# Efficient certification of numeric solutions to eigenproblems

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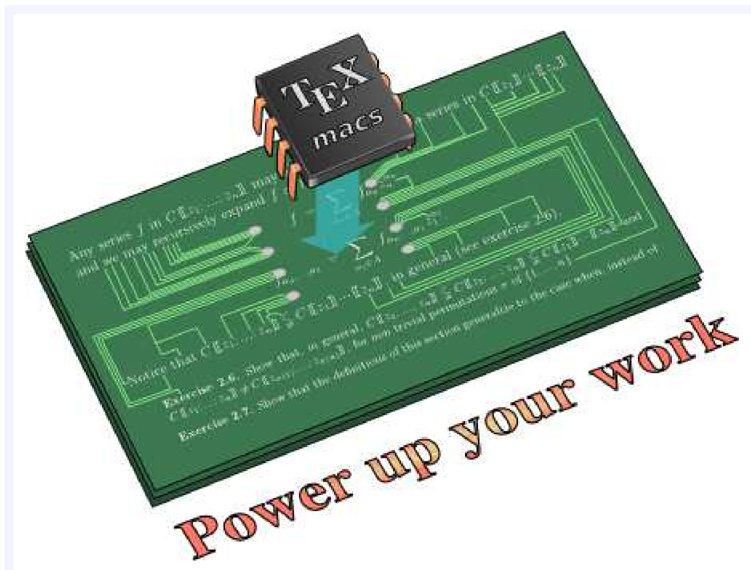


**Vienna, November 15, 2017**

http://www.TeXmacs.org

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$$\Lambda_k = \begin{pmatrix} \lambda_{i_k} & & \\ & \ddots & \\ & & \lambda_{i_{k+1}-1} \end{pmatrix} + E_k, \qquad \lambda_{i_k} \approx \dots \approx \lambda_{i_{k+1}-1}, \qquad E_k \text{ “small”}$$

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$$M = \begin{pmatrix} M_{1,1} & \cdots & M_{1,n} \\ \vdots & & \vdots \\ M_{n,1} & \cdots & M_{n,n} \end{pmatrix} = \begin{pmatrix} \mathscr{B}(c_{1,1}, r_{1,1}) & \cdots & \mathscr{B}(c_{1,n}, r_{1,n}) \\ \vdots & & \vdots \\ \mathscr{B}(c_{n,1}, r_{n,1}) & \cdots & \mathscr{B}(c_{n,n}, r_{n,n}) \end{pmatrix}$$

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such that for every $M \in \boldsymbol{M}$ (e.g. $|M_{i,j} - c_{i,j}| \leqslant r_{i,j}$), there exists a $T \in \boldsymbol{T}$ with

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*Given an approximate numerical solution to an eigenproblem, find an efficient algorithm for certifying the solution.*

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Only use a finite number of matrix multiplications at the current working precision.

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## Asymptotic complexity

Two $n \times n$ matrices with $p$-bit integer entries can be multiplied in time

$$\mathrm{MM}(n,p) = O(n^2\,\mathrm{I}(p) + n^{\omega}\,p\,2^{O(\lg^* p - \lg^* n)}\,\mathrm{I}(\lg n)/\lg n) \quad \text{[Harvey–vdH 2014].}$$

$\mathrm{I}(p)$: cost of $p$-bit integer multiplication, $O(n^{\omega})$: cost of $n \times n$ matrix multiplication

$$\mathrm{I}(p) = O(p \lg p\, K^{\lg^* p}), \quad K \leqslant \begin{array}{ll} 6 & \text{[Harvey 2017]} \\ 4^* & \text{[Harvey–vdH–Lecerf 2014]} \end{array}$$

$$\omega < 2.3728639 \quad \text{[LeGall 2014]}$$

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## Practical complexity

Two $n \times n$ matrices with $p$-bit integer entries can be multiplied in time

$$\mathrm{MM}(n, p) = O(n^3 p).$$

**Individual eigenvectors for single eigenvalues** $O(n^4 p)$

- T. Yamamoto 1980
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## Individual eigenvectors for multiple eigenvalues $O(n^4 p)$

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**All eigenvectors, no multiple eigenvalues** $O(n^3 p)$

- vdH–Mourrain–Trébuchet 2009
- Miyajima 2010 (eigenvalues only)

Assume that

$$M = D + H, \qquad D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

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We thus obtain a simple approximate solution

$$\Delta_{i,i} = H_{i,i}$$

$$E_{i,j} = \frac{H_{i,j}}{\lambda_j - \lambda_i}, \qquad (i \neq j)$$

We have reduced our problem

$$T^{-1}(D+H)\,T \;=\; \text{diagonal matrix}$$
$$\downarrow$$
$$(T')^{-1}(D'+H')\,T' \;=\; \text{diagonal matrix}',$$

where

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**Theorem.** *This "Newton iteration" converges quadratically to a solution.*

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**Theorem.** *We can bound $\|T-1\|$ as a function of $\|D\|$, $\|H\|$ and $\sigma := \max_{i\neq j}|\lambda_i-\lambda_j|^{-1}$.*

## Clustering

Partition $\{1, \ldots, n\} = I_1 \amalg \cdots \amalg I_p$ and define $i \sim j \Longleftrightarrow (\exists k, \{i, j\} \subseteq I_k)$
Idea: $i \sim j \Longleftrightarrow (|\lambda_i - \lambda_j|$ is small$)$
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## Modified Newton iteration

