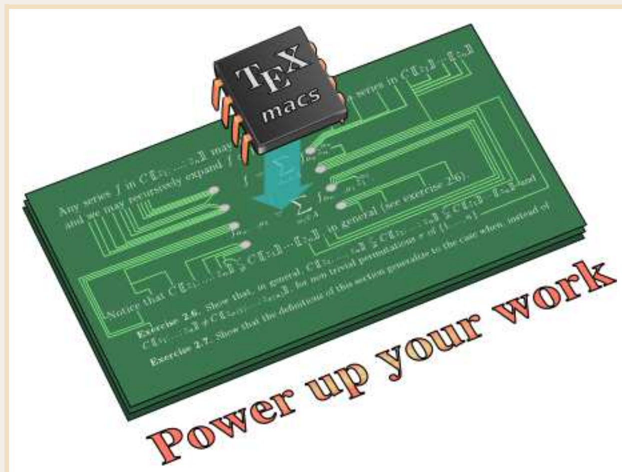
# The art of guessing

# Longue et heureuse vie à Marc !

Joris van der Hoeven




**Gecko/Tera, École Polytechnique, 2008**

http://www.TeXmacs.org

# Guess what

$$0.142857142857142857142857142857142857142857\cdots$$

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# Guess what

$$\frac{1}{7}$$

# Guess what

**Applications:**

- More comprehensive output.
- New unexpected insights.
- Guide decisions in algorithms.

# Efficient algorithm

$$x_0 = a_0 + \frac{1}{x_1} \qquad a_0 = \lfloor x_0 \rfloor$$

$$x_l = \frac{p_l\, x_0 + q_l}{r_l\, x_0 + s_l} \qquad p_l, q_l, r_l, s_l \in \mathbb{Z}$$

$$\begin{pmatrix} p_{k,l+m} & q_{k,l+m} \\ r_{k,l+m} & s_{k,l+m} \end{pmatrix} = \begin{pmatrix} p_{k+l,m} & q_{k+l,m} \\ r_{k+l,m} & s_{k+l,m} \end{pmatrix} \begin{pmatrix} p_{k,l} & q_{k,l} \\ r_{k,l} & s_{k,l} \end{pmatrix}$$

$$\begin{aligned} \mathsf{F}(p) &= 2\,\mathsf{F}(p/2) + O(\mathsf{I}(p)) \\ &= O(\mathsf{I}(p) \log p) \\ &= O(p \log^2 p \log\log p) \end{aligned}$$

# Efficient algorithm

$$x_0 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{x_2}} \qquad a_1 = \lfloor x_1 \rfloor$$

$$x_l = \frac{p_l\, x_0 + q_l}{r_l\, x_0 + s_l} \qquad p_l, q_l, r_l, s_l \in \mathbb{Z}$$

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# Efficient algorithm

$$x_0 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{l-2} + \cfrac{1}{a_{l-1} + x_l}}}}$$

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$$x_0 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{l-2} + \cfrac{1}{a_{l-1} + x_l}}}}$$

$$x_{k+l} = \frac{p_{k,l}\,x_l + q_{k,l}}{r_{k,l}\,x_l + s_{k,l}} \qquad p_{k,l}, q_{k,l}, r_{k,l}, s_{k,l} \in \mathbb{Z}$$

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# Guessing $\mathbb{Z}$-linear dependencies

**Question:** given $z_1, \ldots, z_n \in \mathbb{R}$, find $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}$ with

$$\lambda_1 z_1 + \cdots + \lambda_n z_n = 0$$

**Example:**

- $z_1 = 1, \ldots, z_n = \alpha^{n-1}$ for $\alpha \in \mathbb{R}$.
- $z_1 = \mathrm{Li}_{3,3}$, $z_2 = \mathrm{Li}_6$, $z_3 = \mathrm{Li}_{4,2}$, $z_4 = \mathrm{Li}_{2,2,2}$.

**Lattice reduction:**

$$M = \begin{pmatrix} z_1 & \varepsilon & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ z_n & 0 & 0 & \varepsilon \end{pmatrix}$$

# LLL-algorithm (sketch)

## Ingredient 1: orthogonal projection

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} \\ a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} & a_{5,6} \end{pmatrix}$$

to be reduced

reduced

## Ingredient 2: flipping

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} \\ a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} & a_{5,6} \end{pmatrix}$$

$|a_3| < \lambda\,|\alpha_4|$

reduced

# LLL-algorithm (sketch)

## Ingredient 1: orthogonal projection

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & \alpha_4 & \alpha_5 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} \\ a'_{3,1} & a'_{3,2} & a'_{3,3} & a'_{3,4} & a'_{3,5} & a'_{3,6} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} \\ a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} & a_{5,6} \end{pmatrix}$$

$a'_3 \perp V(a_4, a_5)$

reduced

## Ingredient 2: flipping

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} \\ a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} & a_{5,6} \end{pmatrix}$$

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reduced

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# Improved LLL-algorithms

**Ingredient 1: block algorithms**

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} & a_{1,7} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} & a_{2,7} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} & a_{3,7} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} & a_{4,7} \\ a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} & a_{5,6} & a_{5,7} \\ a_{6,1} & a_{6,2} & a_{6,3} & a_{6,4} & a_{6,5} & a_{6,6} & a_{6,7} \end{pmatrix}$$

**Ingredient 2: binary splitting**

$$A = A^{\text{hi}} + A^{\text{lo}} \, 2^{-p/2}$$

$$P^{\text{hi}} A^{\text{hi}} = R^{\text{hi}}$$

$$P^{\text{hi}} A = B$$

$$P^{\text{lo}} B = R$$

$$P = P^{\text{lo}} P^{\text{hi}}$$

**Practical complexity?** $O(\mathsf{IM}(n, p) \log p) = O(n^{2.38} \, p \log p + n^2 \log^2 p \log \log p)$

# Improved LLL-algorithms

## Ingredient 1: block algorithms

$$\begin{pmatrix} 1 & & & p & q & r \\ & 1 & & s & t & u \\ & & 1 & v & w & x \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} a'_{1,1} & a'_{1,2} & a'_{1,3} & a'_{1,4} & a'_{1,5} & a'_{1,6} & a'_{1,7} \\ a'_{2,1} & a'_{2,2} & a'_{2,3} & a'_{2,4} & a'_{2,5} & a'_{2,6} & a'_{2,7} \\ a'_{3,1} & a'_{3,2} & a'_{3,3} & a'_{3,4} & a'_{3,5} & a'_{3,6} & a'_{3,7} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} & a_{4,7} \\ a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} & a_{5,6} & a_{5,7} \\ a_{6,1} & a_{6,2} & a_{6,3} & a_{6,4} & a_{6,5} & a_{6,6} & a_{6,7} \end{pmatrix}$$

