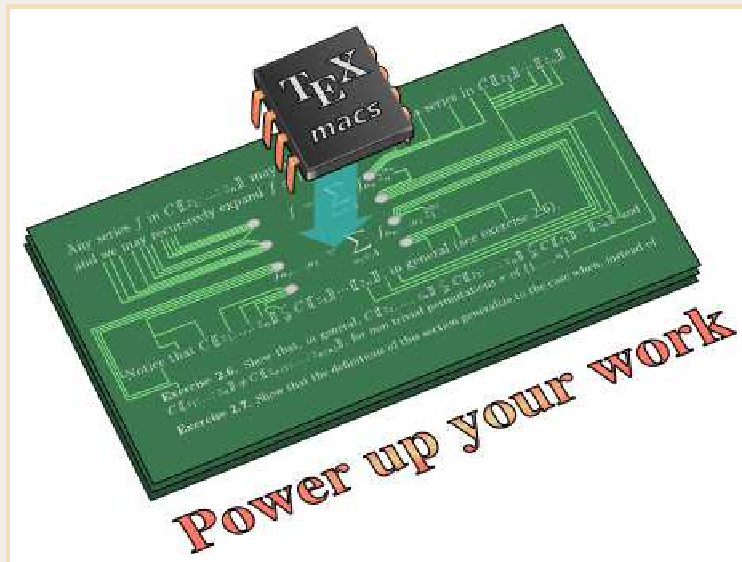
Meta-expansion of transseries

In honour of Maurice Pouzet



Mahdia, 2008

http://www.TeXmacs.org

Challenges

- **Asymptotic behaviour for $x \to \infty$**

$$f = \log \log (x\, e^{x\, e^{x}} + 1) - \exp \exp \left(\log \log x + \frac{1}{x} \right)$$

- **Asymptotic resolution of differential equations**

$$(\log x)\, f^2 - f' + f + e^{-x} = 0$$

- **Asymptotic resolution of functional equations**

$$f = \frac{1}{x} + f'(x^2) + f(e^{\log^2 x})$$

Orders at infinity

- **Du Bois-Reymond (1875–1877)**
  - Notations $\preccurlyeq$, $\prec$, $\asymp$, etc.
  - First appearance of diagonal argument?
- **Hardy (1910–1911)**
  - L-functions : formed from $\mathbb{R}(x)^{\mathrm{alg}}$, $+$, $-$, $\times$, $\exp$ and $\log$
  - The germs of L-functions at $+\infty$ form a totally ordered field, which is stable under differentiation

Transfinite series

- **Hahn (1907)**

  C: field of constants

  $\mathfrak{M}$: totally ordered group of monomials for $\preccurlyeq$

$$C[[\mathfrak{M}]] = \left\{ \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \, \mathfrak{m} \middle| \operatorname{supp} f \text{ well-ordered for } \preccurlyeq^{\mathrm{op}} \right\}.$$

  $C[[\mathfrak{M}]]$: field of well-based series

  Example for $\mathfrak{M} = x^{\mathbb{Z}} \mathrm{e}^{\mathbb{Z} x}$:

$$\frac{1}{1 - \frac{1}{x}} \frac{1}{1 - \frac{1}{\mathrm{e}^x}} = 1 + \frac{1}{x} + \frac{1}{x^2} + \cdots + \frac{1}{\mathrm{e}^x} + \frac{1}{x \, \mathrm{e}^x} + \cdots + \frac{1}{\mathrm{e}^{2x}} + \cdots \cdots$$

- **Transseries: L-fuction with $+ \rightsquigarrow \sum$**

  - Dahn-Göring (1984–1986): model theory of real exponentiel fields
  - Écalle (1992): Hilbert's 16-th problem & Dulac's conjecture

- Computer algebra (1988–1994)

$$\mathrm{e}^{\mathrm{e}^{x}+\frac{\mathrm{e}^{x}}{x}+\frac{\mathrm{e}^{x}}{x^{2}}+\cdots}+\frac{\mathrm{e}^{\mathrm{e}^{x}+\frac{\mathrm{e}^{x}}{x}+\frac{\mathrm{e}^{x}}{x^{2}}+\cdots}}{\log x}+\frac{\mathrm{e}^{\mathrm{e}^{x}+\frac{\mathrm{e}^{x}}{x}+\frac{\mathrm{e}^{x}}{x^{2}}+\cdots}}{\log^{2} x}+\cdots$$

Transfinite series

- **Higman (1952)**

  C: field of constants

  $\mathfrak{M}$: partially ordered monoid of monomials for $\preccurlyeq$

$$C[[\mathfrak{M}]] = \left\{ \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m} \middle| \operatorname{supp} f \text{ well-quasi-ordered for } \preccurlyeq^{\mathrm{op}} \right\}.$$

  $C[[\mathfrak{M}]]$: ring of well-based series, $1+\varepsilon$ invertible for $\operatorname{supp} \varepsilon \prec 1$

- **Transseries: L-fuction with $+ \rightsquigarrow \sum$**

  - Dahn-Göring (1984–1986): model theory of real exponentiel fields
  - Écalle (1992): Hilbert's 16-th problem & Dulac's conjecture
  - Computer algebra (1988–1994)

