Transserial Hardy fields

Joris van der Hoeven

CNRS, France

(partly joint work with M. Aschenbrenner and L. van den Dries)

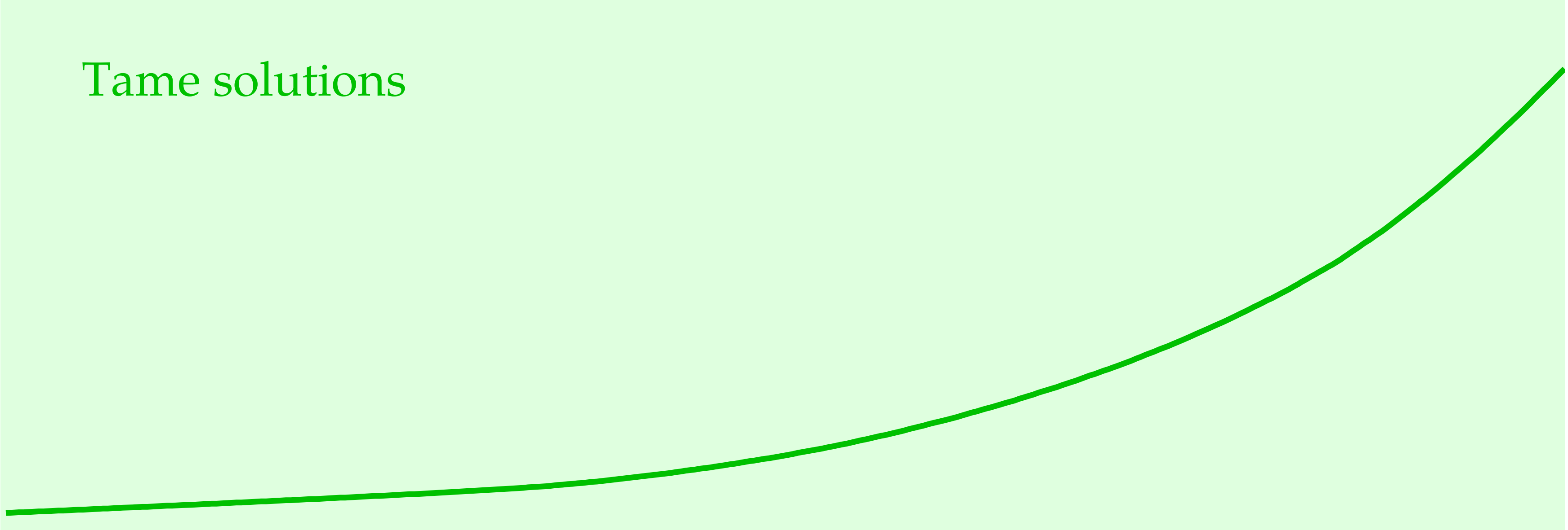
ICM 2018
RIO DE JANEIRO

Motivation

$$y(x)^7 + e^{e^x} y(x)^4 y'(x)\, y''(x) - \frac{y'''(x)}{\log x} = e^{x^2 \sqrt{\log x}} \qquad (x \to \infty)$$

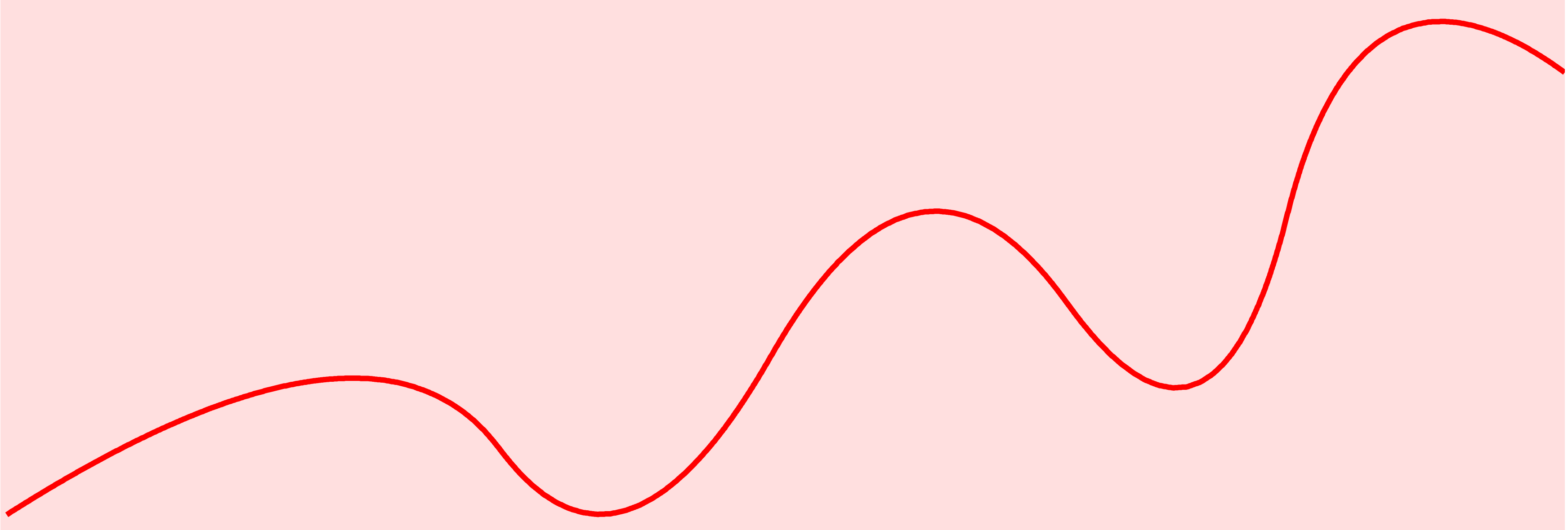
Motivation

$$y(x)^7 + \mathrm{e}^{\mathrm{e}^x} y(x)^4 y'(x)\, y''(x) - \frac{y'''(x)}{\log x} = \mathrm{e}^{x^2 \sqrt{\log x}} \qquad (x \to \infty)$$



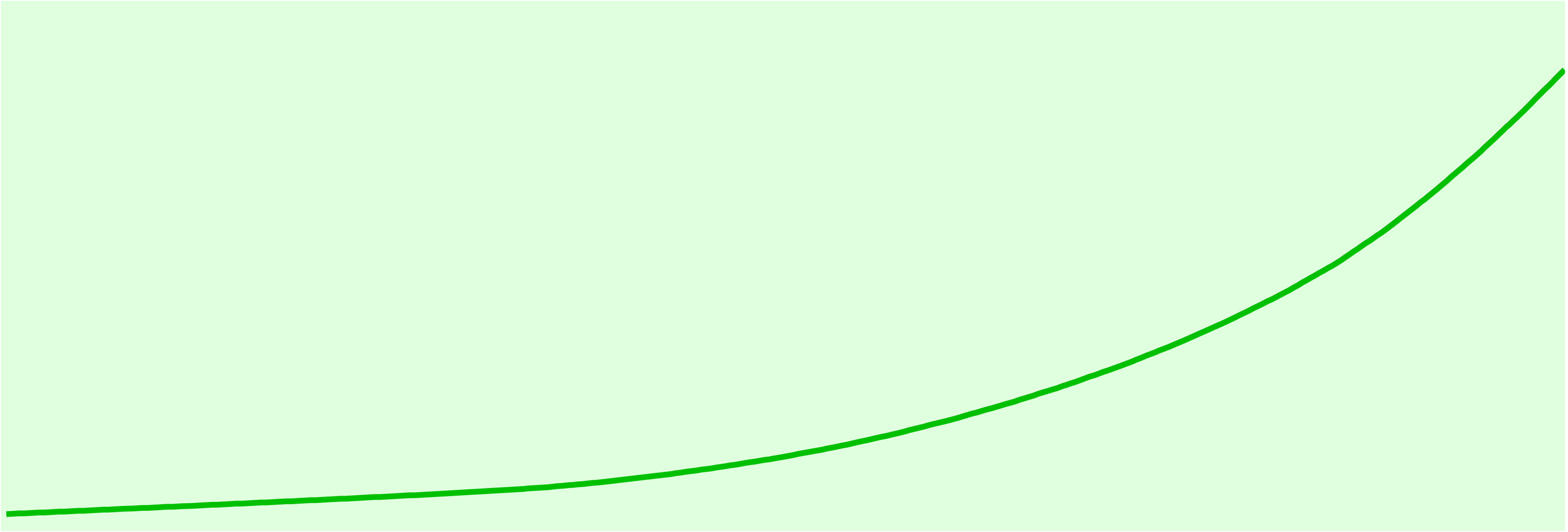
Motivation

$$y(x)^7 + e^{e^x} y(x)^4 y'(x)\, y''(x) - \frac{y'''(x)}{\log x} = e^{x^2 \sqrt{\log x}} \qquad (x \to \infty)$$