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# Improved LLL-algorithms

## Ingredient 1: block algorithms

$$\begin{pmatrix} 1 & & & \lfloor p \rceil & \lfloor q \rceil & \lfloor r \rceil \\ & 1 & & \lfloor s \rceil & \lfloor t \rceil & \lfloor u \rceil \\ & & 1 & \lfloor v \rceil & \lfloor w \rceil & \lfloor x \rceil \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} a''_{1,1} & a''_{1,2} & a''_{1,3} & a''_{1,4} & a''_{1,5} & a''_{1,6} & a''_{1,7} \\ a''_{2,1} & a''_{2,2} & a''_{2,3} & a''_{2,4} & a''_{2,5} & a''_{2,6} & a''_{2,7} \\ a''_{3,1} & a''_{3,2} & a''_{3,3} & a''_{3,4} & a''_{3,5} & a''_{3,6} & a''_{3,7} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} & a_{4,7} \\ a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} & a_{5,6} & a_{5,7} \\ a_{6,1} & a_{6,2} & a_{6,3} & a_{6,4} & a_{6,5} & a_{6,6} & a_{6,7} \end{pmatrix}$$

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**Practical complexity?** $O(\mathsf{IM}(n, p) \log p) = O(n^{2.38} \, p \log p + n^2 \log^2 p \log \log p)$

# Power series analogues

## Guessing rationality

Problem: given $f \in \mathbb{K}[[z]]$, guess $P, Q \in \mathbb{K}[z]$ with $f = \frac{P}{Q}$

Solution: Padé approximation

## Polynomial dependencies

Problem: given $f_1, \ldots, f_n \in \mathbb{K}[[z]]$, guess $P_1, \ldots, P_n \in \mathbb{K}[z]$ with

$$P_1 f_1 + \cdots + P_n f_n = 0.$$

Solution: Padé-Hermite approximation (Beckermann-Labahn, Derksen, Gfun)

## Applications

- $(f_n)_{n \in \mathbb{N}} = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$
- $f_1 = \varphi, f_2 = \varphi', \ldots, f_n = \varphi^{(n-1)}$, with $\varphi \in \mathbb{K}[[z]]$

# Guessing singular dependencies

**Example:** number $s_n$ of alcohols of the form $C_n H_{2n+1} O H$

$$s(z) = 1 + z\,\frac{s(z)^3 + 2\,s(z^3)}{3}.$$

Dominant **algebraic** singularity at $r = 0.304218409\cdots$. Setting

$$\begin{aligned}\varphi(z) &= s(r+z)\\ \psi(z) &= 2\,s((r+z)^3) + 3,\end{aligned}$$

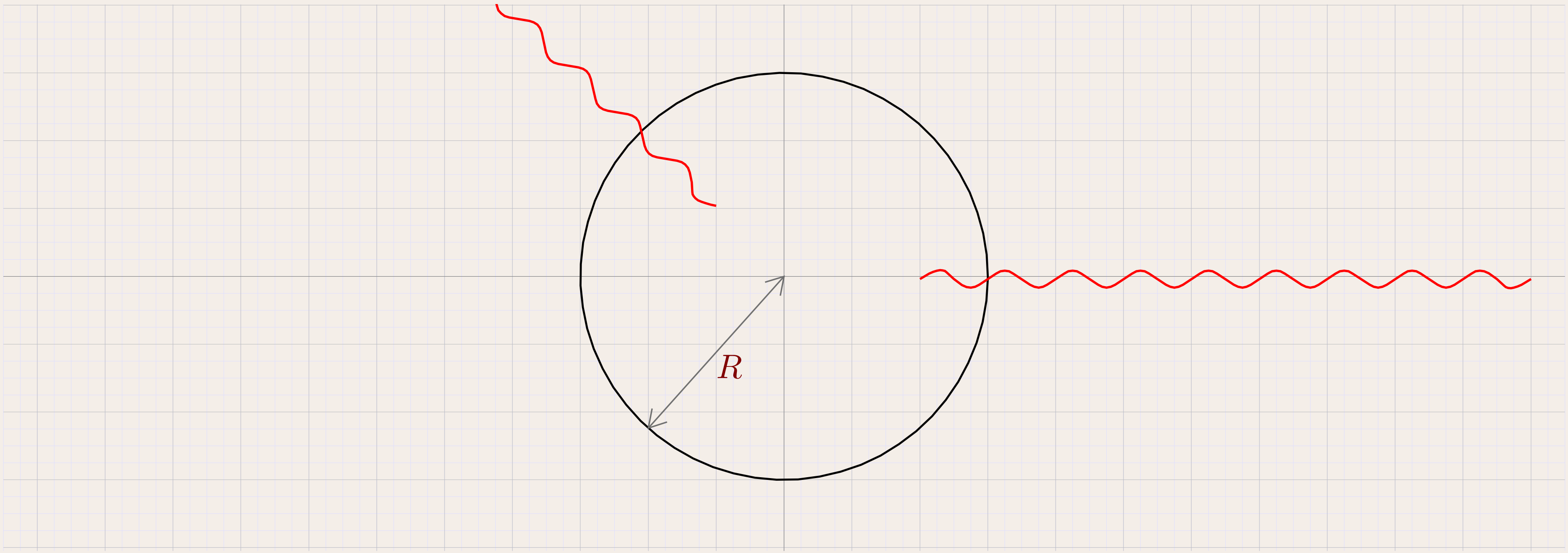
we have

$$(r+z)\,\varphi(z)^3 - 3\,\varphi(z) + \psi(z) \;=\; 0.$$

**Problem:** given $f_1, \ldots, f_d \in \mathbb{C}[[z]]$ and $R$, guess relations

$$g_1\,f_1 + \cdots + g_d\,f_d = h \qquad (g_1, \ldots, g_d, h \in \mathbb{A}_R;\ \text{i.e. analytic on } \bar{\mathcal{D}}_R)$$

