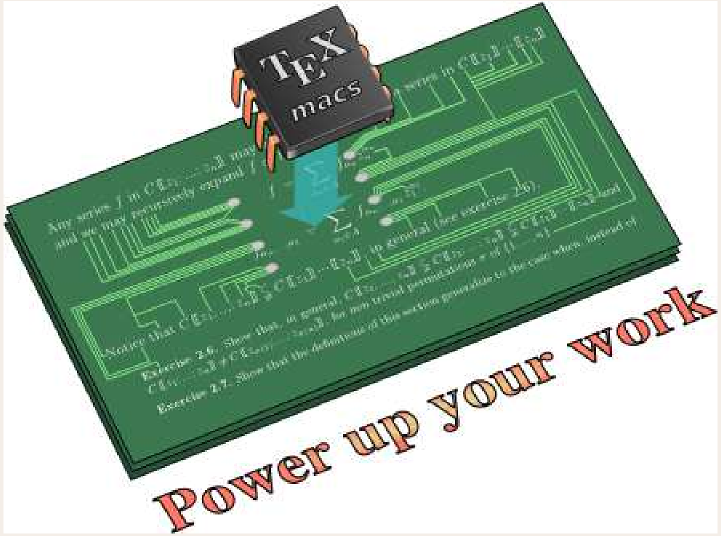
Hardy field solutions to algebraic differential equations



Joris van der Hoeven, Pisa 2007
http://www.TEXmacs.org

A missing subject?

Algebraic geometry	$\longrightarrow$	Real algebraic geometry + Valuation theory
$\downarrow$		$\downarrow$
Differential algebra	$\longrightarrow$	?

- Hardy fields: Hardy, Rosenlicht, Boshernitzan, Singer, etc.
- Pfaff systems: Khovanskii, Wilkie, van den Dries, Rolin, etc.

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- **LNM 1888**: Transseries and Real Differential Algebra
- Other work on `http://www.math.u-psud.fr/~vdhoeven`

Sufficiently closed models

Algebraic geometry	$\longrightarrow$	Real algebraic geometry + Valuation theory
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Sufficiently closed models

$\mathbb{C}$	$\longrightarrow$	Real algebraic geometry + Valuation theory
$\downarrow$		$\downarrow$
Differential algebra	$\longrightarrow$	Real differential algebra + Asymptotic differential algebra

Sufficiently closed models

$\mathbb{C}$	$\longrightarrow$	$\mathbb{R}$ + Valuation theory
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Differential algebra	$\longrightarrow$	Real differential algebra + Asymptotic differential algebra

Sufficiently closed models

$\mathbb{C}$	$\longrightarrow$	$\mathbb{R}$ + $\mathbb{C}[[z^{\mathbb{Q}}]]$
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Differential algebra	$\longrightarrow$	Real differential algebra + Asymptotic differential algebra

Sufficiently closed models

$\mathbb{C}$	$\longrightarrow$	$\mathbb{R}$ + $\mathbb{C}[[z^{\mathbb{Q}}]]$
$\downarrow$		$\downarrow$
Wild	$\longrightarrow$	Real differential algebra + Asymptotic differential algebra

Sufficiently closed models

$\mathbb{C}$	$\longrightarrow$	$\mathbb{R}$ + $\mathbb{C}[[z^{\mathbb{Q}}]]$
$\downarrow$		$\downarrow$
Wild	$\longrightarrow$	Maximal Hardy field (?) + Asymptotic differential algebra

Sufficiently closed models

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \begin{array}{c}\mathbb{R} \\ + \\ \mathbb{C}[[z^{\mathbb{Q}}]]\end{array} \\ \downarrow & & \downarrow \\ \textit{Wild} & \longrightarrow & \begin{array}{c}\mathbb{R}[[[x]]] \\ + \\ \text{Asymptotic differential algebra}\end{array} \end{array}$$

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What is a transseries?

