Effective Analytic Functions

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Written using GNU \textsc{Texmacs} (www.texmacs.org)
Computation with transcendental functions

Functions of interest

- “Simple functions” and closure under $+, \times, \partial, \int, \circ$, etc.
- Solutions to algebraic differential equations.

Testing functional identities

- $\sin^2 x + \cos^2 x = 1$
- $\log(x^x + e^{x\log x}) - x^x \log x = \log (1 + x^x(1-x^{x-1}))$
- $3\sqrt[3]{5\sqrt{32/5} - 5\sqrt{27/5}} = (1 + 5\sqrt{3} - 5\sqrt{9}) / 5\sqrt{25}$

Functional identities = constant identities + power series identities

Evaluating functions wherever they are defined

- Fast algorithms for safe numerical evaluation (relaxed power series).
- Analytic continuation.
Theorem 1. [DL] Given a power series \( f = \sum f_n z^n \) with rational coefficients, which is the unique solution of an algebraic differential equation

\[
P(z, f, \ldots, f^{(l)}) = 0,
\]

with rational coefficients and rational initial conditions, one cannot in general decide whether the radius of convergence \( \rho(f) \) of \( f \) is \(<1 \) or \( \geq 1 \).

Columbus
Effective germs

Effective complex number

Instance $z$ of abstract data type Complex, with method

- approximate: $\mathbb{Z} \to (\mathbb{Z} + i \mathbb{Z}) 2^{\mathbb{Z}}; k \mapsto \tilde{z}$, with $|\tilde{z} - z| < 2^k$.

Effective series

Instance $f$ of abstract data type Series($\mathbb{R}$), with method

- expand: $\mathbb{N} \to \mathbb{R}[z]; n \mapsto f_0 + \cdots + f_{n-1} z^{n-1}$.

Effective germ (at the origin)

Instance $f$ of abstract data type Germ which extends Series(Complex) with the following additional methods:

- radius: $() \to \mathbb{R} \cup \{+\infty\}$ returns a lower bound $\rho(f) > 0$ for $\rho(f)$. This bound is called the effective radius of convergence of $f$.

- norm: $\mathbb{R} \to \mathbb{R}$, which, given $0 < r < \rho(f)$, returns an upper bound $|f|_r$ for $|\bar{f}|_r$. We assume that $|f|_r$ is increasing in $r$. 
Algorithm \text{evaluate-approx}(f, z, \varepsilon)

Input \( f \in \text{Germ} \) and \( z \in \text{Complex} \) with \( |z| < \rho(f) \), and \( \varepsilon \in 2^\mathbb{Z} \)

Output an approximation \( \tilde{f} \) for \( f(z) \) with \( |\tilde{f} - f(z)| < \varepsilon \)

Step 1 [Compute expansion order]
Let \( r = (\rho(f) + |z|)/2 \) and \( M = |f|_r \)
Let \( n \in \mathbb{N} \) be smallest such that
\[
\frac{Mr}{r - |z|} \left( \frac{|z|}{r} \right)^n < \frac{\varepsilon}{2}
\]

Step 2 [Approximate the series expansion]
Compute \( \hat{f} = f_0 + f_1 z + \cdots + f_{n-1} z^{n-1} \in \text{Complex} \)
Return \( \hat{f} < \varepsilon/2 \)

Proof
\[
|\hat{f} - f(z)| \leq \frac{|z|^n}{2\pi} \left| \int_{|w| = r} \frac{M}{(r - |z|) r^n} \, dw \right| = \frac{Mr}{r - |z|} \left( \frac{|z|}{r} \right)^n < \frac{\varepsilon}{2}.
\]
Effective analytic functions

Quasi-effective analytic function

Instance $f$ of the abstract data type `AnFunc`, which extends `Germ` with

- continue: `Complex→AnFunc` computes $f_{+\delta}$ for all $\delta$ with $|\delta|<\rho(f)$.

Effective domain of $f$: effective paths $\gamma=[\delta_1,\ldots,\delta_i] \in \text{Path}$ with

$$|\delta_i|<\rho(f_{+\delta_1,\ldots,\delta_{i-1}}).$$

Homotopy condition

If $\gamma, \gamma+[\delta_1], \gamma+[\delta_1,\delta_2], \gamma+[\delta_1+\delta_2] \in \text{Dom } f$, then

$$f_{+\gamma,+(\delta_1+\delta_2)} \equiv f_{+\gamma,+\delta_1,\delta_2}.$$  

Continuity condition

Locally: $\delta \mapsto \rho(f_{+\delta})$ and $(\delta, r) \mapsto |f_{+\delta}|_r$ extend by continuity.

