Ordering infinities

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Based on joint work with M. Aschenbrenner, L. van den Dries, V. Bagayoko, E. Kaplan

August 27, 2020

In honour of the 75th birthday of Maurice Pouzet
Back to the dark unordered ages

Georg Cantor
Back to the dark unordered ages

Georg Cantor

Cardinal numbers
Georg Cantor

Cardinal numbers

Ordinal numbers

0, 1, 2, ...
Georg Cantor

Cardinal numbers

Ordinal numbers

0, 1, 2, ..., ω
Georg Cantor

Cardinal numbers

Ordinal numbers

$0, 1, 2, \ldots, \omega, \omega + 1, \ldots$
Georg Cantor

Cardinal numbers

Ordinal numbers

$0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \omega \cdot 2, \omega \cdot 2 + 1, \ldots$
Georg Cantor

Cardinal numbers

Ordinal numbers

\[0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \omega \cdot 2, \omega \cdot 2 + 1, \ldots, \omega^2\]
Back to the dark unordered ages

Georg Cantor

Cardinal numbers

Ordinal numbers

$0, 1, 2, ..., \omega, \omega + 1, ..., \omega \cdot 2, \omega \cdot 2 + 1, ..., \omega^2, ..., \omega^3$
Georg Cantor

Cardinal numbers

Ordinal numbers

\[0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \omega \cdot 2, \omega \cdot 2 + 1, \ldots, \omega^2, \ldots, \omega^3, \ldots, \omega^\omega\]
Georg Cantor

Cardinal numbers

\[0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \omega \cdot 2, \omega \cdot 2 + 1, \ldots, \omega^2, \ldots, \omega^3, \ldots, \omega^\omega, \ldots, \aleph_1, \ldots\]
Georg Cantor

Cardinal numbers

Ordinal numbers

0, 1, 2, ..., ω, ω + 1, ..., ω², ω² + 1, ..., ω³, ..., ω², ..., ω³, ..., ω⁴, ..., \( \aleph_0 \), ..., \( \aleph_1 \), ....

Cantor normal form

\[ \omega^{\omega^2 \cdot 3 + \omega^8 \cdot 7 + \omega \cdot 3 + 2} \cdot 9 + \omega^{\omega^1 \cdot 3} + \omega^{\omega \cdot 7 \cdot 5} + \omega^8 + \omega^2 \cdot 111 + 2020 \]
Back to the dark unordered ages

Paul du Bois-Reymond
Paul du Bois-Reymond

Precursor of asymptotic calculus

$$\log x < \frac{x}{2} < \frac{x^2}{10} \quad (x \to \infty)$$
Paul du Bois-Reymond

Precursor of asymptotic calculus

$$\log x < \frac{x}{2} < \frac{x^2}{10} \quad (x \to \infty)$$

Diagonal argument

$$\exists f, \quad x < e^x < e^{e^x} < e^{e^{e^x}} < \cdots < f$$
Introduction

Three intimately related topics...

(surreal) **Numbers**

**Germs**
(in HARDY fields)

**Transseries**
HARDY fields
Let $C^1$ be the ring of germs at $+\infty$ of continuously differentiable functions $(a, \infty) \to \mathbb{R}$ $(a \in \mathbb{R})$.

We denote the germ at $+\infty$ of a function $f$ also by $f$, relying on context.

**Definition**

A **HARDY field** is a subring of $C^1$ which is a field that contains with each germ of a function $f$ also the germ of its derivative $f'$ (where $f'$ might be defined on a smaller interval than $f$).