$(x \succ 1)$

$$e^{e^x + \frac{e^x}{x} + \frac{e^x}{x^2} + \cdots} + \frac{2}{\log x} e^{e^x + \frac{e^x}{x} + \frac{e^x}{x^2} + \cdots} + e^{\sqrt{x} + e^{\sqrt{\log x} + e^{\sqrt{\log \log x} + \cdots}}} + \cdots$$

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- Dahn & Göring
- Écalle

Examples of transseries

$$\frac{1}{1 - x^{-1} - x^{-\mathrm{e}}} = 1 + x^{-1} + x^{-2} + x^{-\mathrm{e}} + x^{-3} + x^{-\mathrm{e}-1} + \cdots$$

$$\frac{1}{1 - x^{-1} - \mathrm{e}^{-x}} = 1 + \frac{1}{x} + \frac{1}{x^2} + \cdots + \mathrm{e}^{-x} + 2\,\frac{\mathrm{e}^{-x}}{x} + \cdots + \mathrm{e}^{-2x} + \cdots$$

$$-\mathrm{e}^x \int \frac{\mathrm{e}^{-x}}{x} = \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{6}{x^4} + \frac{24}{x^5} - \frac{120}{x^6} + \cdots$$

$$\Gamma(x) = \frac{\sqrt{2\,\pi}\,\mathrm{e}^{x\,(\log x - 1)}}{x^{1/2}} + \frac{\sqrt{2\,\pi}\,\mathrm{e}^{x\,(\log x - 1)}}{12\,x^{3/2}} + \frac{\sqrt{2\,\pi}\,\mathrm{e}^{x\,(\log x - 1)}}{288\,x^{5/2}} + \cdots$$

$$\zeta(x) = 1 + 2^{-x} + 3^{-x} + 4^{-x} + \cdots$$

$$\varphi(x) = \frac{1}{x} + \varphi(x^{\pi}) = \frac{1}{x} + \frac{1}{x^{\pi}} + \frac{1}{x^{\pi^2}} + \frac{1}{x^{\pi^3}} + \cdots$$

$$\psi(x) = \frac{1}{x} + \psi(\mathrm{e}^{\log^2 x}) = \frac{1}{x} + \frac{1}{\mathrm{e}^{\log^2 x}} + \frac{1}{\mathrm{e}^{\log^4 x}} + \frac{1}{\mathrm{e}^{\log^8 x}} + \cdots$$

The field $\mathbb{T}$ of grid-based transseries

- $\mathbb{T} = \mathbb{R}[[\mathfrak{T}]]$, where $\mathfrak{T}$ is a totally ordered monomial group.
- $\mathbb{R}[[\mathfrak{T}]]$: series $f = \sum_{\mathfrak{m} \in \mathfrak{T}} f_{\mathfrak{m}} \mathfrak{m} \in \mathbb{R}[[\mathfrak{T}]]$ with **grid-based support**:

$$\operatorname{supp} f \subseteq \{\mathfrak{m}_1, \ldots, \mathfrak{m}_m\}^* \mathfrak{n}, \qquad \mathfrak{m}_1, \ldots, \mathfrak{m}_m \prec 1$$

- $\mathbb{T}$ is a totally ordered, real closed field.
- $\mathbb{T}$ is stable under $\exp$, $\log$, ∂, $\int$, $\circ$ and ${}^{\mathrm{inv}}$.

Intermediate value theorem

Theorem. *(2000) Given* $P \in \mathbb{T}\{F\}$ *and* $f < g \in \mathbb{T}$ *with* $P(f)\, P(g) < 0$*. Then there exists an* $h \in \mathbb{T}$ *with* $f < h < g$ *and* $P(h) = 0$*.*

1. Calculus with **cuts** $\hat{f} \in \hat{\mathbb{T}}$.
2. Classification of cuts and behaviour of $P(f)$ near a cut.
3. Newton polygon method for shrinking interval on which a sign change occurs and whose end-points are cuts.

