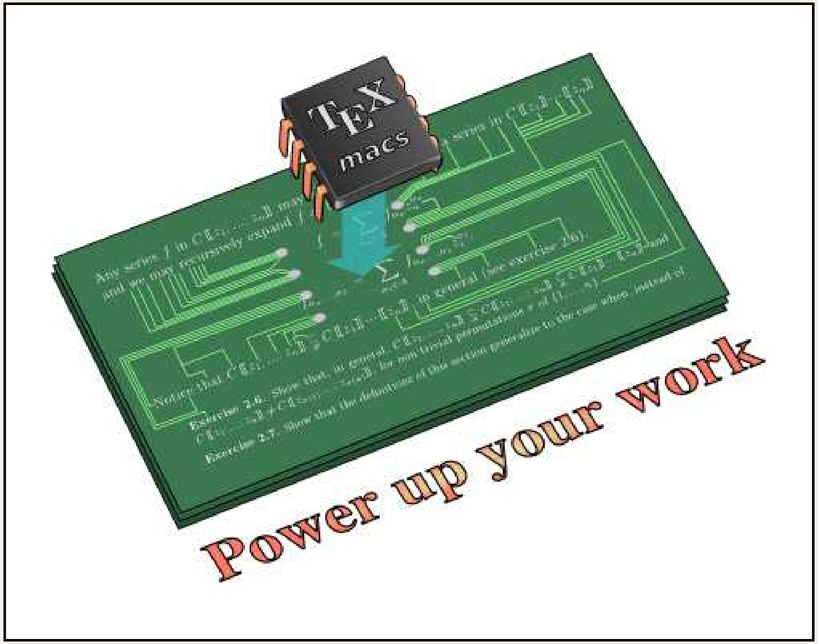
Asymptotic differential equations

Lecture 1: transseries and asymptotic differential equations



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Asymptotic differential algebra

$$\begin{array}{ccc} \text{Algebraic geometry} & \longrightarrow & \begin{array}{c}\text{Real algebraic geometry}\\ +\\ \text{Valuation theory}\end{array} \\ \downarrow & & \downarrow \\ \text{Differential algebra} & \longrightarrow & ? \end{array}$$

- Hardy fields: Rosenlicht, Boshernitzan, Singer, etc.
- Pfaff systems: Khovanskii, Wilkie, etc.

Asymptotic differential algebra

$$\begin{tikzcd} \text{Algebraic geometry} \arrow[r] \arrow[d] & \begin{array}{c}\text{Real algebraic geometry}\\ +\\ \text{Valuation theory}\end{array} \arrow[d] \\ \text{Differential algebra} \arrow[r] & \begin{array}{c}\text{Real differential algebra}\\ +\\ \text{Asymptotic differential algebra}\end{array} \end{tikzcd}$$

- Hardy fields: Rosenlicht, Boshernitzan, Singer, etc.
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Asymptotic differential algebra

$$\begin{array}{ccc}
\text{Algebraic geometry} & \longrightarrow & \begin{array}{c}\text{Real algebraic geometry}\\ +\\ \text{Valuation theory}\end{array}\\
\downarrow & & \downarrow\\
\text{Differential algebra} & \longrightarrow & \begin{array}{c}\text{Real differential algebra}\\ +\\ \text{Asymptotic differential algebra}\end{array}
\end{array}$$

- LNM 1888: Transseries and Real Differential Algebra
- Upcoming book with Matthias Aschenbrenner and Lou van den Dries

Sufficiently closed models

$$\begin{array}{ccc}
\text{Algebraic geometry} & \longrightarrow & \begin{array}{c}\text{Real algebraic geometry}\\ +\\ \text{Valuation theory}\end{array}\\
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\end{array}$$

Sufficiently closed models

$$
\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[d] & \begin{array}{c}\text{Real algebraic geometry}\\ +\\ \text{Valuation theory}\end{array} \arrow[d] \\
\text{Differential algebra} \arrow[r] & \begin{array}{c}\text{Real differential algebra}\\ +\\ \text{Asymptotic differential algebra}\end{array}
\end{tikzcd}
$$

Sufficiently closed models

$$
\begin{array}{ccc}
\mathbb{C} & \longrightarrow & \begin{array}{c}\mathbb{R} \\ + \\ \text{Valuation theory}\end{array} \\
\downarrow & & \downarrow \\
\text{Differential algebra} & \longrightarrow & \begin{array}{c}\text{Real differential algebra} \\ + \\ \text{Asymptotic differential algebra}\end{array}
\end{array}
$$

Sufficiently closed models

$$\begin{array}{ccc}
\mathbb{C} & \longrightarrow & \begin{array}{c}\mathbb{R}\\ +\\ \mathbb{C}[[z^{\mathbb{Q}}]]\end{array}\\
\downarrow & & \downarrow\\
\text{Differential algebra} & \longrightarrow & \begin{array}{c}\text{Real differential algebra}\\ +\\ \text{Asymptotic differential algebra}\end{array}
\end{array}$$

Sufficiently closed models

$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[d] & \begin{array}{c}\mathbb{R}\\ + \\ \mathbb{C}\llbracket z^{\mathbb{Q}}\rrbracket\end{array} \arrow[d] \\ \textit{Wild} \arrow[r] & \begin{array}{c}\text{Real differential algebra}\\ + \\ \text{Asymptotic differential algebra}\end{array} \end{tikzcd}$$

Sufficiently closed models

$$
\begin{array}{ccc}
\mathbb{C} & \longrightarrow & \begin{array}{c}\mathbb{R}\\ +\\ \mathbb{C}[[z^{\mathbb{Q}}]]\end{array}\\
\big\downarrow & & \big\downarrow\\
\textit{Wild} & \longrightarrow & \begin{array}{c}\text{Maximal Hardy field}\\ +\\ \text{Asymptotic differential algebra}\end{array}
\end{array}
$$

Sufficiently closed models

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \begin{array}{c}\mathbb{R}\\ + \\ \mathbb{C}\llbracket z^{\mathbb{Q}} \rrbracket\end{array} \\ \downarrow & & \downarrow \\ \textit{Wild} & \longrightarrow & \begin{array}{c}\mathbb{R}[[[x]]]\\ + \\ \text{Asymptotic differential algebra}\end{array}\end{array}$$

Sufficiently closed models

$$
\begin{array}{ccc}
\mathbb{C} & \longrightarrow & \begin{array}{c}\mathbb{R}\\ \text{+}\\ \mathbb{C}[[z^{\mathbb{Q}}]]\end{array}\\
\downarrow & & \downarrow\\
\textit{Wild} & \longrightarrow & \begin{array}{c}\mathbb{R}[[[x]]]\\ \text{+}\\ \mathbb{C}[[[z]]]\end{array}
\end{array}
$$

