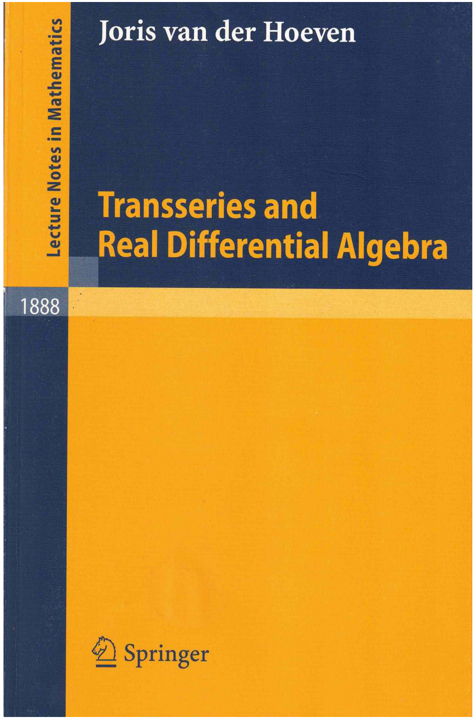
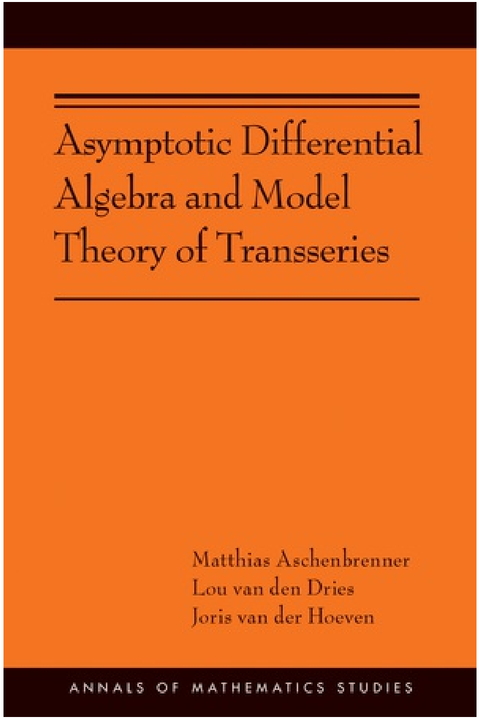
Model theory of asymptotic differential algebra

Joris van der Hoeven

IMS summer school
Singapore, July 10, 2023

Background material





MAXIMAL HARDY FIELDS

MATTHIAS ASCHENBRENNER, LOU VAN DEN DRIES, AND JORIS VAN DER HOEVEN

To the memory of Michael Boshernitzan (1950–2019)

ABSTRACT. We show that all maximal Hardy fields are elementarily equivalent as differential fields, and give various applications of this result and its proof. We also answer some questions on Hardy fields posed by Boshernitzan.

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Date: April, 2023.

Main goal

An introduction to three topics

Lessons 1 – 2	Hardy fields
Lessons 3 – 7	Transseries
Lessons 8 – 10	H-fields and model theory

What is asymptotic differential algebra?

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Short answer

The formal study of asymptotic properties of solutions to differential equations.

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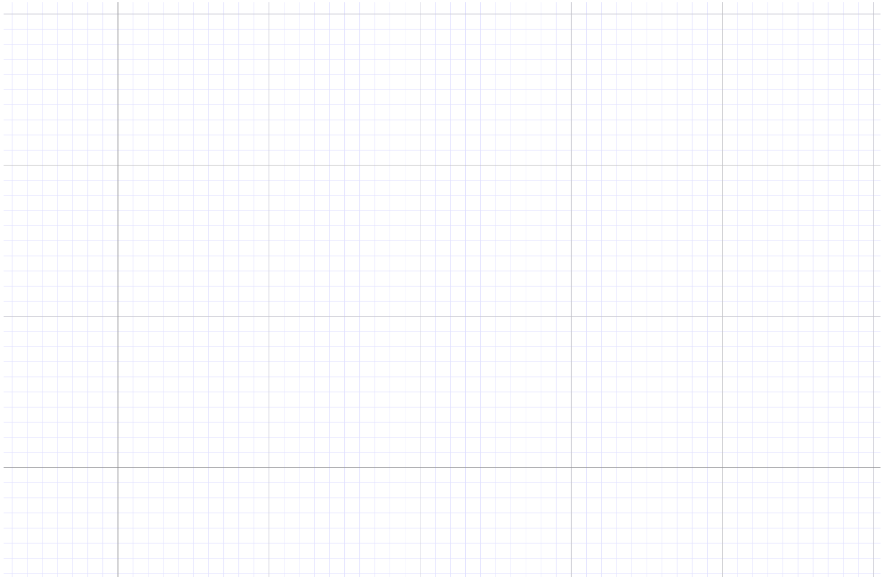

