Lesson 7 — Algebraic differential equations over $\mathbb{T}$

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IMS summer school

Singapore, July 13, 2023

Differential polynomials over $\mathbb{T}$

Differential polynomials as series. $P \in \mathbb{T}\{Y\} \subseteq \mathbb{R}\{Y\}[[\mathfrak{T}]]$

$\operatorname{supp} P$	support of P
$\mathfrak{d}(P) \in \mathfrak{T}$	dominant monomial of P
$D(P) \in \mathbb{R}\{Y\}$	dominant coefficient or "part" of P
$\preccurlyeq, \prec, \asymp, \ldots$	extend to $\mathbb{T}\{Y\}$

Standard decomposition

Standard decomposition. $P \in \mathbb{T}\{Y\}$ of order r.

$$P = \sum_{\boldsymbol{i}=(i_0,\ldots,i_r)\in\mathbb{N}^{r+1}} P_{\boldsymbol{i}} Y^{\boldsymbol{i}}, \qquad Y^{\boldsymbol{i}} := Y^{i_0}(Y')^{i_1}\cdots(Y^{(r)})^{i_r}.$$

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Degree and valuation.

$$\deg P := \max\{|\boldsymbol{i}| : P_{\boldsymbol{i}} \neq 0\}$$
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Decomposition in homogeneous parts. $P \in \mathbb{T}\{Y\}$ of degree d

$$P = P_d + \cdots + P_0, \qquad P_k := \sum_{|\boldsymbol{i}|=k} P_{\boldsymbol{i}} Y^{\boldsymbol{i}}, \qquad |\boldsymbol{i}| := i_0 + \cdots + i_r.$$

Additive conjugation

$P \in \mathbb{T}\{Y\}, \varphi \in \mathbb{T}$

$$P_{+\varphi}(y) \;=\; P(y+\varphi)$$

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If P has order r, then for any $\boldsymbol{i} = (i_0, \ldots, i_r) \in \mathbb{N}^{r+1}$,

$$\begin{aligned} P_{+\varphi,\boldsymbol{i}} \;&=\; P^{(\boldsymbol{i})}(\varphi) \;=\; \frac{\partial P}{\partial Y^{i_0} \cdots (\partial Y^{(r)})^{i_r}}(\varphi) \\ &=\; \sum_{\boldsymbol{j} \geqslant \boldsymbol{i}} \binom{\boldsymbol{j}}{\boldsymbol{i}} \varphi^{\boldsymbol{j}-\boldsymbol{i}} P_{\boldsymbol{j}} \;=\; \sum_{j_0 \geqslant i_0, \ldots, j_r \geqslant i_r} \binom{j_0}{i_0} \cdots \binom{j_r}{i_r} \varphi^{i_0} (\varphi')^{i_1} \cdots (\varphi^{(r)})^{i_r} P_{\boldsymbol{j}}. \end{aligned}$$

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Proposition

If $\varphi = c + \varepsilon$ with $c \in \mathbb{R}$ and $\varepsilon \prec 1$, then

$$\begin{aligned} P_{+\varphi} &\asymp P \\ D(P_{+\varphi}) &= D(P)_{+c}. \end{aligned}$$

Decomposition by orders

Decomposition by orders. P of order r and degree d.

$$P = \sum_{\substack{\boldsymbol{\omega}=(\omega_1,\dots,\omega_l)\\ l\leqslant d}} P_{[\boldsymbol{\omega}]} Y^{[\boldsymbol{\omega}]}, \qquad Y^{[\boldsymbol{\omega}]} := Y^{(\omega_1)} \dots Y^{(\omega_l)}.$$

Here we assume that $P_{[\boldsymbol{\omega}]} = P_{[\boldsymbol{\tau}]}$ if $\boldsymbol{\tau} = (\omega_{\sigma(1)}, \dots, \omega_{\sigma(l)})$ for some permutation σ.

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Decomposition into isobaric parts. P of weight w

$$P = P_{[w]} + \cdots + P_{[0]}, \qquad P_{[k]} := \sum_{|\omega|=k} P_{[\omega]} Y^{[\omega]}.$$

Multiplicative conjugation

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Proposition

If $\varphi \gg x$, *then* $\dfrac{\mathfrak{d}(P_{\times\varphi})}{\mathfrak{d}(P)} \;\preccurlyeq\; \varphi$.

If $\varphi \gg x$ *and* P *is homogeneous of degree* d, *then* $\dfrac{\mathfrak{d}(P_{\times\varphi})}{\varphi^d\,\mathfrak{d}(P)} \;\ll\; \varphi$.

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where

$$s_{\tau,\omega} \;=\; s_{\tau_1,\omega_1} \cdots s_{\tau_l,\omega_l} \;\in\; \mathbb{Z}, \qquad f(\log x)^{(j)} \;=\; \sum_{0 \leqslant i \leqslant j} \frac{s_{i,j}}{x^j} f^{(j)}(\log x).$$

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Proposition

We have $\dfrac{\mathfrak{d}(P{\uparrow})}{\mathfrak{d}(P){\uparrow}} \;\leqq\; \mathrm{e}^x.$

If P *is isobaric of weight* w*, then* $\dfrac{\mathfrak{d}(P{\uparrow})}{\mathrm{e}^{-wx}\,\mathfrak{d}(P){\uparrow}} \;\ll\; \mathrm{e}^x.$

Getting rid of logarithms

Proposition

If $P \in \mathbb{T}\{Y\}$ *has level* l*, then* $P\uparrow$ *has level at least* $\min(l+1, 1)$.

