Hardy field solutions to algebraic differential equations

Joris van der Hoeven, Fields Institute 2009
http://www.TEXMACS.org
Real algebraic geometry
+ Valuation theory

Differential algebra

- Hardy fields: Hardy, Rosenlicht, Boshernitzan, Singer, etc.
- Pfaff systems: Khovanskii, Wilkie, van den Dries, Rolin, etc.
Algebraic geometry

→

Real algebraic geometry
+ Valuation theory

↓

Real differential algebra
+ Asymptotic differential algebra

- Hardy fields: Hardy, Rosenlicht, Boshernitzan, Singer, etc.
- Pfaff systems: Khovanskii, Wilkie, van den Dries, Rolin, etc.
Algebraic geometry

Real algebraic geometry
+ Valuation theory

Differential algebra

Real differential algebra
+ Asymptotic differential algebra

- **LNM 1888**: Transseries and Real Differential Algebra
- Other work on [http://www.math.u-psud.fr/~vdhoeven](http://www.math.u-psud.fr/~vdhoeven)
Sufficiently closed models

Algebraic geometry → Real algebraic geometry
+ Valuation theory

Differential algebra → Real differential algebra
+ Asymptotic differential algebra
Sufficiently closed models

\[ \mathbb{C} \rightarrow \]

Real algebraic geometry + Valuation theory

\[ \downarrow \]

Real differential algebra + Asymptotic differential algebra

\[ \rightarrow \]

Differential algebra
Sufficiently closed models

\[ \mathbb{C} \rightarrow \mathbb{R} \]

Valuation theory

\[ \downarrow \]

Differential algebra

\[ \rightarrow \]

Real differential algebra

\[ \rightarrow \]

Asymptotic differential algebra
Suciently closed models

\[ \mathbb{C} \xrightarrow{\rightarrow} \mathbb{R} + \mathbb{C}[[z^\mathbb{Q}]] \]

Differential algebra

Real differential algebra

Asymptotic differential algebra
Sufficiently closed models

\[ \mathbb{C} \rightarrow \mathbb{R} + \mathbb{C}[z^Q] \]

\[ \downarrow \]

\[ \text{Real differential algebra} \rightarrow + \]

\[ \text{Asymptotic differential algebra} \]
Sufficiently closed models

\[ \mathbb{C} \longrightarrow \mathbb{R} + \mathbb{C}[z \mathbb{Q}] \]

\[ \downarrow \]

Maximal Hardy field (?) + Asymptotic differential algebra

Wild \[ \longrightarrow \]
Sufficiently closed models

\[ \mathbb{C} \hookrightarrow \mathbb{R} \oplus \mathbb{C}[z^Q] \]

\[ \downarrow \]

\[ \text{Wild} \hookrightarrow \mathbb{R} \oplus x \mathbb{R} \]

Asymptotic differential algebra
Sufficiently closed models

\[
\begin{array}{c}
\mathbb{C} \\
\uparrow \\
Wild
\end{array}
\quad \rightarrow 
\quad \begin{array}{c}
\mathbb{R} \\
+ \\
\mathbb{C}[[z^Q]]
\end{array}
\]

\[
\begin{array}{c}
\mathbb{R}[[x]] \\
+ \\
\mathbb{C}[[z]]
\end{array}
\]
What is a transseries?

\( (x > 1) \)

\[ e^x + \frac{e^x}{x} + \frac{e^x}{x^2} + \cdots + \frac{2}{\log x} e^x + \frac{e^x}{x} + \frac{e^x}{x^2} + \cdots + e^{\sqrt{x}} + e^{\sqrt{\log x}} + e^{\sqrt{\log \log x}} + \cdots + \cdots \]
What is a transseries?

\[(x > 1)\]

\[e^x + \frac{e^x}{x} + \frac{e^x}{x^2} + \cdots + \frac{2}{\log x} e^x + \frac{e^x}{x} + \frac{e^x}{x^2} + \cdots + e^{\sqrt{x}} + e^{\sqrt{\log x}} + e^{\sqrt{\log \log x}} + \cdots + \cdots \]

