Lesson 6 — Linear differential equations over $\mathbb{T}$

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Transbases and differentiation

Let $\mathfrak{B} = (\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$ be a transbasis of level $l \in \mathbb{Z}$, i.e. $\mathfrak{b}_1 = \exp_l x$.

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Proof. Recall that $f\lll g \Leftrightarrow \log f\prec \log g \Leftrightarrow (\log f)'\prec(\log g)' \Leftrightarrow f^{\dagger}\prec g^{\dagger}$, for $f,g\in\mathbb{T}^{\neq 1}$.

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If $l = 1$, then $f^\dagger \succcurlyeq 1$ for all $f \in \mathbb{R}[[\mathfrak{B}^{\mathbb{R}}]]^{\not\asymp 1}$.

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If $f \not\asymp 1$, then $f \mathrel{\underline{\asymp}} \mathfrak{b}_i$ for some i, whence $f^\dagger \asymp \mathfrak{b}_i^\dagger \succcurlyeq \mathfrak{b}_1^\dagger = 1$. □

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Proposition

$\mathfrak{B}\uparrow$ and $\mathfrak{B}\downarrow$ are transbases of levels $l + 1$ and $l - 1$.

Upward shifting

Proposition

Given $L = L_r \partial^r + \cdots + L_0 \in \mathbb{T}[\partial]$*, there exists a unique* $L{\uparrow} \in \mathbb{T}[\partial]$ *such that, for all* $f \in \mathbb{T}$*,*

$$(L{\uparrow})(f{\uparrow}) \;=\; (Lf){\uparrow}.$$

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(\partial^2{\uparrow})(g) &= (g \circ \log)'' \circ \exp = \left(\frac{g'' \circ \log - g' \circ \log}{x^2}\right) \circ \exp = \mathrm{e}^{-2x} (\partial^2 g - \partial g) \\
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\partial^4\!\uparrow &= \mathrm{e}^{-4x}(\partial^4 - 6\partial^3 - 11\partial^2 + 6\partial)\\
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(The coefficients are Stirling numbers of the first kind.) □

Getting rid of logarithms

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$$L = (\log x)\,\partial^2 + x^x\,\partial - e^x$$

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Proposition

If $\exp_l x, \exp_{l+1} x, \ldots, x \in \mathbb{R}[[\mathfrak{B}^{\mathbb{R}}]]$ *and* $L \in \mathbb{R}[[\mathfrak{B}^{\mathbb{R}}]][\partial]$ *for* $\mathfrak{B}$ *of level* $l \leqslant 0$*,*
then $\exp_{l+1} x, \ldots, x \in \mathbb{R}[[\mathfrak{B}{\uparrow}^{\mathbb{R}}]]$ *and* $L{\uparrow} \in \mathbb{R}[[\mathfrak{B}{\uparrow}^{\mathbb{R}}]][\partial]$ *for* $\mathfrak{B}{\uparrow}$ *of level* $l+1$.

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Proposition

If $L \in \mathbb{R}[[\mathfrak{B}^{\mathbb{R}}]][\partial]$ *for* $\mathfrak{B} = (\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$ *of level* 1,
then $L{\uparrow} \in \mathbb{R}[[\hat{\mathfrak{B}}^{\mathbb{R}}]][\partial]$ *for* $\hat{\mathfrak{B}} = (\mathrm{e}^x, \mathfrak{b}_1{\uparrow}, \ldots, \mathfrak{b}_n{\uparrow})$ *of level* 1.

Riccati polynomials

Notation. $K\{Y\} := K[Y, Y', \ldots]$ ring of differential polynomials over differential field K.

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For any $L = L_r \partial^r + \cdots + L_0 \in \mathbb{T}[\partial]$, *there exists a unique* $R_L \in \mathbb{T}\{W\}$ *with*

$$LY = Y R_L(Y^\dagger).$$

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Proof. We have $R_L = L_r R_r + \cdots + L_0 R_0$, with $R_k := R_{\partial^k} \in \mathbb{Q}\{W\}$ as follows:

$$Y = Y \qquad\qquad R_0 = 1$$

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$$\begin{aligned} Y &= Y \\ Y' &= Y Y^\dagger \\ Y'' &= Y((Y^\dagger)^2 + (Y^\dagger)') \\ Y''' &= Y((Y^\dagger)^3 + 3Y^\dagger (Y^\dagger)' + (Y^\dagger)'') \end{aligned} \qquad \begin{aligned} R_0 &= 1 \\ R_1 &= W \\ R_2 &= W^2 + W' \\ R_3 &= W^3 + 3WW' + W'' \end{aligned}$$

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$$\begin{aligned} Y &= Y & R_0 &= 1 \\ Y' &= Y Y^\dagger & R_1 &= W \\ Y'' &= Y((Y^\dagger)^2 + (Y^\dagger)') & R_2 &= W^2 + W' \\ Y''' &= Y((Y^\dagger)^3 + 3Y^\dagger (Y^\dagger)' + (Y^\dagger)'') & R_3 &= W^3 + 3WW' + W'' \\ Y^{(k)} &= Y R_k(Y^\dagger) & R_k &= W R_{k-1} + R'_{k-1} \end{aligned}$$

□

Riccati polynomials

Notation. $K\{Y\} := K[Y, Y', \ldots]$ ring of differential polynomials over differential field K.