Sufficiently closed models

$(x \succ 1)$

$$\mathrm{e}^{\mathrm{e}^{x} + \frac{\mathrm{e}^{x}}{x} + \frac{\mathrm{e}^{x}}{x^2} + \cdots} + \frac{2}{\log x} \mathrm{e}^{\mathrm{e}^{x} + \frac{\mathrm{e}^{x}}{x} + \frac{\mathrm{e}^{x}}{x^2} + \cdots} + \mathrm{e}^{\sqrt{x} + \mathrm{e}^{\sqrt{\log x} + \mathrm{e}^{\sqrt{\log \log x} + \cdots}}} + \cdots$$

Sufficiently closed models

$(x \succ 1)$

$$\mathrm{e}^{\mathrm{e}^{x} + \frac{\mathrm{e}^{x}}{x} + \frac{\mathrm{e}^{x}}{x^2} + \cdots} + \frac{2}{\log x} \mathrm{e}^{\mathrm{e}^{x} + \frac{\mathrm{e}^{x}}{x} + \frac{\mathrm{e}^{x}}{x^2} + \cdots} + \mathrm{e}^{\sqrt{x} + \mathrm{e}^{\sqrt{\log x} + \mathrm{e}^{\sqrt{\log \log x} + \cdots}}} + \cdots$$

- Dahn & Göring
- Écalle

Examples of transseries

$$
\begin{aligned}
\frac{1}{1-x^{-1}-x^{-\mathrm{e}}} &= 1+x^{-1}+x^{-2}+x^{-\mathrm{e}}+x^{-3}+x^{-\mathrm{e}-1}+\cdots \\
\frac{1}{1-x^{-1}+\mathrm{e}^{-x}} &= 1+\frac{1}{x}+\frac{1}{x^2}+\cdots+\mathrm{e}^{-x}+2\,\frac{\mathrm{e}^{-x}}{x}+\cdots+\mathrm{e}^{-2x}+\cdots \\
-\mathrm{e}^{x}\int\frac{\mathrm{e}^{-x}}{x} &= \frac{1}{x}-\frac{1}{x^2}+\frac{2}{x^3}-\frac{6}{x^4}+\frac{24}{x^5}-\frac{120}{x^6}+\cdots \\
\Gamma(x) &= \frac{\sqrt{2\,\pi}\,\mathrm{e}^{x\,(\log x-1)}}{x^{1/2}}+\frac{\sqrt{2\,\pi}\,\mathrm{e}^{x\,(\log x-1)}}{12\,x^{3/2}}+\frac{\sqrt{2\,\pi}\,\mathrm{e}^{x\,(\log x-1)}}{288\,x^{5/2}}+\cdots \\
\zeta(x) &= 1+2^{-x}+3^{-x}+4^{-x}+\cdots \\
\varphi(x) &= \frac{1}{x}+\varphi(x^{\pi})=\frac{1}{x}+\frac{1}{x^{\pi}}+\frac{1}{x^{\pi^2}}+\frac{1}{x^{\pi^3}}+\cdots \\
\psi(x) &= \frac{1}{x}+\psi(\mathrm{e}^{\log^2 x})=\frac{1}{x}+\frac{1}{\mathrm{e}^{\log^2 x}}+\frac{1}{\mathrm{e}^{\log^4 x}}+\frac{1}{\mathrm{e}^{\log^8 x}}+\cdots
\end{aligned}
$$

```
Mmx] use "asymptotix"
```

```
Mmx] x == infinity ('x);
```

```
Mmx] 1 / (x + 1)
```

$$\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x^4} + O\left(\frac{1}{x^5}\right)$$

```
Mmx] 1 / (exp x + x + 1)
```

$$\frac{1}{\mathrm{e}^{x}}-\frac{x}{\mathrm{e}^{2x}}-\frac{1}{\mathrm{e}^{2x}}+\frac{x^2}{\mathrm{e}^{3x}}+\frac{2x}{\mathrm{e}^{3x}}+\frac{1}{\mathrm{e}^{3x}}-\frac{x^3}{\mathrm{e}^{4x}}-\frac{3x^2}{\mathrm{e}^{4x}}-\frac{3x}{\mathrm{e}^{4x}}-\frac{1}{\mathrm{e}^{4x}}+O\left(\frac{x^4}{\mathrm{e}^{5x}}\right)$$

```
Mmx] lengthen (exp (x^4 / (x + 1)), 7)
```

$$\frac{\mathrm{e}^{x^3-x^2+x}}{\mathrm{e}}+\frac{\mathrm{e}^{x^3-x^2+x}}{\mathrm{e}\,x}-\frac{\mathrm{e}^{x^3-x^2+x}}{2\,\mathrm{e}\,x^2}+O\left(\frac{\mathrm{e}^{x^3-x^2+x}}{x^7}\right)$$

```
Mmx] integrate (exp (x^2), x)
```

$$\frac{\mathrm{e}^{x^2}}{2x}+\frac{\mathrm{e}^{x^2}}{4x^3}+\frac{3\,\mathrm{e}^{x^2}}{8x^5}+\frac{15\,\mathrm{e}^{x^2}}{16x^7}+O\left(\frac{\mathrm{e}^{x^2}}{x^9}\right)$$

```
Mmx] integrate (x^x, x)
```

$$\frac{\mathrm{e}^{x\log(x)}}{\log(x)}-\frac{\mathrm{e}^{x\log(x)}}{\log(x)^2}+\frac{\mathrm{e}^{x\log(x)}}{\log(x)^3}-\frac{\mathrm{e}^{x\log(x)}}{\log(x)^4}+O\left(\frac{\mathrm{e}^{x\log(x)}}{\log(x)^5}\right)+\frac{\mathrm{e}^{x\log(x)}}{x\log(x)^3}-\frac{3\,\mathrm{e}^{x\log(x)}}{x\log(x)^4}+\frac{6\,\mathrm{e}^{x\log(x)}}{x\log(x)^5}+$$
$$O\left(\frac{\mathrm{e}^{x\log(x)}}{x\log(x)^6}\right)+\frac{\mathrm{e}^{x\log(x)}}{x^2\log(x)^4}-\frac{\mathrm{e}^{x\log(x)}}{x^2\log(x)^5}+O\left(\frac{\mathrm{e}^{x\log(x)}}{x^2\log(x)^6}\right)+\frac{2\,\mathrm{e}^{x\log(x)}}{x^3\log(x)^5}+O\left(\frac{\mathrm{e}^{x\log(x)}}{x^4\log(x)^6}\right)$$

```
Mmx] fixed_point_transseries (f :-> 1/x + f @ (x^2) + f @ (exp x))
```