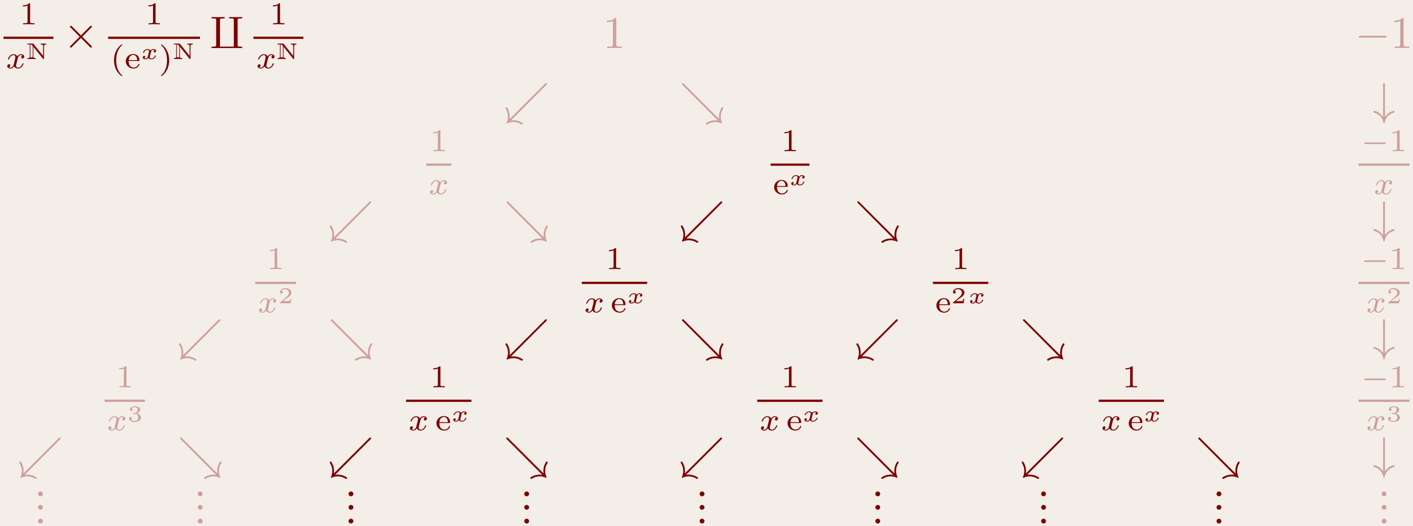
$$\mathrm{e}^{\mathrm{e}^x + \frac{\mathrm{e}^x}{x} + \frac{\mathrm{e}^x}{x^2} + \cdots} + \frac{\mathrm{e}^{\mathrm{e}^x + \frac{\mathrm{e}^x}{x} + \frac{\mathrm{e}^x}{x^2} + \cdots}}{\log x} + \frac{\mathrm{e}^{\mathrm{e}^x + \frac{\mathrm{e}^x}{x} + \frac{\mathrm{e}^x}{x^2} + \cdots}}{\log^2 x} + \cdots$$

- **Infinite cancellations and zero-testing**

$$\frac{1}{1-\frac{1}{x}}\,\frac{1}{1-\frac{1}{e^x}}-\frac{1}{1-\frac{1}{x}} \;=\; 1+\frac{1}{x}+\frac{1}{x^2}+\cdots+\frac{1}{e^x}+\frac{1}{x\,e^x}+\cdots+\frac{1}{e^{2x}}+\cdots$$
$$-1-\frac{1}{x}-\frac{1}{x^2}+\cdots$$

- **Syntax-oriented total expansions & well-based indexations**

$$\frac{1}{x^{\mathbb{N}}}\times\frac{1}{(e^x)^{\mathbb{N}}}\amalg\frac{1}{x^{\mathbb{N}}}$$



Higman's theorem

- **Inversion of a series**

$$\varepsilon = \sum_{\mathfrak{m}\in\mathfrak{M}} \varepsilon_{\mathfrak{m}}\,\mathfrak{m}, \qquad \operatorname{supp}\varepsilon \prec 1$$

$$\frac{1}{1-\varepsilon} = \sum_{\mathfrak{m}_1\cdots\mathfrak{m}_k\in\mathfrak{M}^*} \varepsilon_{\mathfrak{m}_1}\cdots\varepsilon_{\mathfrak{m}_k}\,\mathfrak{m}_1\cdots\mathfrak{m}_k$$

- **Non-commutative operators**

$$\begin{aligned} f &= \tfrac{1}{x} + \Phi_1(f) + \Phi_2(f) = \tfrac{1}{x} + f(x^2) + f(\mathrm{e}^{\log^2 x}) \\ &= \sum_{\Phi_{i_1}\cdots\Phi_{i_k}\in\{\Phi_1,\Phi_2\}^*} (\Phi_{i_1}\circ\cdots\circ\Phi_{i_k})\left(\tfrac{1}{x}\right) \end{aligned}$$