- Dahn & Göring
- Écalle
Examples of transseries

\[ \frac{1}{1 - x^{-1} - x^{-e}} = 1 + x^{-1} + x^{-2} + x^{-e} + x^{-3} + x^{-e-1} + \ldots \]

\[ \frac{1}{1 - x^{-1} - e^{-x}} = 1 + \frac{1}{x} + \frac{1}{x^2} + \ldots + e^{-x} + 2 \frac{e^{-x}}{x} + \ldots + e^{-2x} + \ldots \]

\[ -e^x \int \frac{e^{-x}}{x} = \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{6}{x^4} + \frac{24}{x^5} - \frac{120}{x^6} + \ldots \]

\[ \Gamma(x) = \frac{\sqrt{2\pi} e^x (\log x - 1)}{x^{1/2}} + \frac{\sqrt{2\pi} e^x (\log x - 1)}{12 x^{3/2}} + \frac{\sqrt{2\pi} e^x (\log x - 1)}{288 x^{5/2}} + \ldots \]

\[ \zeta(x) = 1 + 2^{-x} + 3^{-x} + 4^{-x} + \ldots \]

\[ \varphi(x) = \frac{1}{x} + \varphi(x^{\pi}) = \frac{1}{x} + \frac{1}{x^{\pi}} + \frac{1}{x^{\pi^2}} + \frac{1}{x^{\pi^3}} + \ldots \]

\[ \psi(x) = \frac{1}{x} + \psi(e^{\log^2 x}) = \frac{1}{x} + \frac{1}{e^{\log^2 x}} + \frac{1}{e^{\log^4 x}} + \frac{1}{e^{\log^8 x}} + \ldots \]
The field $\mathbb{T}$ of grid-based transseries

- $\mathbb{T} = \mathbb{R}[\mathcal{T}]$, where $\mathcal{T}$ is a totally ordered monomial group.
- $\mathbb{R}[\mathcal{T}]$: series $f = \sum_{m \in \mathcal{T}} f_m m \in \mathbb{R}[\mathcal{T}]$ with grid-based support:
  \[ \text{supp } f \subseteq \{m_1, ..., m_m\}^* n, \quad m_1, ..., m_m < 1 \]

- $\mathbb{T}$ is a totally ordered, real closed field.
- $\mathbb{T}$ is stable under $\exp$, $\log$, $\partial$, $\int$, $\circ$ and $\text{inv}$. 
Theorem. (2000) Given $P \in \mathbb{T}\{F\}$ and $f < g \in \mathbb{T}$ with $P(f)P(g) < 0$. Then there exists an $h \in \mathbb{T}$ with $f < h < g$ and $P(h) = 0$.

- Calculus with cuts $\hat{f} \in \hat{\mathbb{T}}$.
- Classification of cuts and behaviour of $P(f)$ near a cut.
- Newton polygon method for shrinking interval on which a sign change occurs and whose end-points are cuts.
**Theorem.** (2000) Given $P \in \mathbb{T} \{F\}$ and $f < g \in \mathbb{T}$ with $P(f)P(g) < 0$. Then there exists an $h \in \mathbb{T}$ with $f < h < g$ and $P(h) = 0$.

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**Corollary.** Any $P \in \mathbb{T} \{F\}$ of odd degree admits a root in $\mathbb{T}$. 
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- Calculus with cuts $\hat{f} \in \hat{\mathbb{T}}$.
- Classification of cuts and behaviour of $P(f)$ near a cut.
- Newton polygon method for shrinking interval on which a sign change occurs and whose end-points are cuts.

Example. The following equation admits a solution in $\mathbb{T}$:

$$\frac{1}{x} f''' (f')^2 f^{24} + e^x (f'')^{27} - \Gamma(\Gamma(\log x)) f^2 = \frac{e^{e^x+x^2}}{\Gamma(e^x+x)}.$$
**Theorem.** (2000) Given $P \in \mathbb{T}\{F\}$ and $f < g \in \mathbb{T}$ with $P(f)P(g) < 0$. Then there exists an $h \in \mathbb{T}$ with $f < h < g$ and $P(h) = 0$.

- Calculus with **cuts** $\hat{f} \in \hat{\mathbb{T}}$.
- Classification of cuts and behaviour of $P(f)$ near a cut.
- Newton polygon method for shrinking interval on which a sign change occurs and whose end-points are cuts.

**Corollary.** Any monic $L \in \mathbb{T}[\partial]$ admits a factorization with factors

$$\partial - a \quad \text{or}$$

$$\partial^2 - (2a + b^\dagger) \partial + (a^2 + b^2 - a' + ab^\dagger) = (\partial - (a - b i + b^\dagger)) (\partial - (a + b i))$$
Theorem. (2001) Every asymptotic differential equation over $\mathbb{T}$ of Newton degree $d$ admits at least $d$ solutions (when counting with multiplicities). Moreover, it suffices to add iterated logarithms to the asymptotic scale.
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Warning. $\mathbb{T}$ is not differentially algebraically closed

\[ f^3 + (f')^2 + f = 0 \]
\[ f^3 + f \neq 0 \]

→ Desingularization of vector fields (Cano, Panazzolo, ...)