$$\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^4}+\frac{1}{x^8}+O\left(\frac{1}{x^{16}}\right)+\frac{1}{\mathrm{e}^{x}}+\frac{1}{\mathrm{e}^{2x}}+\frac{1}{\mathrm{e}^{4x}}+O\left(\frac{1}{\mathrm{e}^{8x}}\right)+\frac{1}{\mathrm{e}^{x^2}}+\frac{1}{\mathrm{e}^{2x^2}}+O\left(\frac{1}{\mathrm{e}^{4x^2}}\right)+\frac{1}{\mathrm{e}^{x^4}}+$$
$$O\left(\frac{1}{\mathrm{e}^{2x^4}}\right)+\frac{1}{\mathrm{e}^{\mathrm{e}^{x}}}+\frac{1}{\mathrm{e}^{2\mathrm{e}^{x}}}+O\left(\frac{1}{\mathrm{e}^{4\mathrm{e}^{x}}}\right)+\frac{1}{\mathrm{e}^{\mathrm{e}^{2x}}}+O\left(\frac{1}{\mathrm{e}^{2\mathrm{e}^{2x}}}\right)+\frac{1}{\mathrm{e}^{\mathrm{e}^{x^2}}}+O\left(\frac{1}{\mathrm{e}^{2\mathrm{e}^{x^2}}}\right)+\frac{1}{\mathrm{e}^{\mathrm{e}^{\mathrm{e}^{x}}}}+O\left(\frac{1}{\mathrm{e}^{2\mathrm{e}^{\mathrm{e}^{x}}}}\right)$$

```
Mmx] lengthen (product (x, x), 4)
```

$$\frac{e^{x\log(x)-x}}{\mathrm{sqrt}(x)}+\frac{e^{x\log(x)-x}}{12\,x^{\frac{3}{2}}}+\frac{e^{x\log(x)-x}}{288\,x^{\frac{5}{2}}}-\frac{139\,e^{x\log(x)-x}}{51840\,x^{\frac{7}{2}}}+O\left(\frac{e^{x\log(x)-x}}{x^{\frac{9}{2}}}\right)$$

```
Mmx] lengthen (product (log x, x), 2)
```

$$e^{x\log(\log(x))-\frac{x}{\log(x)}-\frac{x}{\log(x)^2}-\frac{2x}{\log(x)^3}+O\left(\frac{x}{\log(x)^4}\right)-\frac{\log(\log(x))}{2}}+\frac{e^{x\log(\log(x))-\frac{x}{\log(x)}-\frac{x}{\log(x)^2}-\frac{2x}{\log(x)^3}+O\left(\frac{x}{\log(x)^4}\right)-\frac{\log(\log(x))}{2}}}{12\,x\log(x)}+\frac{e^{x\log(\log(x))-\frac{x}{\log(x)}-\frac{x}{\log(x)^2}-\frac{2x}{\log(x)^3}+O\left(\frac{x}{\log(x)^4}\right)-\frac{\log(\log(x))}{2}}}{288\,x^2\log(x)^2}-\frac{e^{x\log(\log(x))-\frac{x}{\log(x)}-\frac{x}{\log(x)^2}-\frac{2x}{\log(x)^3}+O\left(\frac{x}{\log(x)^4}\right)-\frac{\log(\log(x))}{2}}}{360\log(x)\,x^3}-\frac{e^{x\log(\log(x))-\frac{x}{\log(x)}-\frac{x}{\log(x)^2}-\frac{2x}{\log(x)^3}+O\left(\frac{x}{\log(x)^4}\right)-\frac{\log(\log(x))}{2}}}{240\,x^3\log(x)^2}+O\left(\frac{e^{x\log(\log(x))-\frac{x}{\log(x)}-\frac{x}{\log(x)^2}-\frac{2x}{\log(x)^3}+O\left(\frac{x}{\log(x)^4}\right)-\frac{\log(\log(x))}{2}}}{x^3\log(x)^3}\right)$$

```
Mmx] eval (integrate (exp (x^2), x), x, 100.0)
```

$4.40362931632092710468e4340$

```
Mmx]
```

Generalized power series

- C: constant field
- $(\mathfrak{M}, \preccurlyeq)$: totally ordered group of monomials

  I.e. $\log \mathfrak{M}$ is a value group with $\mathfrak{m} \preccurlyeq \mathfrak{n} \Leftrightarrow v(\log \mathfrak{m}) \geqslant v(\log \mathfrak{n})$
- $C[[\mathfrak{M}]]$: Hahn field of $f = \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}$ with **well-based support**.

$$\mathfrak{m}_1 \prec \mathfrak{m}_2 \prec \cdots \quad \text{with} \quad \mathfrak{m}_1, \mathfrak{m}_2, \ldots \in \operatorname{supp} f \quad \text{is impossible}$$

- $C[\![\mathfrak{M}]\!]$: field of $f = \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}$ with **grid-based support**.

$$\operatorname{supp} f \subseteq \mathfrak{m}_1^{\mathbb{N}} \cdots \mathfrak{m}_m^{\mathbb{N}} \mathfrak{n}, \qquad \mathfrak{m}_1, \ldots, \mathfrak{m}_m \prec 1$$

- $\mathscr{S} \subseteq \mathscr{P}(\mathfrak{M})$ closed under $\cup, \cdot$ and power products of infinitesimal sets, with $\{\mathfrak{m}\} \in \mathscr{S}$ for all $\mathfrak{m} \in \mathfrak{M}$.

Generalized power series

- C: constant ring (or set)
- $(\mathfrak{M}, \preccurlyeq)$: partially ordered monoid (or set) of monomials
- $C[[\mathfrak{M}]]$: ring of $f = \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}$ with **well-based support**.

$$\mathfrak{m}_1 \prec \mathfrak{m}_2 \prec \cdots \quad \text{with} \quad \mathfrak{m}_1, \mathfrak{m}_2, \ldots \in \operatorname{supp} f \quad \text{is impossible}$$
$$\operatorname{supp} f \text{ contains no infinite antichains}$$

- $C[\![\mathfrak{M}]\!]$: ring of $f = \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}$ with **grid-based support**.

$$\operatorname{supp} f \subseteq \mathfrak{m}_1^{\mathbb{N}} \cdots \mathfrak{m}_m^{\mathbb{N}} \{\mathfrak{n}_1, \ldots, \mathfrak{n}_n\}, \qquad \mathfrak{m}_1, \ldots, \mathfrak{m}_m \prec 1$$