Motivation

$$y(x)^7 + \mathrm{e}^{\mathrm{e}^x} y(x)^4 y'(x)\, y''(x) - \frac{y'''(x)}{\log x} = \mathrm{e}^{x^2 \sqrt{\log x}} \qquad (x \to \infty)$$

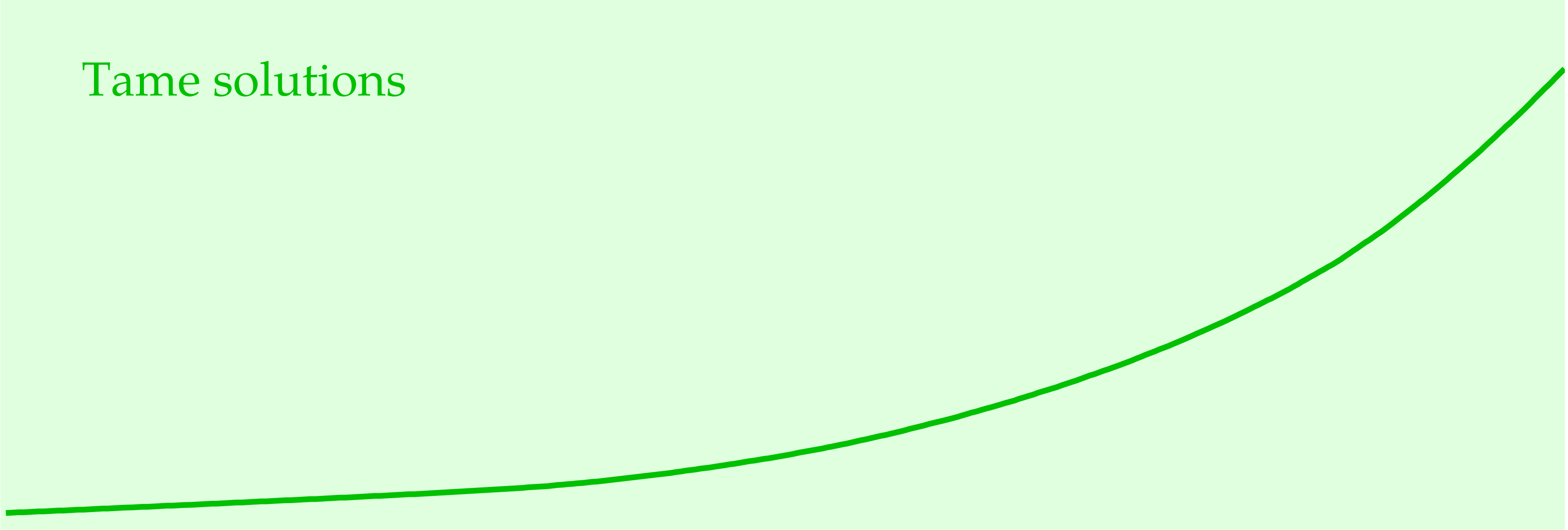


Motivation

$$y(x)^7 + e^{e^x} y(x)^4 y'(x) y''(x) - \frac{y'''(x)}{\log x} = e^{x^2 \sqrt{\log x}} \qquad (x \to \infty)$$

Motivation

$$y(x)^7 + \mathrm{e}^{\mathrm{e}^x} y(x)^4 y'(x)\, y''(x) - \frac{y'''(x)}{\log x} = \mathrm{e}^{x^2 \sqrt{\log x}} \qquad (x \to \infty)$$



Motivation

$$y(x)^7 + \mathrm{e}^{\mathrm{e}^x} y(x)^4 y'(x)\, y''(x) - \frac{y'''(x)}{\log x} = \mathrm{e}^{x^2 \sqrt{\log x}} \qquad (x \to \infty)$$

Transseries

- Formal solutions

Motivation

$$y(x)^7 + e^{e^x} y(x)^4 y'(x)\, y''(x) - \frac{y'''(x)}{\log x} = e^{x^2 \sqrt{\log x}} \qquad (x \to \infty)$$

Transseries

- Formal solutions
- Algorithm to compute solutions

Motivation

$$y(x)^7 + e^{e^x} y(x)^4 y'(x)\, y''(x) - \frac{y'''(x)}{\log x} = e^{x^2 \sqrt{\log x}} \qquad (x \to \infty)$$

Transseries

- Formal solutions
- Algorithm to compute solutions

Hardy fields

- Analytic solutions

Motivation

$$y(x)^7 + e^{e^x} y(x)^4 y'(x)\, y''(x) - \frac{y'''(x)}{\log x} = e^{x^2 \sqrt{\log x}} \qquad (x \to \infty)$$

Transseries

- Formal solutions
- Algorithm to compute solutions

Hardy fields

- Analytic solutions
- Less effective

Motivation

$$y(x)^7 + e^{e^x} y(x)^4 y'(x) y''(x) - \frac{y'''(x)}{\log x} = e^{x^2 \sqrt{\log x}} \qquad (x \to \infty)$$

Transseries

- Formal solutions
- Algorithm to compute solutions

Hardy fields

- Analytic solutions
- Less effective

Central question

Let $\mathbb{T}^{\mathrm{da}(\mathbb{R})}$ be the field of differentially algebraic transseries over $\mathbb{R}$.

How to "incarnate" $\mathbb{T}^{\mathrm{da}(\mathbb{R})}$ as a Hardy field?

Approach I: Borel summation

$$\tilde{f}(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(n-1)!}{x^n}$$

Approach I: Borel summation

$$\tilde{f}(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(n-1)!}{x^n}$$

$\tilde{\mathcal{B}}$

Formal Borel transform

$$\hat{f}(\zeta) = \sum_{n=1}^{\infty} (-1)^{n-1} \zeta^{n-1} = \frac{1}{1+\zeta}$$

Approach I: Borel summation

$$\tilde{f}(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(n-1)!}{x^n} \dashrightarrow f(x) = \int_0^{\infty} \frac{e^{-\zeta x}}{1+\zeta} d\zeta$$

$\tilde{\mathcal{B}}$ Formal Borel transform

$\mathcal{L}$ Analytic Laplace transform

$$\hat{f}(\zeta) = \sum_{n=1}^{\infty} (-1)^{n-1} \zeta^{n-1} = \frac{1}{1+\zeta}$$

Approach I: Borel summation

$$\tilde{f}(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(n-1)!}{x^n} \dashrightarrow f(x) = \int_0^{\infty} \frac{e^{-\zeta x}}{1+\zeta} d\zeta$$

$\tilde{\mathcal{B}}$

Formal Borel transform

$\mathcal{L}$

Analytic Laplace transform

$$\hat{f}(\zeta) = \sum_{n=1}^{\infty} (-1)^{n-1} \zeta^{n-1} = \frac{1}{1+\zeta}$$

Problem: how to proceed when $\tilde{f}(x) \longrightarrow \tilde{f}(x) + \tilde{f}(x^2)\,\tilde{f}(x^3) - \tilde{f}(x^x)$?

Approach I: accelero-summation (ÉCALLE)

$\tilde{f}(x)$

Approach I: accelero-summation (ÉCALLE)

$$\tilde{f}(x)$$

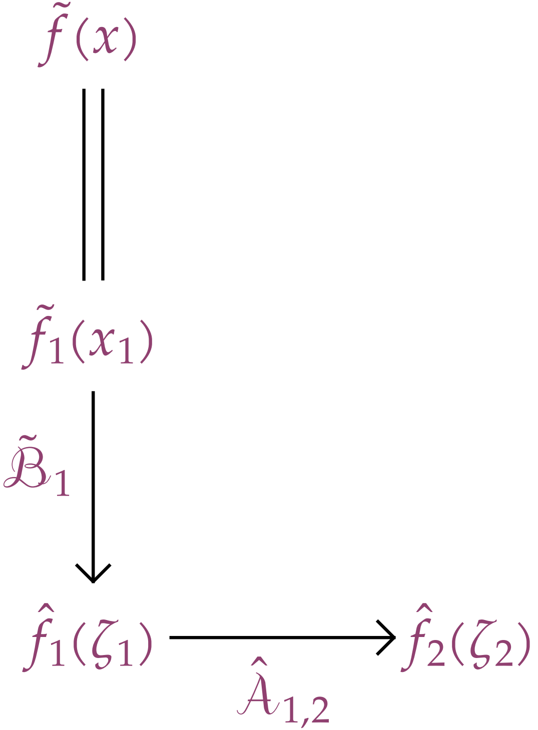
$$\Vert$$

$$\tilde{f}_1(x_1)$$

Approach I: accelero-summation (ÉCALLE)

$$\begin{array}{c} \tilde{f}(x) \\ \Big\| \\ \tilde{f}_1(x_1) \\ \tilde{\mathcal{B}}_1 \Big\downarrow \\ \hat{f}_1(\zeta_1) \end{array}$$

