - $\theta_{\mathfrak{d}(\log_i \mathfrak{m})} = 0$.

  the construction is (algebraically) unique modulo "turn-flips":

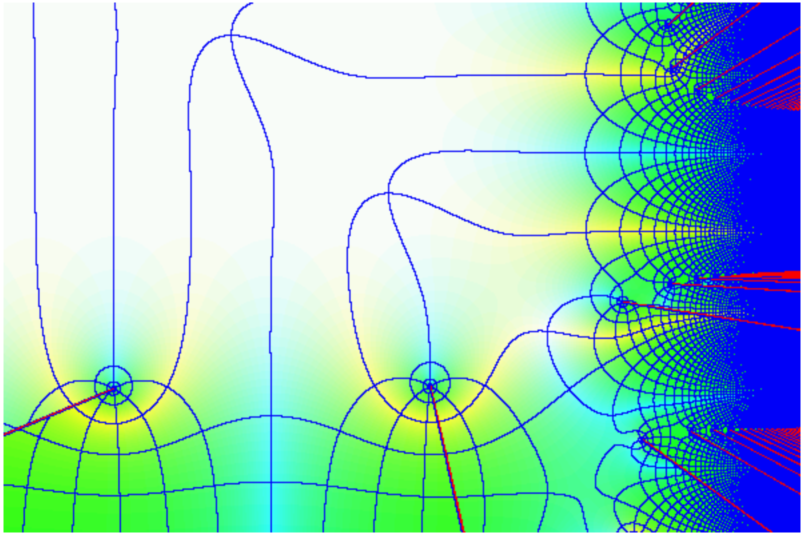
$$\begin{aligned}\varphi\colon \sum_{\mathfrak{m}} f_{\mathfrak{m}}\, \mathfrak{m} &\longmapsto \sum_{\mathfrak{m}} e^{i\xi_{\mathfrak{m}}}\, \iota_{\epsilon_{\mathfrak{m}} \varsigma_{\mathfrak{m}}}(f_{\mathfrak{m}}\, e^{-i\theta_{\mathfrak{m}}})\, \mathfrak{m}\,;\\ \hat{\varphi}\colon \sum_{\mathfrak{m}} f_{\mathfrak{m}}\, \mathfrak{m} &\longmapsto \sum_{\mathfrak{m}} e^{i\xi_{\mathfrak{m}}}\, \iota_{\epsilon_{\mathfrak{m}} \varsigma_{\mathfrak{m}}}(f_{\mathfrak{m}}\, e^{-i\theta_{\mathfrak{m}}})\, e^{\varphi(\log \mathfrak{m})},\end{aligned}$$

  where $\iota_1(z) = z$ and $\iota_{-1}(z) = \bar{z}$.

An explicit example

Expansion of the exp-log function

$$f = \log\left(e^{e^z + iz} + e^{ie^z}\right)$$



If $e^z \succ 1$ and $e^{e^z+iz} \succ e^{ie^z}$, then

$$\begin{aligned} f &= e^z + i\,z + \log\left(1 + e^{(i-1)e^z - iz}\right) \\ &= e^z + i\,z + e^{(i-1)e^z - iz} + \tfrac{1}{2}\,e^{2(i-1)e^z - 2iz} + \cdots \in \mathbb{C}\,[\![z; e^z; e^{e^z+iz}]\!]. \end{aligned}$$

If $e^z \succ 1$ and $e^{e^z+iz} \prec e^{ie^z}$, then

$$\begin{aligned} f &= i\,e^z + \log\left(1 + e^{(1-i)e^z + iz}\right) \\ &= i\,e^z + e^{(1-i)e^z + iz} + \tfrac{1}{2}\,e^{2(1-i)e^z + 2iz} + \cdots \in \mathbb{C}\,[\![z; e^z; e^{e^z+iz}]\!]. \end{aligned}$$

If $e^z \prec 1$ and $e^{iz} \succ 1$, then

$$\begin{aligned} f &= i\,z + \log\left(1 + (e^{e^z} - 1) + e^{-iz}\,e^{ie^z}\right) \\ &= i\,z + e^z + e^{-iz} + (i-1)\,e^{(1-i)z} - \tfrac{1}{2}\,e^{-2iz} + \cdots \in \mathbb{C}\,[\![z; e^z]\!]. \end{aligned}$$