Examples

- $C[[z]] = C[[z^{\mathbb{N}}]] = C\,[\![z^{\mathbb{N}}]\!]$
- $C((z)) = C[[z^{\mathbb{Z}}]] = C\,[\![z^{\mathbb{Z}}]\!]$
- $C[[z_1, z_2]] = C\big[\big[z^{\mathbb{N}^2}\big]\big]$
- $C[[z_1]][[z_2]] = C[[z^{\mathbb{N} \dot{\times} \mathbb{N}}]]$
- $C\,[\![z^{\mathbb{Q}}]\!]$ Puiseux series. $C[[z^{\mathbb{Q}}]] \supsetneq C\,[\![z^{\mathbb{Q}}]\!]$

Strong summability

Strong summability

We say that $(f_i)_{i \in I} \in C[[\mathfrak{M}]]^I$ is (strongly) summable if

1. $\bigcup_{i \in I} \operatorname{supp} f_i$ is grid-based (or well-based if $(f_i)_{i \in I} \in C[[\mathfrak{M}]]^I$)
2. $\{i \in I : \mathfrak{m} \in \operatorname{supp} f_i\}$ is finite for each $\mathfrak{m} \in \mathfrak{M}$

Then $g = \sum f \in C[[\mathfrak{M}]]$ with $g_{\mathfrak{m}} = \sum_{i \in I} f_{i,\mathfrak{m}}$ is well-defined.

Properties

- $\sum (f_{\sigma(i)})_{i \in I} = \sum (f_i)_{i \in I}$
- $\sum F \amalg G = \sum F + \sum G$
- For $F = \coprod_{j \in J} G_j$, we have $\sum_{j \in J} \sum G_j = \sum F$
- More properties

Multiplication

Definition

$$\left(\sum_{\mathfrak{m}\in \operatorname{supp} f} f_{\mathfrak{m}}\,\mathfrak{m}\right)\left(\sum_{\mathfrak{n}\in \operatorname{supp} g} g_{\mathfrak{n}}\,\mathfrak{n}\right)=\sum_{(\mathfrak{m},\mathfrak{n})\in \operatorname{supp} f\times \operatorname{supp} g} f_{\mathfrak{m}}\,g_{\mathfrak{n}}\,\mathfrak{m}\,\mathfrak{n}$$

- $\operatorname{supp} f, \operatorname{supp} g$ well/grid-based $\Rightarrow$ $\operatorname{supp} f \times \operatorname{supp} g$ well/grid-based
- $(\mathfrak{m},\mathfrak{n}) \mapsto \mathfrak{m}\,\mathfrak{n}$ is increasing

Associativity

$$f\,g\,h \;=\; \sum_{(\mathfrak{m},\mathfrak{n},\mathfrak{v})\in \operatorname{supp} f\times \operatorname{supp} g\times \operatorname{supp} h} f_{\mathfrak{m}}\,g_{\mathfrak{n}}\,h_{\mathfrak{v}}\,\mathfrak{m}\,\mathfrak{n}\,\mathfrak{v}$$

Inversion

$$\begin{aligned} f &= c_f \, \mathfrak{d}_f \, (1-\varepsilon), \quad \varepsilon \prec 1 \\ f^{-1} &= c_f^{-1} \, \mathfrak{d}_f^{-1} \, \frac{1}{1-\varepsilon} \end{aligned}$$

$$\frac{1}{1-\varepsilon} = \sum_{(\mathfrak{m}_1,\ldots,\mathfrak{m}_l) \in (\operatorname{supp} \varepsilon)^*} \varepsilon_{\mathfrak{m}_1} \cdots \varepsilon_{\mathfrak{m}_l} \, \mathfrak{m}_1 \cdots \mathfrak{m}_l$$

- Set of words $(\operatorname{supp} \varepsilon)^*$ carries natural partial ordering
- Higman: $\operatorname{supp} \varepsilon$ well-based $\Rightarrow$ $(\operatorname{supp} \varepsilon)^*$ well/grid-based

More generally: for $f \in C[[t]]$ and $\varepsilon \prec 1$, we may define $f(\varepsilon)$

Strong linearity

Definition

Linear $\varphi: C[\![\mathfrak{M}]\!] \longrightarrow C[\![\mathfrak{N}]\!]$ is strongly linear if, for all summable $(f_i)_{i \in I}$, $(\varphi(f_i))_{i \in I}$ is summable and

$$\varphi\left(\sum f_i\right) = \sum \varphi(f_i)$$

Extension by strong linearity

If $\check{\varphi}: \mathfrak{M} \rightarrow C[\![\mathfrak{N}]\!]$ sends grid-based subsets to summable families, then $\check{\varphi}$ admits a unique strongly linear extension

Applications

- For $\varepsilon \prec 1$ and $f, g \in C[[t]]$ with $g_0 = 0$, $f(g(\varepsilon)) = (f \circ g)(\varepsilon)$
- $(1 - \varepsilon) \frac{1}{1 - \varepsilon} = 1$

Henselian equations

$$\begin{aligned} f &= P_0 + P_1 f + P_2 f^2 + \cdots, \qquad f \prec 1 \\ P_i &\prec 1 \end{aligned}$$

$$f = \sum_{T \in \mathrm{Tree}(\operatorname{supp} f)} c_T \, \mathfrak{m}_T$$

$$T = \begin{tikzcd}[column sep=tiny] & & & \mathfrak{m}_0 \arrow[dll, dash] \arrow[dr, dash] \arrow[drr, dash] & & \\ & \mathfrak{m}_1 \arrow[dl, dash] \arrow[dr, dash] & & & \mathfrak{m}_2 & \mathfrak{m}_3 \arrow[d, dash] \\ \mathfrak{m}_4 & & \mathfrak{m}_5 & & & \mathfrak{m}_6 \end{tikzcd}$$

$$\begin{aligned} c_T &= P_{3,\mathfrak{m}_0} P_{2,\mathfrak{m}_1} P_{0,\mathfrak{m}_2} P_{1,\mathfrak{m}_3} P_{0,\mathfrak{m}_4} P_{0,\mathfrak{m}_5} P_{0,\mathfrak{m}_6} \\ \mathfrak{m}_T &= \mathfrak{m}_0 \, \mathfrak{m}_1 \, \mathfrak{m}_2 \, \mathfrak{m}_3 \, \mathfrak{m}_4 \, \mathfrak{m}_5 \, \mathfrak{m}_6 \end{aligned}$$

Kruskal: $\mathrm{Tree}(\operatorname{supp} f)$ carries well-based partial ordering

Abstract fields of transseries

Totally ordered field $\mathbb{T} = \mathbb{R}[[\mathfrak{T}]]$ with a logarithm such that

T1. $\operatorname{dom}\log = \mathbb{T}^{>}$.

T2. $\log\mathfrak{m} \in \mathbb{T}_{\succ}$, for all $\mathfrak{m} \in \mathfrak{T}$, i.e. $\forall\mathfrak{n} \in \operatorname{supp}(\log\mathfrak{m}), \mathfrak{n} \succ 1$.