Proposition

For any $L = L_r \partial^r + \cdots + L_0 \in \mathbb{T}[\partial]$, *there exists a unique* $R_L \in \mathbb{T}\{W\}$ *with*

$$LY = Y R_L(Y^\dagger).$$

Proof. We have $R_L = L_r R_r + \cdots + L_0 R_0$, with $R_k := R_{\partial^k} \in \mathbb{Q}\{W\}$ as follows:

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If $L_r \neq 0$, then R_L has order $\max(r-1, 0)$ and degree r.

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If P has degree d, then this yields $P(w) \preccurlyeq w^{d[c]}$, since $w \succcurlyeq 1$. $\square$

Multiplicative conjugation

Proposition

Given $L \in \mathbb{T}[\partial]$ *and* $\varphi \in \mathbb{T}^{\neq 0}$*, there exists a unique* $L_{\times\varphi} \in \mathbb{T}[\partial]$ *such that, for all* $f \in \mathbb{T}$*,*

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If $L = L_r \partial^r + \cdots + L_0$, then $L_{\times\varphi} = L_r \partial^r_{\times\varphi} + \cdots + L_0 1_{\times\varphi}$.

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1_{\times\varphi} &= \varphi \\
\partial_{\times\varphi} &= \varphi \partial + \varphi' \\
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\partial^3_{\times\varphi} &= \varphi \partial^3 + 3\varphi' \partial^2 + 3\varphi'' \partial + \varphi''' \\
&\vdots
\end{aligned}$$

Multiplicative conjugation — continued

Notation. $\mathfrak{d}(L)$ and $D(L)$:= dominant monomial and coefficient of $L \in \mathbb{T}[\partial] \subseteq \mathbb{R}[\partial][[\mathfrak{T}]]$.

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For some $n \in \mathbb{N}$, we have $R_i(\varphi^{\dagger}) \preccurlyeq (\varphi^{\dagger})^n$ for $i = 0, \ldots, r$. Hence

$$\mathfrak{d}(L_{\times\varphi}) \preccurlyeq \varphi\,\mathfrak{d}(L)\,(\varphi^{\dagger})^n.$$

$$\mathfrak{d}(L) = \mathfrak{d}(L_{\times\varphi,\times\varphi^{-1}}) \preccurlyeq \varphi^{-1}\mathfrak{d}(L_{\times\varphi})\,(-\varphi^{\dagger})^n.$$

□

Equalizing

$$Ly = (e^{-x}\partial^2 + \partial + 3)\,y = e^{e^x}$$

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$$L_{\times \mathrm{e}^{\mathrm{e}^x - x}} v = \mathrm{e}^{\mathrm{e}^x}(\mathrm{e}^{-2x}\partial^2 + (3\mathrm{e}^{-x} - 2\mathrm{e}^{-2x})\partial + 2 + \mathrm{e}^{-x} + \mathrm{e}^{-2x})\,v = \mathrm{e}^{\mathrm{e}^x}$$

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Equalizing

$$Ly \;=\; (\mathrm{e}^{-x}\partial^2+\partial+3)\,y \;=\; \mathrm{e}^{\mathrm{e}^x}$$

$$\mathfrak{d}(L) \;:=\; \mathfrak{d}_L \;=\; 1, \qquad y \;=\; \mathrm{e}^{\mathrm{e}^x}u$$

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The equalizer lemma

If $L \in \mathbb{R}[[\mathfrak{B}^{\mathbb{R}}]][\partial]$ and $\varphi \in \mathbb{R}[[\mathfrak{M}^{\mathbb{R}}]]$ for $\mathfrak{B}$ of level 1, then $L_{\times\varphi} \in \mathbb{R}[[\mathfrak{B}^{\mathbb{R}}]][\partial]$.

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Proposition

Let $\mathfrak{B}$ be a transbasis of level 1 and $L \in \mathbb{R}[[\mathfrak{B}^{\mathbb{R}}]][\partial]$. Then we have the increasing bijection

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- If $i = 0$, then $\mathfrak{v} = 1$ and we take $\mathfrak{m} := 1$.
- Otherwise, $\mathfrak{n} / \mathfrak{d}(L_{\times \mathfrak{v}}) = \mathfrak{v}\, \mathfrak{d}(L) / \mathfrak{d}(L_{\times \mathfrak{v}}) \preccurlyeq \mathfrak{v}^{\dagger} \lll \mathfrak{v}$.

