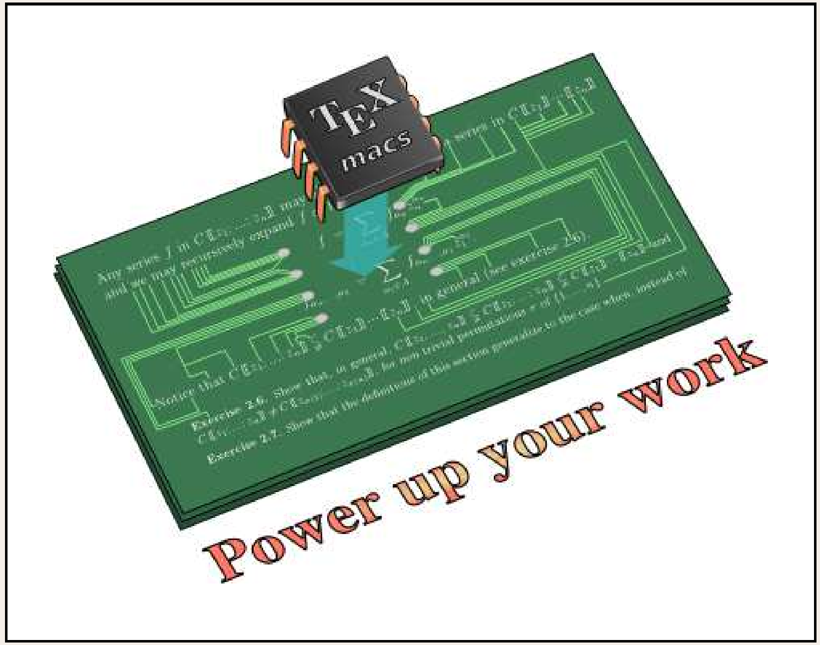
Hardy field solutions to algebraic differential equations



Joris van der Hoeven, Fields Institute 2009
http://www.TEXMACS.org

A missing subject ?

Algebraic geometry	$\longrightarrow$	Real algebraic geometry + Valuation theory
$\downarrow$		$\downarrow$
Differential algebra	$\longrightarrow$	?

- Hardy fields: Hardy, Rosenlicht, Boshernitzan, Singer, etc.
- Pfaff systems: Khovanskii, Wilkie, van den Dries, Rolin, etc.

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- **LNM 1888**: Transseries and Real Differential Algebra
- Other work on `http://www.math.u-psud.fr/~vdhoeven`

Sufficiently closed models

Algebraic geometry	$\longrightarrow$	Real algebraic geometry + Valuation theory
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Sufficiently closed models

$\mathbb{C}$	$\longrightarrow$	Real algebraic geometry + Valuation theory
$\downarrow$		$\downarrow$
Differential algebra	$\longrightarrow$	Real differential algebra + Asymptotic differential algebra

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \begin{array}{c}\mathbb{R}\\ + \\ \text{Valuation theory}\end{array} \\ \downarrow & & \downarrow \\ \text{Differential algebra} & \longrightarrow & \begin{array}{c}\text{Real differential algebra}\\ + \\ \text{Asymptotic differential algebra}\end{array} \end{array}$$

$\mathbb{C}$	$\longrightarrow$	$\mathbb{R}$ + $\mathbb{C}[[z^{\mathbb{Q}}]]$
$\downarrow$		$\downarrow$
Differential algebra	$\longrightarrow$	Real differential algebra + Asymptotic differential algebra

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Sufficiently closed models

$$
\begin{array}{ccc}
\mathbb{C} & \longrightarrow & \begin{array}{c} \mathbb{R} \\ + \\ \mathbb{C}[[z^{\mathbb{Q}}]] \end{array} \\
\downarrow & & \downarrow \\
\textit{Wild} & \longrightarrow & \begin{array}{c} \text{Maximal Hardy field (?)} \\ + \\ \text{Asymptotic differential algebra} \end{array}
\end{array}
$$