T3. $\log(1+\varepsilon) = \varepsilon - \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 + \cdots$, for all $\varepsilon \in \mathbb{T}_{\prec}$.

Example. $\mathbb{L} = \mathbb{R}[[\mathfrak{L}]] = \mathbb{R}[[x^{\mathbb{R}}(\log x)^{\mathbb{R}}(\log_2 x)^{\mathbb{R}}\cdots]]$ with

$$\begin{aligned} \log(x^{\alpha_0}\cdots(\log_k x)^{\alpha_k}) &= \alpha_0\log x + \cdots + \alpha_k\log_{k+1} x \\ \log(f) = \log(c_f\,\mathfrak{d}_f\,(1+\delta_f)) &= \log\mathfrak{d}_f + \log c_f + \log(1+\delta_f) \end{aligned}$$

The field of grid-based transseries in x

Exponential extensions

$\mathbb{T} = \mathbb{R}[[\mathfrak{T}]]$ field of transseries $\Longrightarrow \mathbb{T}_{\exp} = \mathbb{R}[[\mathfrak{T}_{\exp}]] \supseteq \mathbb{T}$ also

$$\mathfrak{T}_{\exp} = \exp(\mathbb{R}[[\mathfrak{T}]]_{\succ})$$

Example. $\mathrm{e}^{x^2 + \frac{x^2}{\log x} + \frac{x^2}{\log^2 x} + \cdots + x + \log \log x} \in \mathfrak{L}_{\exp}$

Closure

Increasing limits of fields of grid-based transseries are fields of transseries

$$\mathbb{T} = \mathbb{L} \cup \mathbb{L}_{\exp} \cup \mathbb{L}_{\exp,\exp} \cup \cdots$$

The field of grid-based transseries in x

Exponential extensions

$\mathbb{T} = \mathbb{R}[[\mathfrak{T}]]$ field of transseries $\Longrightarrow$ $\mathbb{T}_{\exp} = \mathbb{R}[[\mathfrak{T}_{\exp}]] \supseteq \mathbb{T}$ also

$$\mathfrak{T}_{\exp} = \exp(\mathbb{R}[[\mathfrak{T}]]_{\succ})$$

Closure fails in well-based case

$$\begin{aligned} f_1 &= x^2 \\ f_{\alpha+1} &= f_\alpha - \mathrm{e}^{f_\alpha \circ \log x} \\ f_\lambda &= \operatorname*{stat\,lim}_{\alpha<\lambda} f_\alpha \end{aligned}$$

The field of grid-based transseries in x

Exponential extensions

$\mathbb{T} = \mathbb{R}[[\mathfrak{T}]]$ field of transseries $\Longrightarrow$ $\mathbb{T}_{\exp} = \mathbb{R}[[\mathfrak{T}_{\exp}]] \supseteq \mathbb{T}$ also

$$\mathfrak{T}_{\exp} = \exp(\mathbb{R}[[\mathfrak{T}]]_{\succ})$$

Closure fails in well-based case

$$\begin{aligned}
f_1 &= x^2 \\
f_2 &= x^2 - \mathrm{e}^{\log^2 x} \\
&\vdots \\
f_\omega &= x^2 - \mathrm{e}^{\log^2 x} - \mathrm{e}^{\log^2 x - \mathrm{e}^{\log\log^2 x}} - \cdots \\
f_{\omega+1} &= x^2 - \mathrm{e}^{\log^2 x} - \mathrm{e}^{\log^2 x - \mathrm{e}^{\log\log^2 x}} - \cdots - \mathrm{e}^{\log^2 x - \mathrm{e}^{\log\log^2 x} - \cdots} \\
&\vdots
\end{aligned}$$

Differentiation

There exists a unique strong exp-log differentiation on $\mathbb{T}$ with $x' = 1$

AD1. $f \prec g \Rightarrow f' \prec g'$, for all $f, g \in \mathbb{T}$ with $g \not\asymp 1$.

AD2. $f \succ 1 \Rightarrow (f > 0 \Rightarrow f' > 0)$, for all $f \in \mathbb{T}$.

Logarithmic transseries

- $\partial \colon \mathfrak{m} = x^{i_0} \cdots (\log_l x)^{i_l} \mapsto \left(\frac{i_0}{x} + \cdots + \frac{i_l}{x \cdots \log_l x}\right) \mathfrak{m}$ is grid-based
- Hence ∂ admits a unique strongly linear extension on $\mathbb{L}$

Exponential extension of strongly linear ∂ on $\mathbb{T} = C[[\mathfrak{T}]]$

- $\partial \colon \mathfrak{m} \in \mathfrak{T}_{\exp} = \mathrm{e}^{\varphi} \mapsto \varphi' \, \mathfrak{m}$ is grid-based
- Hence ∂ admits a unique strongly linear extension on $\mathfrak{T}_{\exp}$

Composition and inversion

Composition and inversion

Given $g \in \mathbb{T}^{>,\succ}$, there exists a unique strong exp-log postcomposition $\delta = \circ_g$ with g such that $\delta x = g$ and

AΔ1. $f \prec 1 \Rightarrow \delta(f) \prec 1$, for all $f \in \mathbb{T}$.

AΔ2. $f \geqslant 0 \Rightarrow \delta(f) \geqslant 0$, for all $f \in \mathbb{T}$.

Taylor rule

$f, \varepsilon \in \mathbb{T}$ with $\delta \prec x$ and $\mathfrak{m}^{\dagger} \delta \prec 1$ for all $\mathfrak{m} \in \operatorname{supp} f$. Then

$$f \circ (x + \delta) = f + f' \delta + \frac{1}{2} f'' \delta^2 + \cdots$$

Inversion

There exists a unique $g^{\mathrm{inv}} \in \mathbb{T}^{>,\succ}$ with $\delta(g^{\mathrm{inv}}) = x$

Possible to compute g^{inv} using "Translagrange formula"

Well-based operators

Well-based operator $\Phi: C[[\mathfrak{M}]] \rightarrow C[[\mathfrak{M}]]$

$$\begin{aligned} \Phi &= \Phi_0 + \Phi_1 + \Phi_2 + \cdots \text{ (strongly)} \\ \Phi_i(f) &= \check{\Phi}_i(f, \ldots, f) \\ \check{\Phi}_i & \quad \text{strongly } i\text{-linear} \end{aligned}$$

Fixed point theorem

If Φ is *strictly extensive*, then

$$f = \Phi(f)$$

admits a unique solution in $C[[\mathfrak{M}]]$

- Requires additional support condition in grid-based case
- Generalizes to equations $f = \Phi(f, g)$, obtaining $f = \Psi(g)$