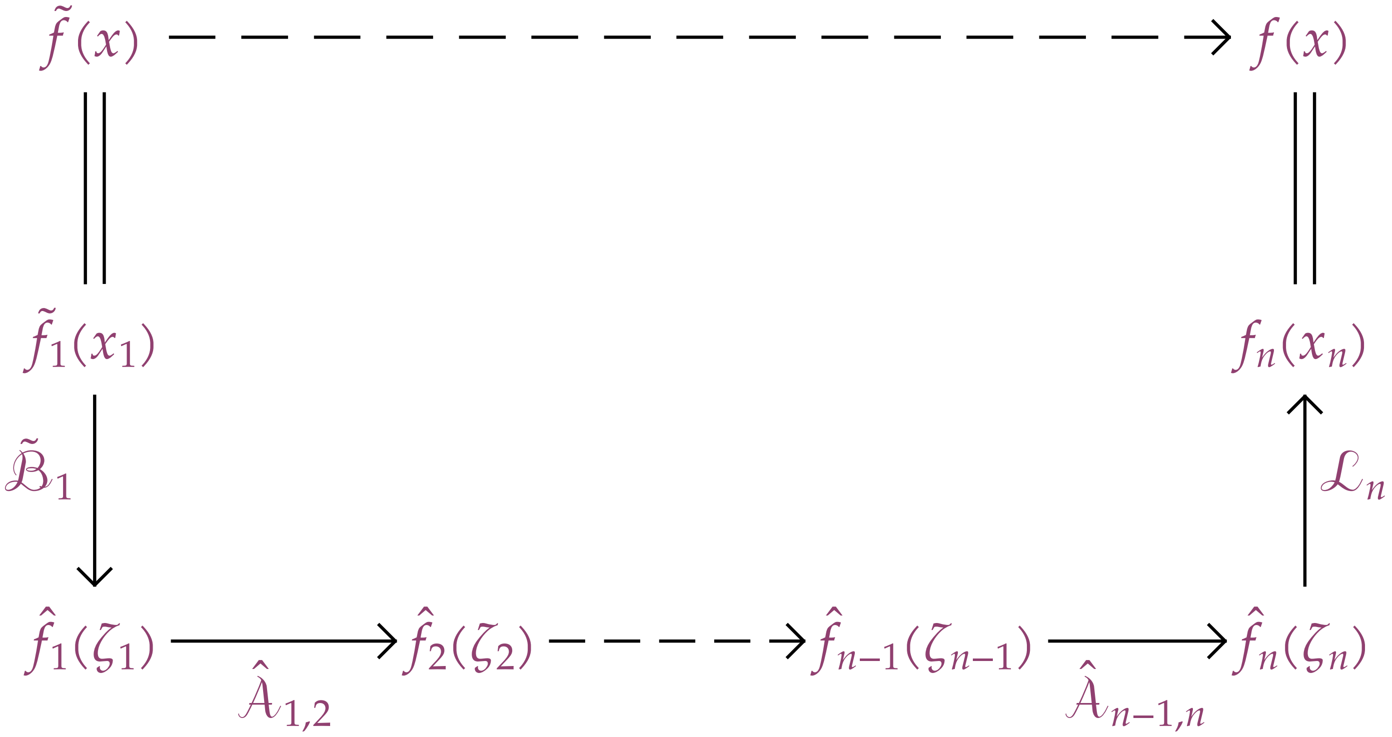
Approach I: accelero-summation (ÉCALLE)



Approach I: accelero-summation (Écalle)

$$
\begin{array}{ccccccc}
\tilde{f}(x) & & & & & & \\
\Vert & & & & & & \\
\tilde{f}_1(x_1) & & & & & & \\
\tilde{\mathcal{B}}_1 \downarrow & & & & & & \\
\hat{f}_1(\zeta_1) & \xrightarrow[\hat{\mathcal{A}}_{1,2}]{} & \hat{f}_2(\zeta_2) & \dashrightarrow & \hat{f}_{n-1}(\zeta_{n-1}) & \xrightarrow[\hat{\mathcal{A}}_{n-1,n}]{} & \hat{f}_n(\zeta_n)
\end{array}
$$

Approach I: accelero-summation (ÉCALLE)



Approach II: model-theoretic (sketch)

Input: $\mathbb{K} \subseteq \mathbb{T}^{\mathrm{da}(\mathbb{R})}$ with $\phi\colon \mathbb{K} \hookrightarrow H$ for HARDY field H

Approach II: model-theoretic (sketch)

Input: $\mathbb{K} \subseteq \mathbb{T}^{\mathrm{da}(\mathbb{R})}$ with $\phi\colon \mathbb{K} \hookrightarrow H$ for HARDY field H

while $\mathbb{K} \neq \mathbb{T}^{\mathrm{da}(\mathbb{R})}$ **do**

Approach II: model-theoretic (sketch)

Input: $\mathbb{K} \subseteq \mathbb{T}^{\mathrm{da}(\mathbb{R})}$ with $\phi \colon \mathbb{K} \hookrightarrow H$ for HARDY field H

while $\mathbb{K} \neq \mathbb{T}^{\mathrm{da}(\mathbb{R})}$ **do**

Pick $y \in \mathbb{T}^{\mathrm{da}(\mathbb{R})} \setminus \mathbb{K}$ of minimal *complexity*

Approach II: model-theoretic (sketch)

Input: $\mathbb{K} \subseteq \mathbb{T}^{\mathrm{da}(\mathbb{R})}$ with $\phi: \mathbb{K} \hookrightarrow H$ for HARDY field H

while $\mathbb{K} \neq \mathbb{T}^{\mathrm{da}(\mathbb{R})}$ **do**

Pick $y \in \mathbb{T}^{\mathrm{da}(\mathbb{R})} \setminus \mathbb{K}$ of minimal *complexity*, i.e. $P(y, \ldots, y^{(r)}) = 0$ with

- P of minimal order r;
- P of minimal degree d in $y^{(r)}$;
- P of minimal degree e in $y, \ldots, y^{(r)}$.

Approach II: model-theoretic (sketch)

Input: $\mathbb{K} \subseteq \mathbb{T}^{\mathrm{da}(\mathbb{R})}$ with $\phi\colon \mathbb{K} \hookrightarrow H$ for HARDY field H

while $\mathbb{K} \neq \mathbb{T}^{\mathrm{da}(\mathbb{R})}$ **do**

Pick $y \in \mathbb{T}^{\mathrm{da}(\mathbb{R})} \setminus \mathbb{K}$ of minimal *complexity*, i.e. $P(y,\ldots,y^{(r)}) = 0$ with

- P of minimal order r;
- P of minimal degree d in $y^{(r)}$;
- P of minimal degree e in $y,\ldots,y^{(r)}$.

Construct $\hat{\phi}\colon \mathbb{K}(y,\ldots,y^{(r)}) \hookrightarrow \hat{H}$

Set $\mathbb{K} := \mathbb{K}(y,\ldots,y^{(r)})$, $\phi := \hat{\phi}$

Approach II: model-theoretic (sketch)

Input: $\mathbb{K} \subseteq \mathbb{T}^{\mathrm{da}(\mathbb{R})}$ with $\phi: \mathbb{K} \hookrightarrow H$ for HARDY field H

while $\mathbb{K} \neq \mathbb{T}^{\mathrm{da}(\mathbb{R})}$ **do**

Pick $y \in \mathbb{T}^{\mathrm{da}(\mathbb{R})} \setminus \mathbb{K}$ of minimal *complexity*, i.e. $P(y, \ldots, y^{(r)}) = 0$ with

- P of minimal order r;
- P of minimal degree d in $y^{(r)}$;
- P of minimal degree e in $y, \ldots, y^{(r)}$.

Construct $\hat{\phi}: \mathbb{K}(y, \ldots, y^{(r)}) \hookrightarrow \hat{H}$

Set $\mathbb{K} := \mathbb{K}(y, \ldots, y^{(r)})$, $\phi := \hat{\phi}$

....**ZORN**

Approach II: model-theoretic (sketch)

Input: $\mathbb{K} \subseteq \mathbb{T}^{\mathrm{da}(\mathbb{R})}$ with $\phi: \mathbb{K} \hookrightarrow H$ for HARDY field H

while $\mathbb{K} \neq \mathbb{T}^{\mathrm{da}(\mathbb{R})}$ **do**

Pick $y \in \mathbb{T}^{\mathrm{da}(\mathbb{R})} \setminus \mathbb{K}$ of minimal *complexity*, i.e. $P(y, \ldots, y^{(r)}) = 0$ with

- P of minimal order r;
- P of minimal degree d in $y^{(r)}$;
- P of minimal degree e in $y, \ldots, y^{(r)}$.