$\mathbb{C}$	$\longrightarrow$	$\mathbb{R}$ + $\mathbb{C}[[z^{\mathbb{Q}}]]$
$\downarrow$		$\downarrow$
Wild	$\longrightarrow$	$\mathbb{R}[[[x]]]$ + Asymptotic differential algebra

$\mathbb{C}$	$\longrightarrow$	$\mathbb{R}$ + $\mathbb{C}[[z^{\mathbb{Q}}]]$
$\downarrow$		$\downarrow$
Wild	$\longrightarrow$	$\mathbb{R}[[[x]]]$ + $\mathbb{C}[[[z]]]$

$(x \succ 1)$

$$\mathrm{e}^{\mathrm{e}^{x}+\frac{\mathrm{e}^{x}}{x}+\frac{\mathrm{e}^{x}}{x^{2}}+\cdots}+\frac{2}{\log x}\,\mathrm{e}^{\mathrm{e}^{x}+\frac{\mathrm{e}^{x}}{x}+\frac{\mathrm{e}^{x}}{x^{2}}+\cdots}+\mathrm{e}^{\sqrt{x}+\mathrm{e}^{\sqrt{\log x}+\mathrm{e}^{\sqrt{\log\log x}+\cdots}}}+\cdots$$

$(x \succ 1)$

$$\mathrm{e}^{\mathrm{e}^{x}+\frac{\mathrm{e}^{x}}{x}+\frac{\mathrm{e}^{x}}{x^2}+\cdots} + \frac{2}{\log x}\,\mathrm{e}^{\mathrm{e}^{x}+\frac{\mathrm{e}^{x}}{x}+\frac{\mathrm{e}^{x}}{x^2}+\cdots} + \mathrm{e}^{\sqrt{x}+\mathrm{e}^{\sqrt{\log x}+\mathrm{e}^{\sqrt{\log\log x}+\cdots}}} + \cdots$$

- Dahn & Göring
- Écalle

$$\frac{1}{1-x^{-1}-x^{-\mathrm{e}}} = 1+x^{-1}+x^{-2}+x^{-\mathrm{e}}+x^{-3}+x^{-\mathrm{e}-1}+\cdots$$

$$\frac{1}{1-x^{-1}-\mathrm{e}^{-x}} = 1+\frac{1}{x}+\frac{1}{x^2}+\cdots+\mathrm{e}^{-x}+2\,\frac{\mathrm{e}^{-x}}{x}+\cdots+\mathrm{e}^{-2x}+\cdots$$

$$-\mathrm{e}^{x}\int\frac{\mathrm{e}^{-x}}{x} = \frac{1}{x}-\frac{1}{x^2}+\frac{2}{x^3}-\frac{6}{x^4}+\frac{24}{x^5}-\frac{120}{x^6}+\cdots$$

$$\Gamma(x) = \frac{\sqrt{2\,\pi}\,\mathrm{e}^{x(\log x-1)}}{x^{1/2}}+\frac{\sqrt{2\,\pi}\,\mathrm{e}^{x(\log x-1)}}{12\,x^{3/2}}+\frac{\sqrt{2\,\pi}\,\mathrm{e}^{x(\log x-1)}}{288\,x^{5/2}}+\cdots$$

$$\zeta(x) = 1+2^{-x}+3^{-x}+4^{-x}+\cdots$$

$$\varphi(x) = \frac{1}{x}+\varphi(x^{\pi}) = \frac{1}{x}+\frac{1}{x^{\pi}}+\frac{1}{x^{\pi^2}}+\frac{1}{x^{\pi^3}}+\cdots$$

$$\psi(x) = \frac{1}{x}+\psi(\mathrm{e}^{\log^2 x}) = \frac{1}{x}+\frac{1}{\mathrm{e}^{\log^2 x}}+\frac{1}{\mathrm{e}^{\log^4 x}}+\frac{1}{\mathrm{e}^{\log^8 x}}+\cdots$$

- $\mathbb{T} = \mathbb{R}[[\mathfrak{T}]]$, where $\mathfrak{T}$ is a totally ordered monomial group.
- $\mathbb{R}[[\mathfrak{T}]]$: series $f = \sum_{\mathfrak{m} \in \mathfrak{T}} f_{\mathfrak{m}} \mathfrak{m} \in \mathbb{R}[[\mathfrak{T}]]$ with **grid-based support**:

$$\operatorname{supp} f \subseteq \{\mathfrak{m}_1, \ldots, \mathfrak{m}_m\}^* \mathfrak{n}, \qquad \mathfrak{m}_1, \ldots, \mathfrak{m}_m \prec 1$$

- $\mathbb{T}$ is a totally ordered, real closed field.
- $\mathbb{T}$ is stable under $\exp$, $\log$, ∂, $\int$, $\circ$ and ${}^{\mathrm{inv}}$.