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If $P \in \mathbb{R}[[\mathfrak{B}^{\mathbb{R}}]]\{Y\}$ *for a transbasis of level* $l \leqslant 0$ *and* $\exp_l x, \exp_{l-1} x, \ldots, x \in \mathbb{R}[[\mathfrak{B}^{\mathbb{R}}]]$*, then* $P\uparrow \in \mathbb{R}[[\mathfrak{B}\uparrow^{\mathbb{R}}]]\{Y\}$*, where* $\mathfrak{B}\uparrow$ *has level* $l+1$ *and* $\exp_{l-1} x, \ldots, x \in \mathbb{R}[[\mathfrak{B}\uparrow^{\mathbb{R}}]]$.

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Proposition

If $P \in \mathbb{R}[[\mathrm{e}^x = \mathfrak{b}_1; \ldots; \mathfrak{b}_n]]\{Y\}$*, then* $P{\uparrow} \in \mathbb{R}[[\mathrm{e}^x; \mathfrak{b}_1{\uparrow}; \ldots; \mathfrak{b}_n{\uparrow}]]\{Y\}$.

Logarithmic decomposition

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y &= y \\
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Proposition

Let $\boldsymbol{i}$ be largest for $\leqslant_{\mathrm{lex}}$ on $\mathbb{N}^{r+1}$ with $P_{\langle \boldsymbol{i} \rangle} \neq 0$. Then for $y \to \infty$, we have $P(y) \sim P_{\langle \boldsymbol{i} \rangle} y^{\langle \boldsymbol{i} \rangle} \neq 0$.

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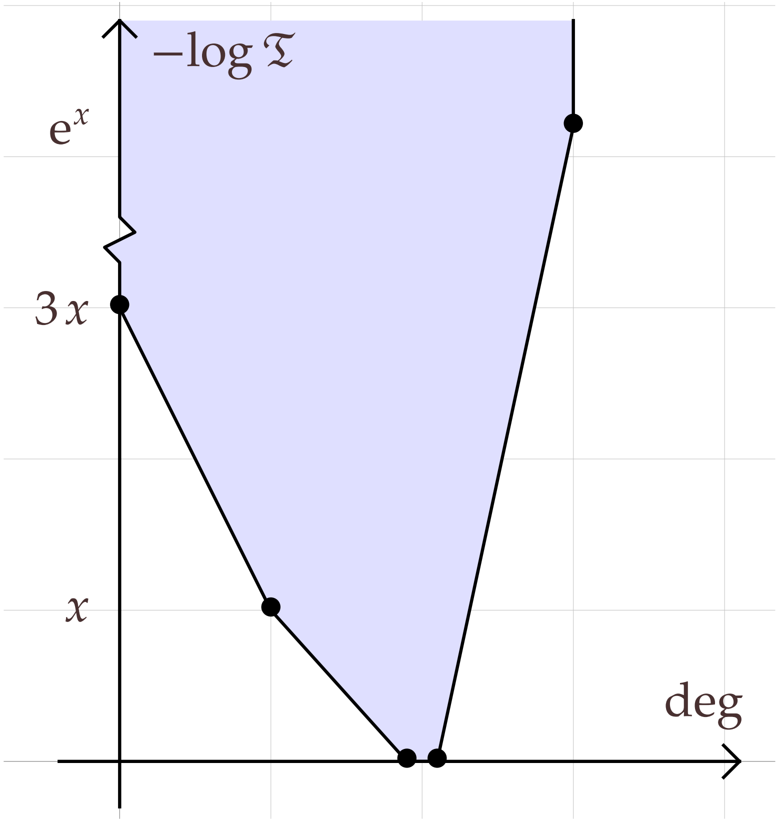
Proof. For large y, we have $y \gg y^\dagger \gg y^{\dagger\dagger} \gg \cdots$. □

Differential Newton polygons

$$\mathrm{e}^{-\mathrm{e}^{x}} y^3 + y y'' - (y')^2 + \mathrm{e}^{-x} y' + \mathrm{e}^{-3x} = 0$$



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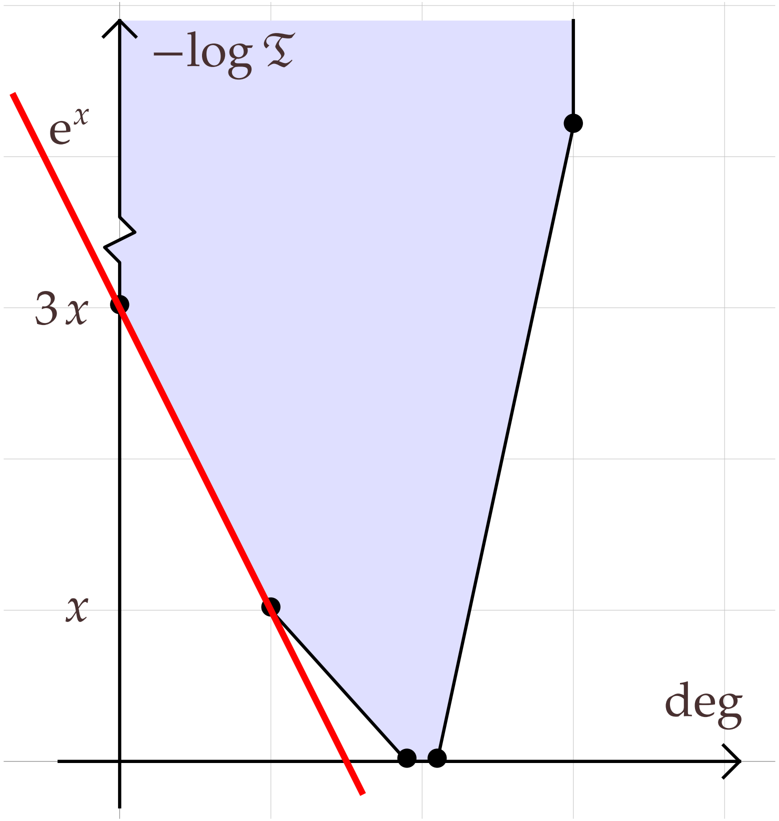
Algebraic starting monomials.