If $e^z \prec 1$ and $e^{iz} \prec 1$, then

$$\begin{aligned} f &= \log\left(1 + (e^{ie^z} - 1) + e^{iz}\,e^{e^z}\right) \\ &= i\,e^z - e^{iz} + (1+i)\,e^{(1+i)z} - \tfrac{1}{2}\,e^{2iz} + \cdots \in \mathbb{C}\,[\![z; e^z]\!]. \end{aligned}$$

Algebraic differential equations

$\mathbb{T}$ good candidate for an existentially closed H-field (without ordering):

Theorem 1. *Consider an asymptotic algebraic differential equation*

$$P(f) = 0 \quad (f \prec \mathfrak{m}) \tag{1}$$

of Newton degree d, with coefficients in $\mathbb{C}[[\mathfrak{b}_1; \ldots; \mathfrak{b}_n]] \subseteq \mathbb{T}$. Then there exist at least d solutions when counting with multiplicities. Moreover, these solutions are all in $\mathbb{C}[[\log_l \mathfrak{b}_1; \ldots, \log \mathfrak{b}_1; \mathfrak{b}_1; \ldots; \mathfrak{b}_n]]$ for some l.

Corollary 2. *The field of complex transseries is Picard-Vessiot closed (but not differentially closed).*

Theorem 3. *There exists an algorithm to find the general solution to (1) in the field of complex transseries (which depends on parameters satisfying real algebraic constraints). The logarithmic depth of this general solution is uniformly bounded in terms of the complexity of the equation.*

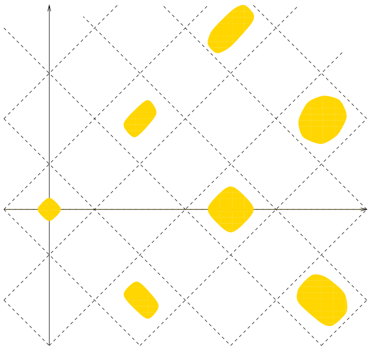
Multiple Stokes phenomenon

Consider the linear differential equation

$$f' = e^{-e^{(1-i)z}}\left(1 + e^{-e^{(1+i)z}}\right) + f^2$$

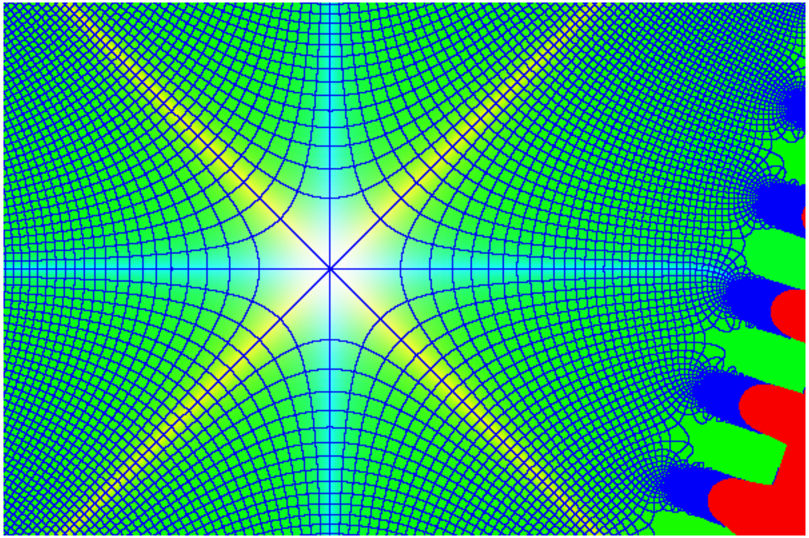
Four types of regions:

	$e^{e^{(1-i)z}} \succ 1$	$e^{e^{(1-i)z}} \prec 1$
$e^{e^{(1+i)z}} \succ 1$	R_1	R_2
$e^{e^{(1+i)z}} \prec 1$	R_3	R_4