Theorem. *(2000) Given* $P \in \mathbb{T}\{F\}$ *and* $f < g \in \mathbb{T}$ *with* $P(f)\,P(g) < 0$*. Then there exists an* $h \in \mathbb{T}$ *with* $f < h < g$ *and* $P(h) = 0$*.*

- Calculus with **cuts** $\hat{f} \in \hat{\mathbb{T}}$.
- Classification of cuts and behaviour of $P(f)$ near a cut.
- Newton polygon method for shrinking interval on which a sign change occurs and whose end-points are cuts.

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Corollary. *Any* $P \in \mathbb{T}\{F\}$ *of odd degree admits a root in* $\mathbb{T}$*.*

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Example. The following equation admits a solution in $\mathbb{T}$:

$$\frac{1}{x}\,f'''\,(f')^2\,f^{24} + \mathrm{e}^x\,(f'')^{27} - \Gamma(\Gamma(\log x))\,f^2 = \frac{\mathrm{e}^{\mathrm{e}^x + x^2}}{\Gamma(\mathrm{e}^x + x)}.$$

Theorem. *(2000) Given* $P \in \mathbb{T}\{F\}$ *and* $f < g \in \mathbb{T}$ *with* $P(f)\,P(g) < 0$*. Then there exists an* $h \in \mathbb{T}$ *with* $f < h < g$ *and* $P(h) = 0$*.*

- Calculus with **cuts** $\hat{f} \in \hat{\mathbb{T}}$.
- Classification of cuts and behaviour of $P(f)$ near a cut.
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Corollary. *Any monic* $L \in \mathbb{T}[\partial]$ *admits a factorization with factors*

$$\partial - a \quad or$$

$$\partial^2 - (2\,a + b^\dagger)\,\partial + (a^2 + b^2 - a' + a\,b^\dagger) = (\partial - (a - b\,\mathrm{i} + b^\dagger))\,(\partial - (a + b\,\mathrm{i}))$$

Theorem. *(2001) Every asymptotic differential equation over* $\mathbb{T}$ *of Newton degree* d *admits at least* d *solutions (when counting with multiplicities). Moreover, it suffices to add iterated logarithms to the asymptotic scale.*

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Warning. $\mathbb{T}$ is not differentially algebraically closed

$$\begin{aligned} f^3 + (f')^2 + f &= 0 \\ f^3 + f &\neq 0 \end{aligned}$$

$\longrightarrow$ Desingularization of vector fields (Cano, Panazzolo, ...)

Theorem. *(2001) Every asymptotic differential equation over* $\mathbb{T}$ *of Newton degree* d *admits at least* d *solutions (when counting with multiplicities). Moreover, it suffices to add iterated logarithms to the asymptotic scale.*

Corollary. $\mathbb{T}$ *is Picard-Vessiot closed.*

Remark. $\exists$ algorithm for computing the solutions of a given equation.

Remark. Zero-test algorithm for polynomials in power series solutions to algebraic differential equations.

1: Accelero-summation

$$\begin{array}{ccccccccc} \tilde{f} & & & & & & & & f \\ \tilde{\mathcal{B}}_{z_1}\big\downarrow & & & & & & & & \big\uparrow \hat{\mathcal{L}}_{z_p}^{\theta_p} \\ \hat{f}_1 & \xrightarrow[\hat{\mathcal{A}}_{z_1 \to z_2}^{\theta_1}]{} & \hat{f}_2 & \longrightarrow & \cdots & \longrightarrow & \hat{f}_{p-1} & \xrightarrow[\hat{\mathcal{A}}_{z_{p-1} \to z_p}^{\theta_{p-1}}]{} & \hat{f}_p \end{array}$$

2: Transserial Hardy fields

$$\mathbb{T} \supseteq \mathcal{T} \overset{\rho}{\hookrightarrow} \mathcal{G}$$

- $\mathcal{G}$: ring of infinitely differentiable real germs at $+\infty$.