Generalization

- **Problem with more general operations**

$$\left(\mathrm{e}^{\mathrm{e}^x+\frac{\mathrm{e}^x}{x}+\frac{\mathrm{e}^x}{x^2}+\cdots}\right)' = \left(\mathrm{e}^x + \frac{\mathrm{e}^x}{x} - \frac{\mathrm{e}^x}{x^3} - 2\,\frac{\mathrm{e}^x}{x^4} + \cdots\right)\mathrm{e}^{\mathrm{e}^x+\frac{\mathrm{e}^x}{x}+\frac{\mathrm{e}^x}{x^2}+\cdots} \notin \mathfrak{M}$$

- **Well-based operators**

$$\begin{aligned}\Phi(f) &= \Phi_0(f) + \Phi_1(f) + \Phi_2(f) + \cdots\\ \Phi_i(f) &= \check{\Phi}_i(f, \overset{i\times}{\ldots}, f)\end{aligned}$$

$$\begin{aligned}\check{\Phi}_i &= \sum_{\mathfrak{m}_1,\ldots,\mathfrak{m}_i,\mathfrak{n}} \check{\Phi}_{i,(\mathfrak{m}_1,\ldots,\mathfrak{m}_i;n)}\,\mathrm{A}_{\mathfrak{m}_1,\ldots,\mathfrak{m}_i;\mathfrak{n}}\\ \mathrm{A}_{\mathfrak{m}_1,\ldots,\mathfrak{m}_i;\mathfrak{n}}(f_1,\ldots,f_i) &= f_{1,\mathfrak{m}_1}\cdots f_{i,\mathfrak{m}_i}\,\mathfrak{n}\end{aligned}$$

- **Fixed point theorem**

Any strictly extensive well-based operator $\Phi\colon C[[\mathfrak{M}]] \to C[[\mathfrak{M}]]$ admits a unique fixed point in $C[[\mathfrak{M}]]$.

Meta-expansion of transseries

- **Approximators**

$$f = \operatorname*{stat\,lim}_{n\to\infty} \check{f}_{;n}$$
$$\check{f}_{;n} \in C[\mathfrak{M}]$$

$$f = \exp\frac{1}{x}$$
$$\check{f}_{;n} = 1 + \frac{1}{x} + \frac{1}{2\,x^2} + \cdots + \frac{1}{n!\,x^n}$$

$$f = g\,h$$
$$f_{;n} = g_{;n}\,h_{;n}$$

- **Expanders**

$$\begin{aligned} f &= \check{f}_0 + \check{f}_1 + \check{f}_2 + \cdots \\ \check{f}_n &\in C[\mathfrak{M}] \\ \check{f} &= \check{f}_0 + \check{f}_1\, z + \check{f}_2\, z^2 + \cdots \end{aligned}$$

$$\begin{aligned} f &= \exp \frac{1}{x} \\ \check{f} &= \exp \frac{z}{x} \end{aligned}$$

$$\begin{aligned} f &= g\, h \\ \check{f} &= \check{g}\, \check{h} \end{aligned}$$

- **Comparison**

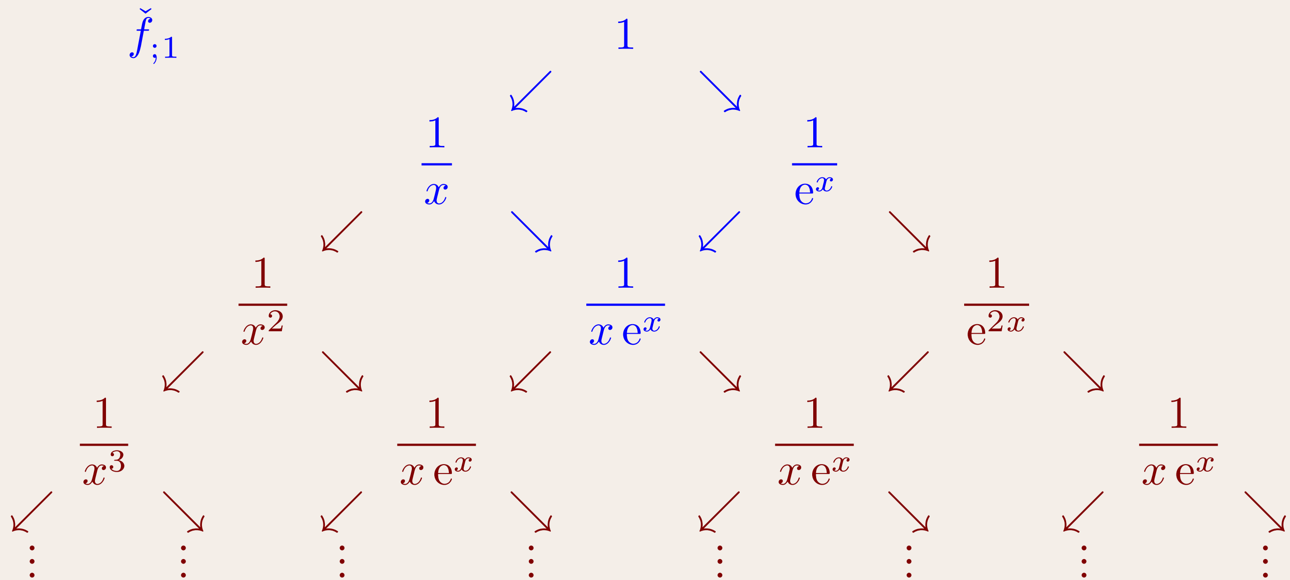
$$\begin{aligned}\check{f}_{;n} &= \check{f}_0 + \cdots + \check{f}_n \\ \check{f}_n &= \check{f}_{;n} - \check{f}_{;n-1}\end{aligned}$$