Construct $\hat{\phi}: \mathbb{K}(y, \ldots, y^{(r)}) \hookrightarrow \hat{H}$

Set $\mathbb{K} := \mathbb{K}(y, \ldots, y^{(r)})$, $\phi := \hat{\phi}$

....**ZORN**

Output: $\tilde{\phi}: \mathbb{T}^{\mathrm{da}(\mathbb{R})} \hookrightarrow \tilde{H}$ for HARDY field extension $\tilde{H}$ of original H

$\mathbb{T}$:= closure of $\mathbb{R} \cup \{x\}$ under exp, log and infinite summation

$$e^{e^{x}+\cdots} - 3\,e^{x^2} + 5\,(\log x)^{\pi} + 42 + x^{-1} + \cdots + e^{-x}$$

Here one should think of x as a positive infinite indeterminate.

Transseries

$\mathbb{T} :=$ closure of $\mathbb{R} \cup \{x\}$ under exp, log and infinite summation

$$\mathrm{e}^{\mathrm{e}^{x}+\mathrm{e}^{x/2}+\cdots} - 3\,\mathrm{e}^{x^2} + 5\,(\log x)^{\pi} + 42 + x^{-1} + 2\,x^{-2} + \cdots + \mathrm{e}^{-x}$$

Here one should think of x as a positive infinite indeterminate.

Transseries

$\mathbb{T} :=$ closure of $\mathbb{R} \cup \{x\}$ under $\exp$, $\log$ and infinite summation

$$e^{e^x + e^{x/2} + e^{x/3} + \cdots} - 3\,e^{x^2} + 5\,(\log x)^{\pi} + 42 + x^{-1} + 2\,x^{-2} + 6\,x^{-3} + \cdots + e^{-x}$$

Here one should think of x as a positive infinite indeterminate.

$\mathbb{T} :=$ closure of $\mathbb{R} \cup \{x\}$ under $\exp$, $\log$ and infinite summation

$$\mathrm{e}^{\mathrm{e}^{x}+\mathrm{e}^{x/2}+\mathrm{e}^{x/3}+\cdots} - 3\,\mathrm{e}^{x^2} + 5\,(\log x)^{\pi} + 42 + x^{-1} + 2\,x^{-2} + 6\,x^{-3} + 24\,x^{-4} + \cdots + \mathrm{e}^{-x}$$

Here one should think of x as a positive infinite indeterminate.

$\mathbb{T} :=$ closure of $\mathbb{R} \cup \{x\}$ under exp, log and infinite summation

$$\sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m} = \mathrm{e}^{\mathrm{e}^{x} + \mathrm{e}^{x/2} + \cdots} - 3\,\mathrm{e}^{x^2} + 5\,(\log x)^{\pi} + 42 + x^{-1} + 2\,x^{-2} + 6\,x^{-3} + \cdots + \mathrm{e}^{-x}$$

Here one should think of x as a positive infinite indeterminate.

Transseries

$\mathbb{T}$:= closure of $\mathbb{R} \cup \{x\}$ under exp, log and infinite summation

$$\sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m} = \mathrm{e}^{\mathrm{e}^{x} + \mathrm{e}^{x/2} + \cdots} - 3\,\mathrm{e}^{x^2} + 5\,(\log x)^{\pi} + 42 + x^{-1} + 2\,x^{-2} + 6\,x^{-3} + \cdots + \mathrm{e}^{-x}$$

Here one should think of x as a positive infinite indeterminate.

$$\begin{aligned} \mathbb{T} &= \mathbb{R}[[\mathfrak{M}]] \\ f_{\mathfrak{m}} &\rightarrow \text{coefficient} \\ \mathfrak{m} &\rightarrow \text{monomial} \\ \operatorname{supp} f &= \{\mathfrak{m} \in \mathfrak{M} : f_{\mathfrak{m}} \neq 0\} \end{aligned}$$

Different types of supports

$$\mathbb{T}^{\mathrm{wb}} \ni x + \log x + \log\log x + \cdots \qquad \text{(well-based)}$$

$$\ni x^{-1} + \mathrm{e}^{-x} + \mathrm{e}^{-\mathrm{e}^{x}} + \cdots$$

Different types of supports

$$\mathbb{T}^{\mathrm{wb}} \ni x + \log x + \log \log x + \cdots \qquad \text{(well-based)}$$

$$\ni x^{-1} + \mathrm{e}^{-x} + \mathrm{e}^{-\mathrm{e}^{x}} + \cdots$$

$$\mathbb{T}^{\mathrm{std}} \ni x^{-1} + \mathrm{e}^{-(\log x)^2} + \mathrm{e}^{-(\log x)^4} + \cdots \qquad \text{(finite exp-log depth)}$$

Different types of supports

$$\mathbb{T}^{\mathrm{wb}} \ni x + \log x + \log \log x + \cdots \qquad \text{(well-based)}$$

$$\ni x^{-1} + \mathrm{e}^{-x} + \mathrm{e}^{-\mathrm{e}^x} + \cdots$$

$$\mathbb{T}^{\mathrm{std}} \ni x^{-1} + \mathrm{e}^{-(\log x)^2} + \mathrm{e}^{-(\log x)^4} + \cdots \qquad \text{(finite exp-log depth)}$$

$$\mathbb{T}^{\mathrm{gb}} \ni \frac{1}{(1 - x^{-1})\,(1 - \mathrm{e}^{-x})} = 1 + x^{-1} + x^{-2} + \cdots + \mathrm{e}^{-x} + \cdots$$

$$\ni \Gamma(x) = \sqrt{2\pi}\,\sqrt{x}\,\mathrm{e}^{-x}\,x^x\,(1 + {}^1\!/_{12}\,x^{-1} + \cdots) \qquad \text{(grid-based)}$$

$$\operatorname{supp} f \subseteq \mathfrak{m}\,\mathfrak{e}_1^{\mathbb{N}} \cdots \mathfrak{e}_k^{\mathbb{N}}, \qquad \mathfrak{e}_1, \ldots, \mathfrak{e}_k \prec 1$$

Different types of supports

$$\mathbb{T}^{\mathrm{wb}} \ni x + \log x + \log \log x + \cdots \qquad \text{(well-based)}$$

$$\ni x^{-1} + \mathrm{e}^{-x} + \mathrm{e}^{-\mathrm{e}^{x}} + \cdots$$

$$\mathbb{T}^{\mathrm{std}} \ni x^{-1} + \mathrm{e}^{-(\log x)^2} + \mathrm{e}^{-(\log x)^4} + \cdots \qquad \text{(finite exp-log depth)}$$

$$\mathbb{T}^{\mathrm{finr}} \ni \phi(x) = x + \sqrt{x} + \sqrt{\sqrt{x}} + \cdots$$

$$\ni \phi(x) + \phi(\mathrm{e}^{x}) \qquad \text{(finite archimedean rank)}$$

$$\mathbb{T}^{\mathrm{gb}} \ni \frac{1}{(1 - x^{-1})(1 - \mathrm{e}^{-x})} = 1 + x^{-1} + x^{-2} + \cdots + \mathrm{e}^{-x} + \cdots$$

$$\ni \Gamma(x) = \sqrt{2\pi}\sqrt{x}\,\mathrm{e}^{-x} x^{x} (1 + {}^{1}\!/_{12}\, x^{-1} + \cdots) \qquad \text{(grid-based)}$$

$$\operatorname{supp} f \subseteq \mathfrak{m}\, \mathfrak{e}_1^{\mathbb{R}} \cdots \mathfrak{e}_k^{\mathbb{R}}, \qquad \mathfrak{e}_1, \ldots, \mathfrak{e}_k \prec 1$$