- $y \asymp \mathrm{e}^{-2x}$



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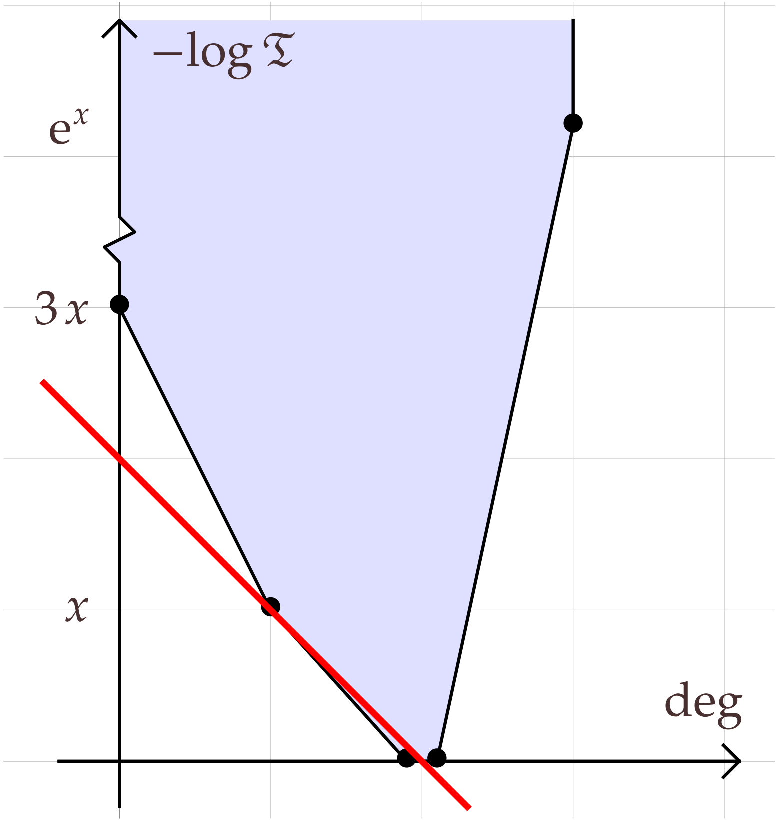

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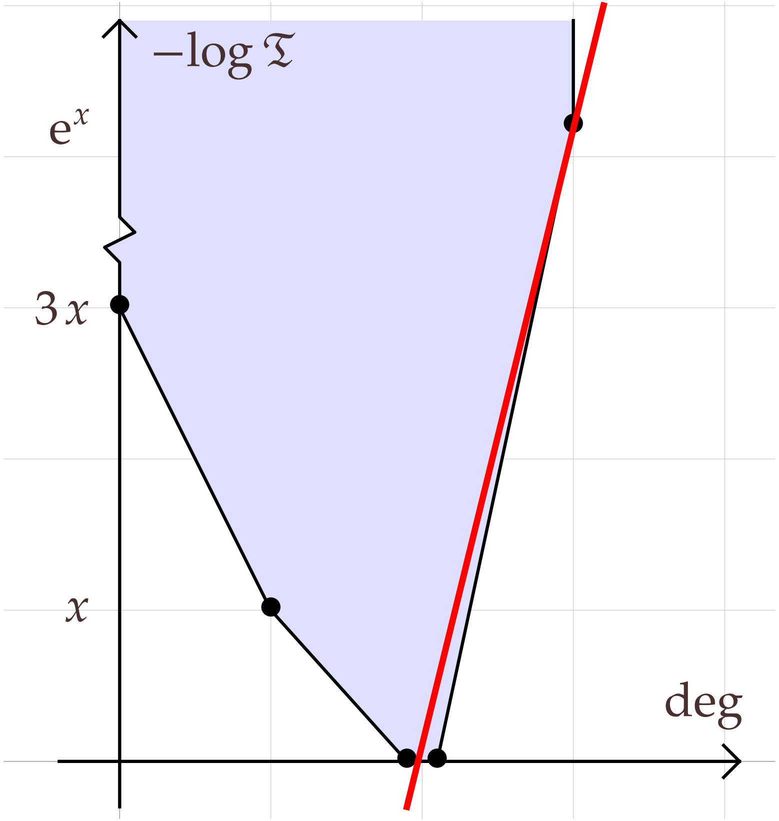