Meta-expansion of transseries

- **Comparison**

$$
\check{f}_{;0}
\qquad
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & \swarrow & & \searrow & & & & & & \\
 & & & & & \frac{1}{x} & & & & \frac{1}{\mathrm{e}^x} & & & & & \\
 & & & & \swarrow & & \searrow & & \swarrow & & \searrow & & & & \\
 & & & \frac{1}{x^2} & & & & \frac{1}{x\,\mathrm{e}^x} & & & & \frac{1}{\mathrm{e}^{2x}} & & & \\
 & & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & & \\
 & \frac{1}{x^3} & & & & \frac{1}{x\,\mathrm{e}^x} & & & & \frac{1}{x\,\mathrm{e}^x} & & & & \frac{1}{x\,\mathrm{e}^x} & \\
\swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

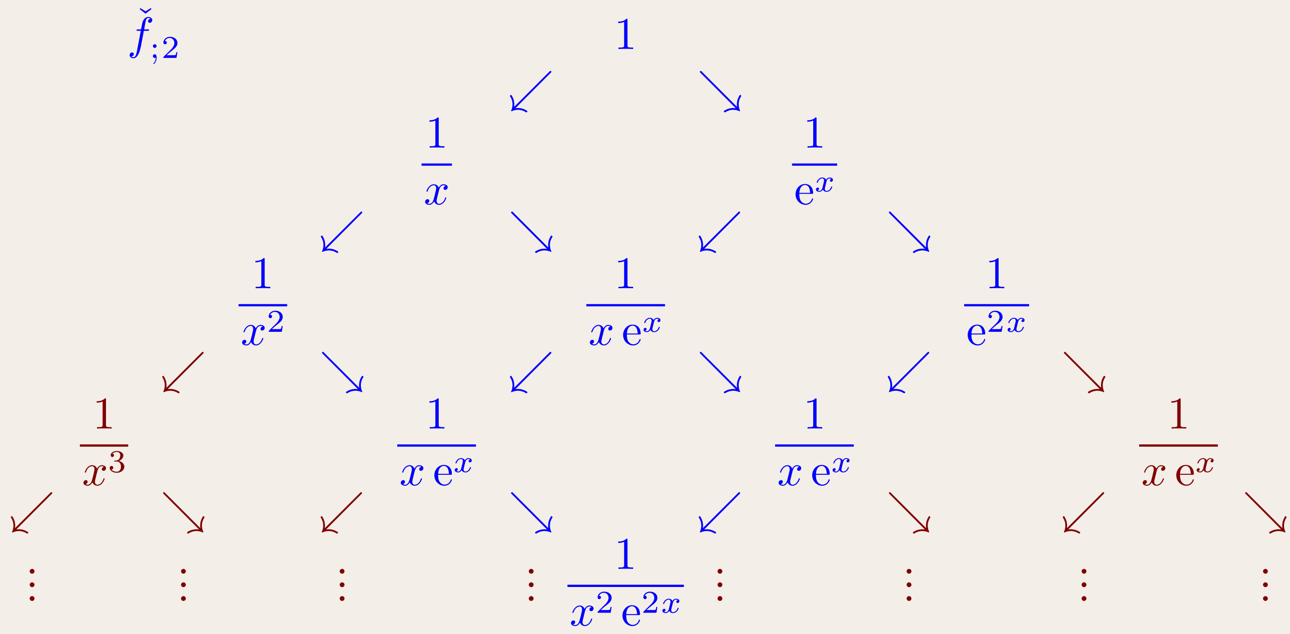
- **Comparison**



$\check{f}_{;1}$
1
$\frac{1}{x}$
$\frac{1}{\mathrm{e}^x}$
$\frac{1}{x^2}$
$\frac{1}{x\,\mathrm{e}^x}$
$\frac{1}{\mathrm{e}^{2x}}$
$\frac{1}{x^3}$
$\frac{1}{x\,\mathrm{e}^x}$
$\frac{1}{x\,\mathrm{e}^x}$
$\frac{1}{x\,\mathrm{e}^x}$