Different types of supports

$$
\begin{aligned}
\mathbb{T}^{\mathrm{wb}} &\ni x+\log x+\log\log x+\cdots && \text{(well-based)}\\
&\ni x^{-1}+\mathrm{e}^{-x}+\mathrm{e}^{-\mathrm{e}^{x}}+\cdots\\
\mathbb{T}^{\mathrm{std}} &\ni x^{-1}+\mathrm{e}^{-(\log x)^2}+\mathrm{e}^{-(\log x)^4}+\cdots && \text{(finite exp-log depth)}\\
\mathbb{T}^{\mathrm{finr}} &\ni \phi(x)=x+\sqrt{x}+\sqrt{\sqrt{x}}+\cdots\\
&\ni \phi(x)+\phi(\mathrm{e}^{x}) && \text{(finite archimedean rank)}\\
\mathbb{T}^{\mathrm{nacc}} &\ni \zeta(x)=1^{-x}+2^{-x}+3^{-x}+\cdots\\
&\ni \zeta(x)+\zeta(\mathrm{e}^{x}) && \text{(accumulation-free)}\\
\mathbb{T}^{\mathrm{gb}} &\ni \frac{1}{(1-x^{-1})\,(1-\mathrm{e}^{-x})}=1+x^{-1}+x^{-2}+\cdots+\mathrm{e}^{-x}+\cdots\\
&\ni \Gamma(x)=\sqrt{2\pi}\,\sqrt{x}\,\mathrm{e}^{-x}\,x^{x}\,(1+{}^{1}\!/_{12}\,x^{-1}+\cdots) && \text{(grid-based)}
\end{aligned}
$$

$$
\operatorname{supp} f\subseteq \mathfrak{m}\,\mathfrak{e}_1^{\mathbb{R}}\cdots\mathfrak{e}_k^{\mathbb{R}},\qquad \mathfrak{e}_1,\ldots,\mathfrak{e}_k\prec 1
$$

Closure properties

Theorem (Ann. of Math. Stud. vol. 195)

The elementary theory of $\mathbb{T}^{\mathrm{wb}}$ *is completely axiomatized by:*

1. $\mathbb{T}$ *is a* LIOUVILLE *closed H-field with small derivation;*
2. $\mathbb{T}$ *satisfies the intermediate value property for differential polynomials.*

Closure properties

Theorem (Ann. of Math. Stud. vol. 195)

The elementary theory of $\mathbb{T}^{\mathrm{wb}}$ *is completely axiomatized by:*

1. $\mathbb{T}$ *is a* LIOUVILLE *closed H-field with small derivation;*
2. $\mathbb{T}$ *satisfies the intermediate value property for differential polynomials.*

In fact…

$$\mathbb{T}^{\mathrm{da}(\mathbb{R})} \subsetneq \mathbb{T}^{\mathrm{gb}} \subsetneq \mathbb{T}^{\mathrm{nacc}} \subsetneq \mathbb{T}^{\mathrm{finr}} \subsetneq \mathbb{T}^{\mathrm{std}}$$

and these fields satisfy the same elementary theory; we call them ***H-closed***.

Closure properties

Theorem (Ann. of Math. Stud. vol. 195)

The elementary theory of $\mathbb{T}^{\mathrm{wb}}$ *is completely axiomatized by:*

1. $\mathbb{T}$ *is a* LIOUVILLE *closed H-field with small derivation;*
2. $\mathbb{T}$ *satisfies the intermediate value property for differential polynomials.*

In fact…

$$\mathbb{T}^{\mathrm{da}(\mathbb{R})} \subsetneq \mathbb{T}^{\mathrm{gb}} \subsetneq \mathbb{T}^{\mathrm{nacc}} \subsetneq \mathbb{T}^{\mathrm{finr}} \subsetneq \mathbb{T}^{\mathrm{std}}$$

and these fields satisfy the same elementary theory; we call them ***H-closed***.

Corollary

$$\mathbb{T}^{\mathrm{da}(\mathbb{R})} \equiv \mathbb{T}^{\mathrm{gb}} \equiv \mathbb{T}^{\mathrm{nacc}} \equiv \mathbb{T}^{\mathrm{finr}} \equiv \mathbb{T}^{\mathrm{std}}.$$

Closure properties

Let H be an H-closed H-field such as $\mathbb{T}^{\mathrm{gb}}$. Write $H\{F\} = H[F, F', F'', \ldots]$

Newtonianity

Any quasi-linear equation $Lf = E(f)$, $f \prec 1$ (with $L \in H[\partial]$ and $E \in H\{F\}$ sufficiently small) admits a solution in H.

Closure properties

Let H be an H-closed H-field such as $\mathbb{T}^{\mathrm{gb}}$. Write $H\{F\} = H[F, F', F'', \ldots]$

Newtonianity

Any quasi-linear equation $Lf = E(f)$, $f \prec 1$ *(with* $L \in H[\partial]$ *and* $E \in H\{F\}$ *sufficiently small) admits a solution in* H.

Splitting of linear differential operators

Any linear differential operator $L \in H[\partial]$ *can be factored into (i) operators of order one or two in* $H[\partial]$*; (ii) operators of order one in* $H[\mathrm{i}][\partial]$.

Closure properties

Let H be an H-closed H-field such as $\mathbb{T}^{\mathrm{gb}}$. Write $H\{F\}=H[F,F',F'',\ldots]$

Newtonianity

Any quasi-linear equation $Lf=E(f)$, $f\prec 1$ *(with* $L\in H[\partial]$ *and* $E\in H\{F\}$ *sufficiently small) admits a solution in* H.

Splitting of linear differential operators

Any linear differential operator $L\in H[\partial]$ *can be factored into (i) operators of order one or two in* $H[\partial]$*; (ii) operators of order one in* $H[\mathrm{i}][\partial]$.

Weak differential closedness

Given $P\in K\{F\}$*, there exists an* $f\in K[\mathrm{i}]$ *with* $P(f)=0$.

The incomplete transbasis theorem

Definition

$\mathfrak{B} = \{\mathfrak{b}_1, \ldots, \mathfrak{b}_n\} \subseteq (\mathbb{T}^{\mathrm{gb}})^{\succ 1}$ with $\mathfrak{b}_1 \lll \cdots \lll \mathfrak{b}_n$ is a **transbasis** if

TB1. $\mathfrak{b}_1 = \log_d x$;

TB2. $\log \mathfrak{b}_i \in \mathbb{R}[[\mathfrak{b}_1^{\mathbb{R}}; \ldots; \mathfrak{b}_{i-1}^{\mathbb{R}}]]^{\mathrm{gb}}$, for $i = 2, \ldots, n$.

$$\log x, \qquad x, \qquad \mathrm{e}^x, \qquad x^x, \qquad \mathrm{e}^{x^{2x}\mathrm{e}^{-x} + x^x \mathrm{e}^{-x} + \mathrm{e}^{-x} + \cdots}$$

The incomplete transbasis theorem

Definition

$\mathfrak{B} = \{\mathfrak{b}_1, \ldots, \mathfrak{b}_n\} \subseteq (\mathbb{T}^{\mathrm{gb}})^{\succ 1}$ with $\mathfrak{b}_1 \prec\!\!\prec \cdots \prec\!\!\prec \mathfrak{b}_n$ is a **transbasis** if

TB1. $\mathfrak{b}_1 = \log_d x$;

TB2. $\log \mathfrak{b}_i \in \mathbb{R}[[\mathfrak{b}_1^{\mathbb{R}}; \ldots; \mathfrak{b}_{i-1}^{\mathbb{R}}]]^{\mathrm{gb}}$, for $i = 2, \ldots, n$.

$$\log x, \qquad x, \qquad \mathrm{e}^x, \qquad x^x, \qquad \mathrm{e}^{x^{2x}\mathrm{e}^{-x} + x^x \mathrm{e}^{-x} + \mathrm{e}^{-x} + \cdots}$$

Incomplete transbasis theorem

Let $\mathfrak{B} = \{\mathfrak{b}_1, \ldots, \mathfrak{b}_n\}$ be a transbasis and $f \in \mathbb{T}^{\mathrm{gb}}$. Then there exists a transbasis $\hat{\mathfrak{B}} = \{\hat{\mathfrak{b}}_1, \ldots, \hat{\mathfrak{b}}_{\hat{n}}\} \supseteq \mathfrak{B}$ with $f \in \mathbb{R}[[\hat{\mathfrak{b}}_1^{\mathbb{R}}; \ldots; \hat{\mathfrak{b}}_{\hat{n}}^{\mathbb{R}}]]^{\mathrm{gb}}$.

Transserial Hardy fields

$\mathscr{G}$:= ring of infinitely differentiable real germs at infinity.

Definition

A **transserial Hardy** field is a differential subfield $\mathscr{H}$ of $\mathbb{T}$, together with a monomorphism $\rho: \mathscr{H} \hookrightarrow \mathscr{C}$ of ordered differential $\mathbb{R}$-algebras, such that

TH1. $\forall f \in \mathscr{H}, \quad \operatorname{supp} f \subseteq \mathscr{H}$.