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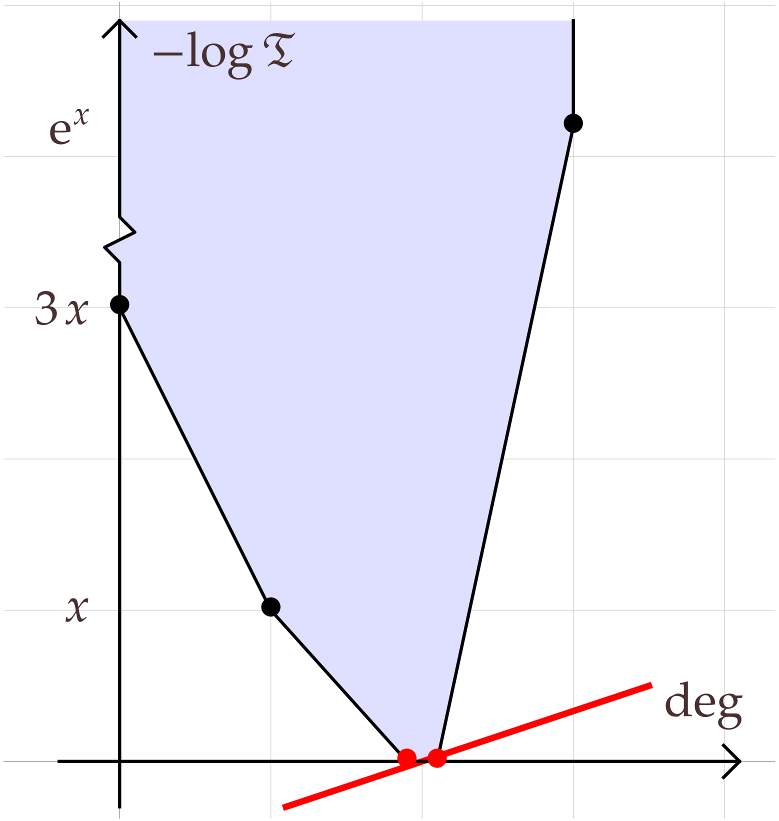

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Differential starting monomials.

- $y \asymp e^{\lambda x}, \lambda > -1$

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$$\begin{aligned} P\uparrow &= \mathrm{e}^{-2x} Y^5 + \mathrm{e}^{-3x} (Y')^3 + \mathrm{e}^{-x} \\ D(P\uparrow) &= 1. \end{aligned}$$

Differential Newton polynomials

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P{\uparrow\uparrow\uparrow} &= e^{-2e^{e^x}}Y^5 + e^{-3e^{e^x}-3e^x-3x}(Y')^3 + e^{-e^{e^x}} & D(P{\uparrow\uparrow\uparrow}) &= 1\\
&\vdots & &\vdots
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Example continued

$$\begin{array}{lcl} P & = & x^{-2}Y^5+(Y')^3+x^{-1} \\ P\uparrow & = & \mathrm{e}^{-2x}Y^5+\mathrm{e}^{-3x}(Y')^3+\mathrm{e}^{-x} \\ P\uparrow\uparrow & = & \mathrm{e}^{-2\mathrm{e}^x}Y^5+\mathrm{e}^{-3\mathrm{e}^x-3x}(Y')^3+\mathrm{e}^{-\mathrm{e}^x} \\ P\uparrow\uparrow\uparrow & = & \mathrm{e}^{-2\mathrm{e}^{\mathrm{e}^x}}Y^5+\mathrm{e}^{-3\mathrm{e}^{\mathrm{e}^x}-3\mathrm{e}^x-3x}(Y')^3+\mathrm{e}^{-\mathrm{e}^{\mathrm{e}^x}} \\ & \vdots & \end{array}$$

$$\begin{array}{lcl} D(P) & = & (Y')^3 \\ D(P\uparrow) & = & 1 \\ D(P\uparrow\uparrow) & = & 1 \\ D(P\uparrow\uparrow\uparrow) & = & 1 \\ & \vdots & \end{array}$$

Theorem DNP

Given $P\in\mathbb{T}\{Y\}$, there exists $l_0\in\mathbb{N}$ and $N(P)\in\mathbb{R}[Y](Y')^{\mathbb{N}}$ with

$$D(P\uparrow_l) = N(P), \qquad \text{for all } l\geqslant l_0.$$

*We call $N(P)$ the **differential Newton polynomial** of P.*

Dominant parts and upward shifting

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Assume that $P \in (\mathbb{E} \circ \exp)\{Y\}$. *Then*

$$D(P{\uparrow}) \;=\; D(D(P){\uparrow}).$$

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$$\begin{aligned} P &= D(P) + O(\mathrm{e}^{-\sqrt{x}}) && (P \in (\mathbb{E} \circ \exp)\{Y\}) \\ P\uparrow &= D(P)\uparrow + O(\mathrm{e}^{-\mathrm{e}^{x/2}}) && (\text{by } (\star)) \\ P\uparrow &\lesssim \mathrm{e}^{x} && (\text{by } (\star)) \end{aligned}$$

Dominant parts and upward shifting

$$P{\uparrow}_{[\omega]} \;=\; \sum_{\tau \geqslant \omega} s_{\tau,\omega} \mathrm{e}^{-|\tau| x} P_{[\tau]}{\uparrow} \qquad (\star)$$