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Corollary. *Any* $P \in \mathbb{T}\{F\}$ *of odd degree admits a root in* $\mathbb{T}$.

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Example. The following equation admits a solution in $\mathbb{T}$:

$$\frac{1}{x} f'''\,(f')^2\, f^{24} + \mathrm{e}^x\,(f'')^{27} - \Gamma(\Gamma(\log x))\, f^2 = \frac{\mathrm{e}^{\mathrm{e}^x + x^2}}{\Gamma(\mathrm{e}^x + x)}.$$

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Corollary. *Any monic* $L \in \mathbb{T}[\partial]$ *admits a factorization with factors*

$$\partial - a \quad \textit{or}$$

$$\partial^2 - (2\,a + b^{\dagger})\,\partial + (a^2 + b^2 - a' + a\,b^{\dagger}) = (\partial - (a - b\,\mathrm{i} + b^{\dagger}))\,(\partial - (a + b\,\mathrm{i}))$$

Complex transseries

Theorem. *(2001) Every asymptotic differential equation over* $\mathbb{T}$ *of Newton degree* d *admits at least* d *solutions (when counting with multiplicities). Moreover, it suffices to add iterated logarithms to the asymptotic scale.*

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Warning. $\mathbb{T}$ is not differentially algebraically closed

$$\begin{aligned} f^3 + (f')^2 + f &= 0 \\ f^3 + f &\neq 0 \end{aligned}$$

$\longrightarrow$ Desingularization of vector fields (Cano, Panazzolo, ...)

Complex transseries

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Corollary. $\mathbb{T}$ *is Picard-Vessiot closed.*

Remark. $\exists$ algorithm for computing the solutions of a given equation.

Remark. Zero-test algorithm for polynomials in power series solutions to algebraic differential equations.

Real transseries solutions $\longrightarrow$ analytic germs

1: Accelero-summation

$$\begin{array}{ccccccccc}
\tilde{f} & & & & & & & & f \\
\downarrow \tilde{\mathcal{B}}_{z_1} & & & & & & & & \uparrow \hat{\mathcal{L}}_{z_p}^{\theta_p} \\
\hat{f}_1 & \xrightarrow[\hat{\mathcal{A}}_{z_1 \to z_2}^{\theta_1}]{} & \hat{f}_2 & \longrightarrow & \cdots & \longrightarrow & \hat{f}_{p-1} & \xrightarrow[\hat{\mathcal{A}}_{z_{p-1} \to z_p}^{\theta_{p-1}}]{} & \hat{f}_p
\end{array}$$

2: Transserial Hardy fields

$$\mathbb{T} \supseteq \mathcal{T} \overset{\rho}{\hookrightarrow} \mathcal{G}$$

- $\mathcal{G}$: ring of infinitely differentiable real germs at $+\infty$.

Real transseries solutions $\longrightarrow$ analytic germs

1: Accelero-summation

Advantages	Disadvantages
Canonical after choosing average Preserves composition Classification local vector fields Differential Galois theory	Requires many different tools Not yet written down

2: Transserial Hardy fields

Advantages	Disadvantages
Less hypotheses on coefficients Might generalize to other models Written down	Not canonical No preservation of composition

Transserial Hardy fields

A **transserial Hardy** field is a differential subfield $\mathcal{T}$ of $\mathbb{T}$, together with a monomorphism $\rho: \mathcal{T} \rightarrow \mathcal{G}$ of ordered differential $\mathbb{R}$-algebras, such that

TH1. $\forall f \in \mathcal{T}$: $\operatorname{supp} f \subseteq \mathcal{T}$.

TH2. $\forall f \in \mathcal{T}$: $f_{\prec} \in \mathcal{T}$. $f_{\prec} = \sum_{\mathfrak{m} \prec 1} f_{\mathfrak{m}} \mathfrak{m}$

TH3. $\exists d \in \mathbb{Z}$: $\forall \mathfrak{m} \in \mathfrak{T} \cap \mathcal{T}$: $\log \mathfrak{m} \in \mathcal{T} + \mathbb{R} \log_d x$.