1: Accelero-summation

Advantages	Disadvantages
Canonical after choosing average Preserves composition Classification local vector fields Differential Galois theory	Requires many different tools Not yet written down

2: Transserial Hardy fields

Advantages	Disadvantages
Less hypotheses on coefficients Might generalize to other models Written down	Not canonical No preservation of composition

A **transserial Hardy** field is a differential subfield $\mathcal{T}$ of $\mathbb{T}$, together with a monomorphism $\rho: \mathcal{T} \to \mathcal{G}$ of ordered differential $\mathbb{R}$-algebras, such that

TH1. $\forall f \in \mathcal{T}$: $\quad \operatorname{supp} f \subseteq \mathcal{T}$.

TH2. $\forall f \in \mathcal{T}$: $\quad f_{\prec} \in \mathcal{T}$. $\qquad f_{\prec} = \sum_{\mathfrak{m} \prec 1} f_{\mathfrak{m}} \mathfrak{m}$

TH3. $\exists d \in \mathbb{Z}$: $\quad \forall \mathfrak{m} \in \mathfrak{T} \cap \mathcal{T}$: $\quad \log \mathfrak{m} \in \mathcal{T} + \mathbb{R} \log_d x$.

TH4. $\mathfrak{T} \cap \mathcal{T}$ is stable under taking real powers.

TH5. $\forall f \in \mathcal{T}^{>}$: $\quad \log f \in \mathcal{T} \Rightarrow \rho(\log f) = \log \rho(f)$.

Example. $\mathcal{T} = \mathbb{R}\{\{x^{-\mathbb{R}}\}\}$.

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$$\frac{x\,\mathrm{e}^x}{1 - x^{-1} - \mathrm{e}^{-x}}$$
$$\|$$
$$x\,\mathrm{e}^x + \mathrm{e}^x + x^{-1}\,\mathrm{e}^x + \cdots + x + 1 + x^{-1} + \cdots + x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} + x^{-1}\,\mathrm{e}^{-x} + \cdots\cdots$$

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$$\|$$

$$x \operatorname{e}^x + \operatorname{e}^x + x^{-1} \operatorname{e}^x + \cdots + x + 1 + x^{-1} + \cdots + x \operatorname{e}^{-x} + \operatorname{e}^{-x} + x^{-1} \operatorname{e}^{-x} + \cdots\cdots$$

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Definitions. $\mathcal{T}$ transserial Hardy field, $f \in \mathbb{T}$, $\hat{f} \in \mathcal{G}$

$$f \sim \hat{f} \quad \Longleftrightarrow \quad (\exists \varphi \in \mathcal{T}\colon\ f \sim_{\mathbb{T}} \varphi \sim_{\mathcal{G}} \hat{f})$$

$$f \textit{ asympt. equiv.} \text{ to } \hat{f} \text{ over } \mathcal{T} \quad \Longleftrightarrow \quad (\forall \varphi \in \mathcal{T}\colon\ f - \varphi \sim \hat{f} - \varphi)$$

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Lemma. *Let $f \in \mathbb{T} \setminus \mathcal{T}$ and $\hat{f} \in \mathcal{G} \setminus \mathcal{T}$ be such that*

- f *is a serial cut over $\mathcal{T}$.*
- f *and $\hat{f}$ are asymptotically equivalent over $\mathcal{T}$.*
- f *and $\hat{f}$ are differentially equivalent over $\mathcal{T}$.*

Then $\exists!$ transserial Hardy field extension $\rho\colon \mathcal{T}\langle f \rangle \to \mathcal{G}$ with $\rho(f) = \hat{f}$.