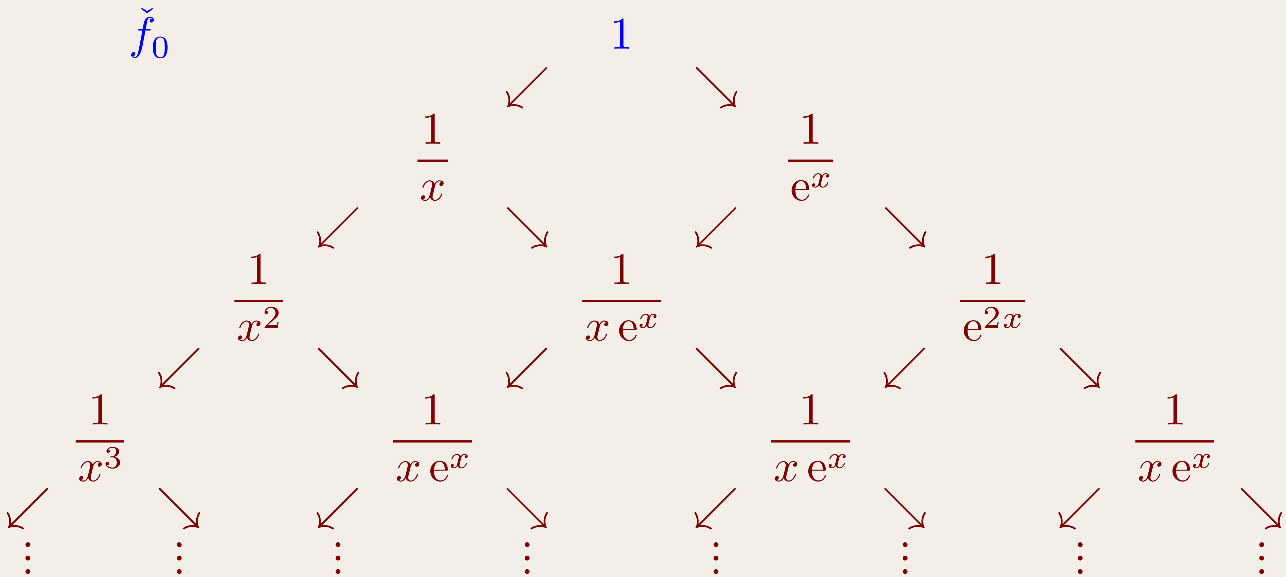
Meta-expansion of transseries

- **Comparison**



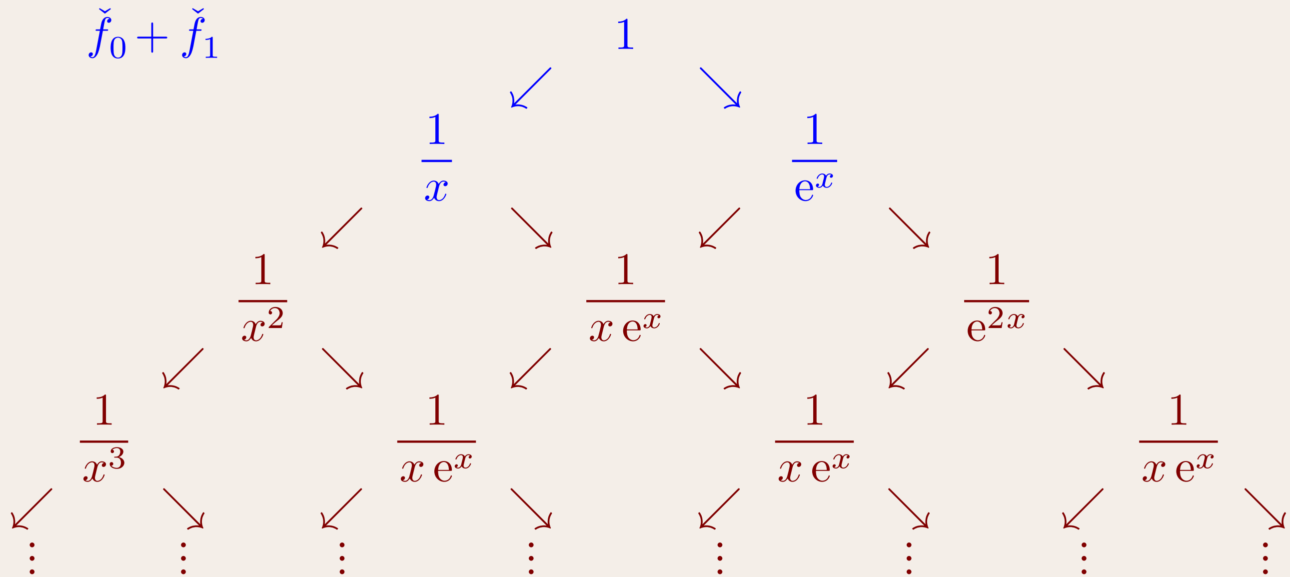
Meta-expansion of transseries

- **Comparison**



Meta-expansion of transseries

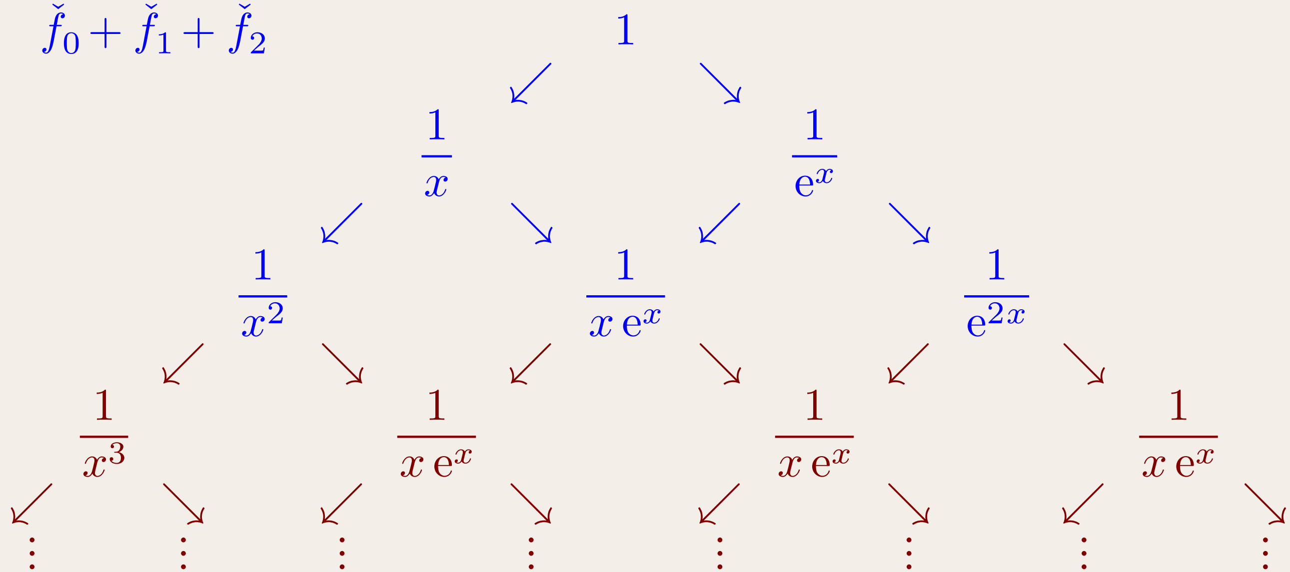
- **Comparison**



Meta-expansion of transseries

- **Comparison**

$\check{f}_0 + \check{f}_1 + \check{f}_2$



Meta-expansion of transseries

- **Other operations**

$$\begin{aligned} g &= \varphi \circ f & \varphi \in C[[t]], f \prec 1 \\ \check{g}(z) &= \varphi(z\check{f}(z)) \end{aligned}$$

$$\begin{aligned} g &= f' \\ \check{g}_n &= \sum_{p+q=n} \sum_{\mathfrak{m} \in \operatorname{supp} \check{f}_p} \check{f}_{p,\mathfrak{m}} \, (\widecheck{\mathfrak{m}'})_q \end{aligned}$$

- **Meta-operations**

$$\check{f} \longmapsto z\check{f} \qquad \text{Shorten}$$

$$\check{f} \longmapsto \check{f}_0 + \frac{1}{z}(\check{f} - \check{f}_0) \qquad \text{Lengthen}$$