TH2. $\forall f \in \mathscr{H}, \quad f_{\prec} \in \mathscr{H}$. $\qquad f_{\prec} = \sum_{\mathfrak{m} \prec 1} f_{\mathfrak{m}} \mathfrak{m}$

$\mathscr{G}$:= ring of infinitely differentiable real germs at infinity.

Definition

A **transserial Hardy** field is a differential subfield $\mathscr{H}$ of $\mathbb{T}$, together with a monomorphism $\rho\colon \mathscr{H} \hookrightarrow \mathscr{C}$ of ordered differential $\mathbb{R}$-algebras, such that

TH1. $\forall f \in \mathscr{H}, \quad \operatorname{supp} f \subseteq \mathscr{H}$.

TH2. $\forall f \in \mathscr{H}, \quad f_{\prec} \in \mathscr{H}$.

$$f_{\prec} = \sum_{\mathfrak{m} \prec 1} f_{\mathfrak{m}} \mathfrak{m}$$

$$\frac{x\,\mathrm{e}^x}{1 - x^{-1} - \mathrm{e}^{-x}}$$

$$\|$$

$$x\,\mathrm{e}^x + \mathrm{e}^x + x^{-1}\mathrm{e}^x + \cdots + x + 1 + x^{-1} + \cdots + x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} + x^{-1}\mathrm{e}^{-x} + \cdots\cdots$$

$\mathscr{G}$:= ring of infinitely differentiable real germs at infinity.

Definition

A **transserial Hardy** field is a differential subfield $\mathscr{H}$ of $\mathbb{T}$, together with a monomorphism $\rho: \mathscr{H} \hookrightarrow \mathscr{C}$ of ordered differential $\mathbb{R}$-algebras, such that

TH1. $\forall f \in \mathscr{H}, \quad \operatorname{supp} f \subseteq \mathscr{H}$.

TH2. $\forall f \in \mathscr{H}, \quad f_{\prec} \in \mathscr{H}$. $\qquad f_{\prec} = \sum_{\mathfrak{m} \prec 1} f_{\mathfrak{m}} \mathfrak{m}$

$$\left(\frac{x\,\mathrm{e}^x}{1 - x^{-1} - \mathrm{e}^{-x}}\right)_{\prec}$$
$$\|$$
$$x\,\mathrm{e}^x + \mathrm{e}^x + x^{-1}\mathrm{e}^x + \cdots + x + 1 + x^{-1} + \cdots + x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} + x^{-1}\mathrm{e}^{-x} + \cdots\cdots$$

Transserial Hardy fields

$\mathscr{G}$:= ring of infinitely differentiable real germs at infinity.

Definition

A **transserial Hardy** field is a differential subfield $\mathscr{H}$ of $\mathbb{T}$, together with a monomorphism $\rho: \mathscr{H} \hookrightarrow \mathscr{C}$ of ordered differential $\mathbb{R}$-algebras, such that

TH1. $\forall f \in \mathscr{H}, \quad \operatorname{supp} f \subseteq \mathscr{H}$.

TH2. $\forall f \in \mathscr{H}, \quad f_{\prec} \in \mathscr{H}$. $\qquad f_{\prec} = \sum_{\mathfrak{m} \prec 1} f_{\mathfrak{m}} \mathfrak{m}$

Transserial Hardy fields

$\mathscr{G}$:= ring of infinitely differentiable real germs at infinity.

Definition

A **transserial Hardy** field is a differential subfield $\mathscr{H}$ of $\mathbb{T}$, together with a monomorphism $\rho\colon \mathscr{H} \hookrightarrow \mathscr{C}$ of ordered differential $\mathbb{R}$-algebras, such that

TH1. $\forall f \in \mathscr{H}, \quad \operatorname{supp} f \subseteq \mathscr{H}$.

TH2. $\forall f \in \mathscr{H}, \quad f_{\prec} \in \mathscr{H}$. $\qquad f_{\prec} = \sum_{\mathfrak{m} \prec 1} f_{\mathfrak{m}} \mathfrak{m}$

TH3. $\exists d \in \mathbb{Z}: \quad \forall \mathfrak{m} \in \mathfrak{M} \cap \mathscr{H}: \quad \log \mathfrak{m} \in \mathscr{H} + \mathbb{R} \log_d x$.

TH4. $\mathfrak{M} \cap \mathscr{H}$ is stable under taking real powers.

TH5. $\forall f \in \mathscr{H}^{>}: \quad \log f \in \mathscr{H} \Rightarrow \rho(\log f) = \log \rho(f)$.

Transserial Hardy fields

$\mathscr{G}$:= ring of infinitely differentiable real germs at infinity.

Definition

A **transserial Hardy** field is a differential subfield $\mathscr{H}$ of $\mathbb{T}$, together with a monomorphism $\rho: \mathscr{H} \hookrightarrow \mathscr{C}$ of ordered differential $\mathbb{R}$-algebras, such that

TH1. $\forall f \in \mathscr{H}, \quad \operatorname{supp} f \subseteq \mathscr{H}$.

TH2. $\forall f \in \mathscr{H}, \quad f_{\prec} \in \mathscr{H}$. $\qquad f_{\prec} = \sum_{\mathfrak{m} \prec 1} f_{\mathfrak{m}} \mathfrak{m}$

TH3. $\exists d \in \mathbb{Z}: \quad \forall \mathfrak{m} \in \mathfrak{M} \cap \mathscr{H}: \quad \log \mathfrak{m} \in \mathscr{H} + \mathbb{R} \log_d x$.

TH4. $\mathfrak{M} \cap \mathscr{H}$ is stable under taking real powers.

TH5. $\forall f \in \mathscr{H}^{>}: \quad \log f \in \mathscr{H} \Rightarrow \rho(\log f) = \log \rho(f)$.

Examples

$\mathscr{H}_1 = \mathbb{R}\{\{x^{-\mathbb{R}}\}\}^{\text{gb}}, \qquad \mathscr{H}_2 = \mathbb{R}\{\{x^{-\mathbb{R}}; \mathrm{e}^{\mathbb{R}x}\}\}^{\text{gb}} \qquad \mathscr{H}_3 = \mathbb{R}\{\{x^{-\mathbb{R}}; \Gamma(x)^{\mathbb{R}}\}\}^{\text{gb}}$

Basic extension theorems

$\mathscr{H}$:= transserial Hardy field of depth $d < \infty$.

Theorem (real algebraic extensions)

$\exists$ *unique transserial Hardy field structure on* $\mathscr{H}^{\mathrm{rcl}}$ *that extends the one of* $\mathscr{H}$.

Theorem (adding logarithms)

$\exists$ *unique transserial Hardy field structure* $\rho \colon \mathscr{H}((\log_d x)^{\mathbb{R}}) \to \mathscr{G}$ *on* $\mathscr{H}((\log_d x)^{\mathbb{R}})$ *with* $\rho((\log_d x)^{\lambda}) = (\log_d x)^{\lambda}$ *for all* $\lambda \in \mathbb{R}$.

Theorem (adding exponentials)

Let $\varphi \in \mathscr{H}_{>}$ *be such that* $\mathrm{e}^{\varphi} \notin \mathscr{H}$. $\exists$ *unique transserial Hardy field structure* $\rho \colon \mathscr{H}(\mathrm{e}^{\mathbb{R}\varphi}) \to \mathscr{G}$ *on* $\mathscr{H}(\mathrm{e}^{\mathbb{R}\varphi})$ *with* $\rho(\mathrm{e}^{\lambda\varphi}) = \mathrm{e}^{\lambda\rho(\varphi)}$ *for all* $\lambda \in \mathbb{R}$.