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Theorem. *Let $\mathcal{T}$ be a transserial Hardy field. Then its real closure $\mathcal{T}^{\mathrm{rcl}}$ admits a unique transserial Hardy field structure which extends the one of $\mathcal{T}$.*

Theorem. *Let $\mathcal{T}$ be a transserial Hardy field and let $\varphi \in \mathcal{T}_{\succ}$ be such that $\mathrm{e}^{\varphi} \notin \mathcal{T}$. Then the set $\mathcal{T}(\mathrm{e}^{\mathbb{R}\varphi})$ carries the structure of a transserial Hardy field for the unique differential morphism $\rho: \mathcal{T}(\mathrm{e}^{\mathbb{R}\varphi}) \to \mathcal{G}$ over $\mathcal{T}$ with $\rho(\mathrm{e}^{\lambda\varphi}) = \mathrm{e}^{\lambda\rho(\varphi)}$ for all $\lambda \in \mathbb{R}$.*

Theorem. *Let $\mathcal{T}$ be a transserial Hardy field of depth $d < \infty$. Then $\mathcal{T}((\log_d x)^{\mathbb{R}})$ carries the structure of a transserial Hardy field for the unique differential morphism $\rho: \mathcal{T}((\log_d x)^{\mathbb{R}}) \to \mathcal{G}$ over $\mathcal{T}$ with $\rho((\log_d x)^{\lambda}) = (\log_d x)^{\lambda}$ for all $\lambda \in \mathbb{R}$.*

Step 1. A given algebraic differential equation

$$f^2 - f' + \frac{x}{e^x} = 0$$

Step 2. Put equation in integral form

$$f = \int \left(\frac{x}{e^x} + f^2\right)$$

Step 3. Integral transseries solution

Step 1. A given algebraic differential equation

$$f^2 - \mathrm{e}^x f' + \frac{\mathrm{e}^{2x}}{x} = 0$$

Step 2. Put equation in integral form

$$f = \int \left(\frac{\mathrm{e}^x}{x} + \frac{f^2}{\mathrm{e}^x} \right)$$

Step 3. Integrate from a fixed point $x_0 < \infty$

Step 1. A general algebraic differential equation

$$P(f) = 0$$

Step 2. Equation in split-normal form

$$(\partial - \varphi_1) \cdots (\partial - \varphi_r) f = P(f) \qquad \text{with } P(f) \text{ small}$$

Attention: $\varphi_1, \ldots, \varphi_r \in \mathcal{T}[\mathrm{i}]$, even though $(\partial - \varphi_1) \cdots (\partial - \varphi_r) \in \mathcal{T}[\partial]$.

Step 3. Solve the split-normal equation using the fixed-point technique.

Lemma. *The operator* $J=(\partial-\varphi)^{-1}_{x_0}$, *defined by*

$$(Jf)(x) = \begin{cases} \mathrm{e}^{\Phi(x)} \int_{\infty}^{x} \mathrm{e}^{-\Phi(t)} f(t)\,\mathrm{d}t & \textit{(repulsive case)} \\ \mathrm{e}^{\Phi(x)} \int_{x_0}^{x} \mathrm{e}^{-\Phi(t)} f(t)\,\mathrm{d}t & \textit{(attractive case)} \end{cases}$$

and

$$\Phi(x)=\begin{cases} \int_{\infty}^{x} \varphi(t)\,\mathrm{d}t & \textit{(repulsive case)} \\ \int_{x_0}^{x} \varphi(t)\,\mathrm{d}t & \textit{(attractive case)} \end{cases}$$

is a continuous right-inverse of $L=\partial-\varphi$ *on* $\mathcal{G}^{\preccurlyeq}_{x_0}[\mathrm{i}]$, *with*

$$|||J|||_{x_0} \leqslant \left\| \frac{1}{\operatorname{Re}\varphi} \right\|_{x_0}.$$