**Examples.** $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}(x)$, $\mathbb{R}(x, e^x)$, $\mathbb{R}(x, e^x, \log x)$, $\mathbb{R}(x, e^{x^2}, \text{erf } x)$
HARDY fields capture the somewhat vague notion of functions with “regular growth” at infinity (BOREL, DU BOIS-REYMOND, ...):

Let $H$ be a HARDY field and $f \in H$. Then

$$f \neq 0 \implies \frac{1}{f} \in H \implies \begin{cases} f(x) > 0, \text{eventually, or} \\ f(x) < 0, \text{eventually.} \end{cases}$$

Consequently,

- $H$ carries an ordering making $H$ an **ordered field**:
  $$f > 0 \iff f(x) > 0 \text{ eventually;}$$

- $f$ is **eventually monotonic**, and
  $$\lim_{x \to +\infty} f(x) \in \mathbb{R} \cup \{\pm \infty\}.$$
Transseries

(surreal) Numbers

Germs (in HARDY fields)
Transseries

(surreal) Numbers

Germs (in HARDY fields)
The field $\mathbb{T}$ of transseries

$\mathbb{T} := \text{closure of } \mathbb{R} \cup \{x\} \text{ under exp, log and infinite summation}$

$e^x + e^{x/2} + e^{x/3} + \cdots - 3e^x + 5(\log x)\pi + 42 + x^{-1} + 2x^{-2} + 6x^{-3} + \cdots + e^{-x}$
The field $\mathbb{T}$ of transseries

$\mathbb{T} = \mathbb{R}[[\mathbb{M}]] :=$ closure of $\mathbb{R} \cup \{x\}$ under $\exp$, $\log$ and infinite summation

$$\sum_{m} f_{m} \cdot m = e^{x} + e^{x/2} + \cdots - 3e^{x^2} + 5(\log x)^{\pi} + 42 + x^{-1} + 2x^{-2} + 6x^{-3} + \cdots + e^{-x}$$

$x$: positive infinite indeterminate \hspace{1cm} $f_{m}$: coefficient \hspace{1cm} $m$: transmonomial

$\text{supp } f$: well-based subset of $\mathbb{M}$

disallow $x + \log x + \log \log x + \cdots$ and $e^{-x} + e^{-e^{x}} + e^{-e^{e^{x}}} + \cdots$
• With the natural ordering of transseries (via the leading coefficient), \( \mathbb{T} \) is a \textit{real closed ordered field} extension of \( \mathbb{R} \).

• Each \( f \in \mathbb{T} \) can be \textit{differentiated} term by term (with \( x' = 1 \)):

\[
\left( \sum_{n=0}^{\infty} \frac{n!}{x^n} e^x \right)' = \sum_{n=0}^{\infty} n! \left( \frac{e^x}{x^n} \right)' = \sum_{n=0}^{\infty} n! \left( \frac{e^x}{x^n} - n \frac{e^x}{x^{n+1}} \right) = e^x/x
\]

• This yields a \textit{derivation} \( f \mapsto f' \) on the field \( \mathbb{T} \):

\[
(f + g)' = f' + g', \quad (f \cdot g)' = f' \cdot g + f \cdot g'
\]

Its constant field is \( \{ f \in \mathbb{T} : f' = 0 \} = \mathbb{R} \).

• Given \( f, g \in \mathbb{T} \), the equation \( y' + fy = g \) admits a solution \( y \neq 0 \) in \( \mathbb{T} \).
Surreal numbers

(surreal) Numbers

Germs (in HARDY fields)

Transseries
Surreal numbers

(surreal)
Numbers

Germs

Transseries
Class On of ordinal numbers

For any set $L$ of ordinal numbers, there is a smallest ordinal number $\alpha > L$. 
Class On of ordinal numbers

For any set $L$ of ordinal numbers, there is a smallest ordinal number $\alpha > L$.

Class No of surreal numbers (CONWAY)

For any sets $L < R$ of surreal numbers, there is a simplest surreal number $\{L \mid R\}$ such that $L < \{L \mid R\} < R$. 
Class On of ordinal numbers

For any set $L$ of ordinal numbers, there is a smallest ordinal number $\alpha > L$.

Class No of surreal numbers (CONWAY)

For any sets $L < R$ of surreal numbers, there is a simplest surreal number $\{L|R\}$ such that $L < \{L|R\} < R$.

