Effective Analytic Functions

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MEGA'03, Kaiserslautern, 14/6/2003

Written using GNU TeXmacs (www.texmacs.org)

Computation with transcendental functions

Functions of interest

- "Simple functions" and closure under $+, \times, \partial, \int, \circ$, etc.
- Solutions to algebraic differential equations.

Testing functional identities

- $\sin^2 x + \cos^2 x = 1$
- $\log(x^{x^x} + e^{x \log x}) - x^x \log x = \log\left(1 + x^{x(1 - x^{x-1})}\right)$
- $\sqrt[3]{\sqrt[5]{32/5} - \sqrt[5]{27/5}} = (1 + \sqrt[5]{3} - \sqrt[5]{9}) / \sqrt[5]{25}$

functional identities = constant identities + power series identities

Evaluating functions whereever they are defined

- Fast algorithms for safe numerical evaluation (relaxed power series).
- Analytic continuation.

Warning

Theorem 1. [DL] *Given a power series* $f = \sum f_n z^n$ *with rational coefficients, which is the unique solution of an algebraic differential equation*

$$P(z, f, \ldots, f^{(l)}) = 0,$$

with rational coefficients and rational initial conditions, one cannot in general decide whether the radius of convergence $\rho(f)$ *of* f *is* <1 *or* $\geqslant 1$.

Columbus

Effective germs

Effective complex number

Instance z of abstract data type $\mathrm{Complex}$, with method

- approximate: $\mathbb{Z} \rightarrow (\mathbb{Z} + \mathrm{i}\, \mathbb{Z})\, 2^{\mathbb{Z}}; k \longmapsto \tilde{z}$, with $|\tilde{z} - z| < 2^k$.

Effective series

Instance f of abstract data type $\mathrm{Series}(\mathrm{R})$, with method

- expand: $\mathbb{N} \rightarrow \mathrm{R}[z]; n \longmapsto f_0 + \cdots + f_{n-1}\, z^{n-1}$.

Effective germ (at the origin)

Instance f of abstract data type Germ which extends $\mathrm{Series}(\mathrm{Complex})$ with the following additional methods:

- radius: $() \rightarrow \mathrm{Real} \cup \{+\infty\}$ returns a lower bound $\rho(f) > 0$ for $\rho(\bar{f})$. This bound is called the *effective radius of convergence* of f.
- norm: $\mathrm{Real} \rightarrow \mathrm{Real}$, which, given $0 < r < \rho(f)$, returns an upper bound $\lceil f \rceil_r$ for $|\bar{f}|_r$. We assume that $\lceil f \rceil_r$ is increasing in r.

Evaluation of effective germs

Algorithm evaluate-approx(f, z, ε)

Input $f \in \mathrm{Germ}$ and $z \in \mathrm{Complex}$ with $|z| < \rho(f)$, and $\varepsilon \in 2^{\mathbb{Z}}$

Output an approximation $\tilde{f}$ for $f(z)$ with $|\tilde{f} - f(z)| < \varepsilon$

Step 1 [Compute expansion order]

Let $r = (\rho(f) + |z|)/2$ and $M = |f|_r$

Let $n \in \mathbb{N}$ be smallest such that

$$\frac{M\,r}{r - |z|}\left(\frac{|z|}{r}\right)^n < \frac{\varepsilon}{2}$$

Step 2 [Approximate the series expansion]

Compute $\hat{f} = f_0 + f_1\,z + \cdots + f_{n-1}\,z^{n-1} \in$ Complex

Return $\hat{f}_{<\varepsilon/2}$

Proof

$$|\hat{f} - f(z)| \leqslant \frac{|z|^n}{2\,\pi}\left|\oint_{|w|=r} \frac{M}{(r - |z|)\,r^n}\,\mathrm{d}w\right| = \frac{M\,r}{r - |z|}\left(\frac{|z|}{r}\right)^n < \frac{\varepsilon}{2}.$$

Effective analytic functions

Quasi-effective analytic function

Instance f of the abstract data type AnFunc, which extends Germ with

- continue: $\mathrm{Complex} \rightarrow \mathrm{AnFunc}$ computes $f_{+\delta}$ for all δ with $|\delta| < \rho(f)$.

Effective domain of f: effective paths $\gamma = [\delta_1, \ldots, \delta_l] \in \mathrm{Path}$ with

$$|\delta_i| < \rho(f_{+\delta_1, \ldots, +\delta_{i-1}}).$$

Homotopy condition

If $\gamma, \gamma + [\delta_1], \gamma + [\delta_1, \delta_2], \gamma + [\delta_1 + \delta_2] \in \mathrm{Dom}\, f$, then

$$f_{+\gamma, +(\delta_1 + \delta_2)} \equiv f_{+\gamma, +\delta_1, +\delta_2}.$$

Continuity condition

Locally: $\delta \mapsto \rho(f_{+\delta})$ and $(\delta, r) \mapsto |f_{+\delta}|_r$ extend by continuity.