Elementary extensions, definitions

Definitions

$\mathscr{H}$ transserial Hardy field, $\quad \mathbb{T} \ni f \xrightarrow{???} \hat{f} \in \mathscr{G}$

Equivalence (over $\mathscr{H}$)

$$f \sim \hat{f} \quad \Longleftrightarrow \quad (\exists \varphi \in \mathscr{H} \colon\ f \sim_{\mathbb{T}} \varphi \sim_{\mathscr{G}} \hat{f})$$

Definitions

$\mathscr{H}$ transserial Hardy field, $\quad \mathbb{T} \ni f \xrightarrow{???} \hat{f} \in \mathscr{G}$

Equivalence (over $\mathscr{H}$)

$$f \sim \hat{f} \quad \Longleftrightarrow \quad (\exists \varphi \in \mathscr{H} \colon f \sim_{\mathbb{T}} \varphi \sim_{\mathscr{G}} \hat{f})$$

Asymptotic equivalence (over $\mathscr{H}$)

$$f \approx \hat{f} \quad \Longleftrightarrow \quad (\forall \varphi \in \mathscr{H}, f - \varphi \sim \hat{f} - \varphi)$$

Elementary extensions, definitions

Definitions

$\mathscr{H}$ transserial Hardy field, $\mathbb{T} \ni f \xrightarrow{???} \hat{f} \in \mathscr{G}$

Equivalence (over $\mathscr{H}$)

$$f \sim \hat{f} \quad \Longleftrightarrow \quad (\exists \varphi \in \mathscr{H} : f \sim_{\mathbb{T}} \varphi \sim_{\mathscr{G}} \hat{f})$$

Asymptotic equivalence (over $\mathscr{H}$)

$$f \approx \hat{f} \quad \Longleftrightarrow \quad (\forall \varphi \in \mathscr{H}, f - \varphi \sim \hat{f} - \varphi)$$

Differential equivalence (over $\mathscr{H}$)

$$f \approxeq \hat{f} \quad \Longleftrightarrow \quad (\forall P \in \mathscr{H}\{F\}, P(f) = 0 \Leftrightarrow P(\hat{f}) = 0)$$

Fundamental Lemma (elementary extensions)

Let $f \in \mathbb{T} \setminus \mathscr{H}$ *and* $\hat{f} \in \mathscr{G} \setminus \mathscr{H}$ *be such that*

- f *is a serial cut over* $\mathscr{H}$.
- f *and* $\hat{f}$ *are asymptotically equivalent over* $\mathscr{H}$.
- f *and* $\hat{f}$ *are differentially equivalent over* $\mathscr{H}$. *(minimal cuts* $\Rightarrow$ *OK)*

Then $\exists!$ *transserial Hardy field extension* $\rho: \mathscr{H}\langle f \rangle \to \mathscr{G}$ *with* $\rho(f) = \hat{f}$.

Differential equations (main ideas)

Step 1. A given algebraic differential equation

$$f^2 - f' + \frac{x}{e^x} = 0$$

Differential equations (main ideas)

Step 1. A given algebraic differential equation

$$f^2 - f' + \frac{x}{e^x} = 0$$

Step 2. Put equation in integral form

$$f = \int \left(\frac{x}{e^x} + f^2 \right)$$

Differential equations (main ideas)

Step 1. A given algebraic differential equation

$$f^2 - f' + \frac{x}{e^x} = 0$$

Step 2. Put equation in integral form

$$f = \int \left(\frac{x}{e^x} + f^2 \right)$$

Step 3. Integral transseries solution

$$f = \int \frac{x}{e^x} + \int \left(\int \frac{x}{e^x} \right)^2 + 2 \int \left(\int \frac{x}{e^x} \right) \left(\int \left(\int \frac{x}{e^x} \right)^2 \right) + \cdots$$

Differential equations (main ideas)

Step 1. A given algebraic differential equation

$$f^2 - f' + \frac{x}{e^x} = 0$$

Step 2. Put equation in integral form

$$f = \int \left(\frac{x}{e^x} + f^2 \right)$$

Step 3. Integral transseries solution ... has a natural analytic meaning

$$f = \int_\infty \frac{x}{e^x} + \int_\infty \left(\int_\infty \frac{x}{e^x} \right)^2 + 2 \int_\infty \left(\int_\infty \frac{x}{e^x} \right) \left(\int_\infty \left(\int_\infty \frac{x}{e^x} \right)^2 \right) + \cdots$$

Step 1. A given algebraic differential equation

$$f^2 - e^x f' + \frac{e^{2x}}{x} = 0$$

Differential equations (main ideas)

Step 1. A given algebraic differential equation

$$f^2 - e^x f' + \frac{e^{2x}}{x} = 0$$

Step 2. Put equation in integral form

$$f = \int \left(\frac{e^x}{x} + \frac{f^2}{e^x} \right)$$

Differential equations (main ideas)

Step 1. A given algebraic differential equation

$$f^2 - e^x f' + \frac{e^{2x}}{x} = 0$$

Step 2. Put equation in integral form

$$f = \int \left(\frac{e^x}{x} + \frac{f^2}{e^x} \right)$$

Step 3. Integrate from a fixed point $x_0 < \infty$

$$f = \int_{x_0} \frac{e^x}{x} + \int_{x_0} \frac{1}{e^x} \Big(\int_{x_0} \frac{e^x}{x} \Big)^2 + 2 \int_{x_0} \frac{1}{e^{2x}} \Big(\int_{x_0} \frac{e^x}{x} \Big) \Big(\int_{x_0} \frac{1}{e^x} \Big(\int_{x_0} \frac{e^x}{x} \Big)^2 \Big) + \cdots$$

Differential equations (main ideas)

Step 1. A general algebraic differential equation

$$P(f) = 0$$

Differential equations (main ideas)

Step 1. A general algebraic differential equation

$$P(f) = 0$$

Step 2. Using minimality, put equation in **split-normal form**

$$(\partial - \varphi_1) \cdots (\partial - \varphi_r) f \; = \; E(f) \qquad \text{(with } E(f) \text{ small)}$$

Attention: $\varphi_1, \ldots, \varphi_r \in \mathscr{H}[\mathrm{i}]$, even though $(\partial - \varphi_1) \cdots (\partial - \varphi_r) \in \mathscr{H}[\partial]$.

Differential equations (main ideas)

Step 1. A general algebraic differential equation

$$P(f) = 0$$

Step 2. Using minimality, put equation in **split-normal form**

$$(\partial - \varphi_1) \cdots (\partial - \varphi_r) f \;=\; E(f) \qquad \text{(with } E(f) \text{ small)}$$

Attention: $\varphi_1, \ldots, \varphi_r \in \mathscr{H}[\mathrm{i}]$, even though $(\partial - \varphi_1) \cdots (\partial - \varphi_r) \in \mathscr{H}[\partial]$.

Step 3. Solve the split-normal equation using the fixed-point technique.

Continuous right inverses (first order)

Lemma

The operator $J = (\partial - \varphi)^{-1}_{x_0}$, *defined by*

$$(Jf)(x) = \begin{cases} e^{\Phi(x)} \int_{\infty}^{x} e^{-\Phi(t)} f(t)\,dt & \text{(repulsive case)} \\ e^{\Phi(x)} \int_{x_0}^{x} e^{-\Phi(t)} f(t)\,dt & \text{(attractive case)} \end{cases}$$

and

$$\Phi(x) = \begin{cases} \int_{\infty}^{x} \varphi(t)\,dt & \text{(repulsive case)} \\ \int_{x_0}^{x} \varphi(t)\,dt & \text{(attractive case)} \end{cases}$$

is a continuous right-inverse of $L = \partial - \varphi$ *on* $\mathscr{G}^{\preccurlyeq}_{x_0}[\mathrm{i}]$, *with*

$$|||J|||_{x_0} \leqslant \left\| \frac{1}{\operatorname{Re} \varphi} \right\|_{x_0}.$$

Continuous right-inverses (higher order)

Lemma

Given a split-normal operator

$$L = (\partial - \varphi_1) \cdots (\partial - \varphi_r),$$

with a factorwise right-inverse $L^{-1} = J_r \cdots J_1$, *the operator*

$$\vartheta^{\nu} J_r \cdots J_1 \colon \mathscr{G}^{\preccurlyeq}_{x_0}[\mathrm{i}] \to \mathscr{G}^{\preccurlyeq}_{x_0;r}[\mathrm{i}]$$

is a continuous operator for every $\nu > r\,\sigma_L$. *Here* $\mathscr{G}^{\preccurlyeq}_{x_0;r}[\mathrm{i}]$ *carries the norm*

$$\|f\|_{x_0;r} = \max\{\|f\|_{x_0}, \ldots, \|f^{(r)}\|_{x_0}\}.$$

Preservation of realness

Lemma

Assume that $L \in \mathscr{H}[\partial]$ *admits a splitting*

$$L = (\partial - \varphi_1) \cdots (\partial - \varphi_r)$$

that formally preserves realness in the sense that it induces a factorization of L *as a product of real differential operators of order one or two.*

Then $J_r \cdots J_1$ *preserves realness in the sense that it maps* $\mathscr{G}_{x_0}^{\preccurlyeq}$ *into itself.*

Theorem

Consider a split-monic equation

$$Lf = E(f), \quad f \prec 1,$$

and let ν *be such that* $r\,\sigma_L < \nu < v_P$. *Then for any sufficiently large* x_0, *there exists a continuous factorwise right-inverse* $J_{r,\times\mathfrak{v}^\nu} \cdots J_{1,\times\mathfrak{v}^\nu}$ *of* $L_{\times\mathfrak{v}^\nu}$, *such that the operator*

$$\Xi\colon f \longmapsto (J_r \cdots J_1)(E(f))$$

admits a unique fixed point

$$f = \lim_{n\to\infty} \Xi^{(n)}(0) \in \mathscr{B}\left(\mathscr{G}^{\preccurlyeq}_{x_0;r}, \frac{1}{2}\right).$$

Preservation of asymptotics

Theorem

Let $\mathscr{H}$ be a transserial Hardy field of span $\mathfrak{v} \succcurlyeq e^x$. Consider a monic split-normal quasi-linear equation

$$Lf = E(f), \quad f \prec 1, \tag{1}$$

over $\mathscr{H}$ without solutions in $\mathscr{H}$. Assume that one of the following holds:

- *$\mathcal{T}$ is (1,1,1)-differentially closed in $\mathbb{T}_{\preccurlyeq \mathfrak{v}}$ and (1) is first order.*
- *$\mathcal{T}[i]$ is (1,1,1)-differentially closed in $\mathbb{T}[i]_{\preccurlyeq \mathfrak{v}}$.*

Then there exist solutions $f \in \mathscr{G}$ and $\tilde{f} \in \hat{\mathscr{H}}$ to (1), such that f and $\tilde{f}$ are asymptotically equivalent over $\mathscr{H}$.