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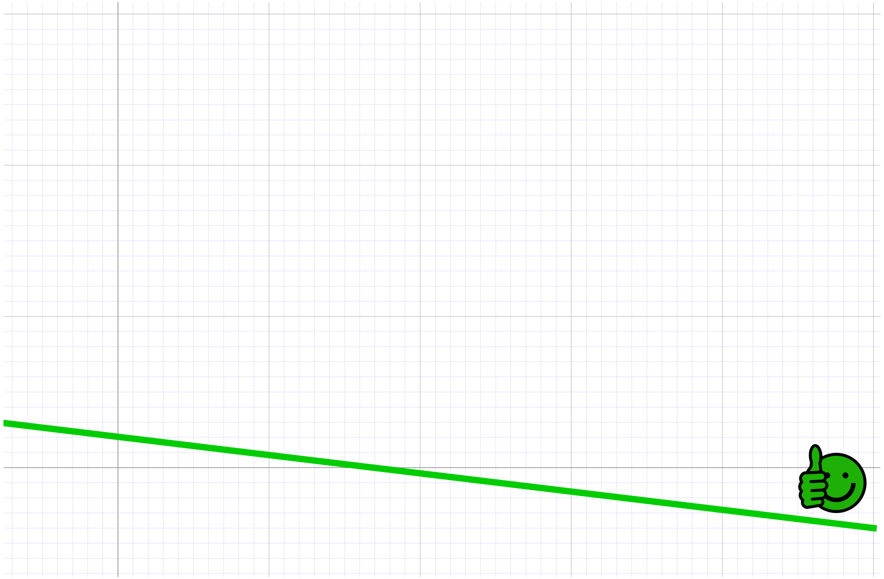

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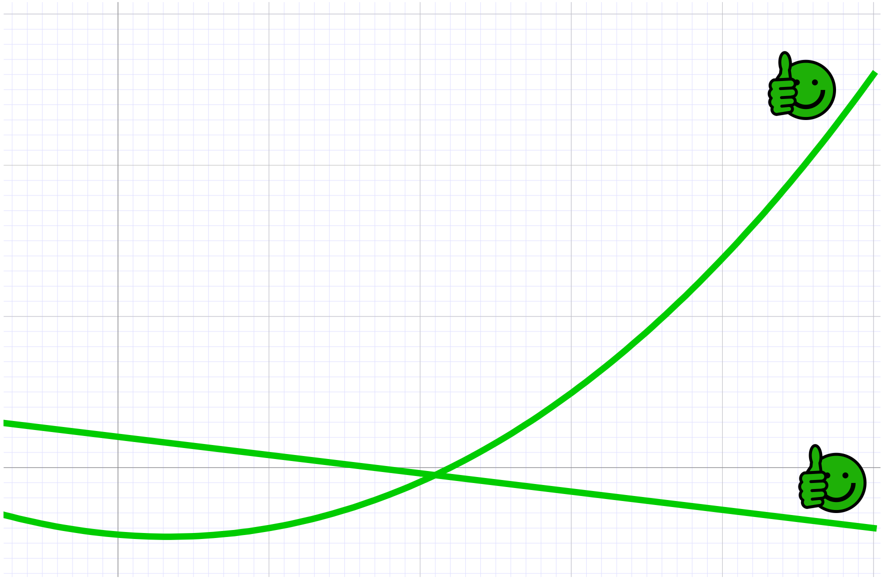