Lemma. *Given a split-normal operator*

$$L = (\partial - \varphi_1) \cdots (\partial - \varphi_r), \tag{1}$$

with a factorwise right-inverse $L^{-1} = J_r \cdots J_1$*, the operator*

$$\mathfrak{v}^{\nu} J_r \cdots J_1 : \mathcal{G}_{x_0}^{\preccurlyeq}[\mathrm{i}] \to \mathcal{G}_{x_0;r}^{\preccurlyeq}[\mathrm{i}]$$

is a continuous operator for every $\nu > r\,\sigma_L$*. Here* $\mathcal{G}_{x_0;r}^{\preccurlyeq}[\mathrm{i}]$ *carries the norm*

$$\| f \|_{x_0;r} = \max \{ \| f \|_{x_0}, \ldots, \| f^{(r)} \|_{x_0} \}.$$

Lemma. *If* $L \in \mathcal{T}[\partial]$ *and the splitting* (1) *(formally) preserves realness, then* $J_r \cdots J_1$ *preserves realness in the sense that it maps* $\mathcal{G}_{x_0}^{\preccurlyeq}$ *into itself.*

Theorem. *Consider a split-monic equation*

$$L f = P(f), \quad f \prec 1,$$

and let ν *be such that* $r\,\sigma_L < \nu < \upsilon_P$. *Then for any sufficiently large* x_0, *there exists a continuous factorwise right-inverse* $J_{r,\ltimes \mathfrak{v}^\nu} \cdots J_{1,\ltimes \mathfrak{v}^\nu}$ *of* $L_{\ltimes \mathfrak{v}^\nu}$, *such that the operator*

$$\Xi \colon f \longmapsto (J_r \cdots J_1)(P(f))$$

admits a unique fixed point

$$f \;=\; \lim_{n \to \infty} \Xi^{(n)}(0) \;\in\; \mathcal{B}(\mathcal{G}^{\preccurlyeq}_{x_0;r}, \tfrac{1}{2}).$$

Theorem. *Let $\mathcal{T}$ be a transserial Hardy field of span $\mathfrak{v} \succcurlyeq \mathrm{e}^x$. Consider a monic split-normal quasi-linear equation*

$$L f = P(f), \quad f \prec 1, \tag{2}$$

over $\mathcal{T}$ without solutions in $\mathcal{T}$. Assume that one of the following holds:

- *$\mathcal{T}$ is $(1,1,1)$-differentially closed in $\mathbb{T}_{\preccurlyeq \mathfrak{v}}$ and (2) is first order.*

  i.e. $\mathcal{T}$ is closed under the resolution of linear first order equations.

- *$\mathcal{T}[\mathrm{i}]$ is $(1,1,1)$-differentially closed in $\mathbb{T}[\mathrm{i}]_{\preccurlyeq \mathfrak{v}}$.*

Then there exist solutions $f \in \mathcal{G}$ and $\tilde{f} \in \hat{\mathcal{T}}$ to (2), such that f and $\tilde{f}$ are asymptotically equivalent over $\mathcal{T}$.

Lemma. *Let* $L = \partial - \varphi \in \mathcal{T}[\partial]$ *be a normal operator. Let* $\tilde{f} \in \hat{\mathcal{T}}^{\preccurlyeq}$ *and* $g \in \mathcal{T}^{\preccurlyeq}$ *be such that* $\tilde{f}$ *is transcendental over* $\mathcal{T}$ *and* $L\,\tilde{f} = g$*. Then there exists an* $f \in \mathcal{G}^{\preccurlyeq}$ *with* $L\,f = g$*, such that* f *and* $\tilde{f}$ *are both differentially and asymptotically equivalent over* $\mathcal{T}$*.*

Theorem. *Let* $\mathcal{T}$ *be a transserial Hardy field. Let* $\mathcal{T}^{\mathrm{fo}} \supseteq \mathcal{T}$ *be the smallest differential subfield of* $\mathbb{T}$*, such that for any* $P \in \mathcal{T}^{\mathrm{fo}}\{F\}^{\neq}$ *with* $r_P \leqslant 1$ *and* $f \in \mathbb{T}$ *we have* $P(f) = 0 \Rightarrow f \in \mathcal{T}^{\mathrm{fo}}$*. Then the transserial Hardy field structure of* $\mathcal{T}$ *can be extended to* $\mathcal{T}^{\mathrm{fo}}$*.*