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Hence there exists a $\mathfrak{w} \in \mathfrak{B}^{\mathbb{R}}$ with $\mathfrak{d}(L_{\times \mathfrak{v}, \times \mathfrak{w}}) = \mathfrak{n}$, and we take $\mathfrak{m} := \mathfrak{v}\, \mathfrak{w}$. □

Dominant equation after equalizing

Note. For $L := \partial \in \mathbb{R}[\partial] \subseteq \mathbb{R}[[e^x]][\partial]$, the solutions of $Ly = 1$ live in $\mathbb{R}[x] \subseteq \mathbb{R}[[x; e^x]]$.

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Lemma CC (constant coefficients)

Let $L = L_r \partial^r + \cdots + L_s \partial^s \in \mathbb{R}[\partial]$ *and* $g \in \mathbb{R}[x]$ *be such that* $L_r \neq 0$, $L_s \neq 0$, *and* $r > s$. *Then*

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Now $y = \varphi + \tilde{y} \in x^s \mathbb{R}[x]$ satisfies $Ly = g$ and $\deg y = d + s$. $\square$

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Definition

Let $\mathfrak{M}$ be a totally ordered group and let $L: \mathbb{R}[[\mathfrak{M}]] \to \mathbb{R}[[\mathfrak{M}]]$ be strongly linear. Then

$$\operatorname{supp}_* L \;:=\; \bigcup_{\mathfrak{m} \in \mathfrak{M}} \mathfrak{m}^{-1} \operatorname{supp} L(\mathfrak{m})$$

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Proof. Let $\mathfrak{G} := \operatorname{supp} \mathfrak{b}_1^\dagger \cup \cdots \cup \operatorname{supp} \mathfrak{b}_n^\dagger$. Then we have seen that $\operatorname{supp}_* \partial \subseteq \mathfrak{G}$.
Now if $L = L_r \partial^r + \cdots + L_0$, then $\operatorname{supp}_* L \subseteq (\operatorname{supp} L_r) \mathfrak{G}^r \cup \cdots \cup \operatorname{supp} L_0$. □

Twisting

From now on: $\mathfrak{B} = (\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$ a transbasis of level 1 and $L = L_r \partial^r + \cdots + L_0 \in \mathbb{R}[[\mathfrak{B}^{\mathbb{R}}]][\partial]$.

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Assume $K, L \in \mathbb{R}[[\mathfrak{B}^{\mathbb{R}}]][\partial]$. In the ring $(\mathbb{R}[[\mathfrak{B}^{\mathbb{R}}]][\partial], +, \cdot)$, we have

$$(K \cdot L)_{\ltimes\varphi} \;=\; \varphi^{-1} \cdot K \cdot L \cdot \varphi \;=\; \varphi^{-1} \cdot K \cdot \varphi \cdot \varphi^{-1} \cdot L \cdot \varphi \;=\; K_{\ltimes\varphi} \cdot L_{\ltimes\varphi}.$$

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Thus $L\mapsto L_{\ltimes\varphi}$ is a skew endomorphism with $\partial_{\ltimes\varphi}=\partial+\varphi^{\dagger}$:

$$L_{\ltimes\varphi}(\partial) \;=\; L(\partial+\varphi^{\dagger}) \;=\; L_r(\partial+\varphi^{\dagger})^r+\cdots+L_1(\partial+\varphi^{\dagger})+L_0.$$

Transseries with parameterized coefficients

$$\operatorname{supp}_* L \subseteq \bigcup_{\mathfrak{m} \in \mathfrak{B}^{\mathbb{R}}} \operatorname{supp} L_{\ltimes \mathfrak{m}}$$

$$\begin{aligned} L_{\ltimes \mathfrak{b}_1^{\lambda_1} \cdots \mathfrak{b}_n^{\lambda_n}} &= L(\partial + (\mathfrak{b}_1^{\lambda_1} \cdots \mathfrak{b}_n^{\lambda_n})^{\dagger}) \\ &= L(\partial + \lambda_1 \mathfrak{b}_1^{\dagger} + \cdots + \lambda_n \mathfrak{b}_n^{\dagger}) \\ &= L_r (\partial + \lambda_1 \mathfrak{b}_1^{\dagger} + \cdots + \lambda_n \mathfrak{b}_n^{\dagger})^r + \cdots + L_0 \\ &\in \mathbb{R}[\lambda_1, \ldots, \lambda_n][[\mathfrak{B}^{\mathbb{R}}]][\partial] \end{aligned}$$

Transseries with parameterized coefficients

$$(\alpha^2-\beta)\mathrm{e}^{\mathrm{e}^x} + (\alpha^2-1)\mathrm{e}^x + 3(\alpha^2-1)x^5 + (\alpha+1)^4x^3 + (\alpha+\beta)x^2 + \alpha x + \cdots$$

$\alpha=2, \beta=4$

$$3\mathrm{e}^x + 9x^5 + 81x^3 + 4x^2 + 2x + \cdots$$

$\alpha=1, \beta=2$

$$-\mathrm{e}^{\mathrm{e}^x} + 16x^3 + 2x^2 + x + \cdots$$

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A finiteness lemma

Lemma

The set $\mathfrak{F} := \{\mathfrak{d}(L_{\ltimes\mathfrak{m}}) : \mathfrak{m} \in \mathfrak{B}^{\mathbb{R}}\}$ *is finite.*