Globally: this is the case for all $f_{+\gamma}$ with $\gamma \in \mathrm{Dom}\, f$.

Evaluation modulo subdivision

Extended effective domain

$\mathrm{Dom}^\sharp f = \{\gamma : \gamma' \in \mathrm{Dom}\, f \text{ for some subdivision } \gamma' \text{ of } \gamma\}$.

f *faithful* if $\mathrm{Dom}^\sharp f = \mathrm{Dom}\, \bar{f} \cap \mathrm{Path}$.

Evaluation on extended domain

$0 < \lambda < 1$

Algorithm evaluate-subdiv(f, γ)

Input $f \in \mathrm{AnFunc}$ and $\gamma \in \mathrm{Path}$, such that $\gamma \in \mathrm{Dom}^\sharp f$

Output $f(\gamma)$

Step 1 [Handle trivial cases]

If $\gamma = \epsilon$, then return $f(0)$

Write $\gamma = [\delta] + \gamma'$

If $\delta < \rho(f)$ then return evaluate-subdiv$(f_{+\delta}, \gamma')$

Step 2 [Subdivide path]

Let $\delta_1 = \lambda\, |\rho(f)| \delta / |\delta|$

Return evaluate-subdiv$(f_{+\delta'}, [\delta - \delta'] + \gamma')$

The identity function

Class $\mathrm{IdentityAnFunc} \triangleright \mathrm{AnFunc}$

- $z \in \mathrm{Complex}$
- new: $() \mapsto z := 0$
- new: $\tilde{z} \in \mathrm{Complex} \mapsto z := \tilde{z}$
- radius: $() \mapsto \infty$
- norm: $r \mapsto |z| + r$
- continue: $\delta \mapsto \mathrm{IdentityAnFunc}(z + \delta)$

Integration

Class $\mathrm{IntAnFunc} \triangleright \mathrm{AnFunc}$

- $f \in \mathrm{AnFunc}$
- $c \in \mathrm{Complex}$
- new: $\tilde{f} \mapsto f := \tilde{f}$, $c := 0$
- new: $(\tilde{f} \in \mathrm{AnFunc}, \tilde{c} \in \mathrm{Complex}) \mapsto f := \tilde{f}, c := \tilde{c}$
- radius: $() \mapsto \rho(f)$
- norm: $r \mapsto |c| + r\, |f|_r$
- continue: $\delta \mapsto \mathrm{IntAnFunc}(f_{+\delta}, \text{this}(\delta))$

Inversion

$0 < \lambda < 1$

Class $\mathrm{InvAnFunc} \triangleright \mathrm{AnFunc}$

- $f \in \mathrm{AnFunc}$
- new: $\tilde{f} \in \mathrm{AnFunc} \mapsto f := \tilde{f}$
- radius: $() \mapsto \min\left(\lambda\,\rho(f), \frac{|f(0)|}{|f'|_{\lambda\rho(f)}}\right)$
- norm: $r \mapsto \frac{1}{|f(0)| - |f'|_r r}$
- continue: $\delta \mapsto \mathrm{InvAnFunc}(f_{+\delta})$

Consider the linear differential equation

$$f^{(l)} = L_{l-1} f^{(l-1)} + \cdots + L_0 f. \tag{1}$$

with initial conditions $f(0) = \nu_0, \ldots, f^{(l-1)}(0) = \nu_{l-1}$.

Theorem 2. *Let* $L_0, \ldots, L_{l-1} \in \mathrm{AnFunc}$ *and let* $\nu_0, \ldots, \nu_{l-1} \in \mathrm{Complex}$*. Then there exists an effective analytic solution* f *to* (1) *with*

$$\mathrm{Dom}\, f = \mathrm{Dom}\,(L_1, \ldots, L_l).$$

In particular, if $\mathrm{Dom}\,(L_1, \ldots, L_l)$ *is effective, then so is* $\mathrm{Dom}\, f$*.*

Proof. Uses very sharp version of the majorant method. □

Algebraic differential equations

Consider the system of algebraic differential equations

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix} f_1 \\ \vdots \\ f_l \end{pmatrix} = \begin{pmatrix} P_1(f_1, \ldots, f_l) \\ \vdots \\ P_l(f_1, \ldots, f_l) \end{pmatrix} \tag{2}$$

with initial conditions $f_1(0) = \nu_1, \ldots, f_l(0) = \nu_l$.

Theorem 3. *Let P_1, ..., P_l be polynomials with coefficients in $\mathrm{Complex}$ and let ν_1, ..., $\nu_l \in \mathrm{Complex}$. Then the system* (2) *admits a faithful effective analytic solution $(f_1, \ldots, f_l)$.*

Proof.

- Majorant method.
- Continuity condition + compactness of path $\longrightarrow$ faithfulness. □