Proof. As long as $\mathcal{T}^{\mathrm{fo}} \neq \mathcal{T}$:

- Close off under $\exp$, $\log$ and algebraic equations.
- Choose $P \in \mathcal{T}\{F\}^{\neq}, r_P = 1, f \in \mathbb{T}, P(f) = 0$ such that P has minimal "complexity" (r_P, d_P, t_P) and apply the previous results. $\square$

Lemma. *Let* $L = \partial - \varphi \in \mathcal{T}[\mathrm{i}][\partial]$ *be a normal operator. Let* $\tilde{f} \in \hat{\mathcal{T}}[\mathrm{i}]^{\preccurlyeq}$ *and* $g \in \mathcal{T}[\mathrm{i}]^{\preccurlyeq}$ *be such that* $\operatorname{Re} \tilde{f}$ *has order* 2 *over* $\mathcal{T}$ *and* $L\,\tilde{f} = g$*. Then there exists an* $f \in \mathcal{G}^{\preccurlyeq}[\mathrm{i}]$ *with* $L\,f = g$*, such that* $\operatorname{Re} f$ *and* $\operatorname{Re} \tilde{f}$ *are both differentially and asymptotically equivalent over* $\mathcal{T}$*.*

Theorem. *Let* $\mathcal{T}$ *be a transserial Hardy field. Let* $\mathcal{T}^{\mathrm{dalg}} \supseteq \mathcal{T}$ *be the smallest differential subfield of* $\mathbb{T}$*, such that for any* $P \in \mathcal{T}^{\mathrm{dalg}}\{F\}^{\neq}$ *and* $f \in \mathbb{T}$ *we have* $P(f) = 0 \Rightarrow f \in \mathcal{T}^{\mathrm{dalg}}$*. Then the transserial Hardy field structure of* $\mathcal{T}$ *can be extended to* $\mathcal{T}^{\mathrm{dalg}}$*.*

Corollary. *There exists a transserial Hardy field* $\mathcal{T}$*, such that for any* $P \in \mathcal{T}\{F\}$ *and* $f, g \in \mathcal{T}$ *with* $f < g$ *and* $P(f)\,P(g) < 0$*, there exists a* $h \in \mathcal{T}$ *with* $f < h < g$ *and* $P(h) = 0$*.*

Corollary. *There exists a transserial Hardy field* $\mathcal{T}$*, such that* $\mathcal{T}[\mathrm{i}]$ *is weakly differentially closed.*

Corollary. *There exists a differentially Henselian transserial Hardy field* $\mathcal{T}$*, i.e., such that any quasi-linear differential equation over* $\mathcal{T}$ *admits a solution in* $\mathcal{T}$*.*

Theorem. *Let $\mathcal{T}$ be a transserial Hardy field and $\mathcal{H}$ a differentially algebraic Hardy field extension of $\mathcal{T}$, such that $\mathcal{H}$ is differentially Henselian and stable under exponentiation. Then there exists a transserial Hardy field structure on $\mathcal{H}$ which extends the structure on $\mathcal{T}$.*

Corollary. *Let $\mathcal{T}$ be a transserial Hardy field and $\mathcal{H}$ a differentially algebraic Hardy field extension of $\mathcal{T}$, such that $\mathcal{H}$ is differentially Henselian. Assume that $\mathcal{H}$ admits no non-trivial algebraically differential Hardy field extensions. Then $\mathcal{H}$ satisfies the differential intermediate value property.*

Theorem. *(Boshernitzan 1987) Any solution of the equation*

$$f'' + f = \mathrm{e}^{x^2}$$

is contained in a Hardy field. However, none of these solutions is contained in the intersection of all maximal Hardy fields.

- Embeddability of Hardy fields in differentially Henselian Hardy fields.
- Do maximal Hardy fields satisfy the intermediate value property?
- Restricted analytic (instead of algebraic) differential equations.
- Preservation of composition:
  - $f(x+\varepsilon)$, small ε: expand.
  - $f(q\,x+\varepsilon)$: expand, but more intricate.
  - $f(\varphi(x)), \varphi \succ x$: abstract nonsense.