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Proof. Given $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, we have

$$L_{\ltimes\mathfrak{m}} \; = \; L_r(\partial + \lambda_1 \mathfrak{b}_1^\dagger + \cdots + \lambda_n \mathfrak{b}_n^\dagger) + \cdots + L_0 \; \in \; \mathbb{R}[\lambda_1, \ldots, \lambda_n][[\mathfrak{B}^{\mathbb{R}}]].$$

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Given $\mathfrak{v} \in \mathfrak{B}^{\mathbb{R}}$, the following sets are Zariski closed:

$$\begin{aligned} Z_{\mathfrak{v}} &:= \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \mathfrak{d}(L_{\ltimes \mathfrak{b}_1^{\lambda_1} \cdots \mathfrak{b}_n^{\lambda_n}}) \preccurlyeq \mathfrak{v}\} \\ Z_{\mathfrak{v}}^* &:= \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \mathfrak{d}(L_{\ltimes \mathfrak{b}_1^{\lambda_1} \cdots \mathfrak{b}_n^{\lambda_n}}) \prec \mathfrak{v}\}. \end{aligned}$$

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Clearly, $\mathfrak{v} \prec \mathfrak{w} \Longrightarrow Z_{\mathfrak{v}} \supseteq Z_{\mathfrak{v}}^* \supseteq Z_{\mathfrak{w}} \supseteq Z_{\mathfrak{w}}^*$.

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Since $\mathbb{R}[\lambda_1, \ldots, \lambda_n]$ is noetherian, the set of $\mathfrak{v}$ with $Z_{\mathfrak{v}} \supsetneq Z_{\mathfrak{v}}^{*}$ is finite. □

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$$\mathfrak{H}_L \;\ni\; \mathrm{e}^x \;,\; \mathrm{e}^{-x}$$

$$y \;=\; {}^1\!/_2\, x\, \mathrm{e}^x \;-\; 1 \;-\; {}^1\!/_2\, x\, \mathrm{e}^{-x} \;+\; {}^1\!/_3\, \mathrm{e}^{-2x} \;+\; {}^1\!/_8\, \mathrm{e}^{-3x} \;+\; \cdots$$

Proof of existence

We will show that $\operatorname{supp} y \subseteq \mathfrak{S} := \mathfrak{V}\mathfrak{W}^* \mathfrak{G} \setminus \mathfrak{H}_L$, where $\mathfrak{G} := \operatorname{supp} g$ and

$$\begin{aligned} \mathfrak{V} &:= x^{r-\mathbb{N}} \{\mathfrak{d}(L_{\ltimes\mathfrak{m}})^{-1} : \mathfrak{m} \in \mathfrak{B}^{\mathbb{R}}\} \\ \mathfrak{W} &:= x^{r-\mathbb{N}} \bigcup_{\mathfrak{m} \in \mathfrak{B}^{\mathbb{R}}} \frac{\operatorname{supp} L_{\ltimes\mathfrak{m}} \setminus \{\mathfrak{d}(L_{\ltimes\mathfrak{m}})\}}{\mathfrak{d}(L_{\ltimes\mathfrak{m}})} \prec 1. \end{aligned}$$

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Let $f \in \mathbb{R}[[x^{\mathbb{N}} \mathfrak{B}^{\mathbb{R}}]]$, $h := g - Lf$, $\operatorname{supp} f \subseteq \mathfrak{S}$, $\operatorname{supp} h \subseteq \mathfrak{W}^* \mathfrak{G}$.

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Lemma EQ $\Longrightarrow$ unique $\mathfrak{m} \in \mathfrak{B}^{\mathbb{R}}$ with $\mathfrak{d}(L_{\times\mathfrak{m}}) = \mathfrak{n}$. Note that $\mathfrak{m} = \mathfrak{n} / \mathfrak{d}(L_{\ltimes\mathfrak{m}})$.

Proof of existence

We will show that $\operatorname{supp} y \subseteq \mathfrak{S} := \mathfrak{V}\mathfrak{W}^* \mathfrak{G} \setminus \mathfrak{H}_L$, where $\mathfrak{G} := \operatorname{supp} g$ and

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Lemma CC $\Longrightarrow$ unique $c \in \mathbb{R}[x]$ with $L_{\times \mathfrak{m}} c - h \prec \mathfrak{n}$ and $c_i = 0$ for $i < \mu(L_{\times \mathfrak{m}})$.

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$\operatorname{supp} c \mathfrak{m}$

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$\operatorname{supp} L_{\times\mathfrak{m}}\, c$

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$\operatorname{supp} L_{\times\mathfrak{m}} c = \mathfrak{m} \operatorname{supp} L_{\ltimes\mathfrak{m}} c$

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Let $f \in \mathbb{R}[[x^{\mathbb{N}} \mathfrak{B}^{\mathbb{R}}]]$, $h := g - Lf$, $\operatorname{supp} f \subseteq \mathfrak{S}$, $\operatorname{supp} h \subseteq \mathfrak{W}^* \mathfrak{G}$. Let $x^{\nu} \mathfrak{n} := \mathfrak{d}_h$, $\mathfrak{n} \in \mathfrak{B}^{\mathbb{R}}$.

