Lesson 1 — Hausdorff fields

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IMS summer school

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Spaces of real functions

Notation

k	$k \in \{0, 1, 2, \ldots, \infty\}$
I	Real interval $I \subseteq \mathbb{R}$
$\mathscr{C}^k(I)$	Ring of real-valued functions of class $\mathscr{C}^k$ on I

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Ordered ring structure. For $f, g \in \mathscr{C}^k(I)$, we define a **partial** ordering

$$f \leqslant g \iff (\forall x \in I) \;\; f(x) \leqslant g(x)$$

$$f < g \iff f \leqslant g \;\wedge\; f \neq g$$

Germs of real functions at infinity

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$\mathscr{C}_a^k := \mathscr{C}^k([a, \infty))$	Ring of functions of class $\mathscr{C}^k$ on $[a, \infty)$

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Eventual relations. For $f \in \mathscr{C}_a^k$ and $g \in \mathscr{C}_b^k$, we define

$$f =_\infty g \iff (\exists c > a,b)\ (\forall x \geqslant c)\ \ f(x) = g(x)$$
$$f \leqslant_\infty g \iff (\exists c > a,b)\ (\forall x \geqslant c)\ \ f(x) \leqslant g(x)$$
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$\mathscr{C}_a^k := \mathscr{C}^k([a,\infty))$ Ring of functions of class $\mathscr{C}^k$ on $[a,\infty)$

$f =_\infty g$ Eventual equality of $f, g \in \bigcup_{a\in\mathbb{R}} \mathscr{C}_a^k$

Germs at infinity

$$\mathscr{G}^k \; := \; \left(\bigcup_{a\in\mathbb{R}} \mathscr{C}_a^k\right) / =_\infty$$

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Abuse of notation: identify $f \in \mathscr{G}^k$ with $\check{f} \in \mathscr{C}_a^k$ that eventually represents f

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$$\mathscr{C}_a^k \supsetneq \mathscr{C}_{a+1}^k \supsetneq \mathscr{C}_{a+2}^k \supsetneq \cdots \supsetneq \mathscr{G}^k$$

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$$\mathscr{C}_a^k \supsetneq \mathscr{C}_{a+1}^k \supsetneq \mathscr{C}_{a+2}^k \supsetneq \cdots \supsetneq \mathscr{G}^k$$

$\mathscr{G}^k$ is a **partially** ordered ring for $\leqslant_\infty / =_\infty$

Asymptotic relations

For $f, g \in \mathscr{G}$, we define

$$
\begin{aligned}
f = \mathscr{O}(g) \iff f \preccurlyeq g &\iff (\exists c \in \mathbb{R}^{>0})\ |f| \leqslant c|g| \\
f = \mathscr{o}(g) \iff f \prec g &\iff (\forall c \in \mathbb{R}^{>0})\ c|f| < |g| \\
f \asymp g &\iff f \preccurlyeq g \preccurlyeq f \\
f \sim g &\iff f - g \prec f
\end{aligned}
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Here $|f| \in \mathscr{G}$ is such that $|f|(x) = |f(x)|$, eventually

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Flatness relations. Assuming that $g \succ 1$, we define

$$\begin{aligned} f \preceq\!\!\!\preceq g &\iff (\exists c \in \mathbb{R}^{>0})\ |f| \preccurlyeq |g|^c \\ f \lll g &\iff (\forall c \in \mathbb{R}^{>0})\ |f|^c \prec |g| \end{aligned}$$

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Flatness relations (general definition for g arbitrary)

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Example: with $x := \mathrm{Id}_{\mathbb{R}} / =_{\infty}$, we have $\log x \lll x < 2x \preccurlyeq -x \prec x^2 \lll e^{-x} \underline{\lll} e^{5x} \lll x^x$

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Hausdorff fields

Definition

*A **Hausdorff field** is a subfield of the ring $\mathcal{G}$.*

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Proposition

Let K *be a Hausdorff field. Then* K *is an ordered field.*

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Let K be a Hausdorff field. Then K is an ordered field.