Complex transseries
Theorem. (2001) Every asymptotic differential equation over $\mathbb{T}$ of Newton degree $d$ admits at least $d$ solutions (when counting with multiplicities). Moreover, it suffices to add iterated logarithms to the asymptotic scale.

Corollary. $\mathbb{T}$ is Picard-Vessiot closed.

Remark. $\exists$ algorithm for computing the solutions of a given equation.

1: Accelero-summation

\[ \tilde{f} \xrightarrow{\tilde{B}_{z_1}} \hat{f}_1 \xrightarrow{\hat{A}^{\theta_1}_{z_1 \to z_2}} \hat{f}_2 \cdots \xrightarrow{\hat{A}^{\theta_{p-1}}_{z_{p-1} \to z_p}} \hat{f}_{p-1} \xrightarrow{\hat{A}^{\theta_p}_{z_p}} \hat{f}_p \]

2: Transserial Hardy fields

\[ \mathbb{T} \supseteq \mathcal{T} \xrightarrow{\rho} \mathcal{G} \]

- \( \mathcal{G} \): ring of infinitely differentiable real germs at \(+\infty\).
### 1: Accelero-summation

<table>
<thead>
<tr>
<th>Advantages</th>
<th>Disadvantages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Canonical after choosing average</td>
<td>Requires many different tools</td>
</tr>
<tr>
<td>Preserves composition</td>
<td>Not yet written down</td>
</tr>
<tr>
<td>Classification local vector fields</td>
<td></td>
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<tr>
<td>Differential Galois theory</td>
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### 2: Transserial Hardy fields

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A **transsernal Hardy** field is a differential subfield $\mathcal{T}$ of $\mathbb{T}$, together with a monomorphism $\rho: \mathcal{T} \rightarrow \mathcal{G}$ of ordered differential $\mathbb{R}$-algebras, such that

**TH1.** $\forall f \in \mathcal{T}: \text{ supp } f \subseteq \mathcal{T}$.

**TH2.** $\forall f \in \mathcal{T}: \quad f_{<} \in \mathcal{T}$.

**TH3.** $\exists d \in \mathbb{Z}: \forall m \in \mathcal{I} \cap \mathcal{T}: \quad \log m \in \mathcal{T} + \mathbb{R} \log_d x$.

**TH4.** $\mathcal{I} \cap \mathcal{T}$ is stable under taking real powers.

**TH5.** $\forall f \in \mathcal{T}^\succ: \quad \log f \in \mathcal{T} \Rightarrow \rho(\log f) = \log \rho(f)$.

**Example.** $\mathcal{T} = \mathbb{R}\{x^{-\mathbb{R}}\}$. 
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**TH2.** $\forall f \in \mathcal{T}$: $f_{<} \in \mathcal{T}$.

\[
\begin{align*}
  f_{<} &= \sum_{m < 1} f_m m \\
  \frac{x e^x}{1 - x^{-1} - e^{-x}} &= x e^x + e^x + x^{-1} e^x + \cdots + x + 1 + x^{-1} + \cdots + x e^{-x} + e^{-x} + x^{-1} e^{-x} + \cdots
\end{align*}
\]

**TH3.** $\exists d \in \mathbb{Z}$: $\forall m \in \mathcal{I} \cap \mathcal{T}$: $\log m \in \mathcal{T} + \mathbb{R} \log_d x$.

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**TH1.** $\forall f \in \mathcal{T}: \supp f \subseteq \mathcal{T}$.

**TH2.** $\forall f \in \mathcal{T}: f \prec f \in \mathcal{T}$.

$$f \prec = \sum_{m \prec 1} f_{m} m$$

$$\left( \frac{x e^{x}}{1-x^{-1} - e^{-x}} \right) \prec$$

$$\| x e^{x} + e^{x} + x^{-1} e^{x} + \cdots + x + 1 + x^{-1} + \cdots + x e^{-x} + e^{-x} + x^{-1} e^{-x} + \cdots \|$$

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1. **TH1.** $\forall f \in \mathcal{T}:$ supp $f \subseteq \mathcal{T}$.