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Lemma CC $\Longrightarrow$ unique $c \in \mathbb{R}[x]$ with $L_{\times\mathfrak{m}} c - h \prec \mathfrak{n}$ and $\operatorname{supp} c\,\mathfrak{m} \cap \mathfrak{H}_L = \emptyset$.

$\Longrightarrow \deg c \leqslant \deg h + \mu(L_{\times\mathfrak{m}}) \leqslant \deg h + r = \nu + r$.

$\operatorname{supp} c\,\mathfrak{m} = \operatorname{supp} c\,\mathfrak{n} / \mathfrak{d}(L_{\ltimes\mathfrak{m}}) \subseteq x^{\nu+r-\mathbb{N}} \mathfrak{n} / \mathfrak{d}(L_{\ltimes\mathfrak{m}}) = (x^{\nu} \mathfrak{n})(x^{r-\mathbb{N}} / \mathfrak{d}(L_{\ltimes\mathfrak{m}})) \subseteq \mathfrak{d}_h \mathfrak{V} \subseteq \mathfrak{V}\mathfrak{W}^* \mathfrak{G}$.

$\operatorname{supp} L_{\times\mathfrak{m}} c = \mathfrak{m} \operatorname{supp} L_{\ltimes\mathfrak{m}} c \subseteq (\mathfrak{n} / \mathfrak{d}(L_{\ltimes\mathfrak{m}}))\, x^{\nu+r-\mathbb{N}} (\operatorname{supp} L_{\ltimes\mathfrak{m}} \setminus \{\mathfrak{d}(L_{\ltimes\mathfrak{m}})\}) \subseteq x^{\nu} \mathfrak{n} \mathfrak{W} \subseteq \mathfrak{W}^* \mathfrak{G}$.

Proof of existence

We will show that $\operatorname{supp} y \subseteq \mathfrak{S} := \mathfrak{V}\mathfrak{W}^* \mathfrak{G} \setminus \mathfrak{H}_L$, where $\mathfrak{G} := \operatorname{supp} g$ and

$$\begin{aligned}\mathfrak{V} &:= x^{r-\mathbb{N}} \{\mathfrak{d}(L_{\ltimes\mathfrak{m}})^{-1} : \mathfrak{m} \in \mathfrak{B}^{\mathbb{R}}\}\\ \mathfrak{W} &:= x^{r-\mathbb{N}} \bigcup_{\mathfrak{m}\in\mathfrak{B}^{\mathbb{R}}} \frac{\operatorname{supp} L_{\ltimes\mathfrak{m}} \setminus \{\mathfrak{d}(L_{\ltimes\mathfrak{m}})\}}{\mathfrak{d}(L_{\ltimes\mathfrak{m}})} \prec 1.\end{aligned}$$

Let $f \in \mathbb{R}[[x^{\mathbb{N}} \mathfrak{B}^{\mathbb{R}}]]$, $h := g - Lf$, $\operatorname{supp} f \subseteq \mathfrak{S}$, $\operatorname{supp} h \subseteq \mathfrak{W}^* \mathfrak{G}$. Let $x^{\nu} \mathfrak{n} := \mathfrak{d}_h$, $\mathfrak{n} \in \mathfrak{B}^{\mathbb{R}}$.

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$\Longrightarrow \deg c \leqslant \deg h + \mu(L_{\times\mathfrak{m}}) \leqslant \deg h + r = \nu + r$.

$\operatorname{supp} c\,\mathfrak{m} = \operatorname{supp} c\,\mathfrak{n} / \mathfrak{d}(L_{\ltimes\mathfrak{m}}) \subseteq x^{\nu+r-\mathbb{N}}\, \mathfrak{n} / \mathfrak{d}(L_{\ltimes\mathfrak{m}}) = (x^{\nu}\, \mathfrak{n})\, (x^{r-\mathbb{N}} / \mathfrak{d}(L_{\ltimes\mathfrak{m}})) \subseteq \mathfrak{d}_h \mathfrak{V} \subseteq \mathfrak{V}\mathfrak{W}^* \mathfrak{G}$.

$\operatorname{supp} L_{\times\mathfrak{m}}\, c = \mathfrak{m} \operatorname{supp} L_{\ltimes\mathfrak{m}}\, c \subseteq (\mathfrak{n} / \mathfrak{d}(L_{\ltimes\mathfrak{m}}))\, x^{\nu+r-\mathbb{N}} (\operatorname{supp} L_{\ltimes\mathfrak{m}} \setminus \{\mathfrak{d}(L_{\ltimes\mathfrak{m}})\}) \subseteq x^{\nu}\, \mathfrak{n} \mathfrak{W} \subseteq \mathfrak{W}^* \mathfrak{G}$.