Lemma

Assume that $P \in (\mathbb{E} \circ \exp)\{Y\}$. *Then*

$$D(P{\uparrow}) \;=\; D(D(P){\uparrow}).$$

Proof. Without loss of generality, we may assume that $P \asymp 1$.

$$\begin{aligned} P &= D(P) + O(\mathrm{e}^{-\sqrt{x}}) && (P \in (\mathbb{E} \circ \exp)\{Y\}) \\ P{\uparrow} &= D(P){\uparrow} + O(\mathrm{e}^{-\mathrm{e}^{x/2}}) && (\text{by } (\star)) \\ P{\uparrow} &\lessapprox \mathrm{e}^{x} && (\text{by } (\star)) \\ P{\uparrow} - D(P){\uparrow} &\prec P{\uparrow} && \square \end{aligned}$$

Weight and upward shifting

$$P{\uparrow}_{[\omega]} = \sum_{\tau \geqslant \omega} s_{\tau,\omega} \mathrm{e}^{-|\tau|x} P_{[\tau]}{\uparrow} \qquad (\star)$$

Lemma

a) If $P \in \mathbb{R}\{Y\}$*, then* $\operatorname{wt} P{\uparrow} = \operatorname{wv} P$.

b) If $\operatorname{wv} P{\uparrow} = \operatorname{wv} P$*, then* $P{\uparrow} = \mathrm{e}^{-(\operatorname{wv} P)x} P$ *and* $D(P{\uparrow}) = P$.

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Proof. From $(\star)$, we deduce,

$$P{\uparrow} \;\asymp\; \mathrm{e}^{-(\operatorname{wv} P) x}$$

Weight and upward shifting

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Proof. From $(\star)$, we deduce,

$$P\uparrow \asymp \mathrm{e}^{-(\mathrm{wv}\, P)x}$$
$$D(P\uparrow) = \sum_{|\tau| = \mathrm{wv}\, P,\ \tau \geqslant \omega} s_{\tau,\omega} P_{[\tau]}\uparrow Y^{[\omega]}.$$

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a) If $P \in \mathbb{R}\{Y\}$*, then* $\operatorname{wt} P{\uparrow} = \operatorname{wv} P$.

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$$\begin{aligned} P{\uparrow} &\asymp \mathrm{e}^{-(\operatorname{wv} P) x} \\ D(P{\uparrow}) &= \sum_{|\tau| = \operatorname{wv} P,\ \tau \geqslant \omega} s_{\tau,\omega} P_{[\tau]}{\uparrow} Y^{[\omega]}. \end{aligned}$$

If $\operatorname{wv} P{\uparrow} = \operatorname{wv} P$, then the last formula becomes

$$D(P{\uparrow}) \;=\; \sum_{|\tau| = \operatorname{wv} P,\ \tau = \omega} s_{\tau,\omega} P_{[\tau]}{\uparrow} Y^{[\omega]} \;=\; P. \qquad \square$$

Existence of differential Newton polynomials

$$P\uparrow_{[\omega]} = \sum_{\tau \geqslant \omega} s_{\tau,\omega} \mathrm{e}^{-|\tau| x} P_{[\tau]}\uparrow \qquad (\star)$$

Lemma

Given $P \in \mathbb{T}\{Y\}$, *there exists* $l_0 \in \mathbb{N}$ *and isobaric* $N(P) \in \mathbb{R}\{Y\}$ *with*

$$D(P\uparrow_l) = N(P), \qquad \textit{for all } l \geqslant l_0.$$

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$$P\!\uparrow_{[\omega]} \;=\; \sum_{\tau \geqslant \omega} s_{\tau,\omega} \mathrm{e}^{-|\tau|x} P_{[\tau]}\!\uparrow \qquad (\star)$$

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Proof. The previous two lemmas yield

$$\operatorname{wt} D(P) \;\geqslant\; \operatorname{wv} D(P) \;=\; \operatorname{wt} D(P\!\uparrow) \;\geqslant\; \operatorname{wv} D(P\!\uparrow) \;=\; \operatorname{wt} D(P\!\uparrow\uparrow) \;\geqslant\; \cdots.$$

In other words, $\operatorname{wv} D(P\!\uparrow_l)$ stabilizes for sufficiently large $l \geqslant l_0$.

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When that happens, we have $D(P\uparrow_{l+1}) = D(D(P\uparrow_l)\uparrow) = D(P\uparrow_l)$ for all $l \geqslant l_0$, again by the previous two lemmas

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When that happens, we have $D(P{\uparrow}_{l+1}) = D(D(P{\uparrow}_l){\uparrow}) = D(P{\uparrow}_l)$ for all $l \geqslant l_0$, again by the previous two lemmas, and $D(P{\uparrow}_l)$ is isobaric. □

Special form of differential Newton polynomials

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If $P \in \mathbb{R}\{Y\}$ *is isobaric of weight* w *with* $D(P{\uparrow}) = P$*, then* $P \in \mathbb{R}[Y](Y')^{\mathbb{N}}$.

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Proof. Let $P^* = \sum_{i \in \mathbb{N}} P_{(i,w,0,\ldots,0)} Y^i (Y')^w$.

Assume for contradiction that $\Delta := P - P^* \neq 0$.

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If $P \in \mathbb{R}\{Y\}$ is isobaric of weight w with $D(P\uparrow) = P$, then $P \in \mathbb{R}[Y](Y')^{\mathbb{N}}$.

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Now Δ is isobaric of weight w and $D(\Delta\uparrow) = \Delta$.