$$(\operatorname{stab} f)_{;n} = \sum_{\substack{\mathfrak{m} \in \operatorname{supp} \check{f}_{;n} \\ \check{f}_{;n+1,\mathfrak{m}} = \check{f}_{;n,\mathfrak{m}}}} \check{f}_{;n,\mathfrak{m}} \, \mathfrak{m} \qquad \text{Stabilize}$$

Example of stabilization

$$f = \log\log\left(x\,\mathrm{e}^{x\,\mathrm{e}^{x}} + 1\right) - \exp\exp\left(\log\log x + \tfrac{1}{x}\right).$$

$$\begin{aligned}
\check{f}_{;0} &= \log x \\
\check{f}_{;1} &= \log x + \frac{\log x}{x\,\mathrm{e}^{x}} + \frac{1}{x\,\mathrm{e}^{x}} \\
\check{f}_{;2} &= \frac{\log x}{x\,\mathrm{e}^{x}} + \frac{1}{x\,\mathrm{e}^{x}} - \frac{\log^2 x}{2\,x^2\,\mathrm{e}^{2x}} - \frac{\log x}{x^2\,\mathrm{e}^{2x}} - \frac{1}{2\,x^2\,\mathrm{e}^{2x}} \\
\check{f}_{;3} &= -\frac{\log x}{2\,x} + \frac{\log x}{x\,\mathrm{e}^{x}} + \frac{1}{x\,\mathrm{e}^{x}} - \frac{\log^2 x}{2\,x^2\,\mathrm{e}^{2x}} - \frac{\log x}{x^2\,\mathrm{e}^{2x}} - \frac{1}{2\,x^2\,\mathrm{e}^{2x}} + {} \\
&\quad \frac{\log^3 x}{3\,x^3\,\mathrm{e}^{3x}} + \frac{\log^2 x}{x^3\,\mathrm{e}^{3x}} + \frac{\log x}{x^3\,\mathrm{e}^{3x}} + \frac{1}{3\,x^3\,\mathrm{e}^{3x}}
\end{aligned}$$