TH4. $\mathfrak{T} \cap \mathcal{T}$ is stable under taking real powers.

TH5. $\forall f \in \mathcal{T}^{>}$: $\log f \in \mathcal{T} \Rightarrow \rho(\log f) = \log \rho(f)$.

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$$\frac{x\,\mathrm{e}^x}{1 - x^{-1} - \mathrm{e}^{-x}}$$

$$\|$$

$$x\,\mathrm{e}^x + \mathrm{e}^x + x^{-1}\,\mathrm{e}^x + \cdots + x + 1 + x^{-1} + \cdots + x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} + x^{-1}\,\mathrm{e}^{-x} + \cdots\cdots$$

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$$\left(\frac{x\,\mathrm{e}^x}{1 - x^{-1} - \mathrm{e}^{-x}} \right)_{\prec}$$

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$$x\,\mathrm{e}^x + \mathrm{e}^x + x^{-1}\,\mathrm{e}^x + \cdots + x + 1 + x^{-1} + \cdots + x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} + x^{-1}\,\mathrm{e}^{-x} + \cdots\cdots$$

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Elementary extensions

Definitions. $\mathcal{T}$ transserial Hardy field, $f \in \mathbb{T}$, $\hat{f} \in \mathcal{G}$

$$f \sim \hat{f} \quad \Longleftrightarrow \quad (\exists \varphi \in \mathcal{T}\colon \ f \sim_{\mathbb{T}} \varphi \sim_{\mathcal{G}} \hat{f})$$

$$f \ \textit{asympt. equiv.} \text{ to } \hat{f} \text{ over } \mathcal{T} \quad \Longleftrightarrow \quad (\forall \varphi \in \mathcal{T}\colon \ f - \varphi \sim \hat{f} - \varphi)$$

$$f \ \textit{diff. equiv.} \text{ to } \hat{f} \text{ over } \mathcal{T} \quad \Longleftrightarrow \quad (\forall P \in \mathcal{T}\{F\}\colon \ P(f) = 0 \Leftrightarrow P(\hat{f}) = 0)$$

Lemma. *Let $f \in \mathbb{T} \setminus \mathcal{T}$ and $\hat{f} \in \mathcal{G} \setminus \mathcal{T}$ be such that*

i. f is a serial cut over $\mathcal{T}$.

ii. f and $\hat{f}$ are asymptotically equivalent over $\mathcal{T}$.

iii. f and $\hat{f}$ are differentially equivalent over $\mathcal{T}$.

Then $\exists!$ transserial Hardy field extension $\rho\colon \mathcal{T}\langle f \rangle \rightarrow \mathcal{G}$ with $\rho(f) = \hat{f}$.

Basic extension theorems

Theorem. *Let* $\mathcal{T}$ *be a transserial Hardy field. Then its real closure* $\mathcal{T}^{\mathrm{rcl}}$ *admits a unique transserial Hardy field structure which extends the one of* $\mathcal{T}$.

Theorem. *Let* $\mathcal{T}$ *be a transserial Hardy field and let* $\varphi \in \mathcal{T}_{\succ}$ *be such that* $\mathrm{e}^{\varphi} \notin \mathcal{T}$. *Then the set* $\mathcal{T}(\mathrm{e}^{\mathbb{R}\varphi})$ *carries the structure of a transserial Hardy field for the unique differential morphism* $\rho: \mathcal{T}(\mathrm{e}^{\mathbb{R}\varphi}) \to \mathcal{G}$ *over* $\mathcal{T}$ *with* $\rho(\mathrm{e}^{\lambda\varphi}) = \mathrm{e}^{\lambda\rho(\varphi)}$ *for all* $\lambda \in \mathbb{R}$.