Numerical example

$$f'' = e^z f' + e^{(2+i)z} f + 1$$



Accelero-summation

$$
\begin{array}{ccccc}
\tilde{f}(z)=\tilde{f}(z_1) & & & & f(z_n)=f(z) \\
\downarrow \tilde{\mathcal{B}} & & & & \uparrow \mathcal{L} \\
\hat{f}(\zeta_1) & \xrightarrow{\mathcal{A}} & \cdots & \xrightarrow{\mathcal{A}} & \hat{f}(\zeta_n)
\end{array}
$$

Parallel summation

$$
\tilde{f}(z)=\tilde{f}(z_1,\ldots,z_n) \;\; \begin{array}{c} \nearrow \\ \xrightarrow{\tilde{\mathcal{B}}} \\ \searrow \end{array} \;\; \hat{f}(\zeta_1,\ldots,\zeta_n) \;\; \begin{array}{c} \searrow \\ \xrightarrow{\mathcal{L}} \\ \nearrow \end{array} \;\; f(z_1,\ldots,z_n)=f(z)
$$

Some evidence

Explicit integrals:

$$f(z) = \iint \frac{e^{-\zeta_1 z^{1+i} - \zeta_2 z^{1-i}}}{(1+\zeta_1)(2+\zeta_2)} \, \mathrm{d}\zeta_1 \, \mathrm{d}\zeta_2.$$

Solving differential equations

Usual dictionnary:

$$\begin{array}{rcl} f\,g & \longmapsto & \hat{f} * \hat{g} \\ z_i\, f & \longmapsto & \hat{f}_{\zeta_i} \\ f_{z_i} & \longmapsto & -\zeta_i\, \hat{f} \\ \delta_i\, f & \longmapsto & -\delta_i\, \hat{f} - \hat{f} \end{array}$$

For $z_1 = z^{\tau_1}, \ldots, z_n = z^{\tau_n}$ with $\tau_1, \ldots, \tau_n \in \mathbb{C}^{\neq}$, we get

$$\delta \longmapsto -\tau_1\, \delta_1 - \cdots - \tau_n\, \delta_n - (\tau_1 + \cdots + \tau_n)$$

Resurgence monomials

$$\begin{aligned} f &= e^{k_1 z^{\sigma_1}+\cdots+k_n z^{\sigma_n}} \int e^{k_1 z^{\sigma_1}+\cdots+k_n z^{\sigma_n}} g \\ \delta f &= (k_1 \sigma_1 z^{\sigma_1}+\cdots+k_n \sigma_n z^{\sigma_n})\, f+g \end{aligned}$$

Set $\tau_1=\sigma_n-\sigma_1, \ldots, \tau_{n-1}=\sigma_n-\sigma_{n-1}, \tau_n=\sigma_n$ and rewrite as

$$\begin{aligned} \delta f &= z^{\tau_n}\left(\frac{k_1 \sigma_1}{z^{\tau_1}}+\cdots+\frac{k_{n-1}\sigma_{n-1}}{z^{\tau_{n-1}}}+k_n \sigma_n\right) f+g \\ &= z_n\left(\frac{k_1 \sigma_1}{z_1}+\cdots+\frac{k_{n-1}\sigma_{n-1}}{z_{n-1}}+k_n \sigma_n\right) f+g \end{aligned}$$

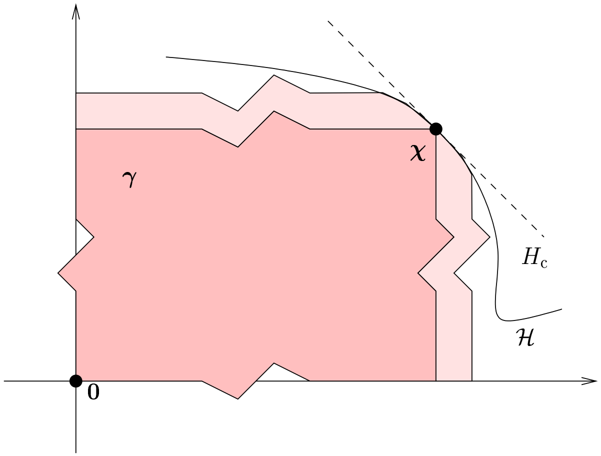
After transformation:

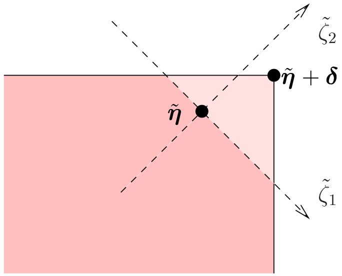
$$\begin{aligned} &\tau_1 \zeta_1 \hat{f}_{\zeta_1}+\cdots+\tau_{n-1}\zeta_{n-1}\hat{f}_{\zeta_{n-1}}+\tau_n(\zeta_n+k_n)\hat{f}_{\zeta_n}+ \\ &(k_1\sigma_1 I_1+\cdots+k_{n-1}\sigma_{n-1}I_{n-1}+(\tau_1+\cdots+\tau_n) I_n)*\hat{f}_{\zeta_n}=-\hat{g} \end{aligned}$$

General equations

$$\begin{aligned} f' &= e^{-z^2}+e^{-z^{1+i}}+f^2 \\ f &= \int e^{-z^2}+\int e^{-z^{1+i}}+\int\left(\int e^{-z^2}+\int e^{-z^{1+i}}\right)^2+\cdots \end{aligned}$$

Localizing the problem





Generalized convolution products

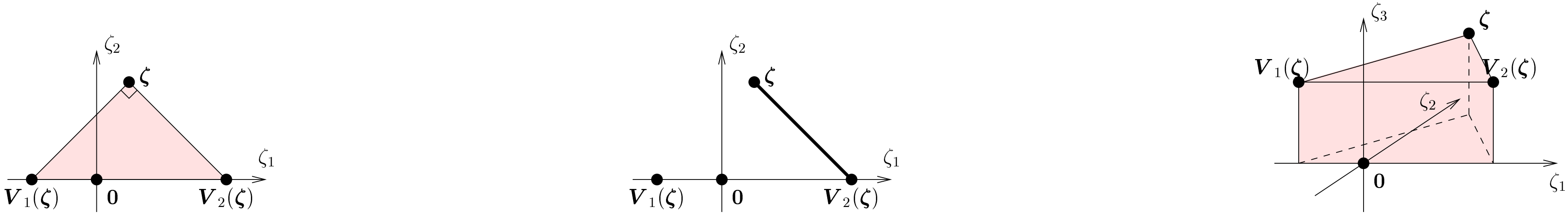
$\mathcal{A}$: real affine subspace of the set of all complex $n \times n$ matrices

$\mathcal{M}$: compact subset of $\mathcal{A}$

$$(g *_{\mathcal{M}} f)(\boldsymbol{\zeta}) = \int_{\mathcal{M} \cdot \boldsymbol{\zeta}} g(\boldsymbol{\zeta} - \boldsymbol{\xi})\, f(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}.$$

Normal convolution: $\mathcal{M} = [0,1]^n$

Convolution products associated to unitary matrix U



Majorants

$$\begin{aligned}\mathbb{b}_{\boldsymbol{\alpha}} &= \frac{1}{1-\langle\boldsymbol{\alpha},\boldsymbol{\zeta}\rangle}\\ \bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\alpha},r} &= \{\boldsymbol{\zeta}\in\mathbb{C}^n : \langle\boldsymbol{\alpha},|\boldsymbol{\zeta}|\rangle\leqslant r\}\end{aligned}$$

Lemma 4. *Let* $\alpha\in(\mathbb{R}^>)^n$ *be such that*

$$\{\boldsymbol{M}\cdot\boldsymbol{\zeta} : \boldsymbol{M}\in\boldsymbol{\mathcal{M}}_{\boldsymbol{U}}\wedge\boldsymbol{\zeta}\in\bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\alpha},1}\}\subseteq\bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\alpha},1}$$

and $0<\varepsilon<1$. *Then*

$$g\trianglelefteqslant\mathbb{b}_{\varepsilon\boldsymbol{\alpha}/2}\wedge f\trianglelefteqslant\mathbb{b}_{\boldsymbol{\alpha}}^p \Longrightarrow g *_{\boldsymbol{U}} f\trianglelefteqslant\frac{\zeta_n^n}{n!\,|U_{n,1}|\cdots|U_{n,n}|(1-\varepsilon)^n}\,\mathbb{b}_{\boldsymbol{\alpha}}^p$$

for all $f,g\in\mathbb{C}[[\boldsymbol{\zeta}]]$ *and* $p\in\mathbb{N}$.