Example of stabilization

$$f = \log\log\left(x\,\mathrm{e}^{x\,\mathrm{e}^{x}} + 1\right) - \exp\exp\left(\log\log x + \tfrac{1}{x}\right).$$

$$\check{f}_{;0} = \log x$$

$$\check{f}_{;1} = \frac{\log x}{x\,\mathrm{e}^{x}} + \frac{1}{x\,\mathrm{e}^{x}}$$

$$\check{f}_{;2} = \frac{\log x}{x\,\mathrm{e}^{x}} + \frac{1}{x\,\mathrm{e}^{x}} - \frac{\log^2 x}{2\,x^2\,\mathrm{e}^{2x}} - \frac{\log x}{x^2\,\mathrm{e}^{2x}} - \frac{1}{2\,x^2\,\mathrm{e}^{2x}}$$

$$\check{f}_{;3} = -\frac{\log x}{2\,x} + \frac{\log x}{x\,\mathrm{e}^{x}} + \frac{1}{x\,\mathrm{e}^{x}} - \frac{\log^2 x}{2\,x^2\,\mathrm{e}^{2x}} - \frac{\log x}{x^2\,\mathrm{e}^{2x}} - \frac{1}{2\,x^2\,\mathrm{e}^{2x}} + \frac{\log^3 x}{3\,x^3\,\mathrm{e}^{3x}} + \frac{\log^2 x}{x^3\,\mathrm{e}^{3x}} + \frac{\log x}{x^3\,\mathrm{e}^{3x}} + \frac{1}{3\,x^3\,\mathrm{e}^{3x}}$$

Example of stabilization

$$f = \log\log\left(x\,\mathrm{e}^{x\,\mathrm{e}^{x}} + 1\right) - \exp\exp\left(\log\log x + \tfrac{1}{x}\right).$$

$$\begin{aligned}
\check{f}_{;0} &= \\
\check{f}_{;1} &= \frac{\log x}{x\,\mathrm{e}^{x}} + \frac{1}{x\,\mathrm{e}^{x}} \\
\check{f}_{;2} &= \frac{\log x}{x\,\mathrm{e}^{x}} + \frac{1}{x\,\mathrm{e}^{x}} - \frac{\log^2 x}{2\,x^2\,\mathrm{e}^{2x}} - \frac{\log x}{x^2\,\mathrm{e}^{2x}} - \frac{1}{2\,x^2\,\mathrm{e}^{2x}} \\
\check{f}_{;3} &= -\frac{\log x}{2\,x} + \frac{\log x}{x\,\mathrm{e}^{x}} + \frac{1}{x\,\mathrm{e}^{x}} - \frac{\log^2 x}{2\,x^2\,\mathrm{e}^{2x}} - \frac{\log x}{x^2\,\mathrm{e}^{2x}} - \frac{1}{2\,x^2\,\mathrm{e}^{2x}} + {} \\
&\qquad \frac{\log^3 x}{3\,x^3\,\mathrm{e}^{3x}} + \frac{\log^2 x}{x^3\,\mathrm{e}^{3x}} + \frac{\log x}{x^3\,\mathrm{e}^{3x}} + \frac{1}{3\,x^3\,\mathrm{e}^{3x}}
\end{aligned}$$

Example of stabilization

$$f = \log\log\left(x\,\mathrm{e}^{x\,\mathrm{e}^{x}} + 1\right) - \exp\exp\left(\log\log x + \tfrac{1}{x}\right).$$

$$\begin{aligned}
\check{f}_{;4} \;=\;& -\frac{\log^2 x}{2\,x} - \frac{\log x}{2\,x} - \frac{\log x}{6\,x^2} + \frac{\log x}{x\,\mathrm{e}^{x}} + \frac{1}{x\,\mathrm{e}^{x}} - \frac{\log^2 x}{2\,x^2\,\mathrm{e}^{2x}} - \frac{\log x}{x^2\,\mathrm{e}^{2x}} - \frac{1}{2\,x^2\,\mathrm{e}^{2x}} + \frac{\log^3 x}{3\,x^3\,\mathrm{e}^{3x}} + \\
& \frac{\log^2 x}{x^3\,\mathrm{e}^{3x}} + \frac{\log x}{x^3\,\mathrm{e}^{3x}} + \frac{1}{3\,x^3\,\mathrm{e}^{3x}} - \frac{\log^4 x}{4\,x^4\,\mathrm{e}^{4x}} - \frac{\log^3 x}{x^4\,\mathrm{e}^{4x}} - \frac{3\log^2 x}{2\,x^4\,\mathrm{e}^{4x}} - \frac{\log x}{x^4\,\mathrm{e}^{4x}} - \frac{1}{4\,x^4\,\mathrm{e}^{4x}} \\
\check{f}_{;5} \;=\;& -\frac{\log^2 x}{2\,x} - \frac{\log x}{2\,x} - \frac{\log^2 x}{2\,x^2} - \frac{\log x}{6\,x^2} - \frac{\log x}{24\,x^3} + \frac{\log x}{x\,\mathrm{e}^{x}} + \frac{1}{x\,\mathrm{e}^{x}} - \frac{\log^2 x}{2\,x^2\,\mathrm{e}^{2x}} - \frac{\log x}{x^2\,\mathrm{e}^{2x}} - \\
& \frac{1}{2\,x^2\,\mathrm{e}^{2x}} + \frac{\log^3 x}{3\,x^3\,\mathrm{e}^{3x}} + \frac{\log^2 x}{x^3\,\mathrm{e}^{3x}} + \frac{\log x}{x^3\,\mathrm{e}^{3x}} + \frac{1}{3\,x^3\,\mathrm{e}^{3x}} - \frac{\log^4 x}{4\,x^4\,\mathrm{e}^{4x}} - \frac{\log^3 x}{x^4\,\mathrm{e}^{4x}} - \frac{3\log^2 x}{2\,x^4\,\mathrm{e}^{4x}} - \\
& \frac{\log x}{x^4\,\mathrm{e}^{4x}} - \frac{1}{4\,x^4\,\mathrm{e}^{4x}} + \frac{\log^5 x}{5\,x^5\,\mathrm{e}^{5x}} + \frac{\log^4 x}{x^5\,\mathrm{e}^{5x}} + \frac{2\log^3 x}{x^5\,\mathrm{e}^{5x}} + \frac{2\log^2 x}{x^5\,\mathrm{e}^{5x}} + \frac{\log x}{x^5\,\mathrm{e}^{5x}} + \frac{1}{5\,x^5\,\mathrm{e}^{5x}}
\end{aligned}$$