We have $\text{On} \subseteq \text{No}$ by taking $R = \emptyset$:

\[
\begin{align*}
0 &= \{\} \\
1 &= \{0\} \\
2 &= \{0, 1\} \\
\omega &= \{0, 1, 2, \ldots\}
\end{align*}
\]
Surreal numbers

0 = \{ | \}
Surreal numbers

\[ 0 = \{ | \} \]

\[ -1 = \{ | 0 \} \quad 1 = \{ 0 | \} \]
Surreal numbers

0 = {}  

-1 = {0|}  

-2 = {-1,0}  

-1/2 = {-1|0}  

1/2 = {0|1}  

1 = {0|}  

2 = {0,1|}
Surreal numbers

\[
\begin{align*}
0 &= \{ | \} \\
-1 &= \{ | 0 \} \\
-2 &= \{ | -1, 0 \} \\
-\frac{1}{2} &= \{ -1 | 0 \} \\
\frac{1}{2} &= \{ 0 | 1 \} \\
1 &= \{ 0 | \} \\
2 &= \{ 0, 1 | \} \\
-3 &= \{ | -1 \} \\
-1 \frac{1}{2} &= \{ | -1 \frac{1}{2} \} \\
-\frac{3}{4} &= \{ | -\frac{3}{4} \} \\
-\frac{1}{4} &= \{ | -\frac{1}{4} \} \\
\frac{1}{4} &= \{ | \frac{1}{4} \} \\
\frac{3}{4} &= \{ | \frac{3}{4} \} \\
1 \frac{1}{2} &= \{ | 1 \frac{1}{2} \} \\
3 &= \{ | 3 \} 
\end{align*}
\]
Surreal numbers

\[
0 = \{ | \}
\]

\[
-1 = \{ | 0 \}
\]

\[
-2 = \{ | -1, 0 \}
\]

\[
-\omega = \{ | \ldots, -1, 0 \}
\]

\[
-\frac{1}{2} = \{ -1 | 0 \}
\]

\[
-\frac{3}{4} = \{ -\frac{1}{4}, \}
\]

\[
\omega = \{ 0, 1, \ldots | \}
\]

\[
\frac{1}{\omega} = \{ 0, \ldots, \frac{1}{2}, 1 | \}
\]

\[
\frac{1}{2} = \{ 0 | 1 \}
\]

\[
\frac{3}{4} = \{ 0, \frac{1}{2}, 1 | \}
\]

\[
2 = \{ 0, 1 | \}
\]

\[
\frac{1}{4} = \{ 0, \frac{1}{2}, 1 | \}
\]

\[
\frac{1}{2} = \{ 0 | 1 \}
\]

\[
\frac{3}{4} = \{ 0, \frac{1}{2}, 1 | \}
\]

\[
3 = \{ 0, 1, \ldots | \}
\]
### Definition

If \( x = \{x^L|x^R\} \) and \( y = \{y^L|y^R\} \), then

\[
x + y := \{x^L + y, x + y^L|x^R + y, x + y^R\}
\]

(Idea: we want \( x^L + y < x + y < x^R + y, \ldots \))
Definition

If \( x = \{x^L | x^R\} \) and \( y = \{y^L | y^R\} \), then

\[
x + y := \{x^L + y, x + y^L | x^R + y, x + y^R\}
\]

(Idea: we want \( x^L + y < x + y < x^R + y, \ldots \))

Definition

If \( x = \{x^L | x^R\} \) and \( y = \{y^L | y^R\} \), then

\[
x y := \{x y + x y - x y, \bar{x} y + x \bar{y} - x \bar{y} | x y + x \bar{y} - x \bar{y}, \bar{x} y + x y - \bar{x} \bar{y}\},
\]

where \( x' \in x_L, x'' \in x_R, y' \in y_L, y'' \in y_R \)
Definition

If \( x = \{x^L|x^R\} \) and \( y = \{y^L|y^R\} \), then

\[
x + y := \{x^L + y, x + y^L|x^R + y, x + y^R\}
\]

(Idea: we want \( x^L + y < x + y < x^R + y, \ldots \))