Proof. Let $f \in K \setminus \{0\}$. Then $f^{-1} \in K \subseteq \mathcal{G}$, so $f(x) \neq 0$, eventually.

Since f is continuous, this means that $f(x) > 0$ or $f(x) < 0$, eventually. □

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Examples.

- Any subfield of $\mathbb{R}$ is a Hausdorff field
- $\mathbb{R}(x)$ is a Hausdorff field, where $x = \mathrm{Id}_{\mathbb{R}} / =_{\infty}$

Monomial groups

Proposition

For any $f \in \mathbb{R}(x)^{\neq 0}$, *we have* $f \sim c\,x^k$, *for some* $c \in \mathbb{R}^{\neq}$ *and* $k \in \mathbb{Z}$.

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Proof. If $f = P \in \mathbb{R}[x]^{\neq 0}$, then

$$\begin{aligned} P &= P_d x^d + \cdots + P_0, && (P_d \neq 0) \\ &= x^d (P_d + P_{d-1} x^{-1} + \cdots + P_0 x^{-d}) \\ &= x^d (P_d + \mathcal{O}(1)) \\ &\sim P_d x^d. \end{aligned}$$

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Proof. If $f = P/Q$ with $P, Q \in \mathbb{R}[x]^{\neq 0}$, then

$$\begin{aligned} P &\sim P_d\, x^d \\ Q &\sim Q_e\, x^e \\ \frac{P}{Q} &\sim \frac{P_d}{Q_e} x^{d-e}, \end{aligned}$$

where $d = \deg P$, $e = \deg Q$. □

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For any $f \in \mathbb{R}(x)^{\neq 0}$*, we have* $f \sim c\, x^k$*, for some* $c \in \mathbb{R}^{\neq}$ *and* $k \in \mathbb{Z}$*.*

Corollary

The Hausdorff field $K = \mathbb{R}(x)$ *has* $x^{\mathbb{Z}}$ *as a monomial group.*

Definition

Let K *be a Hausdorff field. We say that* $\mathfrak{M} \subseteq K^{\neq}$ *is a* ***monomial group*** *if* $\mathfrak{M}$ *is a totally ordered subgroup of* K *for* $\preccurlyeq$ *such that any* $f \in K^{\neq}$ *has a unique decomposition*

$$f = c\,\mathfrak{m} + \delta,$$

where $c \in \mathbb{R}^{\neq}$*,* $\mathfrak{m} \in \mathfrak{M}$*, and* $\delta \in K$ *is such that* $\delta \prec \mathfrak{m}$*.*

Asymptotic expansions

Proposition

Let K be a Hausdorff field with a monomial group $\mathfrak{M}$.

Given $f \in K$ and $n \in \mathbb{N}$, there exists a unique expansion

$$f = c_1 \mathfrak{m}_1 + \cdots + c_k \mathfrak{m}_k + \rho,$$

where

$$c_1, \ldots, c_k \in \mathbb{R}^{\neq 0}$$

$$\mathfrak{m}_1 \succ \cdots \succ \mathfrak{m}_k \succ \rho$$

with $\mathfrak{m}_1, \ldots, \mathfrak{m}_k \in \mathfrak{M}$, $\rho \in K$, and $k \leqslant n$.

Non-archimedean monomial groups

Proposition

If K is a Hausdorff field and $g \in \mathcal{G}^{>\mathbb{R}}$, then $K \circ g := \{f \circ g : f \in K\}$ is a Hausdorff field.

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If K is a Hausdorff field and $g \in \mathcal{G}^{>\mathbb{R}}$, then $K \circ g := \{f \circ g : f \in K\}$ is a Hausdorff field.