# Guessing singular dependencies

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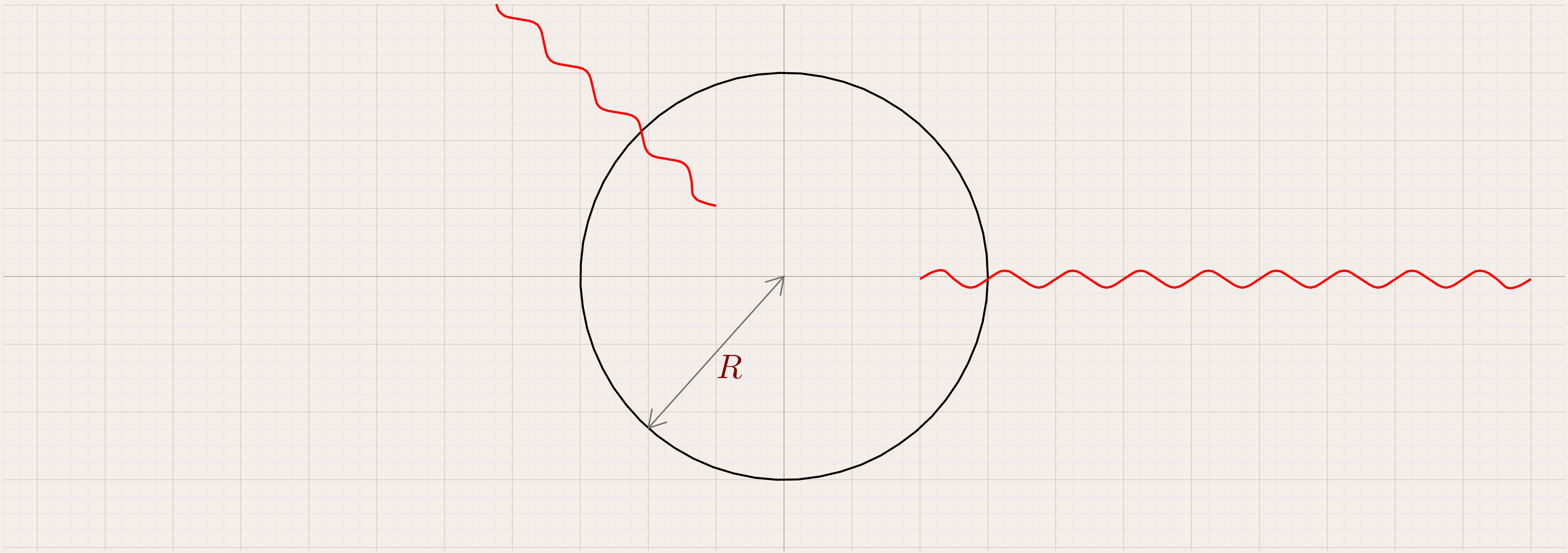



**Problem:** given $f_1, \ldots, f_d \in \mathbb{C}[[z]]$ and $R = 1$, find a "minimal relation"

$$g_1 f_1 + \cdots + g_d f_d = h$$

for the power series norm

$$\|\varphi\| = \sqrt{\varphi_0^2 + \varphi_1^2 + \cdots}.$$

# Truncated problem at order $n$

$$g_1 f_1 + \cdots + g_d f_d = h$$

$$G = \begin{pmatrix} g_{1,n-1} & \cdots & g_{d,n-1} & \cdots & \cdots & g_{1,0} & \cdots & g_{d,0} \end{pmatrix}$$

$$M = \left(\begin{array}{cccccccccccccccc}
f_{1,0} & & & & & 1 & & & & & & & & & & \\
\vdots & & & & & & \ddots & & & & & & & & & \\
f_{d,0} & & & & & & & 1 & & & & & & & & \\
f_{1,1} & f_{1,0} & & & & & & & 1 & & & & & & & \\
\vdots & \vdots & & & & & & & & \ddots & & & & & & \\
f_{d,1} & f_{d,0} & & & & & & & & & 1 & & & & & \\
\vdots & \vdots & \ddots & & & & & & & & & \ddots & & & & \\
\vdots & \vdots & & \ddots & & & & & & & & & \ddots & & & \\
f_{1,n-1} & f_{1,n-2} & \cdots & \cdots & f_{1,0} & & & & & & & & & 1 & & \\
\vdots & \vdots & & & \vdots & & & & & & & & & & \ddots & \\
f_{d,n-1} & f_{d,n-2} & \cdots & \cdots & f_{d,0} & & & & & & & & & & & 1
\end{array}\right).$$

$$G\,M = \begin{pmatrix} h_{n-1} & h_{n-2} & \cdots & \cdots & h_0 & g_{1,n-1} & \cdots & g_{d,n-1} & \cdots & g_{1,0} & \cdots & g_{d,0} \end{pmatrix}$$

# Ultimate correctness

**Definition.** *Given* $n \in \mathbb{N} \cup \{+\infty\}$, *let* $\boldsymbol{\Phi}_{;n}$ *be the set of vectors*

$$\varphi = (h_{n-1}, \ldots, h_0, g_{1,n-1}, \ldots, g_{d,n-1}, \ldots\ldots, g_{1,0}, \ldots, g_{d,0}), \qquad \|\varphi\| < +\infty.$$

*We say that* $\varphi$ *is* ***i-normal*** *if* $g_{i,0} = 1$ *and* $g_{j,0} = 0$ *for all* $j > i$.

**Theorem.**

- *If* $\boldsymbol{\Phi}_{;\infty} \neq 0$, *then* $\boldsymbol{\Phi}_{;\infty}$ *contains a minimal normal relation.*
- *Assume that* $\varphi \in \boldsymbol{\Phi}_{;\infty}$ *is a minimal* $i$*-normal relation. For each* $n \in \mathbb{N}$, *let* $f_{;n}$ *be the truncation of* $f$ *at order* $n$ *and consider the corresponding minimal* $i$*-normal relation* $\varphi_{;n} \in \boldsymbol{\Phi}_{;n}$. *Then the relations* $\varphi_{;n}$ *converge to* $\varphi$.
- *If* $\boldsymbol{\Phi}_{;\infty} = 0$, *then the* $\|\varphi_{;n}\|$ *are unbounded for* $n \to \infty$.