Example of stabilization

$$f = \log\log\left(x\,\mathrm{e}^{x\,\mathrm{e}^{x}} + 1\right) - \exp\exp\left(\log\log x + \tfrac{1}{x}\right).$$

$$\check{f}_{;4} = -\frac{\log^2 x}{2\,x} - \frac{\log x}{2\,x} - \frac{\log x}{6\,x^2} + \frac{\log x}{x\,\mathrm{e}^{x}} + \frac{1}{x\,\mathrm{e}^{x}} - \frac{\log^2 x}{2\,x^2\,\mathrm{e}^{2x}} - \frac{\log x}{x^2\,\mathrm{e}^{2x}} - \frac{1}{2\,x^2\,\mathrm{e}^{2x}} + \frac{\log^3 x}{3\,x^3\,\mathrm{e}^{3x}} + \frac{\log^2 x}{x^3\,\mathrm{e}^{3x}} + \frac{\log x}{x^3\,\mathrm{e}^{3x}} + \frac{1}{3\,x^3\,\mathrm{e}^{3x}} - \frac{\log^4 x}{4\,x^4\,\mathrm{e}^{4x}} - \frac{\log^3 x}{x^4\,\mathrm{e}^{4x}} - \frac{3\log^2 x}{2\,x^4\,\mathrm{e}^{4x}} - \frac{\log x}{x^4\,\mathrm{e}^{4x}} - \frac{1}{4\,x^4\,\mathrm{e}^{4x}}$$

$$\check{f}_{;5} = -\frac{\log^2 x}{2\,x} - \frac{\log x}{2\,x} + O\!\left(\frac{\log^2 x}{2\,x^2}\right) - \frac{\log x}{6\,x^2} + O\!\left(\frac{\log x}{24\,x^3}\right) + \frac{\log x}{x\,\mathrm{e}^{x}} + \frac{1}{x\,\mathrm{e}^{x}} - \frac{\log^2 x}{2\,x^2\,\mathrm{e}^{2x}} - \frac{\log x}{x^2\,\mathrm{e}^{2x}} - \frac{1}{2\,x^2\,\mathrm{e}^{2x}} + \frac{\log^3 x}{3\,x^3\,\mathrm{e}^{3x}} + \frac{\log^2 x}{x^3\,\mathrm{e}^{3x}} + \frac{\log x}{x^3\,\mathrm{e}^{3x}} + \frac{1}{3\,x^3\,\mathrm{e}^{3x}} - \frac{\log^4 x}{4\,x^4\,\mathrm{e}^{4x}} - \frac{\log^3 x}{x^4\,\mathrm{e}^{4x}} - \frac{3\log^2 x}{2\,x^4\,\mathrm{e}^{4x}} - \frac{\log x}{x^4\,\mathrm{e}^{4x}} - \frac{1}{4\,x^4\,\mathrm{e}^{4x}} + O\!\left(\frac{\log^5 x}{5\,x^5\,\mathrm{e}^{5x}}\right)$$

Demonstration inside Mathemagix

```
Mmx ≫ use "numerix"; use "algebramix"; use "analyziz";
      use "symbolix"; use "multimix";
```

```
Mmx ≫ x == infinity ('x);
```

```
Mmx ≫ 1 / (1 - 1/x - exp (-x))
```

```
Mmx ≫ lengthen fixed_point (f :-> (exp x) * integrate (exp (-x) * ((log x) * f * f +
      exp (-x)), x))
```

```
Mmx ≫ fixed_point (f :-> 1/x + derive (f@(x^2), x) + f@exp((log x)^2))
```

```
Mmx ≫ lengthen (product (x, x), 8)
```

```
Mmx ≫
```