Definition

If \( x = \{x^L|x^R\} \) and \( y = \{y^L|y^R\} \), then

\[
x y := \{xy + xy - xy, \bar{x}y + x\bar{y} - \bar{x}\bar{y}|xy + x\bar{y} - \bar{x}\bar{y}, \bar{x}y + x\bar{y} - \bar{x}\bar{y}\},
\]

where \( x' \in x_L, x'' \in x_R, y' \in y_L, y'' \in y_R \)

Theorem (CONWAY)

No is a real closed field.
• In the 1980s, GONSHOR (based on ideas of KRUSKAL) defined an exponential function \( \exp: \mathbb{N}_0 \to \mathbb{N}_0 > 0 \) that extends \( x \mapsto e^x \) on \( \mathbb{R} \).

• In 2006, BERARDUCCI and MANTOVA (using ideas of VDH and SCHMELING) defined a derivation \( \partial_{BM} \) on \( \mathbb{N}_0 \) with

\[
\ker \partial_{BM} = \mathbb{R}, \quad \partial_{BM}(\omega) = 1, \quad \partial_{BM}(\exp(f)) = \partial_{BM}(f) \cdot \exp(f) \quad \text{for} \quad f \in \mathbb{N}_0.
\]

In a certain technical sense, it is the simplest such derivation that satisfies some natural further conditions.

• The BM-derivation on \( \mathbb{N}_0 \) behaves in many ways like the derivation on \( \mathbb{T} \), with \( \omega > \mathbb{R} \) in the role of \( x > \mathbb{R} \). For instance, \( \partial_{BM}(\log \omega) = \frac{1}{\omega} \).
Towards a unified theory

(surreal) Numbers

Germs (in HARDY fields)

Transseries
Towards a unified theory

Numbers (surreal)

Germs (in HARDY fields)

Transseries

?
Towards a unified theory

(surreal) Numbers
Germs (in HARDY fields)
H-fields
Transseries
Towards a unified theory

(surreal) Numbers

H-fields

Germ (in HARDY fields)

Hardy

Hausdorff

Hahn

Transseries
Asymptotic relations

Let $K$ be an ordered differential field with constant field

$$C = \{ f \in K : f' = 0 \}.$$ 

We define

\[
\begin{align*}
  f \preceq g & : \iff |f| \leq c |g| \text{ for some } c \in C^>0 \quad (f \text{ is dominated by } g) \\
  f \prec g & : \iff |f| \leq c |g| \text{ for all } c \in C^>0 \quad (f \text{ is negligible w.r.t. } g) \\
  f \sim g & : \iff f \preceq g \preceq f \quad (f \text{ is asymptotic to } g) \\
  f \sim g & : \iff f - g \preceq g \quad (f \text{ is equivalent to } g)
\end{align*}
\]

Example. In $\mathbb{T}$: $0 < e^{-x} < x^{-10} < 1 \approx 100 < \log x < x^{1/10} < e^x \sim e^x + x < e^{e^x}$
We call \( K \) an **H-field** if

**H1.** \( f > C \implies f' > 0; \)

**H2.** \( f \preceq 1 \implies f \sim c \) for some \( c \in C \).

**Examples.** HARDY fields containing \( \mathbb{R} \); ordered differential subfields of \( \mathbb{T} \) or **No** that contain \( \mathbb{R} \).

\( \mathbb{T} \) admits further elementary properties in addition to being an H-field. It

- has **small derivation**, that is, \( f < 1 \implies f' < 1 \); and

- is **LIOUVILLE closed**, that is, it is real closed and for all \( f, g \), there is some \( y \neq 0 \) with \( y' + fy = g \).
We view $\mathbb{T}$ model-theoretically as a structure with the primitives

$0, 1, +, \times, \partial$ (derivation), $\leq$ (ordering).