# Example I

$$f_1 = \frac{1}{1-\lambda z} \qquad (\lambda > 1)$$

$$\begin{aligned}
g_{;4,1} &= 1.0000 - 1.6000\, z - 0.60000\, z^2 \\
&\quad - 0.20000\, z^3 \\
g_{;16,1} &= 1.0000 - 1.6180\, z - 0.61803\, z^2 - \cdots \\
&\quad - 5.8340\, 10^{-6}\, z^{14} - 1.9447\, 10^{-6}\, z^{15} \\
g_{;64,1} &= 1.0000 - 1.6180\, z - 0.61803\, z^2 - \cdots \\
&\quad - 5.0484\, 10^{-26}\, z^{62} - 1.6828\, 10^{-26}\, z^{63} \\
g_{;256,1} &= 1.0000 - 1.6180\, z - 0.61803\, z^2 - \cdots \\
&\quad - 2.8307\, 10^{-106}\, z^{254} - 9.4357\, 10^{-107}\, z^{255}
\end{aligned}$$

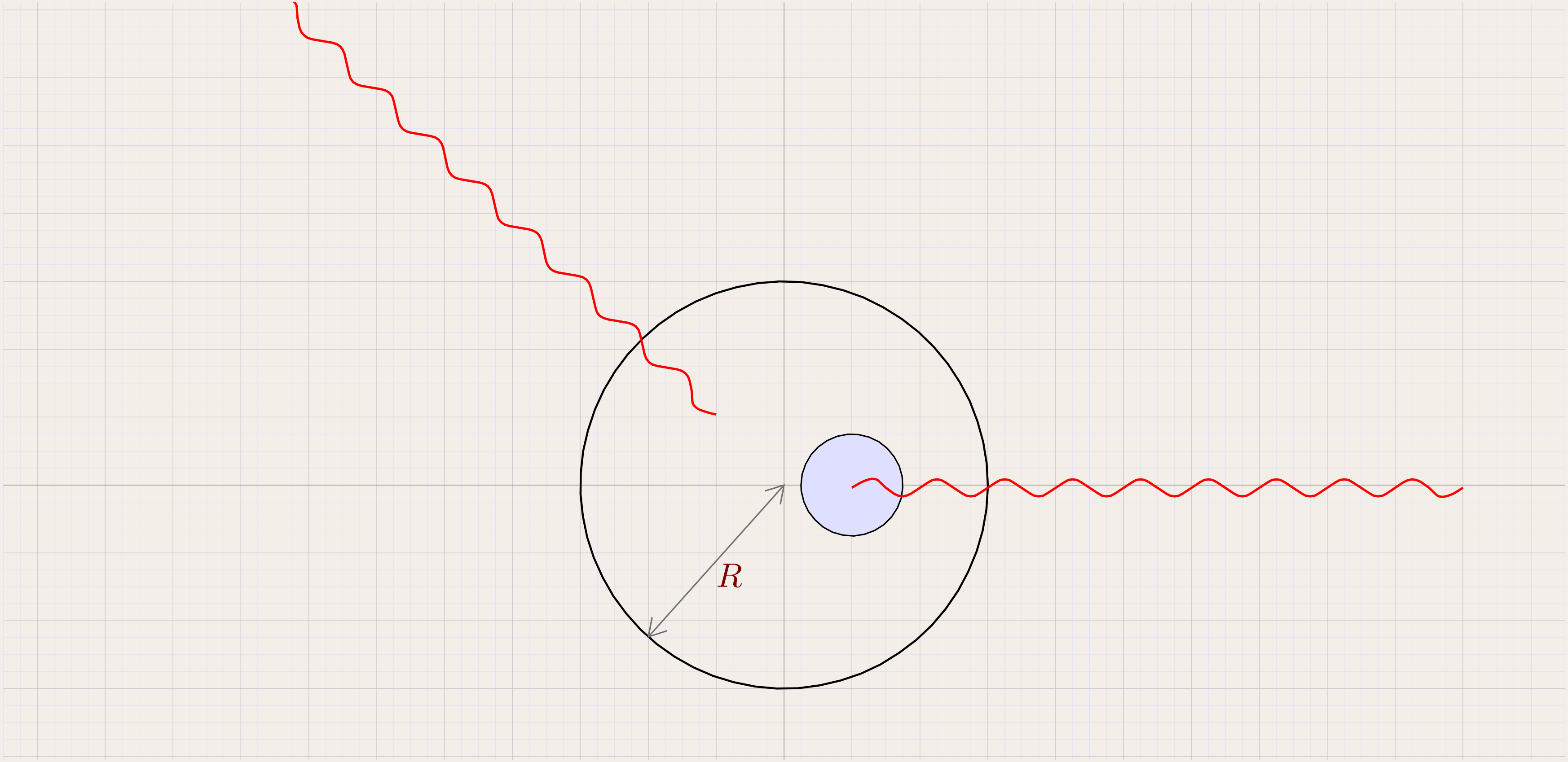
**Fast convergence**, but **not** $g_1 = 1 - \lambda z$. In fact

$$g_1 = \frac{1-\lambda z}{1-\alpha z}$$

$$\alpha + \frac{1}{\alpha} = \lambda + \frac{2}{\lambda} \qquad (\alpha < 1)$$

# Example II

$$f_1 = s(0.15\,(z+0.25))$$
$$f_2 = f_1'$$



# Example II

$$f_1 = s(0.15\,(z+0.25))$$
$$f_2 = f_1'$$

$$g_{;32,1} = 1.6836 - 0.29080\,z + 0.0013079\,z^2 + \cdots + 0.022645\,z^{30} - 0.0029284\,z^{31}$$

$$g_{;32,2} = 1.0000 - 2.4006\,z - 0.88670\,z^2 - \cdots + 0.0061880\,z^{30} - 9.9191 \cdot 10^{-4}\,z^{31}$$

$$g_{;64,1} = 1.7104 - 0.36445\,z + 0.23778\,z^2 - \cdots + 1.6968 \cdot 10^{-4}\,z^{62} - 1.0104 \cdot 10^{-5}\,z^{63}$$

$$g_{;64,2} = 1.0000 - 2.3257\,z - 1.2027\,z^2 - \cdots + 5.2359 \cdot 10^{-5}\,z^{62} - 3.4225 \cdot 10^{-6}\,z^{63}$$

$$g_{;128,1} = 1.7105 - 0.36215\,z + 0.23696\,z^2 - 0.052535\,z^3 + 0.033518\,z^4 - \cdots + 4.6860 \cdot 10^{-9}\,z^{126} - 1.3437 \cdot 10^{-10}\,z^{127}$$

$$g_{;128,2} = 1.0000 - 2.3235\,z - 1.2138\,z^2 - 0.0067226\,z^3 - 0.080434\,z^4 + \cdots + 1.5192 \cdot 10^{-9}\,z^{126} - 4.5516 \cdot 10^{-11}\,z^{127}$$