2. **TH2.** $\forall f \in \mathcal{T}:$ $f_{\prec} \in \mathcal{T}$.

3. **TH3.** $\exists d \in \mathbb{Z}: \forall m \in \mathcal{I} \cap \mathcal{T}:$ log $m \in \mathcal{T} + \mathbb{R} \log d x$.

4. **TH4.** $\mathcal{I} \cap \mathcal{T}$ is stable under taking real powers.

5. **TH5.** $\forall f \in \mathcal{T}^{>}:$ log $f \in \mathcal{T} \Rightarrow \rho($log $f) = \log \rho(f)$.

**Example.** $\mathcal{T} = \mathbb{R}\{x^{-\mathbb{R}}}\}$. 

\[ f_{\prec} = \sum_{m_{\prec} \leq 1} f_m m \]
Definitions. \( \mathcal{T} \) transserial Hardy field, \( f \in \mathcal{T}, \hat{f} \in \mathcal{G} \)

\[
f \sim \hat{f} \iff (\exists \varphi \in \mathcal{T}: f \sim_{\mathcal{T}} \varphi \sim_{\mathcal{G}} \hat{f})
\]

\( f \) asympt. equiv. to \( \hat{f} \) over \( \mathcal{T} \) \( \iff (\forall \varphi \in \mathcal{T}: f - \varphi \sim \hat{f} - \varphi) \)

\( f \) diff. equiv. to \( \hat{f} \) over \( \mathcal{T} \) \( \iff (\forall P \in \mathcal{T}\{F\}: P(f) = 0 \Leftrightarrow P(\hat{f}) = 0) \)

Lemma. Let \( f \in \mathcal{T} \setminus \mathcal{T} \) and \( \hat{f} \in \mathcal{G} \setminus \mathcal{T} \) be such that

- \( f \) is a serial cut over \( \mathcal{T} \).
- \( f \) and \( \hat{f} \) are asymptotically equivalent over \( \mathcal{T} \).
- \( f \) and \( \hat{f} \) are differentially equivalent over \( \mathcal{T} \).

Then \( \exists! \) transserial Hardy field extension \( \rho: \mathcal{T}\langle f \rangle \to \mathcal{G} \) with \( \rho(f) = \hat{f} \).
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Then \( \exists! \) transserial Hardy field extension \( \rho : \mathcal{T} \langle f \rangle \to \mathcal{G} \) with \( \rho(f) = \hat{f} \).
Theorem. Let $\mathcal{T}$ be a transserial Hardy field. Then its real closure $\mathcal{T}^{rcl}$ admits a unique transserial Hardy field structure which extends the one of $\mathcal{T}$.

Theorem. Let $\mathcal{T}$ be a transserial Hardy field and let $\varphi \in \mathcal{T}_{\infty}$ be such that $e^\varphi \notin \mathcal{T}$. Then the set $\mathcal{T}(e^{\mathbb{R}\varphi})$ carries the structure of a transserial Hardy field for the unique differential morphism $\rho: \mathcal{T}(e^{\mathbb{R}\varphi}) \to \mathcal{G}$ over $\mathcal{T}$ with $\rho(e^{\lambda \varphi}) = e^{\lambda \rho(\varphi)}$ for all $\lambda \in \mathbb{R}$.

Theorem. Let $\mathcal{T}$ be a transserial Hardy field of depth $d < \infty$. Then $\mathcal{T}((\log_d x)^{\mathbb{R}})$ carries the structure of a transserial Hardy field for the unique differential morphism $\rho: \mathcal{T}((\log_d x)^{\mathbb{R}}) \to \mathcal{G}$ over $\mathcal{T}$ with $\rho((\log_d x)^{\lambda}) = (\log_d x)^{\lambda}$ for all $\lambda \in \mathbb{R}$. 
Step 1. A given algebraic differential equation

\[ f^2 - f' + \frac{x}{e^x} = 0 \]

Step 2. Put equation in integral form

\[ f = \int \left( \frac{x}{e^x} + f^2 \right) \]

Step 3. Integral transseries solution
Step 1. A given algebraic differential equation

\[ f^2 - e^x f' + \frac{e^{2x}}{x} = 0 \]

Step 2. Put equation in integral form

\[ f = \int \left( \frac{e^x}{x} + \frac{f^2}{e^x} \right) \]

Step 3. Integrate from a fixed point \( x_0 < \infty \)
**Step 1.** A general algebraic differential equation

\[ P(f) = 0 \]

**Step 2.** Equation in split-normal form

\[ (\partial - \varphi_1) \cdots (\partial - \varphi_r) f = P(f) \quad \text{with } P(f) \text{ small} \]

Attention: \( \varphi_1, \ldots, \varphi_r \in T[i] \), even though \( (\partial - \varphi_1) \cdots (\partial - \varphi_r) \in T[\partial] \).