From $(\star)$, it follows that $\Delta\uparrow = \mathrm{e}^{-wx} \Delta$.

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If $P \in \mathbb{R}\{Y\}$ *is isobaric of weight* w *with* $D(P{\uparrow}) = P$, *then* $P \in \mathbb{R}[Y](Y')^{\mathbb{N}}$.

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Now Δ is isobaric of weight w and $D(\Delta{\uparrow}) = \Delta$.

From $(\star)$, it follows that $\Delta{\uparrow} = \mathrm{e}^{-wx} \Delta$.

Consequently $\Delta(x) = \Delta(\mathrm{e}^x) = \Delta(\mathrm{e}^{\mathrm{e}^x}) = \cdots = 0$, which is impossible. □

Differential Newton polynomials — continued

Theorem (DNP)

Given $P \in \mathbb{T}\{Y\}$, *there exists* $l_0 \in \mathbb{N}$ *and* $N(P) \in \mathbb{R}[Y](Y')^{\mathbb{N}}$ *with*

$$D(P{\uparrow_l}) = N(P), \qquad \textit{for all } l \geqslant l_0.$$

We call $N(P)$ *the* ***differential Newton polynomial*** *of* P.

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Newton degree. For $P \in \mathbb{T}\{Y\}^{\neq 0}$ and $\mathfrak{m} \in \mathfrak{T}$, we define

$$\deg_{\preccurlyeq \mathfrak{m}} P := \deg N(P_{\times \mathfrak{m}})$$
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$\mathfrak{m}$ is a **starting monomial** for $P(y) = 0$ if $N(P_{\times \mathfrak{m}})(c) = 0$ for some $c \in \mathbb{C}^{\neq 0}$.

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$\mathfrak{m}$ is a **differential starting monomial** if $N(P_{\times \mathfrak{m}}) \in \mathbb{R}[Y](Y')^{\nu}$ for some $\nu > 0$.

Properties of Newton degree

Proposition

For any $P \in \mathbb{T}\{Y\}^{\neq 0}$, *we have* $N(P\uparrow) = N(P)$.

Proof. By construction. □

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For $P \in \mathbb{T}\{Y\}^{\neq 0}$ *and* $\mathfrak{m} \prec \mathfrak{n}$ *in* $\mathfrak{T}$*, we have*

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Proof. By considering $P_{\times \mathfrak{n}}$ instead of P, we may also arrange that $\mathfrak{m} \prec \mathfrak{n} = 1$.

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By what precedes, we also arrange that $N(P) = D(P)$, $N(P_{\times \mathfrak{m}}) = D(P_{\times \mathfrak{m}})$, and $\mathfrak{m} \gg x$.

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Recall that $\partial(Q_{\times \mathfrak{m}}) / (\mathfrak{m}^i \partial(Q)) \ll \mathfrak{m}$ for $Q \in \mathbb{T}\{Y\}$ homogeneous of degree i.

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Recall that $\mathfrak{d}(Q_{\times \mathfrak{m}}) / (\mathfrak{m}^i \mathfrak{d}(Q)) \lll \mathfrak{m}$ for $Q \in \mathbb{T}\{Y\}$ homogeneous of degree i.

For all $i > d := \deg_{\prec 1} P$, it follows that

$$P_{\times \mathfrak{m}, d} \asymp \phi \mathfrak{m}^d P_d \succ \psi \mathfrak{m}^i P_i \asymp P_{\times \mathfrak{m}, i},$$

for some $\phi, \psi \lll \mathfrak{m}$. Hence, $\deg N(P_{\times \mathfrak{m}}) = \deg D(P_{\times \mathfrak{m}}) \leqslant d$. □

The non-linear equalizer lemma

Lemma EQ

Let $P \in \mathbb{T}\{Y\}$ *and* $i < j$ *with* $P_i \neq 0$, $P_j \neq 0$. *Then there exists a unique* $\mathfrak{e} \in \mathfrak{T}$ *for which* $N(P_{i,\times\mathfrak{e}} + P_{j,\times\mathfrak{e}})$ *is not homogeneous. We call* $\mathfrak{e}$ *the* (i,j)-***equalizer*** *for* P.

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Consider an equation $P(y) = 0, y \prec \mathfrak{v}$ of Newton degree $d := \deg_{\prec\mathfrak{v}} P$, with $P_{<d} \neq 0$. Then its **principal equalizer** is the unique equalizer $\mathfrak{e}_{P,\mathfrak{v}} := \mathfrak{e}$ with $\deg N(P_{\times\mathfrak{e}}) = d$.

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Proof. We first arrange that $P \in \mathbb{R}[[\mathfrak{B}^{\mathbb{R}}]]\{Y\}$ for a transbasis $\mathfrak{B}$ of level 1.

Without loss of generality, we may assume that $P = P_i + P_j$.

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As in the proof of Theorem DNP, one may show that

- $\operatorname{wt} D((P\uparrow_l)_{\times\mathfrak{e}(P\uparrow_l)})$ strictly decreases as a function of $l \in \mathbb{N}$, until stabilization.

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- $\operatorname{wt} D((P\uparrow_l)_{\times\mathfrak{e}(P\uparrow_l)})$ strictly decreases as a function of $l \in \mathbb{N}$, until stabilization.