**Slow convergence:** $\|\varphi_{;32}\| = 4.3276$, $\|\varphi_{;64}\| = 4.3845$, $\|\varphi_{;128}\| = 4.3863$.

# Counter-Example III

$$
\begin{aligned}
f_1 &= \psi_{i,\lambda} \\
\psi_{1,\lambda} &= \log(1-\lambda\, z) \\
\psi_{2,\lambda} &= \sqrt{1-\lambda\, z} \\
\psi_{3,\lambda} &= \mathrm{e}^{\frac{\lambda\, z}{1-\lambda\, z}} \\
\psi_{4,\lambda} &= \mathrm{random}(\lambda\, z)
\end{aligned}
$$

| | 32 | 64 | 128 | 256 |
|---|---|---|---|---|
| $\psi_{1,\sqrt{2}}$ | $2.5897$ | $4.2958$ | $1.1107 \cdot 10^{1}$ | $7.5308 \cdot 10^{1}$ |
| $\psi_{1,2}$ | $1.2324 \cdot 10^{1}$ | $7.7306 \cdot 10^{1}$ | $3.3839 \cdot 10^{3}$ | $6.4461 \cdot 10^{6}$ |
| $\psi_{1,4}$ | $2.7074 \cdot 10^{3}$ | $3.1503 \cdot 10^{6}$ | $5.0806 \cdot 10^{12}$ | $1.3215 \cdot 10^{25}$ |
| $\psi_{1,8}$ | $5.7101 \cdot 10^{6}$ | $1.6078 \cdot 10^{13}$ | $1.2829 \cdot 10^{26}$ | $8.6064 \cdot 10^{51}$ |
| $\psi_{1,16}$ | $6.0964 \cdot 10^{10}$ | $1.7814 \cdot 10^{21}$ | $1.5612 \cdot 10^{42}$ | $1.2279 \cdot 10^{84}$ |
| $\psi_{1,32}$ | $1.1152 \cdot 10^{15}$ | $9.1674 \cdot 10^{29}$ | $4.9020 \cdot 10^{59}$ | $1.6662 \cdot 10^{119}$ |
| $\psi_{2,\sqrt{2}}$ | $2.2960$ | $3.7208$ | $9.7394$ | $6.5890 \cdot 10^{1}$ |
| $\psi_{2,2}$ | $9.3539$ | $6.1551 \cdot 10^{1}$ | $2.6678 \cdot 10^{3}$ | $5.0111 \cdot 10^{6}$ |
| $\psi_{2,4}$ | $1.7232 \cdot 10^{3}$ | $2.1500 \cdot 10^{6}$ | $3.4149 \cdot 10^{12}$ | $8.7128 \cdot 10^{24}$ |
| $\psi_{2,8}$ | $3.5980 \cdot 10^{6}$ | $1.0031 \cdot 10^{13}$ | $7.9396 \cdot 10^{25}$ | $5.2284 \cdot 10^{51}$ |
| $\psi_{3,\sqrt{2}}$ | $6.2155$ | $1.3295 \cdot 10^{1}$ | $4.1722 \cdot 10^{1}$ | $4.2679 \cdot 10^{2}$ |
| $\psi_{3,2}$ | $4.0164 \cdot 10^{1}$ | $3.9310 \cdot 10^{2}$ | $1.4047 \cdot 10^{4}$ | $1.1091 \cdot 10^{7}$ |
| $\psi_{3,4}$ | $6.9272 \cdot 10^{3}$ | $6.8304 \cdot 10^{6}$ | $4.7564 \cdot 10^{11}$ | $5.1308 \cdot 10^{20}$ |
| $\psi_{3,8}$ | $3.6660 \cdot 10^{6}$ | $9.4896 \cdot 10^{11}$ | $4.2745 \cdot 10^{21}$ | $1.1809 \cdot 10^{39}$ |
| $\psi_{4,2}$ | $8.0487 \cdot 10^{4}$ | $3.5565 \cdot 10^{9}$ | $2.1354 \cdot 10^{19}$ | $4.8792 \cdot 10^{38}$ |
| $\psi_{4,4}$ | $5.0774 \cdot 10^{9}$ | $1.1548 \cdot 10^{19}$ | $2.9916 \cdot 10^{38}$ | $1.2335 \cdot 10^{77}$ |
| $\psi_{4,8}$ | $3.2564 \cdot 10^{14}$ | $4.3151 \cdot 10^{28}$ | $5.1348 \cdot 10^{57}$ | $2.9654 \cdot 10^{115}$ |

$$\log \|\varphi_{;n}\| \approx \frac{n}{2} \log \lambda.$$

# Counter-Example III

$$
\begin{aligned}
f_1 &= \psi_{i,\lambda} \\
\psi_{1,\lambda} &= \log(1 - \lambda\, z) \\
\psi_{2,\lambda} &= \sqrt{1 - \lambda\, z} \\
\psi_{3,\lambda} &= \mathrm{e}^{\frac{\lambda\, z}{1 - \lambda\, z}} \\
\psi_{4,\lambda} &= \mathrm{random}(\lambda\, z)
\end{aligned}
$$

| | 32 | 64 | 128 | 256 |
|---|---|---|---|---|
| $\psi_{1,\sqrt{2}}$ | $2.5897$ | $4.2958$ | $1.1107 \cdot 10^{1}$ | $7.5308 \cdot 10^{1}$ |
| $\psi_{1,2}$ | $1.2324 \cdot 10^{1}$ | $7.7306 \cdot 10^{1}$ | $3.3839 \cdot 10^{3}$ | $6.4461 \cdot 10^{6}$ |
| $\psi_{1,4}$ | $2.7074 \cdot 10^{3}$ | $3.1503 \cdot 10^{6}$ | $5.0806 \cdot 10^{12}$ | $1.3215 \cdot 10^{25}$ |
| $\psi_{1,8}$ | $5.7101 \cdot 10^{6}$ | $1.6078 \cdot 10^{13}$ | $1.2829 \cdot 10^{26}$ | $8.6064 \cdot 10^{51}$ |
| $\psi_{1,16}$ | $6.0964 \cdot 10^{10}$ | $1.7814 \cdot 10^{21}$ | $1.5612 \cdot 10^{42}$ | $1.2279 \cdot 10^{84}$ |
| $\psi_{1,32}$ | $1.1152 \cdot 10^{15}$ | $9.1674 \cdot 10^{29}$ | $4.9020 \cdot 10^{59}$ | $1.6662 \cdot 10^{119}$ |
| $\psi_{2,\sqrt{2}}$ | $2.2960$ | $3.7208$ | $9.7394$ | $6.5890 \cdot 10^{1}$ |
| $\psi_{2,2}$ | $9.3539$ | $6.1551 \cdot 10^{1}$ | $2.6678 \cdot 10^{3}$ | $5.0111 \cdot 10^{6}$ |
| $\psi_{2,4}$ | $1.7232 \cdot 10^{3}$ | $2.1500 \cdot 10^{6}$ | $3.4149 \cdot 10^{12}$ | $8.7128 \cdot 10^{24}$ |
| $\psi_{2,8}$ | $3.5980 \cdot 10^{6}$ | $1.0031 \cdot 10^{13}$ | $7.9396 \cdot 10^{25}$ | $5.2284 \cdot 10^{51}$ |
| $\psi_{3,\sqrt{2}}$ | $6.2155$ | $1.3295 \cdot 10^{1}$ | $4.1722 \cdot 10^{1}$ | $4.2679 \cdot 10^{2}$ |
| $\psi_{3,2}$ | $4.0164 \cdot 10^{1}$ | $3.9310 \cdot 10^{2}$ | $1.4047 \cdot 10^{4}$ | $1.1091 \cdot 10^{7}$ |
| $\psi_{3,4}$ | $6.9272 \cdot 10^{3}$ | $6.8304 \cdot 10^{6}$ | $4.7564 \cdot 10^{11}$ | $5.1308 \cdot 10^{20}$ |
| $\psi_{3,8}$ | $3.6660 \cdot 10^{6}$ | $9.4896 \cdot 10^{11}$ | $4.2745 \cdot 10^{21}$ | $1.1809 \cdot 10^{39}$ |
| $\psi_{4,2}$ | $8.0487 \cdot 10^{4}$ | $3.5565 \cdot 10^{9}$ | $2.1354 \cdot 10^{19}$ | $4.8792 \cdot 10^{38}$ |
| $\psi_{4,4}$ | $5.0774 \cdot 10^{9}$ | $1.1548 \cdot 10^{19}$ | $2.9916 \cdot 10^{38}$ | $1.2335 \cdot 10^{77}$ |
| $\psi_{4,8}$ | $3.2564 \cdot 10^{14}$ | $4.3151 \cdot 10^{28}$ | $5.1348 \cdot 10^{57}$ | $2.9654 \cdot 10^{115}$ |