Proof. The map $f \in K \longmapsto f \circ g \in K \circ g$ is an isomorphism of ordered fields. $\square$

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Corollary

The field $K(\mathrm{e}^x) = K(x) \circ \mathrm{e}^x$ *is a Hausdorff field with monomial group* $\mathrm{e}^{\mathbb{Z}x}$*.*

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The field $\mathbb{R}(x, \mathrm{e}^x) = \mathbb{R}(x)(\mathrm{e}^x)$ *is a Hausdorff field with monomial group* $x^{\mathbb{Z}} \mathrm{e}^{\mathbb{Z}x}$*.*

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Proof. Assume $f = P = P_d \mathrm{e}^{dx} + \cdots + P_i \mathrm{e}^{ix} + \cdots + P_0 \in \mathbb{R}[x, \mathrm{e}^x]^{\neq 0}$ with $0 \neq P_d, \ldots, P_0 \in \mathbb{R}[x]$.

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For each i with $P_i \neq 0$, we have $P_i \sim c_i x^{k_i}$ for some $c_i \in \mathbb{R}^{\neq 0}$ and $k_i \in \mathbb{Z}$.

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If $i < d$, then $P_i \mathrm{e}^{ix} \sim c_i x^{k_i} \mathrm{e}^{ix} \prec c_d x^{k_d} \mathrm{e}^{dx} \sim P_d \mathrm{e}^{dx}$.

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If $i < d$, then $P_i \mathrm{e}^{ix} \sim c_i x^{k_i} \mathrm{e}^{ix} \prec c_d x^{k_d} \mathrm{e}^{dx} \sim P_d \mathrm{e}^{dx}$. Hence $P \sim P_d \mathrm{e}^{dx} \sim c_d x^{k_d} \mathrm{e}^{dx}$.

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The field $\mathbb{R}(x, \mathrm{e}^x) = \mathbb{R}(x)(\mathrm{e}^x)$ is a Hausdorff field with monomial group $x^{\mathbb{Z}} \mathrm{e}^{\mathbb{Z}x}$.

Proof. Assume $f = P = P_d \mathrm{e}^{dx} + \cdots + P_i \mathrm{e}^{ix} + \cdots + P_0 \in \mathbb{R}[x, \mathrm{e}^x]^{\neq 0}$ with $0 \neq P_d, \ldots, P_0 \in \mathbb{R}[x]$.

For each i with $P_i \neq 0$, we have $P_i \sim c_i x^{k_i}$ for some $c_i \in \mathbb{R}^{\neq 0}$ and $k_i \in \mathbb{Z}$.

If $i < d$, then $P_i \mathrm{e}^{ix} \sim c_i x^{k_i} \mathrm{e}^{ix} \prec c_d x^{k_d} \mathrm{e}^{dx} \sim P_d \mathrm{e}^{dx}$. Hence $P \sim P_d \mathrm{e}^{dx} \sim c_d x^{k_d} \mathrm{e}^{dx}$.

For $f = P / Q \in \mathbb{R}(x, \mathrm{e}^x)^{\neq 0}$, it follows $f \sim c\, x^k \mathrm{e}^{lx}$ for $c \in \mathbb{R}^{\neq}$, $k \in \mathbb{Z}$, $l \in \mathbb{Z}$. □

Non-archimedean monomial groups

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Proposition

Let $g_1, \ldots, g_n \in \mathcal{G}^{\neq 0}$ with $g_1 \lll \cdots \lll g_n$.

Then $\mathbb{R}(g_1, \ldots, g_n)$ is a Hausdorff field with monomial group $g_1^{\mathbb{Z}} \cdots g_n^{\mathbb{Z}}$.

Real closure

Theorem (Hausdorff, Boshernitzan)

Let $K \subseteq \mathcal{G}^k$ be a Hausdorff field. Then the set L of germs $y \in \mathcal{G}^k$ that are algebraic over K form a real closed Hausdorff field, which is isomorphic to the real closure K^{rc} of K.