Theorem. *Let* $\mathcal{T}$ *be a transserial Hardy field of depth* $d < \infty$. *Then* $\mathcal{T}((\log_d x)^{\mathbb{R}})$ *carries the structure of a transserial Hardy field for the unique differential morphism* $\rho: \mathcal{T}((\log_d x)^{\mathbb{R}}) \to \mathcal{G}$ *over* $\mathcal{T}$ *with* $\rho((\log_d x)^{\lambda}) = (\log_d x)^{\lambda}$ *for all* $\lambda \in \mathbb{R}$.

Differential equations (main ideas)

Step 1. A given algebraic differential equation

$$f^2 - f' + \frac{x}{\mathrm{e}^x} = 0$$

Step 2. Put equation in integral form

$$f = \int \left(\frac{x}{\mathrm{e}^x} + f^2 \right)$$

Step 3. Integral transseries solution

Differential equations (main ideas)

Step 1. A given algebraic differential equation

$$f^2 - \mathrm{e}^x f' + \frac{\mathrm{e}^{2x}}{x} = 0$$

Step 2. Put equation in integral form

$$f = \int \left(\frac{\mathrm{e}^x}{x} + \frac{f^2}{\mathrm{e}^x} \right)$$

Step 3. Integrate from a fixed point $x_0 < \infty$

Differential equations (main ideas)

Step 1. A general algebraic differential equation

$$P(f) = 0$$

Step 2. Equation in split-normal form

$$(\partial - \varphi_1) \cdots (\partial - \varphi_r) f = P(f) \qquad \text{with } P(f) \text{ small}$$

Attention: $\varphi_1, \ldots, \varphi_r \in \mathcal{T}[\mathrm{i}]$, even though $(\partial - \varphi_1) \cdots (\partial - \varphi_r) \in \mathcal{T}[\partial]$.

Step 3. Solve the split-normal equation using the fixed-point technique.

Continuous right-inverses (first order)

Lemma. *The operator* $J = (\partial - \varphi)_{x_0}^{-1}$, *defined by*

$$(Jf)(x) = \begin{cases} \mathrm{e}^{\Phi(x)} \int_{\infty}^{x} \mathrm{e}^{-\Phi(t)} f(t)\,\mathrm{d}t & \textit{(repulsive case)} \\ \mathrm{e}^{\Phi(x)} \int_{x_0}^{x} \mathrm{e}^{-\Phi(t)} f(t)\,\mathrm{d}t & \textit{(attractive case)} \end{cases}$$

and

$$\Phi(x) = \begin{cases} \int_{\infty}^{x} \varphi(t)\,\mathrm{d}t & \textit{(repulsive case)} \\ \int_{x_0}^{x} \varphi(t)\,\mathrm{d}t & \textit{(attractive case)} \end{cases}$$

is a continuous right-inverse of $L = \partial - \varphi$ *on* $\mathcal{G}^{\preccurlyeq}[\mathrm{i}]$, *with*

$$|||J|||_{x_0} \leqslant \left\| \frac{1}{\operatorname{Re}\varphi} \right\|_{x_0}.$$

Continuous right-inverses (higher order)

Lemma. *Given a split-normal operator*

$$L = (\partial - \varphi_1) \cdots (\partial - \varphi_r), \tag{1}$$

with a factorwise right-inverse $L^{-1} = J_r \cdots J_1$*, the operator*

$$\mathfrak{v}^{\nu} J_r \cdots J_1 : \mathcal{G}_{x_0}^{\preccurlyeq}[\mathrm{i}] \rightarrow \mathcal{G}_{x_0;r}^{\preccurlyeq}[\mathrm{i}]$$

is a continuous operator for every $\nu > r\,\sigma_L$*. Here* $\mathcal{G}_{x_0;r}^{\preccurlyeq}[\mathrm{i}]$ *carries the norm*

$$\| f \|_{x_0;r} = \max \{ \| f \|_{x_0}, \ldots, \| f^{(r)} \|_{x_0} \}.$$

Lemma. *If* $L \in \mathcal{T}[\partial]$ *and the splitting* (1) *(formally) preserves realness, then* $J_r \cdots J_1$ *preserves realness in the sense that it maps* $\mathcal{G}_{x_0}^{\preccurlyeq}$ *into itself.*