The elementary theory of $\mathbb{T}$ is completely axiomatized by:

1. $\mathbb{T}$ is a LIOUVILLE closed H-field with small derivation;
2. $\mathbb{T}$ satisfies the intermediate value property for differential polynomials:
   
   Given $P \in \mathbb{T}[Y, Y', ..., Y^{(r)}]$ and $u < v$ in $\mathbb{T}$ with $P(u)P(v) < 0$, there exists a $y \in \mathbb{T}$ with $u < y < v$ and $P(y) = 0$

In particular: the theory of $\mathbb{T}$ is decidable.

We also prove a quantifier elimination result for $\mathbb{T}$ in a natural expansion of the above language.
H-field elements as germs

(surreal) Numbers

Germ
(in HARDY fields)

H-fields

Transseries
H-field elements as germs

(surreal) Numbers

H-fields

Germs (in HARDY fields)

Transseries
Theorem (HARDY 1910, BOURBAKI 1951)

Any HARDY field has a smallest LIOUVILLE closed HARDY field extension.
## Theorem (Hardy 1910, Bourbaki 1951)

Any HARDY field has a smallest LIOUVILLE closed HARDY field extension.

## Conjecture

Let $H$ be a maximal HARDY field. Then

- $H$ satisfies the differential intermediate value property.
- For countable subsets $L < R$ of $H$, there exists an $h \in H$ with $L < h < R$. 
Closure properties of HARDY fields

**Theorem (HARDY 1910, BOURBAKI 1951)**

Any HARDY field has a smallest LIOUVILLE closed HARDY field extension.

**Conjecture**

Let $H$ be a maximal HARDY field. Then

- **A** $H$ satisfies the differential intermediate value property.
- **B** For countable subsets $L < R$ of $H$, there exists an $h \in H$ with $L < h < R$.

**Corollary**

- **A** $H$ is elementarily equivalent to $T$ as an ordered differential field.
- **B** Under CH, all maximal HARDY fields are isomorphic.
H-field elements as surreal numbers

- (surreal) Numbers
- Germs (in HARDY fields)
- H-fields
- Transseries
Every H-field with small derivation and constant field $\mathbb{R}$ can be embedded as an ordered differential field into $\mathbb{No}$. 

Theorem (JEMS 2019)
<table>
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<td>Let $\kappa$ be an uncountable cardinal. The field $\text{No}(\kappa)$ of surreal numbers of length $&lt;\kappa$ is an elementary submodel of $\text{No}$.</td>
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H-field elements as transseries

(surreal) Numbers

H-fields

Germ (in HARDY fields)

Transseries
H-field elements as transseries

(surreal) Numbers

H-fields

Germs (in HARDY fields)

Transseries
H-field elements as transseries

(surreal) Numbers

H-fields

Germs (in HARDY fields)

Transseries
Definition (Van der Hoeven 2000, Schmeling 2001)

A field $\mathbb{T} = \mathbb{R}[[\mathcal{M}]]$ with $\log: \mathbb{T}^+ \rightarrow \mathbb{T}$ is a field of transseries if ...

A transserial derivation on $\mathbb{T}$ is a derivation $\partial: \mathbb{T} \rightarrow \mathbb{T}$ such that …
Definition (Van der Hoeven 2000, Schmeling 2001)

A field \( T = \mathbb{R}\left[ [M] \right] \) with \( \log: T^+ \rightarrow T \) is a field of transseries if …

A transserial derivation on \( T \) is a derivation \( \partial: T \rightarrow T \) such that …

Theorem (Berarducci–Mantova, 2015)

No is a field of transseries and \( \partial_{BM} \) is a transserial derivation.
**Definition (Van der Hoeven 2000, Schmeling 2001)**

A field $T = \mathbb{R}[[M]]$ with $\log: T^\succ \rightarrow T$ is a **field of transseries** if …

A **transserial derivation** on $T$ is a derivation $\partial: T \rightarrow T$ such that …

**Theorem (Berarducci–Mantova, 2015)**

No is a field of transseries and $\partial_{BM}$ is a transserial derivation.

**Corollary**

Any H-field with constant field $\mathbb{R}$ can be embedded in a field of transseries with a transserial derivation.
What next?