Globally: this is the case for all $f_{+\gamma}$ with $\gamma \in \text{Dom } f$. 

Evaluation modulo subdivision

Extended effective domain

\( \text{Dom}^\# f = \{ \gamma : \gamma' \in \text{Dom} f \text{ for some subdivision } \gamma' \text{ of } \gamma \}. \)

\( f \) faithful if \( \text{Dom}^\# f = \text{Dom} \bar{f} \cap \text{Path} \).

Evaluation on extended domain

\( 0 < \lambda < 1 \)

Algorithm evaluate-subdiv\((f, \gamma)\)

Input \( f \in \text{AnFunc} \) and \( \gamma \in \text{Path} \), such that \( \gamma \in \text{Dom}^\# f \)

Output \( f(\gamma) \)

Step 1 [Handle trivial cases]

If \( \gamma = \epsilon \), then return \( f(0) \)

Write \( \gamma = [\delta] + \gamma' \)

If \( \delta < \rho(f) \) then return \( \text{evaluate-subdiv}(f_+\delta, \gamma') \)

Step 2 [Subdivide path]

Let \( \delta_1 = \lambda |\rho(f)|\delta / |\delta| \)

Return \( \text{evaluate-subdiv}(f_+\delta', [\delta - \delta'] + \gamma') \)
The identity function

Class **IdentityAnFunc** ▷ **AnFunc**

- \( z \in \text{Complex} \)
- **new**: () ↦ \( z := 0 \)
- **new**: \( \tilde{z} \in \text{Complex} \) ↦ \( z := \tilde{z} \)
- **radius**: () ↦ \( \infty \)
- **norm**: \( r \mapsto |z| + r \)
- **continue**: \( \delta \mapsto \text{IdentityAnFunc}(z + \delta) \)
Class \texttt{IntAnFunc} \supset \texttt{AnFunc}

- $f \in \texttt{AnFunc}$
- $c \in \texttt{Complex}$
- \texttt{new}: $\tilde{f} \mapsto f := \tilde{f}$, $c := 0$
- \texttt{new}: $(\tilde{f} \in \texttt{AnFunc}, \tilde{c} \in \texttt{Complex}) \mapsto f := \tilde{f}$, $c := \tilde{c}$
- \texttt{radius}: () $\mapsto \rho(f)$
- \texttt{norm}: $r \mapsto |c| + r \ |f|_r$
- \texttt{continue}: $\delta \mapsto \texttt{IntAnFunc}(f_+\delta, \texttt{this}(\delta))$
$0 < \lambda < 1$

**Class** $\text{InvAnFunc}\uparrow\text{AnFunc}$

- $f \in \text{AnFunc}$
- new: $\tilde{f} \in \text{AnFunc} \rightarrow f := \tilde{f}$
- radius: $() \mapsto \min \left( \lambda \rho(f), \frac{|f(0)|}{|f'|\lambda \rho(f)} \right)$
- norm: $r \mapsto \frac{1}{|f(0)| - |f'|r}$
- continue: $\delta \mapsto \text{InvAnFunc}(f + \delta)$
Consider the linear differential equation

\[ f^{(l)} = L_{l-1} f^{(l-1)} + \cdots + L_0 f. \]  

with initial conditions \( f(0) = \nu_0, \ldots, f^{(l-1)}(0) = \nu_{l-1}. \)

**Theorem 2.** Let \( L_0, \ldots, L_{l-1} \in \text{AnFunc} \) and let \( \nu_0, \ldots, \nu_{l-1} \in \text{Complex} \). Then there exists an effective analytic solution \( f \) to (1) with

\[ \text{Dom} \ f = \text{Dom} \ (L_1, \ldots, L_l). \]

In particular, if \( \text{Dom} \ (L_1, \ldots, L_l) \) is effective, then so is \( \text{Dom} \ f \).

**Proof.** Uses very sharp version of the majorant method.
Consider the system of algebraic differential equations

\[
\frac{d}{dz} \begin{pmatrix} f_1 \\ \vdots \\ f_l \end{pmatrix} = \begin{pmatrix} P_1(f_1, \ldots, f_l) \\ \vdots \\ P_l(f_1, \ldots, f_l) \end{pmatrix}
\]

(2)

with initial conditions \( f_1(0) = \nu_1, \ldots, f_l(0) = \nu_l \).

**Theorem 3.** Let \( P_1, \ldots, P_l \) be polynomials with coefficients in \( \text{Complex} \) and let \( \nu_1, \ldots, \nu_l \in \text{Complex} \). Then the system (2) admits a faithful effective analytic solution \((f_1, \ldots, f_l)\).

**Proof.**

- Majorant method.
- Continuity condition + compactness of path \( \rightarrow \) faithfulness.