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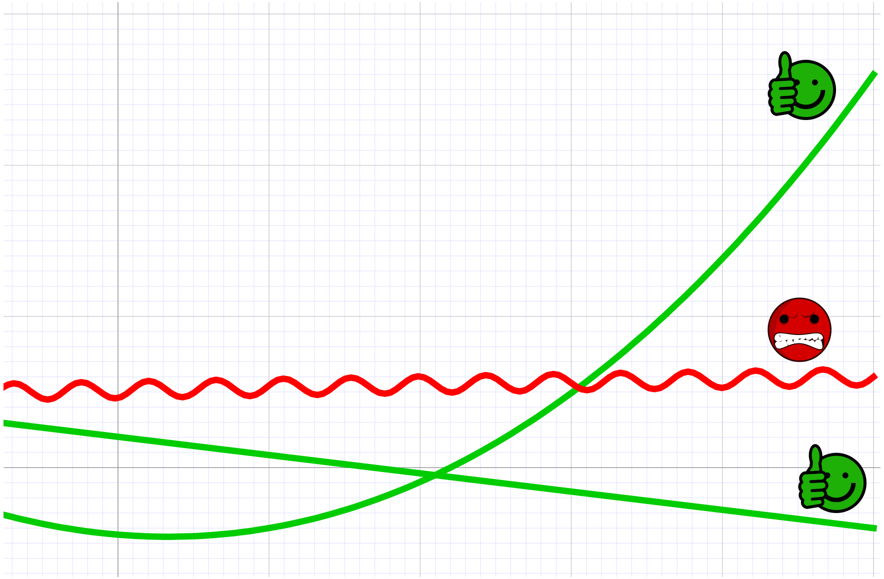

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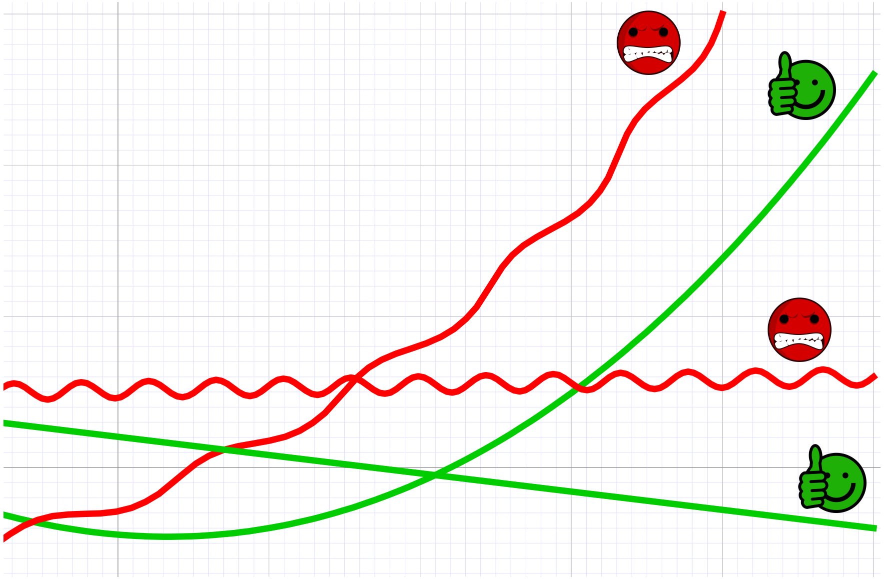

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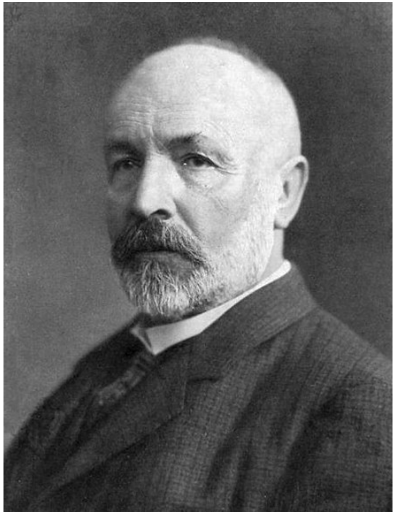
Models for asymptotic differential algebra: Hardy fields and transseries.

Philosophy — grand unification of infinities



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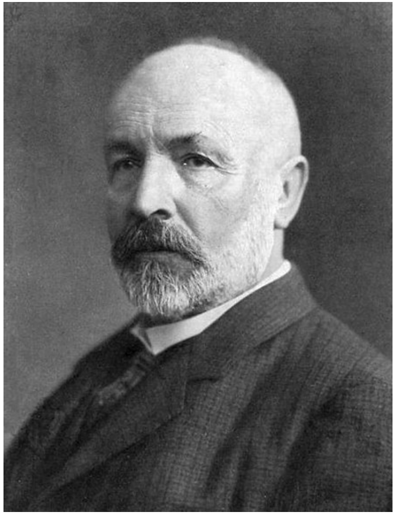