Hence $\tilde{f} := f + c\,\mathfrak{m}$, $\tilde{h} := g - L\tilde{f} = h - L\,(c\,\mathfrak{m})$ satisfy $\operatorname{supp} \tilde{f} \subseteq \mathfrak{S}$, $\operatorname{supp} L\,\tilde{h} \subseteq \mathfrak{W}^* \mathfrak{G}$.

Proof of existence — continued

We will show that $\operatorname{supp} y \subseteq \mathfrak{S} := \mathfrak{V}\mathfrak{W}^* \mathfrak{G} \setminus \mathfrak{H}_L$, where $\mathfrak{G} := \operatorname{supp} g$ and

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We compute $y_{\succ x^{\mathbb{N}}\mathfrak{m}}$ and $y_{\succcurlyeq \mathfrak{m}}$ by transfinite induction on $\mathfrak{m} \in \mathfrak{S}^{\sharp} := (\mathfrak{V}\mathfrak{W}^* \mathfrak{G} \cap \mathfrak{B}^{\mathbb{R}})$.

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Proof of existence — continued

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Given $\mathfrak{m} \in \mathfrak{S}^{\sharp}$, assume that **IH**$_{\mathfrak{v}}$ holds for all $\mathfrak{v} \in \mathfrak{S}^{\sharp}$ with $\mathfrak{v} \succ \mathfrak{m}$.

Proof of existence — continued

We will show that $\operatorname{supp} y \subseteq \mathfrak{S} := \mathfrak{V}\mathfrak{W}^* \mathfrak{G} \setminus \mathfrak{H}_L$, where $\mathfrak{G} := \operatorname{supp} g$ and

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Given $\mathfrak{m} \in \mathfrak{S}^\sharp$, assume that **IH**$_{\mathfrak{v}}$ holds for all $\mathfrak{v} \in \mathfrak{S}^\sharp$ with $\mathfrak{v} \succ \mathfrak{m}$.

Then $y_{\succ\mathfrak{m}} = \sum_{\mathfrak{v}\succ\mathfrak{m}} y_{\mathfrak{v}} \mathfrak{v}$ is known, $\operatorname{supp} L y_{\succ\mathfrak{m}} \subseteq \mathfrak{W}^* \mathfrak{G}$

Proof of existence — continued

We will show that $\operatorname{supp} y \subseteq \mathfrak{S} := \mathfrak{V} \mathfrak{W}^* \mathfrak{G} \setminus \mathfrak{H}_L$, where $\mathfrak{G} := \operatorname{supp} g$ and

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Proof of existence — continued

We will show that $\operatorname{supp} y \subseteq \mathfrak{S} := \mathfrak{V}\mathfrak{W}^* \mathfrak{G} \setminus \mathfrak{H}_L$, where $\mathfrak{G} := \operatorname{supp} g$ and

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\end{aligned}$$

We compute $y_{\succ x^{\mathbb{N}}\mathfrak{m}}$ and $y_{\succcurlyeq\mathfrak{m}}$ by transfinite induction on $\mathfrak{m} \in \mathfrak{S}^\sharp := (\mathfrak{V}\mathfrak{W}^* \mathfrak{G} \cap \mathfrak{B}^{\mathbb{R}})$.

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Given $\mathfrak{m} \in \mathfrak{S}^\sharp$, assume that **IH**$_{\mathfrak{v}}$ holds for all $\mathfrak{v} \in \mathfrak{S}^\sharp$ with $\mathfrak{v} \succ \mathfrak{m}$.

Then $y_{\succ\mathfrak{m}} = \sum_{\mathfrak{v} \succ \mathfrak{m}} y_{\mathfrak{v}} \mathfrak{v}$ is known, $\operatorname{supp} L y_{\succ\mathfrak{m}} \subseteq \mathfrak{W}^* \mathfrak{G}$, and for all $\mathfrak{v} \in \mathfrak{S}^\sharp$ with $\mathfrak{v} \succ \mathfrak{m}$:

$$y_{\succ\mathfrak{m}} - y_{\succcurlyeq\mathfrak{v}} \prec \mathfrak{v} \text{ implies } L(y_{\succ\mathfrak{m}} - y_{\succcurlyeq\mathfrak{v}}) \prec \mathfrak{d}(L_{\times\mathfrak{v}}).$$

Proof of existence — continued

We will show that $\operatorname{supp} y \subseteq \mathfrak{S} := \mathfrak{V}\mathfrak{W}^* \mathfrak{G} \setminus \mathfrak{H}_L$, where $\mathfrak{G} := \operatorname{supp} g$ and

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We compute $y_{\succ x^{\mathbb{N}} \mathfrak{m}}$ and $y_{\succcurlyeq \mathfrak{m}}$ by transfinite induction on $\mathfrak{m} \in \mathfrak{S}^{\sharp} := (\mathfrak{V}\mathfrak{W}^* \mathfrak{G} \cap \mathfrak{B}^{\mathbb{R}})$.

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Given $\mathfrak{m} \in \mathfrak{S}^{\sharp}$, assume that **IH**$_{\mathfrak{v}}$ holds for all $\mathfrak{v} \in \mathfrak{S}^{\sharp}$ with $\mathfrak{v} \succ \mathfrak{m}$.