(surreal) Numbers

H-fields

Germ
(in Nigerian fields)

Transseries
What next?

(surreal) Numbers

Germs (in HARDY fields)

Transseries

beyond H-fields
What next?

(surreal) Numbers

Germs
(in HARDY fields)

= beyond H-fields

Transseries
Iterated exponentials and logarithms

\[ \exp_\omega(x + 1) = \exp \exp_\omega x \]
\[ \exp_{\omega^2}(x + 1) = \exp_\omega \exp_{\omega^2} x \]
...

→ stronger growth than \( e^x, e^{e^x}, ..., \exp_\omega x, e^{\exp_\omega x}, ..., \exp_\omega \exp_\omega x, ... \)
Iterated exponentials and logarithms

\[ \exp_\omega (x + 1) = \exp \exp_\omega x \]
\[ \exp_{\omega^2} (x + 1) = \exp_\omega \exp_{\omega^2} x \]

\[ \vdots \]

→ stronger growth than \( e^x, e^{e^x}, \ldots, \exp_\omega x, e^{\exp_\omega x}, \ldots, \exp_\omega \exp_\omega x, \ldots \)

Functional equations

\[ f(x) = \sqrt{x} + e^{f(\log x)} = \sqrt{x} + e^{\sqrt{\log x} + e^{\sqrt{\log \log x} + \cdots}} \]
Hyperlogarithms and hyperexponentials

\[
\begin{align*}
\exp_\omega (x + 1) &= \exp \exp_\omega x \\
\exp_{\omega^2} (x + 1) &= \exp_\omega \exp_{\omega^2} x \\
\vdots
\end{align*}
\]

\[
\begin{align*}
\log_\omega \log x &= \log_\omega x - 1 \\
\log_{\omega^2} \log_\omega x &= \log_{\omega^2} x - 1 \\
\vdots
\end{align*}
\]
Hyperlogarithms and hyperexponentials

\[ \exp_\omega(x + 1) = \exp \exp_\omega x \]
\[ \exp_\omega^2(x + 1) = \exp_\omega \exp_\omega^2 x \]
\[ \vdots \]
\[ \log_\omega \log x = \log_\omega x - 1 \]
\[ \log_\omega^2 \log_\omega x = \log_\omega^2 x - 1 \]
\[ \vdots \]
\[ \log_\omega x = \int \frac{1}{x \log x \log \log x \ldots} \]
\[ \log_\alpha x = \int \prod_{\beta < \alpha} \frac{1}{\log_\beta x} \]
Hyperlogarithms and hyperexponentials

\[
\begin{align*}
\exp_\omega (x+1) &= \exp \exp_\omega x \\
\exp_\omega^2 (x+1) &= \exp_\omega \exp_\omega^2 x \\
&\vdots \\
\log_\omega \log x &= \log_\omega x - 1 \\
\log_\omega^2 \log_\omega x &= \log_\omega^2 x - 1 \\
&\vdots \\
\log_\omega x &= \int \frac{1}{x \log x \log \log x \cdots} \\
\log_\alpha x &= \int \prod_{\beta < \alpha} \frac{1}{\log_\beta x}
\end{align*}
\]

Nested hyperseries

Solutions de \( f(x) = \sqrt{x} + e^{f(\log x)} \):

\( f_0(x) \)
Hyperlogarithms and hyperexponential

\[ \exp_{\omega}(x+1) = \exp \exp_{\omega} x \]
\[ \exp_{\omega^2}(x+1) = \exp_{\omega} \exp_{\omega^2} x \]
\[ \vdots \]
\[ \log_{\omega \log x} = \log_{\omega} x - 1 \]
\[ \log_{\omega^2 \log_{\omega} x} = \log_{\omega^2} x - 1 \]
\[ \vdots \]
\[ \log_{\omega} x = \int \frac{1}{x \log x \log \log x} \cdots \]
\[ \log_{\alpha} x = \int \prod_{\beta < \alpha} \frac{1}{\log_{\beta} x} \]