Stabilization occurs when $N((P\uparrow_l)_{\times\mathfrak{e}(P\uparrow_l)}) = D((P\uparrow_l)_{\times\mathfrak{e}(P\uparrow_l)})$ and $\mathfrak{e}\uparrow_l := \mathfrak{e}(P\uparrow_l)$. □

Example

$$P = e^{-e^{x}} Y^3 + Y Y'' - (Y')^2 + e^{-x} Y' + e^{-3x}$$

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$$P_{\times e^{-x}\uparrow} = e^{-3e^x} e^{-e^{e^x}} Y^3 + e^{-2x} e^{-2e^x} (Y Y'' - (Y')^2 - Y Y') + e^{-x} e^{-2e^x} Y' - e^{-2e^x} Y + e^{-3e^x}$$

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Let $P \in \mathbb{T}\{Y\}$ be homogeneous of degree d and of order r.

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Proposition

$\mathfrak{m} \in \mathfrak{T}$ is a starting monomial for $P(y) = 0$ if and only if $\deg_{\prec\gamma} R_P > 0$.

Quasi-linear equations

Let $Q \in \mathbb{T}\{Y\}$ and $\mathfrak{m} \in \mathfrak{T}$. We say that

$$Q(y) = 0, \qquad y \prec \mathfrak{m}$$

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Theorem

Any quasi-linear equation as above has a solution in $\mathbb{T}$.

Moreover, there exists a unique solution such that $y_{\mathfrak{d}(\tilde{y}-y)} = 0$ *for any other solution* $\tilde{y}$*; this is called the* ***distinguished solution***.

Quasi-linear equations — the steep case

Lemma

Let $\mathfrak{B} = (\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$ *be of level* 1 *and* $Q \in \mathbb{R}[[\mathfrak{B}^{\mathbb{R}}]]\{Y\}$ *of order* r *and degree* d*.*

Assume that $Q - Q_1 \prec \mathfrak{b}_n^{-\eta} Q_1$ *for some* $\eta \in \mathbb{R}^{>0}$*.*

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Proof. Let $R:=Q_{\neq 1}:=Q-Q_1$ and $L\in\mathbb{T}[\partial]$ be such that $Q_1=LY$. We want to solve

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Let $\mathfrak{V} := \{1\} \cup \operatorname{supp}_* \partial \cup \cdots \cup \operatorname{supp}_* \partial^r$ and $\mathfrak{W} := (\operatorname{supp} R)(\{1\} \cup \mathfrak{V} \cup \cdots \cup \mathfrak{V}^d)$.

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Now $0, L^{-1} R(0), L^{-1} R(L^{-1} R(0)), \ldots$ converges to a solution with $\operatorname{supp} y \subseteq \mathfrak{S}$. □

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For some small $\eta > 0$, we then have $Q_{+y^\sharp,0} \prec \mathfrak{b}_n^{-3\eta}$ and $Q_{+y^\sharp,\times \mathfrak{b}_n^{-\eta},\neq 1} \prec \mathfrak{b}_n^{-\eta} Q_{+y^\sharp,\times \mathfrak{b}_n^{-\eta}}$.

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Now apply the previous lemma $Q_{+y^{\sharp}, \times \mathfrak{b}_n^{-\eta}} \uparrow_k$ and $(\mathrm{e}^x, \ldots, \exp_k x, \mathfrak{b}_1 \uparrow_k, \ldots, \mathfrak{b}_n \uparrow_k)$.

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This yields $u \in \mathbb{R}[[x; \ldots; \exp_k x; \mathfrak{b}_1{\uparrow_k}; \ldots; \mathfrak{b}_n{\uparrow_k}]]^{\prec 1}$ with $Q_{+y^{\sharp},\times \mathfrak{b}_n^{-\eta}}{\uparrow_k}(u) = 0$.

Quasi-linear equations — continued

Lemma

Let $\mathfrak{B} = (\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$ *be of level* 1 *and* $Q \in \mathbb{R}[[\mathfrak{B}^{\mathbb{R}}]]\{Y\}$ *with* $Q - Q_1 \prec Q_1$.
Then $Q(y) = 0,\ y \prec 1$ *has a solution in* $\mathbb{R}[[(\log_k x)^{\mathbb{R}} \cdots x^{\mathbb{R}} \mathfrak{B}^{\mathbb{R}}]]$ *for some* $k \in \mathbb{N}$.

Proof. Without loss of generality, we may assume that $Q \asymp 1$.
We prove the result by induction on n. For $n = 1$ we are done by what precedes.
Let $Q^{\sharp} \in \mathbb{R}[[\mathfrak{b}_1; \ldots; \mathfrak{b}_{n-1}]]\{Y\}$ be the dominant coefficient of Q as a series in $\mathfrak{b}_n^{-1}$.
Induction hypothesis $\rightsquigarrow k \in \mathbb{N}$, $y^{\sharp} \in \mathbb{R}[[(\log_{k-1} x)^{\mathbb{N}} \cdots x^{\mathbb{N}} \mathfrak{B}^{\mathbb{R}}]]^{\prec 1}$ with $Q^{\sharp}(y^{\sharp}) = 0$.
For some small $\eta > 0$, we then have $Q_{+y^{\sharp},0} \prec \mathfrak{b}_n^{-3\eta}$ and $Q_{+y^{\sharp},\times \mathfrak{b}_n^{-\eta},\neq 1} \prec \mathfrak{b}_n^{-\eta} Q_{+y^{\sharp},\times \mathfrak{b}_n^{-\eta}}$.
Now apply the previous lemma $Q_{+y^{\sharp},\times \mathfrak{b}_n^{-\eta}} \uparrow_k$ and $(\mathrm{e}^x, \ldots, \exp_k x, \mathfrak{b}_1 \uparrow_k, \ldots, \mathfrak{b}_n \uparrow_k)$.
This yields $u \in \mathbb{R}[[x; \ldots; \exp_k x; \mathfrak{b}_1 \uparrow_k; \ldots; \mathfrak{b}_n \uparrow_k]]^{\prec 1}$ with $Q_{+y^{\sharp},\times \mathfrak{b}_n^{-\eta}} \uparrow_k (u) = 0$.
Then $y := y^{\sharp} + (u \downarrow_k)\, \mathfrak{b}_n^{-\eta}$ fulfills the requirements. □