Examples

Functional equations

$$f = \frac{1}{x} + f(x^2) + f\left(\mathrm{e}^{\log^2 x} + 1\right)$$

Integration

$$\partial \mathfrak{m} = \Delta \mathfrak{m} + R \mathfrak{m},$$

where $\Delta \mathfrak{m} = c_{\mathfrak{m}'} \mathfrak{d}_{\mathfrak{m}'}$. Δ is strictly $\prec$-increasing on $\mathfrak{T} \setminus \{1\}$, whence Δ and Δ^{-1} extend by strong linearity

$$\int = \Delta^{-1} - \Delta^{-1} R \Delta^{-1} + \Delta^{-1} R \Delta^{-1} R \Delta^{-1} + \cdots$$

Asymptotic algebraic equations

Algebra	Asymptotic algebra

Asymptotic algebraic equations

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$P(f)=0$	

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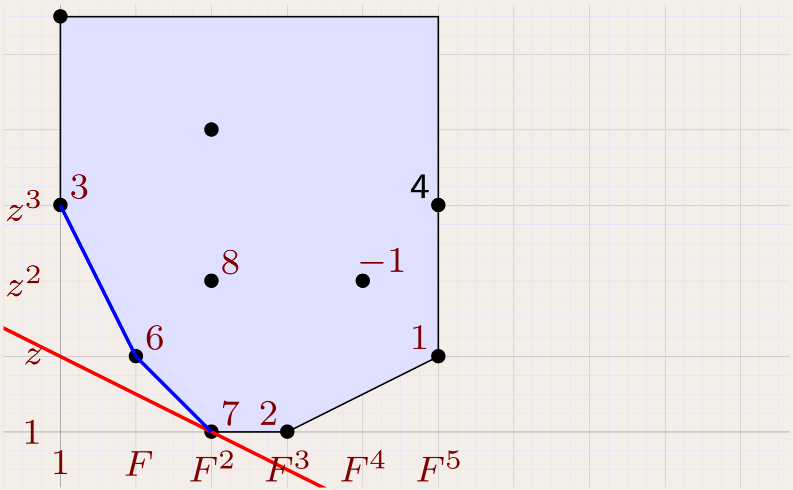
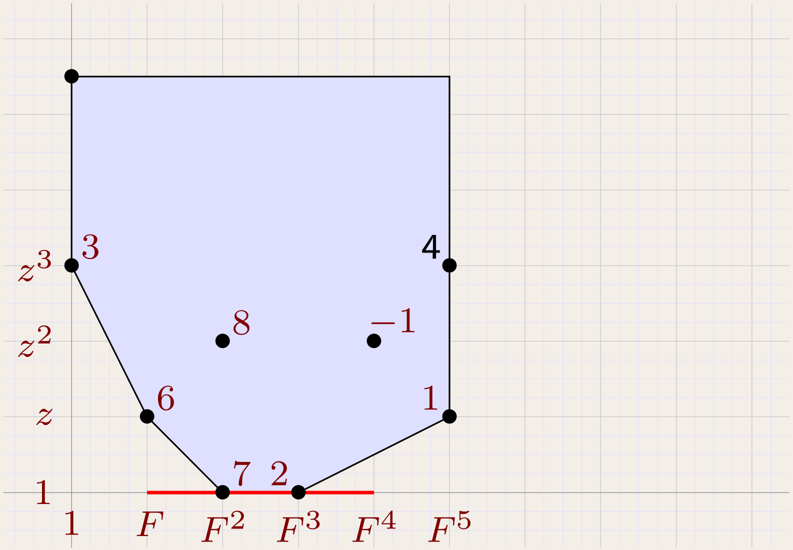


Newton polynomials

- $P \in C[[\mathfrak{M}]][F] \subseteq C[F][[\mathfrak{M}]]$
- $N_P = c_P \in C[F]$

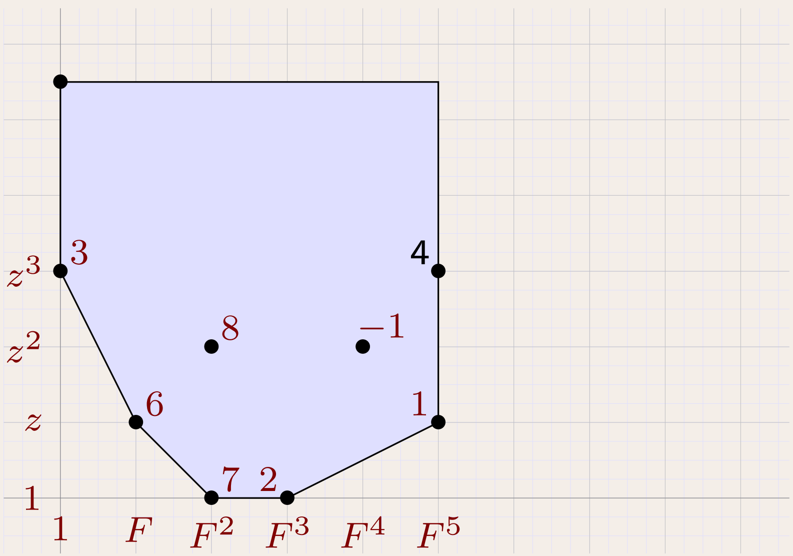


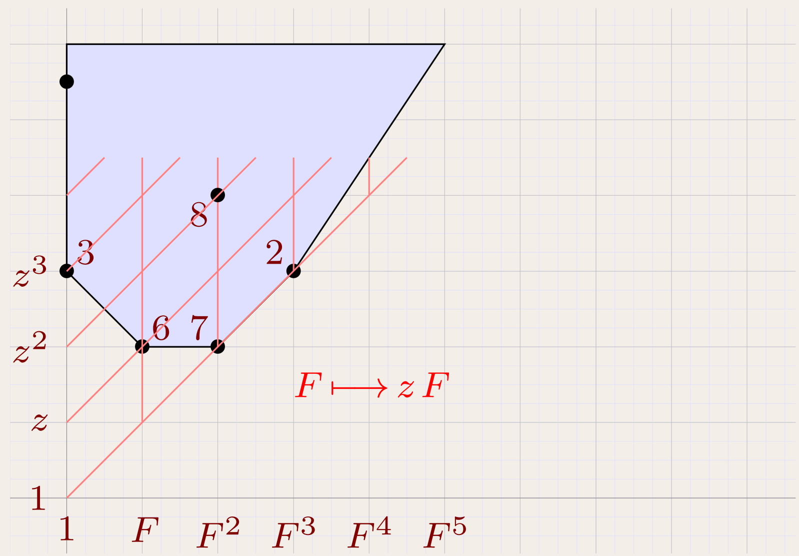
$N_P = 2F^3 + 7F^2$

Starting terms

- $\mathfrak{w} \prec \mathfrak{v}$ is a "starting monomial" $\Longleftrightarrow N_{P_{\times \mathfrak{w}}} \notin C\, F^{\mathbb{N}}$
- $c\,\mathfrak{w}$ is a "starting term" $(c \neq 0) \Longleftrightarrow N_{P_{\times \mathfrak{w}}}(c) = 0$

$$\begin{aligned} P_{\times \varphi}(f) &= P(\varphi\, f) \\ P_{+\varphi}(f) &= P(\varphi + f) \end{aligned}$$