Non-linear equations

Theorem. *Consider a split-monic equation*

$$L f = P(f), \quad f \prec 1,$$

and let ν *be such that* $r\,\sigma_L < \nu < \upsilon_P$*. Then for any sufficiently large* x_0*, there exists a continuous factorwise right-inverse* $J_{r,\ltimes \mathfrak{v}^\nu} \cdots J_{1,\ltimes \mathfrak{v}^\nu}$ *of* $L_{\ltimes \mathfrak{v}^\nu}$*, such that the operator*

$$\Xi \colon f \longmapsto (J_r \cdots J_1)(P(f))$$

admits a unique fixed point

$$f \;=\; \lim_{n \to \infty} \Xi^{(n)}(0) \;\in\; \mathcal{B}(\mathcal{G}^{\preccurlyeq}_{x_0;r}, \tfrac{1}{2}).$$

Preservation of asymptotics

Theorem. *Let $\mathcal{T}$ be a transserial Hardy field of span $\mathfrak{v} \succcurlyeq \mathrm{e}^x$. Consider a monic split-normal quasi-linear equation*

$$L f = P(f), \quad f \prec 1, \tag{2}$$

over $\mathcal{T}$ without solutions in $\mathcal{T}$. Assume that one of the following holds:

a) $\mathcal{T}$ *is* $(1,1,1)$*-differentially closed in* $\mathbb{T}_{\preccurlyeq \mathfrak{v}}$ *and* (2) *is first order.*

i.e. $\mathcal{T}$ *is closed under the resolution of linear first order equations.*

b) $\mathcal{T}[\mathrm{i}]$ *is* $(1,1,1)$*-differentially closed in* $\mathbb{T}[\mathrm{i}]_{\preccurlyeq \mathfrak{v}}$.

Then there exist solutions $f \in \mathcal{G}$ and $\tilde{f} \in \hat{\mathcal{T}}$ to (2)*, such that f and $\tilde{f}$ are asymptotically equivalent over $\mathcal{T}$.*

First order extensions

Lemma. *Let* $L = \partial - \varphi \in \mathcal{T}[\partial]$ *be a normal operator. Let* $\tilde{f} \in \hat{\mathcal{T}}^{\preccurlyeq}$ *and* $g \in \mathcal{T}^{\preccurlyeq}$ *be such that* $\tilde{f}$ *is transcendental over* $\mathcal{T}$ *and* $L\tilde{f} = g$*. Then there exists an* $f \in \mathcal{G}^{\preccurlyeq}$ *with* $Lf = g$*, such that* f *and* $\tilde{f}$ *are both differentially and asymptotically equivalent over* $\mathcal{T}$*.*

Theorem. *Let* $\mathcal{T}$ *be a transserial Hardy field. Let* $\mathcal{T}^{\mathrm{fo}} \supseteq \mathcal{T}$ *be the smallest differential subfield of* $\mathbb{T}$*, such that for any* $P \in \mathcal{T}^{\mathrm{fo}}\{F\}^{\neq}$ *with* $r_P \leqslant 1$ *and* $f \in \mathbb{T}$ *we have* $P(f) = 0 \Rightarrow f \in \mathcal{T}^{\mathrm{fo}}$*. Then the transserial Hardy field structure of* $\mathcal{T}$ *can be extended to* $\mathcal{T}^{\mathrm{fo}}$*.*

Proof. As long as $\mathcal{T}^{\mathrm{fo}} \neq \mathcal{T}$:

- Close off under $\exp$, $\log$ and algebraic equations.
- Choose $P \in \mathcal{T}\{F\}^{\neq}, r_P = 1, f \in \mathbb{T}, P(f) = 0$ such that P has minimal "complexity" (r_P, d_P, t_P) and apply the lemma. □