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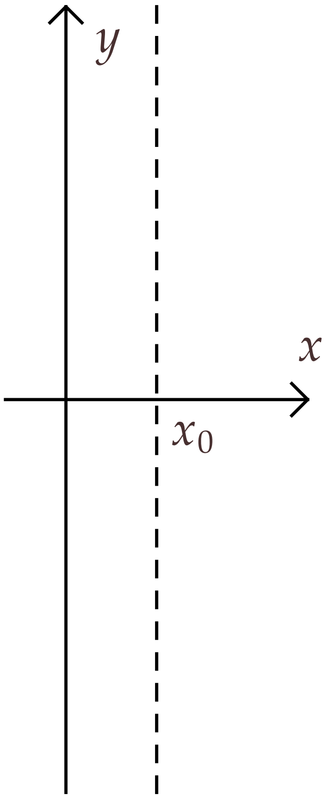
Lemma (continuity of roots)

Let $I \subseteq \mathbb{R}$ *be an interval. Let* $P(X,Y) \in \mathscr{C}^k(I)[Y]$ *with* $\gcd(P, \partial P / \partial Y) = 1$*. Then*

- *The number* n *of solutions of* $P(x,y) = 0$ *in* y *does not depend on* $x \in I$*.*
- *If* $y_1(x) < \cdots < y_n(x)$ *are these solutions, then* $y_i \in \mathscr{C}^k(I)$ *for* $i = 1, \ldots, n$*.*

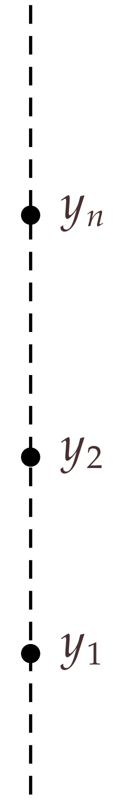
Continuity of roots — proof

Given $x_0 \in I$



Continuity of roots — proof

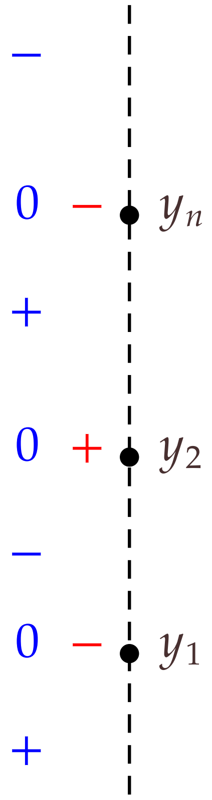
Given $x_0 \in I$, let $y_1 < \cdots < y_n$ be the solutions of $P(x_0, y) = 0$.



Continuity of roots — proof

Given $x_0 \in I$, let $y_1 < \cdots < y_n$ be the solutions of $P(x_0, y) = 0$.

By assumption, $\frac{\partial P}{\partial Y}(x_0, y_i) \neq 0$ for $i = 1, \ldots, n$



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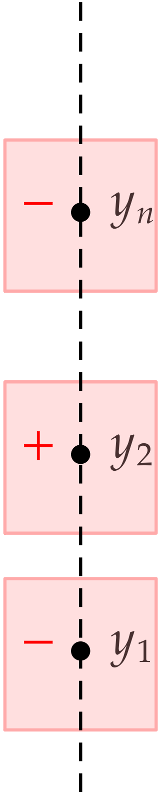
Take $\varepsilon > 0$ with $\left|\frac{\partial P}{\partial Y}(x, y)\right| > 0$ on $[x_0 - \varepsilon, x_0 + \varepsilon] \times [y_i - \varepsilon, y_i + \varepsilon]$



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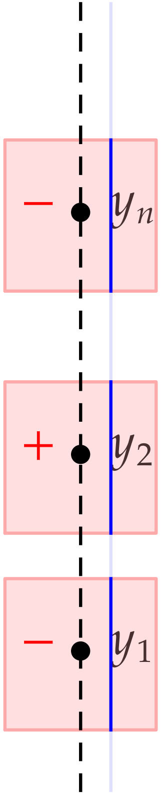
$P_x(y) := P(x, y)$ is strictly monotonic on $[y_i - \varepsilon, y_i + \varepsilon]$



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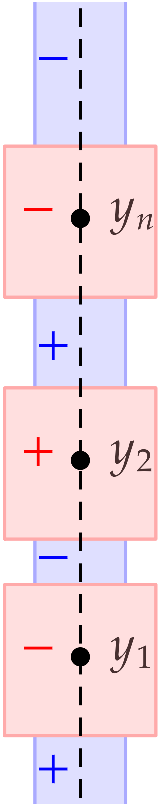
where $\mathscr{Y} := \mathbb{R} \setminus ([y_1 - \varepsilon, y_1 + \varepsilon] \cup \cdots \cup [y_n - \varepsilon, y_n + \varepsilon])$