**Step 3.** Solve the split-normal equation using the fixed-point technique.
Lemma. The operator $J = (\partial - \varphi)_{x_0}^{-1}$, defined by

$$(Jf)(x) = \begin{cases} 
    e^{\Phi(x)} \int_x^\infty e^{-\Phi(t)} f(t) \, dt & \text{(repulsive case)} \\
    e^{\Phi(x)} \int_{x_0}^x e^{-\Phi(t)} f(t) \, dt & \text{(attractive case)}
\end{cases}$$

and

$$\Phi(x) = \begin{cases} 
    \int_x^\infty \varphi(t) \, dt & \text{(repulsive case)} \\
    \int_{x_0}^x \varphi(t) \, dt & \text{(attractive case)}
\end{cases}$$

is a continuous right-inverse of $L = \partial - \varphi$ on $G_{x_0}^{\leq}[i]$, with

$$\|J\|_{x_0} \leq \left\| \frac{1}{\Re \varphi} \right\|_{x_0}.$$
**Lemma.** Given a split-normal operator

\[
L = (\partial - \varphi_1) \cdots (\partial - \varphi_r),
\]

(1)

with a factorwise right-inverse \( L^{-1} = J_r \cdots J_1 \), the operator

\[
\forall^r J_r \cdots J_1 : \mathcal{G}_{x_0}^\leqslant[i] \rightarrow \mathcal{G}_{x_0; r}[i]
\]

is a continuous operator for every \( \nu > r \sigma_L \). Here \( \mathcal{G}_{x_0; r}[i] \) carries the norm

\[
\| f \|_{x_0; r} = \max \{ \| f \|_{x_0}, \ldots, \| f^{(r)} \|_{x_0} \}.
\]

**Lemma.** If \( L \in \mathcal{T}[\partial] \) and the splitting (1) (formally) preserves realness, then \( J_r \cdots J_1 \) preserves realness in the sense that it maps \( \mathcal{G}_{x_0}^\leqslant \) into itself.
**Theorem.** Consider a split-monic equation

\[ Lf = P(f), \quad f < 1, \]

and let \( \nu \) be such that \( r \sigma_L < \nu < \nu_P \). Then for any sufficiently large \( x_0 \), there exists a continuous factorwise right-inverse \( J_{r, \nu} \cdots J_{1, \nu} \) of \( L_{\nu} \), such that the operator

\[ \Xi: f \mapsto (J_r \cdots J_1)(P(f)) \]

admits a unique fixed point

\[ f = \lim_{n \to \infty} \Xi^{(n)}(0) \in \mathcal{B}(G_{x_0; r}^{\frac{1}{2}}). \]
**Theorem.** Let $\mathcal{T}$ be a transserial Hardy field of span $v \succ e^x$. Consider a monic split-normal quasi-linear equation

$$L f = P(f), \quad f \prec 1,$$

over $\mathcal{T}$ without solutions in $\mathcal{T}$. Assume that one of the following holds:

- $\mathcal{T}$ is $(1, 1, 1)$-differentially closed in $\mathcal{T} \preceq v$ and (2) is first order.
  i.e. $\mathcal{T}$ is closed under the resolution of linear first order equations.
- $\mathcal{T}[i]$ is $(1, 1, 1)$-differentially closed in $\mathcal{T}[i] \preceq v$.

Then there exist solutions $f \in \mathcal{G}$ and $\tilde{f} \in \mathcal{\hat{G}}$ to (2), such that $f$ and $\tilde{f}$ are asymptotically equivalent over $\mathcal{T}$.
**Lemma.** Let $L = \partial - \varphi \in \mathcal{T}[\partial]$ be a normal operator. Let $\tilde{f} \in \mathcal{T}^\prec$ and $g \in \mathcal{T}^\prec$ be such that $\tilde{f}$ is transcendental over $\mathcal{T}$ and $L \tilde{f} = g$. Then there exists an $f \in \mathcal{G}^\prec$ with $Lf = g$, such that $f$ and $\tilde{f}$ are both differentially and asymptotically equivalent over $\mathcal{T}$.