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E_{i,j} &= \begin{cases} H_{i,j} / (\lambda_j - \lambda_i) & \text{if } i \nsim j \\ 0 & \text{if } i \sim j \end{cases} \\
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$$\begin{pmatrix} 1.000 & 0.000 & 0.000 \\ 0.000 & 1.000 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{pmatrix} \begin{pmatrix} 2.347 & 0.462 & 0.081 \\ 0.397 & 2.121 & 0.235 \\ 0.959 & 0.097 & 5.559 \end{pmatrix} \begin{pmatrix} 1.000 & 0.000 & 0.000 \\ 0.000 & 1.000 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{pmatrix}$$

$$=$$

$$\begin{pmatrix} 2.347 & 0.462 & 0.081 \\ 0.397 & 2.121 & 0.235 \\ 0.959 & 0.097 & 5.559 \end{pmatrix}$$

$$\begin{pmatrix} 0.993 & -0.001 & -0.025 \\ -0.020 & 0.998 & -0.068 \\ 0.296 & 0.028 & 0.991 \end{pmatrix} \begin{pmatrix} 2.347 & 0.462 & 0.081 \\ 0.397 & 2.121 & 0.235 \\ 0.959 & 0.097 & 5.559 \end{pmatrix} \begin{pmatrix} 1.000 & 0.000 & 0.025 \\ 0.000 & 1.000 & 0.068 \\ -0.299 & -0.028 & 1.000 \end{pmatrix}$$

$$=$$

$$\begin{pmatrix} 2.323 & 0.456 & 0.031 \\ 0.326 & 2.105 & 0.007 \\ 0.002 & 0.136 & 5.599 \end{pmatrix}$$

$$\begin{pmatrix} 0.990 & -0.001 & -0.034 \\ -0.021 & 0.998 & -0.070 \\ 0.296 & 0.067 & 0.988 \end{pmatrix} \begin{pmatrix} 2.347 & 0.462 & 0.081 \\ 0.397 & 2.121 & 0.235 \\ 0.959 & 0.097 & 5.559 \end{pmatrix} \begin{pmatrix} 1.000 & -0.001 & 0.036 \\ -0.000 & 0.997 & 0.070 \\ -0.299 & -0.067 & 0.997 \end{pmatrix}$$

$$=$$

$$\begin{pmatrix} 2.3226 & 0.4551 & 0.0009 \\ 0.3263 & 2.1049 & 0.0030 \\ 0.0127 & 0.0003 & 5.5990 \end{pmatrix}$$

$$\begin{pmatrix} 0.990 & -0.001 & -0.035 \\ -0.021 & 0.998 & -0.071 \\ 0.299 & 0.067 & 0.988 \end{pmatrix} \begin{pmatrix} 2.347 & 0.462 & 0.081 \\ 0.397 & 2.121 & 0.235 \\ 0.959 & 0.097 & 5.559 \end{pmatrix} \begin{pmatrix} 1.000 & -0.001 & 0.035 \\ -0.000 & 0.997 & 0.071 \\ -0.303 & -0.067 & 0.997 \end{pmatrix}$$

$$=$$

$$\begin{pmatrix} 2.32259 & 0.45513 & 0.00040 \\ 0.32627 & 2.10488 & 0.00009 \\ 0.00003 & 0.00176 & 5.59903 \end{pmatrix}$$

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$$=$$

$$\begin{pmatrix} 2.322594 & 0.455127 & 0.000012 \\ 0.326266 & 2.104883 & 0.000039 \\ 0.000165 & 0.000003 & 5.599032 \end{pmatrix}$$

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$$T^{-1} M T \approx D$$

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$$\begin{aligned} \boldsymbol{M} &= \mathscr{B}(M, r) \\ T^{-1} M\, T &\approx D \end{aligned}$$

## Consider

$$\begin{aligned} \boldsymbol{D} &= T^{-1} \boldsymbol{M}\, T \\ \boldsymbol{H} &= \boldsymbol{D} - D \end{aligned}$$

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$$\mathbf{D} = T^{-1} \mathbf{M} T$$
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**Apply theorem** $\Longrightarrow$ bound $\rho > 0$ such that $\forall H \in \mathbf{H}$, $\exists E$ with $\|E\| \leqslant \rho$ and

$$(1 + E)^{-1} (D + H) (1 + E) = \text{block diagonal matrix.}$$

Let $\mathbf{E} = \mathscr{B}(0, \rho)$ be the corresponding ball enclosure for $E$.

## Given

$$\mathbf{M} = \mathscr{B}(M, r)$$
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$$\mathbf{H} = \mathbf{D} - D$$

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Let $\mathbf{E} = \mathscr{B}(0, \rho)$ be the corresponding ball enclosure for $E$.

## Conclude

$$\mathbf{T} = T (1 + \mathbf{E})$$

## High precision

It would be more efficient to have quadratic convergence.

This can be achieved by solving the commutation equations exactly:

$$\begin{array}{rcl} DE - ED & \approx & \Delta - H \\ & \Downarrow & \\ D_{[i]} E_{[i,j]} - E_{[i,j]} D_{[j]} & = & -H_{[i,j]} \end{array}$$

## Certification

We already assume an approximate numeric solution with full precision.

$\Longrightarrow$ linear convergence sufficient for certification

$\Longrightarrow$ more efficient iteration