Newton degree

$$\begin{aligned}
\deg_{\preccurlyeq\mathfrak{v}} P &= \deg N_{P_{\times\mathfrak{v}}}\\
\deg_{\prec\mathfrak{v}} P &= \operatorname{val} N_{P_{\times\mathfrak{v}}}
\end{aligned}$$

$$\begin{aligned}
\deg_{\prec\mathfrak{w}} P &\leqslant \deg_{\prec\mathfrak{v}} P, & \mathfrak{w} \prec \mathfrak{v}\\
\deg_{\prec\mathfrak{v}} P_{+\varphi} &= \deg_{\prec\mathfrak{v}} P, & \varphi \prec \mathfrak{v}\\
\deg_{\prec\mathfrak{v}} P_{\times\mathfrak{w}} &= \deg_{\prec\mathfrak{v}\mathfrak{w}} P\\
\deg_{\prec\mathfrak{v}} (PQ) &= \deg_{\prec\mathfrak{v}} P + \deg_{\prec\mathfrak{v}} Q
\end{aligned}$$

$$\begin{aligned}
\deg_{\prec\varphi} P_{+\varphi} &= \mu(c_\varphi; N_{P_{\times\mathfrak{d}_\varphi}})\\
\mu_{\prec\mathfrak{v}}(f;P) &= \deg_{\prec\mathfrak{v}} P_{+f}
\end{aligned}$$

Newton polygon method

1. $\deg_{\prec\mathfrak{v}} P = d > 0$

   ($P = A_{+g}$ and g root modulo $\prec\mathfrak{v}$ of A)

2. If $d = 1$ then unique solution

3. Determine starting monomial $\mathfrak{m} \prec \mathfrak{v}$

4. Solve $N_{P_{\times\mathfrak{m}}}(c) = 0$ and set $\varphi := c\,\mathfrak{m}$

5. Refine $f = \varphi + \tilde{f}, \tilde{f} \prec \mathfrak{m} \longrightarrow 0 < \deg_{\prec\mathfrak{m}} \tilde{P} \leqslant d$ with $\tilde{P} = P_{+\varphi}$

   ($\tilde{P} = A_{+g+\varphi}$ and $g + \varphi$ root modulo $\prec\mathfrak{m}$ of A)

6. Return to step 1

Newton polygon method

1. $\deg_{\prec \mathfrak{v}} P = d > 0$

   ($P = A_{+g}$ and g root modulo $\prec \mathfrak{v}$ of A)

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   $$\left(f - \frac{1}{1-z} \right)^2 = z^{10000}$$

5. Refine $f = \varphi + \tilde{f}, \tilde{f} \prec \mathfrak{w} \longrightarrow 0 < \deg_{\prec \mathfrak{w}} \tilde{P} \leqslant d$ with $\tilde{P} = P_{+\varphi}$

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Newton polygon method

1. $\deg_{\prec \mathfrak{v}} P = d > 0$

   ($P = A_{+g}$ and g root modulo $\prec \mathfrak{v}$ of A)

2. If $d = 1$ then unique solution

3. Determine starting monomial $\mathfrak{m} \prec \mathfrak{v}$

4. Solve $N_{P_{\times \mathfrak{m}}}(c) = 0$ and set $\varphi := c\,\mathfrak{m}$

   If $\mu_{N_{P_\times}}(c) = d$, then $\varphi :=$ unique solution to $\frac{\partial^{d-1} P}{\partial F^{d-1}}(\varphi) = 0, \varphi \prec \mathfrak{v}$

5. Refine $f = \varphi + \tilde{f}, \tilde{f} \prec \mathfrak{m} \longrightarrow 0 < \deg_{\prec \mathfrak{m}} \tilde{P} \leqslant d$ with $\tilde{P} = P_{+\varphi}$

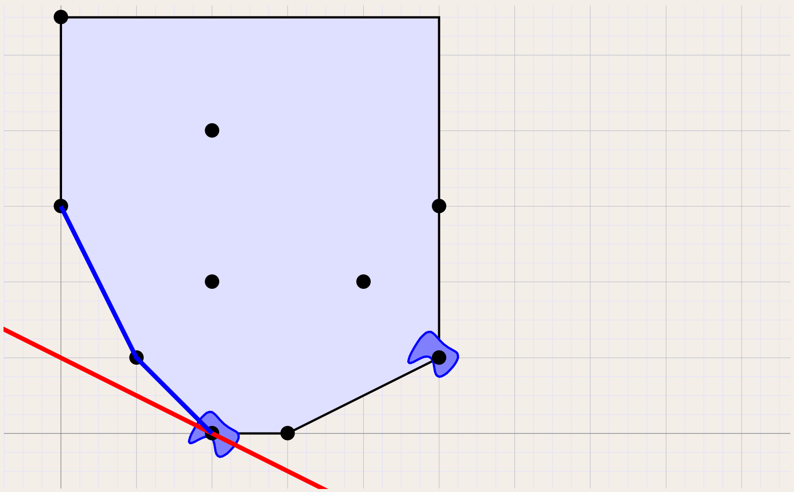
   ($\tilde{P} = A_{+g+\varphi}$ and $g + \varphi$ root modulo $\prec \mathfrak{m}$ of A)

6. Return to step 1

Differential Newton polygon method

$$P(f) = p(f, f', \ldots, f^{(r)}) = 0, \quad f \prec \mathfrak{v}$$

Starting monomials cannot directly be read of from “Newton polygon”



$$P = P_0 + \cdots + P_d$$

Upward shifting

$P{\uparrow}$ unique differential polynomial with

$$(P{\uparrow})(f \circ \mathrm{e}^x) = P(f) \circ \mathrm{e}^x$$

For instance:

$$\begin{aligned}
F'{\uparrow} &= \frac{F'}{\mathrm{e}^x}\\
F''{\uparrow} &= \frac{F'' - F'}{\mathrm{e}^{2x}}\\
F'''{\uparrow} &= \frac{F''' - 3F'' + 2F'}{\mathrm{e}^{3x}}\\
&\vdots
\end{aligned}$$