Distinguished solutions

Theorem

Any quasi-linear equation $Q(y)=0, y \prec 1$ *has a solution in* $\mathbb{T}$.

Moreover, there exists a unique solution such that $y_{\mathfrak{d}(\tilde{y}-y)}=0$ *for any other solution* $\tilde{y}$*; this is called the* ***distinguished solution***.

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Proof. Let $\mathscr{Y}=\{y\in\mathbb{T}^{\prec 1}:Q(y)=0\}$. Previous lemma + upward shifting $\Longrightarrow$ $\mathscr{Y}\neq\emptyset$.

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Uniqueness. If $y,\tilde{y}\in\mathscr{Y}$ with $\tilde{y}\neq y$, then $(\tilde{y}-y)_{\mathfrak{d}(\tilde{y}-y)}\neq 0$, so $\tilde{y}_{\mathfrak{d}(y-\tilde{y})}\neq 0$ or $y_{\mathfrak{d}(\tilde{y}-y)}\neq 0$.

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Proof. Let $\mathscr{Y} = \{y \in \mathbb{T}^{\prec 1} : Q(y) = 0\}$. Previous lemma + upward shifting $\Longrightarrow$ $\mathscr{Y} \neq \emptyset$.

Existence. Let $y, \tilde{y} \in \mathscr{Y}$, $\tilde{y} \neq y$, $\mathfrak{v} := \mathfrak{d}_{\tilde{y}-y}$, and $LY := P_{+y,1}$.

Claim: $\mathfrak{v} \in \mathfrak{H}_L := \{\mathfrak{d}_h : h \in \mathbb{T}, Lh = 0\}$.

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We have $N(P_{+y,\times\mathfrak{v}})=N(P_{+y,\leqslant 1,\times\mathfrak{v}})\in\mathbb{R}^{\neq 0}+\mathbb{R}^{\neq 0}Y$.

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Solutions $u\preccurlyeq\mathfrak{v}$ of $P_{+y,\times\mathfrak{v}}(u)=0$ or $P_{+y,\leqslant 1,\times\mathfrak{v}}(u)=0$ have the same dominant term.

Hence $P_{+y,\times\mathfrak{v}}((\tilde{y}-y)\mathfrak{v})=0$ and $P_{+y,\leqslant 1,\times\mathfrak{v}}(\delta)=0$ has a solution $\delta\sim(\tilde{y}-y)/\mathfrak{v}$

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Repeating this $k\leqslant r$ times, we find $y,\tilde{y},\tilde{\tilde{y}},\ldots,y^{[k]}\in\mathscr{Y}$ with $\mathfrak{H}_{y^{[k]}}^*=\emptyset$. □

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Consider an asymptotic differential equation of Newton degree d

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Given a raveled equation $(\star)$ with $\operatorname{val} P < d$, there exist $\varphi \prec \mathfrak{v}$ and $\tilde{\mathfrak{v}} \prec \mathfrak{v}$ such that

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Sketch of proof. Arrange that $N(P) = (Y - c)^{d-i}(Y')^i$ for $c \in \mathbb{R}^{\neq}$, $1 \prec \mathfrak{v}$, and $i < d$.

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This yields $(\varphi, \tilde{\mathfrak{v}}), (\tilde{\varphi}, \tilde{\tilde{\mathfrak{v}}}), \ldots$ with $\tilde{\mathfrak{v}} / \mathfrak{v} \ggg \tilde{\tilde{\mathfrak{v}}} / \tilde{\mathfrak{v}} \ggg \cdots \Longrightarrow$ termination. $\square$

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Given $P \in \mathbb{T}\{Y\}$ *and* $\mathfrak{v} \in \mathfrak{T}$, *consider*

$$P(y) = 0, \qquad y \prec \mathfrak{v}. \tag{$\star$}$$

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Then $N := N(P_{\times \mathfrak{e}}) = Q(Y)(Y')^i$ for some $i < d$ and $Q(Y)$ is not a $(d-i)$-th power.

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Theorem IVP (intermediate value property)

Let $P \in \mathbb{T}\{Y\}$ *and* $f < g$ *in* $\mathbb{T}$ *be such that* $P(f)\,P(g) < 0$.
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$y(x) := \mathrm{e}^{x} + \mathrm{e}^{\sqrt{x}} + \mathrm{e}^{\sqrt{\sqrt{x}}} + \cdots$ is d-transcendental over $\mathbb{T}\langle\zeta\rangle := \mathbb{T}(\zeta, \zeta', \ldots)$.

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