Then $y_{\succ \mathfrak{m}} = \sum_{\mathfrak{v} \succ \mathfrak{m}} y_{\mathfrak{v}} \mathfrak{v}$ is known, $\operatorname{supp} L y_{\succ \mathfrak{m}} \subseteq \mathfrak{W}^* \mathfrak{G}$, and for all $\mathfrak{v} \in \mathfrak{S}^{\sharp}$ with $\mathfrak{v} \succ \mathfrak{m}$:

- $y_{\succ \mathfrak{m}} - y_{\succcurlyeq \mathfrak{v}} \prec \mathfrak{v}$ implies $L(y_{\succ \mathfrak{m}} - y_{\succcurlyeq \mathfrak{v}}) \prec \mathfrak{d}(L_{\times \mathfrak{v}})$.
- $\mathfrak{d}(L_{\times \mathfrak{v}}) \succ g - L y_{\succcurlyeq \mathfrak{v}} = g - L y_{\succ \mathfrak{m}} + L(y_{\succ \mathfrak{m}} - y_{\succcurlyeq \mathfrak{v}})$, whence $\mathfrak{d}(L_{\times \mathfrak{v}}) \succ g - L y_{\succ \mathfrak{m}}$.

Proof of existence — continued

We will show that $\operatorname{supp} y \subseteq \mathfrak{S} := \mathfrak{V}\mathfrak{W}^* \mathfrak{G} \setminus \mathfrak{H}_L$, where $\mathfrak{G} := \operatorname{supp} g$ and

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We compute $y_{\succ x^{\mathbb{N}} \mathfrak{m}}$ and $y_{\succcurlyeq \mathfrak{m}}$ by transfinite induction on $\mathfrak{m} \in \mathfrak{S}^{\sharp} := (\mathfrak{V}\mathfrak{W}^* \mathfrak{G} \cap \mathfrak{B}^{\mathbb{R}})$.

IH$_{\mathfrak{m}}$: $y_{\succcurlyeq \mathfrak{m}} = \sum_{\mathfrak{v} \succcurlyeq \mathfrak{m}} y_{\mathfrak{v}} \mathfrak{v}$ is known, $\operatorname{supp} L y_{\succcurlyeq \mathfrak{m}} \subseteq \mathfrak{W}^* \mathfrak{G}$, and $\mathfrak{d}(L_{\times \mathfrak{m}}) \succ g - L y_{\succcurlyeq \mathfrak{m}}$.

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Proof of existence — continued

We will show that $\operatorname{supp} y \subseteq \mathfrak{S} := \mathfrak{V}\mathfrak{W}^* \mathfrak{G} \setminus \mathfrak{H}_L$, where $\mathfrak{G} := \operatorname{supp} g$ and

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Let $\tilde{\mathfrak{m}} \in \mathfrak{B}^{\mathbb{R}}$ with $\mathfrak{d}(L_{\times\tilde{\mathfrak{m}}}) \asymp g - L y_{\succ\mathfrak{m}}$. We have shown above that $\tilde{\mathfrak{m}} \in \mathfrak{S}^\sharp$, so $\mathfrak{m} \succcurlyeq \tilde{\mathfrak{m}}$.

Proof of existence — continued

We will show that $\operatorname{supp} y \subseteq \mathfrak{S} := \mathfrak{V}\mathfrak{W}^* \mathfrak{G} \setminus \mathfrak{H}_L$, where $\mathfrak{G} := \operatorname{supp} g$ and

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Given $\mathfrak{m} \in \mathfrak{S}^\sharp$, assume that **IH**$_{\mathfrak{v}}$ holds for all $\mathfrak{v} \in \mathfrak{S}^\sharp$ with $\mathfrak{v} \succ \mathfrak{m}$.

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Let $\tilde{\mathfrak{m}} \in \mathfrak{B}^{\mathbb{R}}$ with $\mathfrak{d}(L_{\times\tilde{\mathfrak{m}}}) \asymp g - L y_{\succ\mathfrak{m}}$. We have shown above that $\tilde{\mathfrak{m}} \in \mathfrak{S}^\sharp$, so $\mathfrak{m} \succcurlyeq \tilde{\mathfrak{m}}$.

If $\mathfrak{m} \succcurlyeq \tilde{\mathfrak{m}}$, then $\mathfrak{d}(L_{\times\mathfrak{m}}) \succ \mathfrak{d}(L_{\times\tilde{\mathfrak{m}}}) \asymp g - L y_{\succ\mathfrak{m}}$, so $y_{\succcurlyeq\mathfrak{m}} := y_{\succ\mathfrak{m}}$ satisfies **IH**$_{\mathfrak{m}}$.