Main theorem

Theorem (vdH 2009)

Given a transserial Hardy field $\mathscr{H}$, the set $\mathbb{T}^{\mathrm{da}(\mathscr{H})}$ of differentially algebraic transseries over $\mathscr{H}$ can be given the structure of a transserial Hardy field (that extends the structure of $\mathscr{H}$).

Main theorem

Theorem (vdH 2009)

Given a transserial Hardy field $\mathscr{H}$, the set $\mathbb{T}^{\mathrm{da}(\mathscr{H})}$ of differentially algebraic transseries over $\mathscr{H}$ can be given the structure of a transserial Hardy field (that extends the structure of $\mathscr{H}$).

Corollary

The subfield $\mathbb{T}^{\mathrm{da}(\mathbb{R})}$ of transseries that satisfy an algebraic differential equation over $\mathbb{R}$ can be embedded (as an ordered differential field) in a HARDY *field.*

Main theorem

Theorem (vdH 2009)

Given a transserial Hardy field $\mathscr{H}$, the set $\mathbb{T}^{\mathrm{da}(\mathscr{H})}$ of differentially algebraic transseries over $\mathscr{H}$ can be given the structure of a transserial Hardy field (that extends the structure of $\mathscr{H}$).

Corollary

The subfield $\mathbb{T}^{\mathrm{da}(\mathbb{R})}$ of transseries that satisfy an algebraic differential equation over $\mathbb{R}$ can be embedded (as an ordered differential field) in a HARDY *field.*

Corollary

There exists an H-closed Hardy field.

Applications

Corollary

There exists a transserial Hardy field $\mathscr{H}$, such that for any $P \in \mathscr{H}\{F\}$ and $f, g \in \mathscr{H}$ with $f < g$ and $P(f)P(g) < 0$, there exists a $h \in \mathscr{H}$ with $f < h < g$ and $P(h) = 0$.

Corollary

There exists a transserial Hardy field $\mathscr{H}$, such that $\mathscr{H}[\mathrm{i}]$ is weakly differentially closed.

Corollary

There exists a newtonian transserial Hardy field $\mathscr{H}$, i.e., such that any quasi-linear differential equation over $\mathscr{H}$ admits a solution in $\mathscr{H}$.

Back to our algorithm

Input: $\mathbb{K} \subseteq \mathbb{T}^{\mathrm{da}(\mathbb{R})}$ with $\phi\colon \mathbb{K} \hookrightarrow H$ for HARDY field H

while $\mathbb{K} \neq \mathbb{T}^{\mathrm{da}(\mathbb{R})}$ **do**

Pick $y \in \mathbb{T}^{\mathrm{da}(\mathbb{R})} \setminus \mathbb{K}$ of minimal *complexity*, i.e. $P(y, \ldots, y^{(r)}) = 0$ with

- P of minimal order r;
- P of minimal degree d in $y^{(r)}$;
- P of minimal degree e in $y, \ldots, y^{(r)}$.

Construct $\hat{\phi}\colon \mathbb{K}(y, \ldots, y^{(r)}) \hookrightarrow \hat{H}$

Set $\mathbb{K} := \mathbb{K}(y, \ldots, y^{(r)})$, $\phi := \hat{\phi}$

....**ZORN**

Output: $\tilde{\phi}\colon \mathbb{T}^{\mathrm{da}(\mathbb{R})} \hookrightarrow \tilde{H}$ for HARDY field extension $\tilde{H}$ of original H

Input: transserial Hardy field H with $\phi: H \hookrightarrow \mathscr{G}$

Set $\mathbb{K} := H$

while $\mathbb{K} \neq H^{\mathrm{da}}$ **do**

Pick $y \in H^{\mathrm{da}} \setminus \mathbb{K}$ of minimal *complexity*, i.e. $P(y, \ldots, y^{(r)}) = 0$ with

- P of minimal order r;
- P of minimal degree d in $y^{(r)}$;
- P of minimal degree e in $y, \ldots, y^{(r)}$.

Construct $\hat{\phi}: \mathbb{K}(y, \ldots, y^{(r)}) \hookrightarrow \mathscr{G}$

Set $\mathbb{K} := \mathbb{K}(y, \ldots, y^{(r)})$, $\phi := \hat{\phi}$

....**ZORN**

Output: transserial Hardy field $\mathbb{K}$ on H^{da} with $\tilde{\phi}: H^{\mathrm{da}} \hookrightarrow \mathscr{G}$

Ongoing work

Input: ω-free H-field H with embedding $\phi: H \hookrightarrow \mathscr{G}$

Set $\mathbb{K} := H$

while $\mathbb{K} \neq H^{\mathrm{da}}$ **do**

Pick $y \in H^{\mathrm{da}} \setminus \mathbb{K}$ of minimal *complexity*, i.e. $P(y, \ldots, y^{(r)}) = 0$ with

- P of minimal order r;
- P of minimal degree d in $y^{(r)}$;
- P of minimal degree e in $y, \ldots, y^{(r)}$.

Construct $\hat{\phi}: \mathbb{K}(y, \ldots, y^{(r)}) \hookrightarrow \mathscr{G}$

Set $\mathbb{K} := \mathbb{K}(y, \ldots, y^{(r)})$, $\phi := \hat{\phi}$

.... **ZORN**

Output: ω-free H-field H^{da} with embedding $\tilde{\phi}: H^{\mathrm{da}} \hookrightarrow \mathscr{G}$

Work in progress

Theorem

Any HARDY *field has an* ω*-free* HARDY *field extension.*

Work in progress

Theorem

Any HARDY *field has an ω-free* HARDY *field extension.*

Theorem in progress

The H-closure H^{da} of a HARDY *field can be given the structure of a* HARDY *field.*

Work in progress

Theorem

Any HARDY *field has an* ω*-free* HARDY *field extension.*

Theorem in progress

The H*-closure* H^{da} *of a* HARDY *field can be given the structure of a* HARDY *field.*

Corollary

Maximal HARDY *fields are H-closed.*

Work in progress

Theorem

Any HARDY *field has an* ω*-free* HARDY *field extension.*

Theorem in progress

The H*-closure* H^{da} *of a* HARDY *field can be given the structure of a* HARDY *field.*

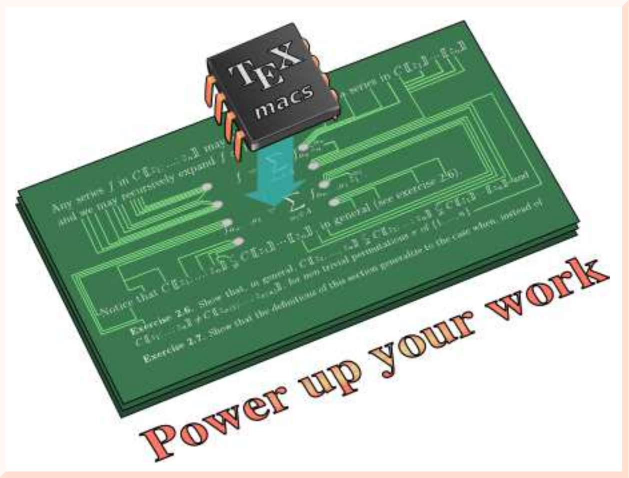
Corollary

Maximal HARDY *fields are H-closed.*

Theorem in progress by other means

For countable $A < B$ *in a maximal* HARDY *field* H*, we can find* $A < y < B$ *in* H*. Under CH, all maximal* HARDY *fields are isomorphic.*

Thank you!



http://www.TeXmacs.org