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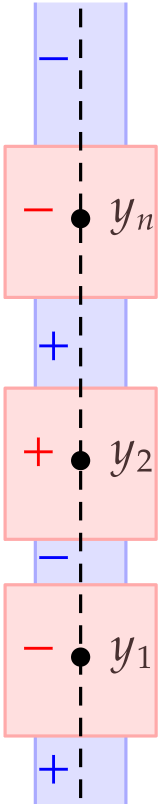
Given $x \in [x_0 - \varepsilon, x_0 + \varepsilon]$ and $i \in \{1, \ldots, n\}$,

$P(x, y_i - \varepsilon)\, P(x, y_i + \varepsilon) < 0 \Longrightarrow (\exists!\, y \in [y_i - \varepsilon, y_i + \varepsilon])\; P(x, y) = 0$

$P(x, y) \neq 0$ whenever $y \in \mathcal{Y}$



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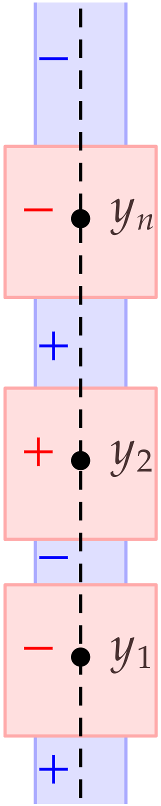
Conclusion: number of roots of P_x constant on $[x_0 - \delta, x_0 + \delta]$

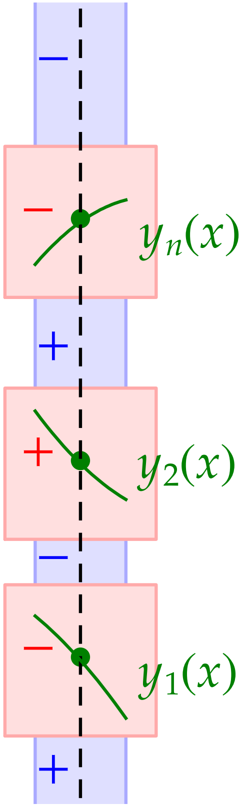


Continuity of roots — proof

Given $x \in [x_0 - \varepsilon, x_0 + \varepsilon]$ and $i \in \{1, \ldots, n\}$,

Sign change $P(x, y_i - \varepsilon)\, P(x, y_i + \varepsilon) < 0$ implies

Unique $y_i(x) \in [y_i - \varepsilon, y_i + \varepsilon]$ with $P(x, y_i(x)) = 0$



- Since we may choose ε arbitrarily small, $y_i(x) \in \mathscr{C}(I)$
- Since $y_i'(x_0) = -\dfrac{\frac{\partial P}{\partial X}(x_0, y_i)}{\frac{\partial P}{\partial Y}(x_0, y_i)}$, we also have $y_i(x) \in \mathscr{C}^1(I)$
- Similarly for higher derivatives

Real closure — proof

Theorem (Hausdorff, Boshernitzan)

Let $K \subseteq \mathcal{G}^k$ be a Hausdorff field. Then the set L of germs $y \in \mathcal{G}^k$ that are algebraic over K form a real closed Hausdorff field, which is isomorphic to the real closure K^{rc} of K.

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Since y^{-1} is algebraic over K, it follows that L is a Hausdorff field.

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If $\deg_Y P(X, Y)$ is odd, then so is $\deg P(a, Y)$. Hence $n \geqslant 1$, so P has a root in $\mathcal{G}^k$.

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If $\deg_Y P(X, Y)$ is odd, then so is $\deg P(a, Y)$. Hence $n \geqslant 1$, so P has a root in $\mathcal{G}^k$.

If $P = Y^2 - f$, $f > 0$, then take a with $f(a) > 0$. Hence $n = 2$, so P has a root in $\mathcal{G}^k$. □