Proof of existence — continued

We will show that $\operatorname{supp} y \subseteq \mathfrak{S} := \mathfrak{V}\mathfrak{W}^* \mathfrak{G} \setminus \mathfrak{H}_L$, where $\mathfrak{G} := \operatorname{supp} g$ and

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IH$_\mathfrak{m}$: $y_{\succcurlyeq \mathfrak{m}} = \sum_{\mathfrak{v} \succcurlyeq \mathfrak{m}} y_\mathfrak{v} \mathfrak{v}$ is known, $\operatorname{supp} L y_{\succcurlyeq \mathfrak{m}} \subseteq \mathfrak{W}^* \mathfrak{G}$, and $\mathfrak{d}(L_{\times \mathfrak{m}}) \succ g - L y_{\succcurlyeq \mathfrak{m}}$.

Given $\mathfrak{m} \in \mathfrak{S}^\sharp$, assume that **IH**$_\mathfrak{v}$ holds for all $\mathfrak{v} \in \mathfrak{S}^\sharp$ with $\mathfrak{v} \succ \mathfrak{m}$.

Then $y_{\succ \mathfrak{m}} = \sum_{\mathfrak{v} \succ \mathfrak{m}} y_\mathfrak{v} \mathfrak{v}$ is known, $\operatorname{supp} L y_{\succ \mathfrak{m}} \subseteq \mathfrak{W}^* \mathfrak{G}$, and $\mathfrak{d}(L_{\times \mathfrak{v}}) \succ g - L y_{\succ \mathfrak{m}}$ for $\mathfrak{v} \succ \mathfrak{m}$.

Let $\tilde{\mathfrak{m}} \in \mathfrak{B}^{\mathbb{R}}$ with $\mathfrak{d}(L_{\times \tilde{\mathfrak{m}}}) \asymp g - L y_{\succ \mathfrak{m}}$. We have shown above that $\tilde{\mathfrak{m}} \in \mathfrak{S}^\sharp$, so $\mathfrak{m} \succcurlyeq \tilde{\mathfrak{m}}$.

If $\mathfrak{m} \succ \tilde{\mathfrak{m}}$, then $\mathfrak{d}(L_{\times \mathfrak{m}}) \succ \mathfrak{d}(L_{\times \tilde{\mathfrak{m}}}) \asymp g - L y_{\succ \mathfrak{m}}$, so $y_{\succcurlyeq \mathfrak{m}} := y_{\succ \mathfrak{m}}$ satisfies **IH**$_\mathfrak{m}$.

If $\mathfrak{m} = \tilde{\mathfrak{m}}$, then taking $f := y_{\succ \mathfrak{m}}$ above, we obtain $y_{\succcurlyeq \mathfrak{m}} := \tilde{f}$ that satisfies **IH**$_\mathfrak{m}$. □

Proof of strong linearity

For any $\mathfrak{m} \in x^{\mathbb{N}} \mathfrak{B}^{\mathbb{R}}$, we have shown that $\operatorname{supp} L^{-1} \mathfrak{m} \subseteq \mathfrak{V} \mathfrak{W}^* \mathfrak{m}$.

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Consider a grid-based subset $\mathfrak{G} \subseteq x^{\mathbb{N}} \mathfrak{B}^{\mathbb{R}}$ and let $\mathfrak{v} \in \mathfrak{V} \mathfrak{W}^* \mathfrak{G}$.

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Consider a grid-based subset $\mathfrak{G} \subseteq x^{\mathbb{N}} \mathfrak{B}^{\mathbb{R}}$ and let $\mathfrak{v} \in \mathfrak{V} \mathfrak{W}^* \mathfrak{G}$.

Then $\{(\mathfrak{m}, \mathfrak{v} / \mathfrak{m}) : \mathfrak{m} \in \mathfrak{G}, \mathfrak{v} \in \operatorname{supp} L^{-1} \mathfrak{m}\}$ is a finite antichain of $\mathfrak{G} \times \mathfrak{V} \mathfrak{W}^*$.

Proof of strong linearity

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We conclude that $L^{-1} = \hat{\Phi}$ is strongly linear. □

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Theorem

Let $L \in \mathbb{T}[\mathrm{i}][\partial] = \mathbb{C}[[\mathfrak{T}]][\partial]$. *Then* $R_L(y^\dagger) = 0$ *has a solution* $y^\dagger \in \mathbb{T}[\mathrm{i}]$.
If $L \in \mathbb{T}[\partial]$ *has odd order, then* $R_L(y^\dagger) = 0$ *has a solution* $y^\dagger \in \mathbb{T}$.

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Any $L\in\mathbb{T}[\mathrm{i}][\partial]$ has a fundamental system of solutions in $\mathbb{T}[\mathrm{i}][\mathrm{e}^{\mathbb{T}_{\succ}[\mathrm{i}]}]$.

Complement — factorization

The skew ring $\mathbb{T}[\partial]$ is an Euclidean ring.

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a) Any $L \in \mathbb{T}[\mathrm{i}][\partial]$ splits into order one factors.

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Application

The following equation has a non-zero solution in $\mathbb{T}$:

$$x^{x^x} y''' - (x^{\Gamma(x)} + 3) y' - (\log \log \log x - 1) y = 0.$$