The local convolution p.d.e.

We will study the generalized convolution equation

$$f_{\zeta_n} = \varphi_1 f_{\zeta_1} + \varphi_2 f_{\zeta_2} + \cdots + \varphi_n f_{\zeta_n} + \psi_1 *_{\boldsymbol{U}} f_{\zeta_1} + \cdots + \psi_n *_{\boldsymbol{U}} f_{\zeta_n} + g, \tag{2}$$

at the origin, where

- $\varphi_1, \ldots, \varphi_n \in \mathbb{C}\{\{\boldsymbol{\zeta}\}\}$ and $\varphi_n(\mathbf{0}) = 0$.
- $\psi_1, \ldots, \psi_n \in \mathbb{C}\{\{\boldsymbol{\zeta}\}\}_{\boldsymbol{U}}^{\sharp}$ and $\psi_{1,\mathfrak{m}} = \cdots = \psi_{n,\mathfrak{m}} = 0$.
- $g \in \mathbb{C}\{\{\boldsymbol{\zeta}\}\}$.

Theorem 5. *For sufficiently large $\boldsymbol{\alpha}$ satisfying the condition from the previous lemma for sufficiently small ε, (2) admits a solution in $\boldsymbol{\mathcal{B}}_{\boldsymbol{\beta},\varepsilon}$ for any initial condition $f_0 \in \mathbb{C}[[\zeta_1, \ldots, \zeta_{n-1}]]$ which converges in $\boldsymbol{\mathcal{B}}_{\boldsymbol{\beta},\varepsilon} \cap (\mathbb{C}^{n-1} \times \{0\})$.*

Local analytic continuation

$\mathcal{D}$ set where f can be continuated.
γ be a hyperpath with endpoint χ.
$\mathcal{B}$ open ball with $\chi \in \bar{\mathcal{B}}$.
$\mathcal{U} \setminus (\mathcal{B}' \setminus \mathcal{B}) \subseteq \mathcal{D}$ for neighbourhood of γ and ball $\mathcal{B}'$ around χ.
$e_1, \ldots, e_n$ and φ cross $\partial \mathcal{B}$ transversally at χ.
$\longrightarrow \chi \in \mathcal{D}$

Global theorem

The solution of the original equation my be continued analytically from a neighbourhood of $\mathbb{C}^{n-1} \times \{0\}$ to the whole Riemann surface $\mathbb{C}^{n-1} \times \mathcal{R}_n$.

Exponential growth: study convolution p.d.e. for $f\, e^{-\mu_1 \zeta_1 - \cdots - \mu_n \zeta_n}$.

Singularities in $\zeta_1, \ldots, \zeta_{n-1}$

$$\begin{aligned}
\varphi_1 f_{\zeta_1} + \varphi_2 f_{\zeta_2} + \psi_1 * f_{\zeta_1} + \psi_2 * f_{\zeta_2} &= g \log \zeta_1 + h \\
\varphi_1 (\Delta f)_{\zeta_1} + \varphi_2 (\Delta f)_{\zeta_2} + \psi_1 * (\Delta f)_{\zeta_1} + \psi_2 * (\Delta f)_{\zeta_2} &= 2 \pi i g
\end{aligned}$$

Complete asymptotic theory for $\mathbb{C}(z)^{\mathbf{PV}}$

Should be possible on non-degenerate regions.
What happens beyond complex transseries?

Towards a multivariate theory of resurgent functions

Minors and majors $\longrightarrow$ left convolution operators

$$\hat{f} \longmapsto [F\colon \hat{\varphi} \rightarrow \hat{f} * \hat{\varphi}]$$

Operations:

$$\begin{aligned}(F * G)(\hat{\varphi}) &= F(G(\hat{\varphi}))\\ (\Delta F)(\hat{\varphi}) &= \Delta(F(\hat{\varphi})) - F(\Delta(\hat{\varphi}))\end{aligned}$$

Non-standard operators: left convolution with non-analytic functions.

Parallel summable multivariate power series

Implicit function theorems?
o-minimality?