$$\log \|\varphi_{;n}\| \approx \frac{n}{d+1} \log \lambda.$$

# Only poles

**Problem:** given $f \in \mathbb{C}[[z]]$, guess $P \in \mathbb{C}[z]$ with

$$P f \in \mathbb{A}_1.$$

**Algorithm** `denom`$(f, N, D)$

INPUT: the first $N > 2D$ coefficients of $f$ and a degree bound $D$

OUTPUT: $\approx$ minimal monic polynomial $P$ with $\deg P \leqslant D$, $P f \in \mathbb{A}_1$, or **failed**

**Step 1.** [Initialize]

$d := 0$

**Step 2.** [Determine $P$]

$$\text{Solve } \begin{pmatrix} f_{N-2d+1} & \cdots & f_{N-d} \\ \vdots & & \vdots \\ f_{N-d} & \cdots & f_{N-1} \end{pmatrix} \begin{pmatrix} P_{d-1} \\ \vdots \\ P_0 \end{pmatrix} + \begin{pmatrix} f_{N-2d} \\ \vdots \\ f_{N-d-1} \end{pmatrix} = 0.$$

Set $P := z^d + P_{d-1} z^{d-1} + \cdots + P_0$

**Step 3.** [Terminate or loop]

If $Pf \in \mathbb{A}_1$ then return $P$

If $d = N$ then return **failed**

Set $d := d + 1$ and go to step 2

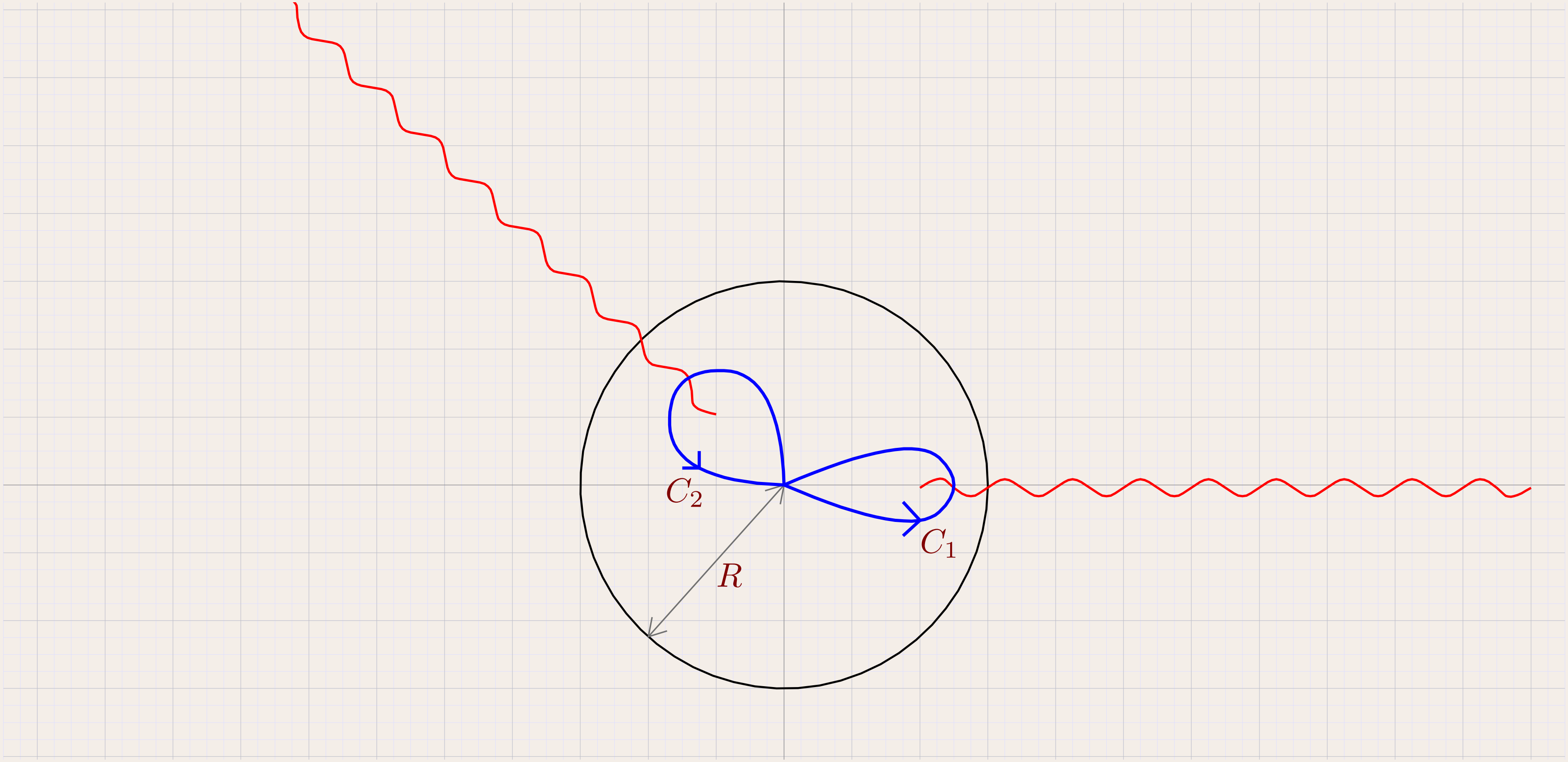
**Theorem 1.** *Exponential convergence in* $N$.