Differential Newton polynomial

Theorem. *There exists a unique* $N_P \in \mathbb{R}\{F\}$*, such that*

$$c_{P \uparrow_l} = N_P$$

for all sufficiently large l *and*

$$N_P \in \mathbb{R}[F]\,(F')^{\mathbb{N}}.$$

Definition. $\mathfrak{m} \prec \mathfrak{v}$ *is a starting monomial* $\Longleftrightarrow N_{P_{\times \mathfrak{m}}} \notin \mathbb{R}\,F^{\mathbb{N}}$

Example

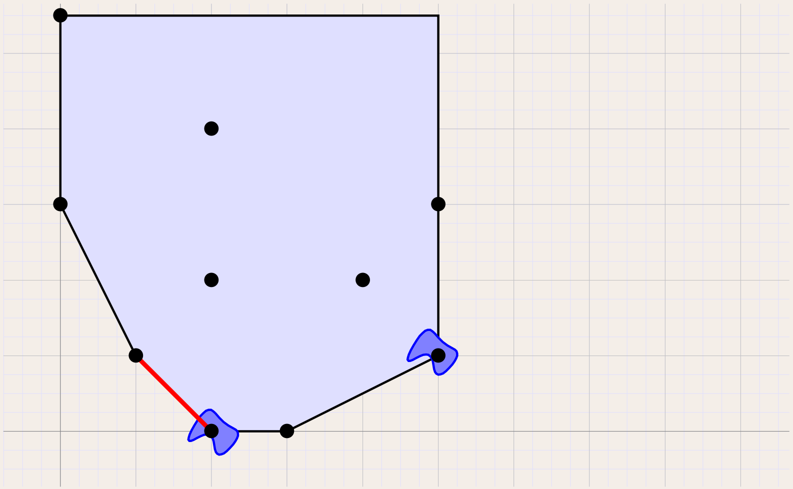
$$
\begin{aligned}
P &= (F')^2 - FF'' \\
P\!\uparrow &= \frac{(F')^2 - FF'' + FF'}{\mathrm{e}^{2x}} \\
P\!\uparrow\uparrow &= \frac{FF'}{\mathrm{e}^{x}\,\mathrm{e}^{2\mathrm{e}^{x}}} + \frac{(F')^2 - FF'' + FF'}{\mathrm{e}^{2x}\,\mathrm{e}^{2\mathrm{e}^{x}}} \\
&\vdots \\
N_P &= FF'
\end{aligned}
$$

Consequence:

$$1 \prec L \prec \log_n x \Longrightarrow P(L) \sim \frac{L\,L'}{x}$$

Starting monomials

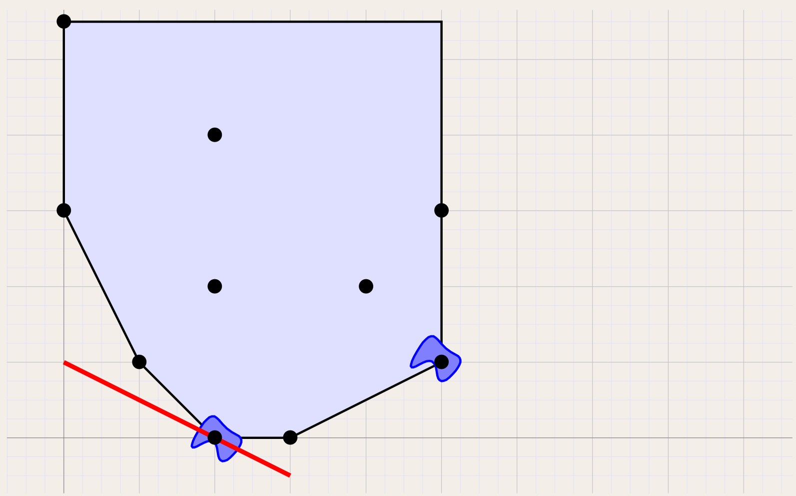
Lemma. *Given $i < j$ with $P_i \neq 0$, $P_j \neq 0$, there exists a unique (i, j)-equalizer $\mathfrak{e} \in \mathfrak{T}$ such that $N_{(P_i + P_j)_{\times \mathfrak{e}}}$ is not homogeneous.*



Starting monomials

Lemma. *Given i with $P_i \neq 0$, we have*

$$\mathfrak{m} \textit{ is a starting monomial for } P_i(f) = 0$$
$$\Updownarrow$$
$$\mathfrak{m}^\dagger = \frac{\mathfrak{m}'}{\mathfrak{m}} \textit{ is a solution to } R_{P_i}(g) = 0 \textit{ modulo } \frac{1}{x \log x \log_2 x \cdots}$$



Solving asymptotic differential equations

Lemma. $\deg_{\prec \mathfrak{v}} P = 1 \Longrightarrow P(f) = 0,\ f \preccurlyeq \mathfrak{v}$ *admits at least one solution.*

Warning. Problem with almost multiple solutions

$$\begin{aligned}
f^2 - 2\, f' + \frac{1}{x^2} + \cdots + \frac{1}{(x \log x \cdots \log_l x)^2} &= 0, \quad (f \prec 1)\\
f^2 - 2\, \mathrm{e}^{-x}\, f' + \frac{1}{\mathrm{e}^{2x}} + \cdots + \frac{1}{(\mathrm{e}^x\, x \cdots \log_{l-1} x)^2} &= 0, \quad (f \prec 1)\\
f^2 - 2\, f' - 2\, f + 1 + \frac{1}{x^2} + \cdots + \frac{1}{(x \log x \cdots \log_{l-1} x)^2} &= 0, \quad (f \prec 1)\\
f^2 - 2\, f' + \frac{1}{x^2} + \cdots + \frac{1}{(x \log x \cdots \log_{l-1} x)^2} &= 0, \quad (f \prec 1)
\end{aligned}$$

Lemma. *"Unravelling process" is finite for grid-based transseries.*

Results

Theorem. *(1997) There exists a theoretical algorithm to find all solutions to an asymptotic algebraic differential equation.*

Theorem. *(1997) Let P be purely exponential of degree d and order r. There exists a constant $C_{r,d}$ such that any solution to $P(f)=0$ involves at most $C_{r,d}$ levels of iterated logarithms.*

Theorem. *(1997) Any general transseries solution to an algebraic differential equation with grid-based coefficients is again grid-based. Generalization of Grigoriev and Singer (1991).*

Corollary. $\zeta(x)$ *and* $f(x)=\frac{1}{x}+\frac{1}{\mathrm{e}^{\log^2 x}}+\frac{1}{\mathrm{e}^{\log^4 x}}+\cdots$ *are differentially transcendental over* $\mathbb{R}$.