**Theorem.** Let $\mathcal{T}$ be a transserial Hardy field. Let $\mathcal{T}^{\text{fo}} \supseteq \mathcal{T}$ be the smallest differential subfield of $\mathcal{T}$, such that for any $P \in \mathcal{T}^{\text{fo}} \{F\} \neq \emptyset$ with $r_P \leq 1$ and $f \in \mathcal{T}$ we have $P(f) = 0 \Rightarrow f \in \mathcal{T}^{\text{fo}}$. Then the transserial Hardy field structure of $\mathcal{T}$ can be extended to $\mathcal{T}^{\text{fo}}$.

**Proof.** As long as $\mathcal{T}^{\text{fo}} \neq \mathcal{T}$:

- Close off under $\exp, \log$ and algebraic equations.

- Choose $P \in \mathcal{T} \{F\} \neq \emptyset$, $r_P = 1$, $f \in \mathcal{T}$, $P(f) = 0$ such that $P$ has minimal “complexity” $(r_P, d_P, t_P)$ and apply the previous results. 

□
Lemma. Let $L = \partial - \varphi \in \mathcal{T}[i][\partial]$ be a normal operator. Let $\tilde{f} \in \hat{\mathcal{T}}[i]$ and $g \in \mathcal{T}[i]$ be such that $\text{Re} \tilde{f}$ has order 2 over $\mathcal{T}$ and $L \tilde{f} = g$. Then there exists an $f \in \mathcal{G}[i]$ with $Lf = g$, such that $\text{Re} f$ and $\text{Re} \tilde{f}$ are both differentially and asymptotically equivalent over $\mathcal{T}$.

Theorem. Let $\mathcal{T}$ be a transserial Hardy field. Let $\mathcal{T}^{\text{dalg}} \supseteq \mathcal{T}$ be the smallest differential subfield of $\mathcal{T}$, such that for any $P \in \mathcal{T}^{\text{dalg}}\{F\} \neq$ and $f \in \mathcal{T}$ we have $P(f) = 0 \Rightarrow f \in \mathcal{T}^{\text{dalg}}$. Then the transserial Hardy field structure of $\mathcal{T}$ can be extended to $\mathcal{T}^{\text{dalg}}$. 
Corollary. There exists a transserial Hardy field $\mathcal{T}$, such that for any $P \in \mathcal{T}\{F\}$ and $f, g \in \mathcal{T}$ with $f < g$ and $P(f)P(g) < 0$, there exists a $h \in \mathcal{T}$ with $f < h < g$ and $P(h) = 0$.

Corollary. There exists a transserial Hardy field $\mathcal{T}$, such that $\mathcal{T}[i]$ is weakly differentially closed.

Corollary. There exists a differentially Henselian transserial Hardy field $\mathcal{T}$, i.e., such that any quasi-linear differential equation over $\mathcal{T}$ admits a solution in $\mathcal{T}$. 
**Theorem.** Let $\mathcal{T}$ be a transserial Hardy field and $\mathcal{H}$ a differentially algebraic Hardy field extension of $\mathcal{T}$, such that $\mathcal{H}$ is differentially Henselian and stable under exponentiation. Then there exists a transserial Hardy field structure on $\mathcal{H}$ which extends the structure on $\mathcal{T}$.

**Corollary.** Let $\mathcal{T}$ be a transserial Hardy field and $\mathcal{H}$ a differentially algebraic Hardy field extension of $\mathcal{T}$, such that $\mathcal{H}$ is differentially Henselian. Assume that $\mathcal{H}$ admits no non-trivial algebraically differential Hardy field extensions. Then $\mathcal{H}$ satisfies the differential intermediate value property.

**Theorem.** (Boshernitzan 1987) Any solution of the equation

$$f'' + f = e^{x^2}$$

is contained in a Hardy field. However, none of these solutions is contained in the intersection of all maximal Hardy fields.
Open problems

- Embeddability of Hardy fields in differentially Henselian Hardy fields.
- Do maximal Hardy fields satisfy the intermediate value property?
- Restricted analytic (instead of algebraic) differential equations.
- Preservation of composition:
  - $f(x + \varepsilon)$, small $\varepsilon$: expand.
  - $f(qx + \varepsilon)$: expand, but more intricate.
  - $f(\varphi(x))$, $\varphi > x$: abstract nonsense.