Higher order extensions

Lemma. *Let* $L = \partial - \varphi \in \mathcal{T}[\mathrm{i}][\partial]$ *be a normal operator. Let* $\tilde{f} \in \hat{\mathcal{T}}[\mathrm{i}]^{\preccurlyeq}$ *and* $g \in \mathcal{T}[\mathrm{i}]^{\preccurlyeq}$ *be such that* $\operatorname{Re} \tilde{f}$ *has order* 2 *over* $\mathcal{T}$ *and* $L \tilde{f} = g$*. Then there exists an* $f \in \mathcal{G}^{\preccurlyeq}[\mathrm{i}]$ *with* $L f = g$*, such that* $\operatorname{Re} f$ *and* $\operatorname{Re} \tilde{f}$ *are both differentially and asymptotically equivalent over* $\mathcal{T}$*.*

Theorem. *Let* $\mathcal{T}$ *be a transserial Hardy field. Let* $\mathcal{T}^{\mathrm{dalg}} \supseteq \mathcal{T}$ *be the smallest differential subfield of* $\mathbb{T}$*, such that for any* $P \in \mathcal{T}^{\mathrm{dalg}}\{F\}^{\neq}$ *and* $f \in \mathbb{T}$ *we have* $P(f) = 0 \Rightarrow f \in \mathcal{T}^{\mathrm{dalg}}$*. Then the transserial Hardy field structure of* $\mathcal{T}$ *can be extended to* $\mathcal{T}^{\mathrm{dalg}}$*.*

Applications

Corollary. *There exists a transserial Hardy field* $\mathcal{T}$*, such that for any* $P \in \mathcal{T}\{F\}$ *and* $f, g \in \mathcal{T}$ *with* $f < g$ *and* $P(f)\,P(g) < 0$*, there exists a* $h \in \mathcal{T}$ *with* $f < h < g$ *and* $P(h) = 0$*.*

Corollary. *There exists a transserial Hardy field* $\mathcal{T}$*, such that* $\mathcal{T}[\mathrm{i}]$ *is weakly differentially closed.*

Corollary. *There exists a differentially Henselian transserial Hardy field* $\mathcal{T}$*, i.e., such that any quasi-linear differential equation over* $\mathcal{T}$ *admits a solution in* $\mathcal{T}$*.*

A partial inverse

Theorem. *Let $\mathcal{T}$ be a transserial Hardy field and $\mathcal{H}$ a differentially algebraic Hardy field extension of $\mathcal{T}$, such that $\mathcal{H}$ is differentially Henselian and stable under exponentiation. Then there exists a transserial Hardy field structure on $\mathcal{H}$ which extends the structure on $\mathcal{T}$.*

Corollary. *Let $\mathcal{T}$ be a transserial Hardy field and $\mathcal{H}$ a differentially algebraic Hardy field extension of $\mathcal{T}$, such that $\mathcal{H}$ is differentially Henselian. Assume that $\mathcal{H}$ admits no non-trivial algebraically differential Hardy field extensions. Then $\mathcal{H}$ satisfies the differential intermediate value property.*

Theorem. *(Boshernitzan 1987) Any solution of the equation*

$$f'' + f = \mathrm{e}^{x^2}$$

is contained in a Hardy field. However, none of these solutions is contained in the intersection of all maximal Hardy fields.

Open questions

1. Embeddability of Hardy fields in differentially Henselian Hardy fields.
2. Do maximal Hardy fields satisfy the intermediate value property?
3. Restricted analytic (instead of algebraic) differential equations.
4. Preservation of composition:
    a. $f(x+\varepsilon)$, small ε: expand.
    b. $f(q\,x+\varepsilon)$: expand, but more intricate.
    c. $f(\varphi(x)), \varphi \succ x$: abstract nonsense.