Nested hyperseries

Solutions de \( f(x) = \sqrt{x} + e^{f(\log x)} \):

\[ f_{-1}(x) < f_0(x) < f_1(x) \]
Hyperlogarithms and hyperexponentials

\[
\begin{align*}
\exp_\omega(x + 1) &= \exp \exp_\omega x \\
\exp_{\omega^2}(x + 1) &= \exp_\omega \exp_{\omega^2} x \\
\vdots
\end{align*}
\]

\[
\begin{align*}
\log_\omega \log x &= \log_\omega x - 1 \\
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\vdots
\end{align*}
\]

\[
\begin{align*}
\log_\omega x &= \int \frac{1}{x \log x \log \log x \ldots} \\
\log_\alpha x &= \int \prod_{\beta < \alpha} \frac{1}{\log_\beta x}
\end{align*}
\]

Nested hyperseries

Solutions de \( f(x) = \sqrt{x} + e^{f(\log x)} \):

\[
\begin{align*}
f_{-2}(x) &< f_{-1}(x) < f_{-\frac{1}{2}}(x) < f_0(x) < f_{\frac{1}{2}}(x) < f_1(x) < f_2(x)
\end{align*}
\]
Hyperlogarithms and hyperexponentials

\[
\begin{align*}
\exp_\omega(x+1) &= \exp \exp_\omega x \\
\exp_{\omega^2}(x+1) &= \exp_\omega \exp_{\omega^2} x \\
&\vdots \\
\log_\omega \log x &= \log_\omega x - 1 \\
\log_{\omega^2} \log_\omega x &= \log_{\omega^2} x - 1 \\
&\vdots \\
\log_\omega x &= \int \frac{1}{x \log x \log \log x} \\
\log_\alpha x &= \int \prod_{\beta<\alpha} \frac{1}{\log_\beta x}
\end{align*}
\]

Nested hyperseries

Solutions de \( f(x) = \sqrt{x} + e^{f(\log x)} \):

\[\cdots < f_{-2}(x) < \cdots < f_{-1}(x) < \cdots < f_0(x) < \cdots < f_{1/2}(x) < \cdots < f_1(x) < \cdots < f_2(x) < \cdots \]
Conjecture (vdH 2006)

For an appropriate definition of the class $\text{Hy}$ of hyperseries, we have $\mathbb{N}_0 \cong \text{Hy}$ for the map $\phi: \text{Hy} \rightarrow \mathbb{N}_0; f \mapsto f(\omega)$. 
For an appropriate definition of the class $\text{Hy}$ of hyperseries, we have $\mathbb{N}_o \cong \text{Hy}$ for the map $\phi: \text{Hy} \rightarrow \mathbb{N}_o; f \mapsto f(\omega)$.

**Proof.** By constructing a Conway bracket $\{\} \}$ on $\text{Hy}$. 
Conjecture (vdH 2006)

For an appropriate definition of the class $\text{Hy}$ of hyperseries, we have $\mathbb{N}o \cong \text{Hy}$ for the map $\phi : \text{Hy} \longrightarrow \mathbb{N}o; f \mapsto f(\omega)$.

**Proof.** By constructing a Conway bracket $\{\|\}$ on $\text{Hy}$.

**Examples:**

\[
\{x, e^x, e^{e^x}, \ldots\} = \exp_\omega x
\]

\[
\{\sqrt{x}, \sqrt{x} + e^{\sqrt{\log x}}, \ldots\|\ldots\|\sqrt{x} + e^{2\sqrt{\log x}}, 2 \sqrt{x}\} = f_0(x)
\]

\[
\{x^2, e^{\log^2 x}, e^{e^{\log^2 x}}, \ldots\|\ldots\|e^{e^{\sqrt{\log \log x}}}, e^{e^{\sqrt{x}}}, e^{\sqrt{x}}\} = \exp_\omega \left(\log_\omega x + \frac{1}{2}\right)
\]
Thank you!

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