# Only algebraic singularities

**Problem:** given $f \in \mathbb{C}[[z]]$, guess $P_d, \ldots, P_0 \in \mathbb{C}[z]$ with

$$P_d f^d + \cdots + P_0 \in \mathbb{A}_1. \quad (1)$$

**Assume:** algorithm for analytic continuation of $f$.



# Only algebraic singularities

**Problem:** given $f \in \mathbb{C}[[z]]$, guess $P_d, \ldots, P_0 \in \mathbb{C}[z]$ with

$$P_d f^d + \cdots + P_0 \in \mathbb{A}_1. \quad (2)$$

**Assume:** algorithm for analytic continuation of $f$.

**Algorithm** `alg_dep`$(f, (\sigma_1, \ldots, \sigma_s), N, D, B)$

INPUT: analytic function $f$ above $\bar{\mathcal{D}}_1 \setminus \{\sigma_1, \ldots, \sigma_s\}$ and bounds $N$, $D$, $B$

OUTPUT: normalized dependency (2) with $d \leqslant B$, $\deg P_d \leqslant D$, or **failed**

**Step 1.** [Initialize]

Set $\Phi := \{f\}$

**Step 2.** [Saturate]

If $C_i \Phi \subseteq \Phi$ for all $i$, then go to step 3

If $\operatorname{card}(\Phi) > N$ then return **failed**

$\Phi := C_1 \Phi \cup \cdots \cup C_s \Phi$

Repeat step 2

**Step 3.** [Terminate]

Denote $\Phi := \{\varphi_1, \ldots, \varphi_k\}$

Compute $Q := (F - \varphi_1) \cdots (F - \varphi_k) = Q_k F^k + \cdots + Q_0$

For each $i \in \{0, \ldots, k\}$, compute $D_i := \mathtt{denom}(Q_i, r, D)$

If $D_i$ = **failed** for some $i$, then return **failed**

Return $\operatorname{lcm}(D_0, \ldots, D_k)\, Q$

# Only Fuchsian singularities

**Problem:** given $f \in \mathbb{C}[[z]]$, guess $L_r, \ldots, L_0 \in \mathbb{C}[z]$ with

$$L_r f^{(r)} + \cdots + L_0 \in \mathbb{A}_1. \qquad (3)$$

**Assume:** algorithm for analytic continuation of $f$.

**Algorithm** `fuch_dep`$(f, (\sigma_1, \ldots, \sigma_s), N, D, B)$

INPUT: analytic function $f$ above $\bar{\mathcal{D}}_1 \setminus \{\sigma_1, \ldots, \sigma_s\}$ and bounds $N$, $D$, $B$

OUTPUT: normalized dependency (3) with $r \leqslant B$, $\deg L_r \leqslant D$, or **failed**

**Step 1.** [Initialize]

Set $\Phi := \{f\}$

**Step 2.** [Saturate]

If $\operatorname{Vect}(C_i \Phi) \subseteq \operatorname{Vect}(\Phi)$ for all $i$, then go to step 3

If $\operatorname{card}(\Phi) > N$ then return **failed**

$\Phi := \Phi \cup \{C_i \varphi\}$ for $i$ and $\varphi \in \Phi$ with $C_i \varphi \notin \operatorname{Vect}(\Phi)$

Repeat step 2

**Step 3.** [Terminate]

Denote $\Phi := \{\varphi_1, \ldots, \varphi_k\}$

Compute $K := \mathrm{lcm}(\partial - \varphi_1^\dagger, \ldots, \partial - \varphi_k^\dagger) = K_k \partial^k + \cdots + K_0$

in the skew polynomial ring $\mathbb{C}((z))[\partial]$, where $\varphi^\dagger$ denotes $\varphi' / \varphi$

For each $i \in \{0, \ldots, k\}$, compute $D_i := \mathtt{denom}(K_i, r, D)$

If $D_i$ = **failed** for some $i$, then return **failed**

Return $\mathrm{lcm}(D_0, \ldots, D_k)\, K$

**Remark.** Only works for Fuchsian singularities, because of

$$f = \mathrm{e}^{\frac{1}{z-\sigma}} + \mathrm{e}^{\frac{-1}{z-\sigma}}.$$

# Guessing asymptotic behaviour

## ⇑ Guessing Stirling's formula

```
Mmx] use "symbolix"; use "multimix"; use "jorix"
Mmx] include "jorix/extrapolate_extra.mmx"
Mmx] bit_precision := 256; significant_digits := 5;
Mmx] v == [ 1.0 * n! | n in 0..1000 ];
Mmx] guess_asymptotics (v, 9)
Mmx]
```

## ⇑ Number of alcohols

```
Mmx] use "symbolix"; use "multimix"; use "jorix"
Mmx] include "jorix/extrapolate_extra.mmx"
Mmx] bit_precision := 256; significant_digits := 15;
Mmx] s == fixed_point_series (f :->
         ((f*f*f + 2*dilate(f,3)) / 3) << 1, 1)
Mmx] s[1000]
Mmx] w == [ 1.0 * s[n] | n in 0..1000 ];
Mmx] guess_asymptotics (w, 9)
Mmx] exp (-1.19000938463478)
Mmx]
```

# Asymptotic extrapolation by repeated stripping

**Given:** $f_0, \ldots, f_N$

**Assume:** $f_n = a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots$

**Phase 0**

$$f_n = a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots$$

**See also:** E-algorithm

# Asymptotic extrapolation by repeated stripping

**Given:** $f_0, \ldots, f_N$

**Assume:** $f_n = a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots$

**Phase 1**

$$
\begin{aligned}
f_n &= a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots \\
(\Delta f)_n = f_{n+1} - f_n &= \frac{-a_1}{n^2} + \frac{a_1 - 2\,a_2}{n^3} + \frac{3\,a_2 - a_1}{n^4} + \cdots
\end{aligned}
$$

**See also:** E-algorithm

# Asymptotic extrapolation by repeated stripping

**Given:** $f_0, \ldots, f_N$

**Assume:** $f_n = a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots$

**Phase 2**

$$
\begin{aligned}
f_n &= a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots \\
(\Delta f)_n &= \frac{-a_1}{n^2} + \frac{a_1 - 2\,a_2}{n^3} + \frac{3\,a_2 - a_1}{n^4} + \cdots \\
(\Delta(n^2\,\Delta f))_n &= \frac{2\,a_2 - a_1}{n^2} + \frac{3\,a_1 - 8\,a_2}{n^3} + \frac{23\,a_2 - 7\,a_1}{n^4} + \cdots
\end{aligned}
$$