Calculus with infinitesimals

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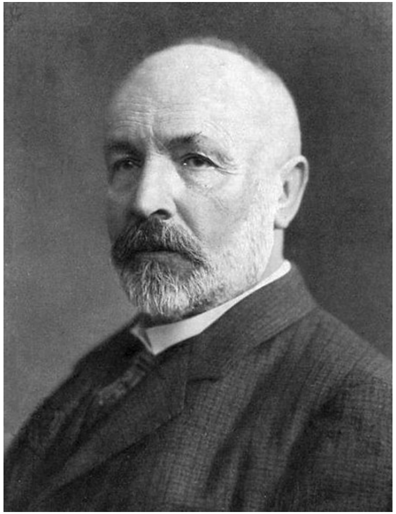

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Growth rates

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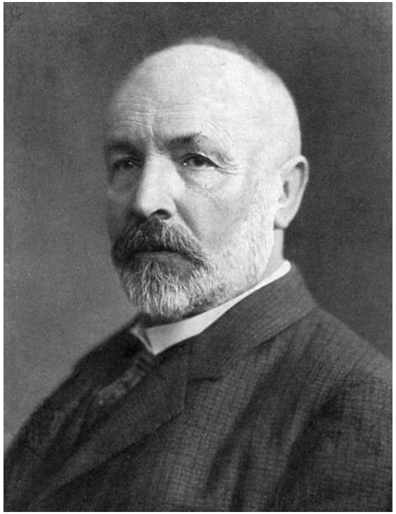

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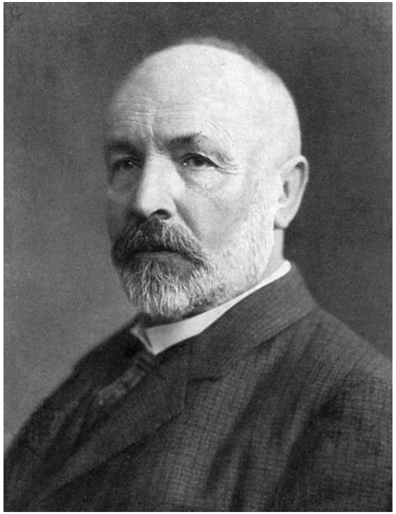

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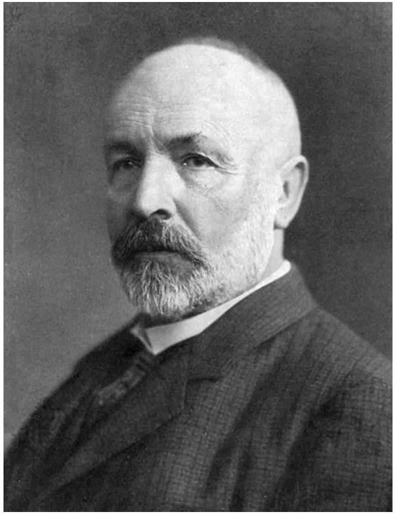

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   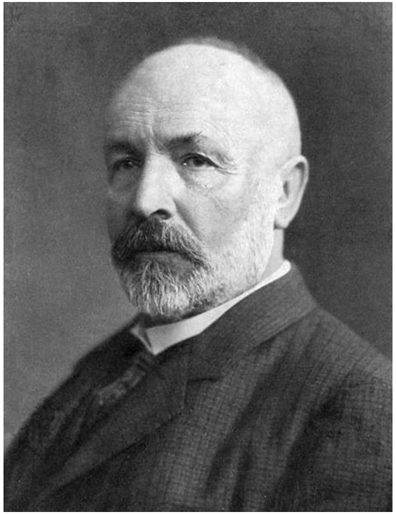

Calculus with infinitesimals — Growth rates — Infinite numbers

Transseries — Hardy fields — Surreal numbers

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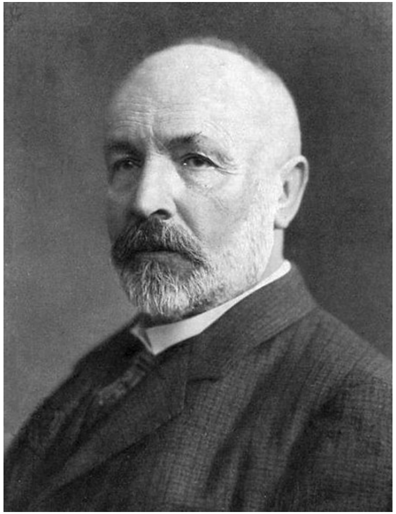

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