**See also:** E-algorithm

# Asymptotic extrapolation by repeated stripping

**Given:** $f_0, \ldots, f_N$

**Assume:** $f_n = a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots$

**Phase 2-bis**

$$
\begin{aligned}
f_n &= a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots \\
(\Delta f)_n &= \frac{-a_1}{n^2} + \frac{a_1 - 2\,a_2}{n^3} + \frac{3\,a_2 - a_1}{n^4} + \cdots \\
(\Delta(n^2\,\Delta f))_n &= \frac{2\,a_2 - a_1}{n^2} + \frac{3\,a_1 - 8\,a_2}{n^3} + \frac{23\,a_2 - 7\,a_1}{n^4} + \cdots \\
n^2\,\Delta(n^2\,\Delta f) &\approx (n^2\,\Delta(n^2\,\Delta f))_{N-2} = c_2
\end{aligned}
$$

**See also:** E-algorithm

# Asymptotic extrapolation by repeated stripping

**Given:** $f_0, \ldots, f_N$

**Assume:** $f_n = a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots$

**Phase 1-bis**

$$\begin{aligned}
f_n &= a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots \\
(\Delta f)_n &= \frac{-a_1}{n^2} + \frac{a_1 - 2\,a_2}{n^3} + \frac{3\,a_2 - a_1}{n^4} + \cdots \\
(\Delta(n^2\,\Delta f))_n &= \frac{2\,a_2 - a_1}{n^2} + \frac{3\,a_1 - 8\,a_2}{n^3} + \frac{23\,a_2 - 7\,a_1}{n^4} + \cdots \\
n^2\,\Delta(n^2\,\Delta f) &\approx c_2 \\
n^2\,\Delta f &\approx [n^2\,\Delta f]_{N-1} + \Delta_{N-1}^{-1}\left(\frac{c_2}{n^2}\right) = c_1 - \frac{c_2}{n} - \frac{c_2}{2\,n^2} - \frac{c_2}{6\,n^3} + \cdots
\end{aligned}$$

**See also:** E-algorithm

# Asymptotic extrapolation by repeated stripping

**Given:** $f_0, \ldots, f_N$

**Assume:** $f_n = a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots$

**Phase 0-bis**

$$\begin{aligned}
f_n &= a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots \\
(\Delta f)_n &= \frac{-a_1}{n^2} + \frac{a_1 - 2\,a_2}{n^3} + \frac{3\,a_2 - a_1}{n^4} + \cdots \\
(\Delta(n^2\,\Delta f))_n &= \frac{2\,a_2 - a_1}{n^2} + \frac{3\,a_1 - 8\,a_2}{n^3} + \frac{23\,a_2 - 7\,a_1}{n^4} + \cdots \\
n^2\,\Delta(n^2\,\Delta f) &\approx c_2 \\
n^2\,\Delta f &\approx c_1 - \frac{c_2}{n} - \frac{c_2}{2\,n^2} - \frac{c_2}{6\,n^3} + \cdots \\
f &\approx f_N + \Delta_N^{-1}\left(\frac{c_1}{n^2} - \frac{c_2}{n^3} - \cdots\right) = c_0 - \frac{c_1}{n} + \frac{c_2 - c_1}{2\,n^2} + \frac{4\,c_2 - c_1}{6\,n^3} + \cdots
\end{aligned}$$

**See also:** E-algorithm

# Constant tactic

**Assumption:**

$$f_n = c + R_n, \qquad R_n \prec 1$$

**Criterion:**

$$\varepsilon_{f, f_N} < \delta$$

$$\varepsilon_{f,\varphi} = \max_{k \in \{L, \ldots, N\}} \frac{|f_n - \varphi_n|}{|f_n| + |\varphi_n|}$$

**Transformation:**

$$f \longmapsto g = \Delta f$$

**Inverse transformation:**

$$\begin{aligned} g_n &\approx \tilde{g}_n \\ f_n &\approx \tilde{f}_n = \left[ f_N - \sum_{0 \leqslant i < N} \tilde{g}_i \right] + \sum_{0 \leqslant i < n} \tilde{g}_i \end{aligned}$$

# Explicit tactic

**Assumption:**

$$f_n = \psi_n (c + R_n), \qquad R_n \prec 1$$

**Criterion:**

$$\varepsilon_{f, \psi f_N / \psi_N} < \delta_\psi$$

$$\varepsilon_{f, \varphi} = \max_{k \in \{L, \ldots, N\}} \frac{|f_n - \varphi_n|}{|f_n| + |\varphi_n|}$$

**Transformation:**

$$f \longmapsto g = \Delta (f / \psi)$$

**Inverse transformation:**

$$\begin{aligned} g_n &\approx \tilde{g}_n \\ f_n &\approx \tilde{f}_n = \psi_n \left( \left[ f_N / \psi_N - \sum_{0 \leqslant i < N} \tilde{g}_i \right] + \sum_{0 \leqslant i < n} \tilde{g}_i \right) \end{aligned}$$

# Exponential tactic

**Assumption:**

$$f_n = \pm \mathrm{e}^{R_n} \qquad (R_n \succ 1).$$

**Criterion:**

$$\frac{f_n}{f_N} > 0 \qquad (n = L, \ldots, N),$$

**Transformation:**

$$f \longmapsto g = \log \frac{f}{\operatorname{sign} f_N}$$

**Inverse transformation:**

$$\begin{aligned} g_n &\approx \tilde{g}_n \\ f_n &\approx \tilde{f}_n = (\operatorname{sign} f_N) \exp \tilde{g}_N \end{aligned}$$

# Asymptotic extrapolation

## Combining tactics

Several tactics with transformations $\Phi_1, \ldots, \Phi_p$ and inverses $\tilde{\Phi}_1, \ldots, \tilde{\Phi}_p$

Compute all valid $\tilde{f}_{i_1,\ldots,i_l} = (\tilde{\Phi}_{i_1} \circ \cdots \circ \tilde{\Phi}_{i_l} \circ \Phi_{i_l} \circ \cdots \circ \Phi_{i_1})(f)$ until specified $l$

Return best $\tilde{f}_{i_1,\ldots,i_l}$

## Inverse transformations using transseries

Inverse transformations $\Delta^{-1}$ and $\exp$ done on "transseries"

Numerical evaluations done using "summing to the least term"